Skew constacyclic codes over a non-chain ring $\mathbb{F}_q[u,v]/\langle f(u), g(v), uv - vu\rangle$

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Abstract

Let $f(u)$ and $g(v)$ be two polynomials of degree k and ℓ respectively, not both linear, which split into distinct linear factors over $\mathbb{F}_q$. Let $\mathcal{R} = \mathbb{F}_q[u,v]/\langle f(u), g(v), uv-vu\rangle$ be a finite commutative non-chain ring. In this paper, we study ψ-skew cyclic and θ_t-skew constacyclic codes over the ring $\mathcal{R}$ where ψ and θ_t are two automorphisms defined on $\mathcal{R}$.

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1 Introduction

Cyclic codes over finite fields have been studied since 1960's because of their algebraic structures as ideals in certain commutative rings. Interest in codes over finite rings increased substantially after a break-through work by Hammons et al. in 1994. In 2007, Boucher et al. [3] generalized the concept of cyclic code over a non-commutative ring, namely skew polynomial ring $\mathbb{F}_q[x;\theta]$, where $\mathbb{F}_q$ is a field with q elements and θ is an automorphism of $\mathbb{F}_q$. In the polynomial ring $\mathbb{F}_q[x;\theta]$, addition is defined as the usual one of polynomials and the multiplication is defined by the rule $ax^i * bx^j = a\theta^i(b)x^{i+j}$ for $a, b \in \mathbb{F}_q$. Boucher and Ulmer [4] constructed some θ-cyclic codes called skew cyclic codes with Hamming distance larger than that of previously known linear codes with the same parameters. Siap et al. [18] investigated structural properties of skew cyclic codes of arbitrary length.

After the first phase of study on skew cyclic codes over fields, the focus of attention moved to skew cyclic codes over rings. Abualrub et.al [1] studied skew cyclic codes over $\mathbb{F}_2 + v\mathbb{F}_2$, where $v^2 = v$ and the automorphism θ was taken as $\theta : v \to v + 1$. Li Jin [13] studied skew cyclic codes over $\mathbb{F}_p + v\mathbb{F}_p$, where $v^2 = 1$ with the automorphism θ taken as $\theta : a + bv \to a - bv$. In 2014, Gursoy

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et al. [10] determined generator polynomials and found idempotent generators of skew cyclic codes over $\mathbb{F}_q + v\mathbb{F}_q$, where $v^2 = v$ and the automorphism θ was defined as $\theta_t : a + bv \to a^{p^t} + b^{p^t} v$. Minjia Shi et al. [16] studied θ_t-skew-cyclic codes over $\mathbb{F}_q + v\mathbb{F}_q + v^2\mathbb{F}_q$, where $v^3 = v$. Later Minjia Shi et al. [17] extended these results to skew cyclic codes over $\mathbb{F}_q + v\mathbb{F}_q + \cdots + v^{m-1}\mathbb{F}_q$, where $v^m = v$. Gao et al. [5] studied skew constacyclic codes over $\mathbb{F}_q + v\mathbb{F}_q$, where $v^2 = v$.

Recently people have started studying skew cyclic codes over finite commutative non-chain rings having 2 or more variables. Yao, Shi and Solé [19] studied skew cyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$, where $u^2 = u, v^2 = v, uv = vu$ and q is a prime power. Ashraf and Mohammad [2] studied skew-cyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$, where $u^2 = u, v^2 = v, uv = vu = 0$. Islam and Prakash [11] studied skew cyclic and skew constacyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$, where $u^2 = u, v^2 = v$ and $uv = vu$. Islam, Verma and Prakash [12] studied skew constacyclic codes of arbitrary length over $\mathbb{F}_{p^m}[v, w]/ < v^2 - 1, w^2 - 1, vw - wv >$. In all these papers θ was taken as $\theta_t : a \to a^{p^t}$ defined on $\mathbb{F}_q$.

In this paper, we study skew cyclic and skew constacyclic codes over a more general ring. Let $f(u)$ and $g(v)$ be two polynomials of degree k and ℓ respectively, which split into distinct linear factors over $\mathbb{F}_q$. We assume that at least one of k and ℓ is ≥ 2. Let $\mathcal{R} = \mathbb{F}_q[u, v]/\langle f(u), g(v), uv - vu\rangle$ be a finite non-chain ring. Cyclic codes over this ring $\mathcal{R}$ were discussed in [8]. A Gray map is defined from $\mathcal{R}^n \to \mathbb{F}_q^{k\ell n}$ which preserves duality. We define two automorphisms ψ and θ_t on $\mathcal{R}$ and discuss ψ-skew cyclic and θ_t-skew α-constacyclic codes over this ring, where α is any unit in $\mathcal{R}$ fixed by the automorphism θ_t, in particular when $\alpha^2 = 1$. Some structural properties, specially generator polynomials and idempotent generators for skew constacyclic codes are determined. We shall show that a skew cyclic code over the ring $\mathcal{R}$ is either a quasi-cyclic code or a cyclic code over $\mathcal{R}$. Further we shall show that Gray image of a θ_t-skew α-constacyclic code of length n over $\mathcal{R}$ is a θ_t-skew α-quasi-twisted code of length $k\ell n$ over $\mathbb{F}_q$ of index $k\ell$. Some examples are also given to illustrate the theory.

In [15], Raka et al. had discussed α-constacyclic codes over the ring $\mathbb{F}_p[u]/\langle u^4 - u\rangle$, $p \equiv 1 (\text{mod } 3)$ for a specific unit $\alpha = (1-2u^3)$. (Note that the unit α here satisfies $\alpha^2 = 1$ in the ring $\mathbb{F}_p[u]/\langle u^4 - u\rangle$). On taking θ_t as identity automorphism, the results on θ_t-skew α-constacyclic codes (Section 4) give the corresponding results for α-constacyclic code over $\mathcal{R}$ which generalize the results of [15].

The results of this paper can easily be extended over the more general ring $\mathbb{F}_q[u_1, u_2, \cdots, u_r]/\langle f_1(u_1), f_2(u_2), \cdots f_r(u_r), u_i u_j - u_j u_i\rangle$ where polynomials $f_i(u_i)$, $1 \leq i \leq r$, split into distinct linear factors over $\mathbb{F}_q$.

The paper is organized as follows: In Section 2, we recall the ring $\mathcal{R} = \mathbb{F}_q[u, v]/\langle f(u), g(v), uv - vu\rangle$ and the Gray map $\Phi : \mathcal{R}^n \to \mathbb{F}_q^{k\ell n}$. In Section 3, we define two automorphisms ψ and θ_t on $\mathcal{R}$, while in Sections 3.1, we discuss skew cyclic codes over $\mathcal{R}$ with respect to ψ. In Section 4, we study skew α-constacyclic codes over the ring $\mathcal{R}$ with respect to the automorphism θ_t.

2 The ring $\mathcal{R}$ and the Gray map

2.1 The ring $\mathcal{R}$

Let q be a prime power, $q = p^s$. Throughout the paper, $\mathcal{R}$ denotes the commutative ring $\mathbb{F}_q[u,v]/\langle f(u), g(v), uv-vu\rangle$, where $f(u)$ and $g(v)$ are polynomials of degree k and ℓ respectively, which split into distinct linear factors over $\mathbb{F}_q$. We assume that at least one of k and ℓ is ≥ 2, otherwise $\mathcal{R} \simeq \mathbb{F}_q$. If $\ell = 1$ or $k = 1$, then the ring $\mathcal{R} = \mathbb{F}_q[u,v]/\langle f(u), g(v), uv - vu\rangle$ is isomorphic to $\mathbb{F}_q[u]/\langle f(u)\rangle$ or $\mathbb{F}_q[v]/\langle g(v)\rangle$. Duadic and triadic cyclic codes, duadic negacyclic codes over $\mathbb{F}_q[u]/\langle f(u)\rangle$ have been discussed by Goyal and Raka in [6, 7]. Further in [8, 9], Goyal and Raka have discussed polyadic cyclic codes and polyadic constacyclic codes over $\mathcal{R} = \mathbb{F}_q[u,v]/\langle f(u), g(v), uv - vu\rangle$.

Let $f(u) = (u-\alpha_1)(u-\alpha_2)...(u-\alpha_k)$, with $\alpha_i \in \mathbb{F}_q$, $\alpha_i \neq \alpha_j$ and $g(v) = (v-\beta_1)(v-\beta_2)...(v-\beta_\ell)$, with $\beta_i \in \mathbb{F}_q$, $\beta_i \neq \beta_j$. $\mathcal{R}$ is a non chain ring of size $q^{k\ell}$ and characteristic p.

For $k \geq 2$ and $\ell \geq 2$, let ϵ_i, $1 \leq i \leq k$ and γ_j, $1 \leq j \leq \ell$, be elements of the ring $\mathcal{R}$ given by

$$\begin{aligned} \epsilon_i = \epsilon_i(u) &= \tfrac{(u-\alpha_1)(u-\alpha_2)\cdots(u-\alpha_{i-1})(u-\alpha_{i+1})\cdots(u-\alpha_k)}{(\alpha_i-\alpha_1)(\alpha_i-\alpha_2)\cdots(\alpha_i-\alpha_{i-1})(\alpha_i-\alpha_{i+1})\cdots(\alpha_i-\alpha_k)} \quad \text{and} \\ \gamma_j = \gamma_j(v) &= \tfrac{(v-\beta_1)(v-\beta_2)\cdots(v-\beta_{j-1})(v-\beta_{j+1})\cdots(v-\beta_\ell)}{(\beta_j-\beta_1)(\beta_j-\beta_2)\cdots(\beta_j-\beta_{j-1})(\beta_j-\beta_{j+1})\cdots(\beta_j-\beta_\ell)}. \end{aligned} \tag{1}$$

If $k \leq 1$, we define $\epsilon_i = 1$ and if $\ell \leq 1$, we take $\gamma_j = 1$.

We note that $\epsilon_i^2 = \epsilon_i,\ \epsilon_i\epsilon_r = 0$ for $1 \leq i, r \leq k,\ i \neq r$ and $\sum_i \epsilon_i = 1$ modulo $f(u)$; $\gamma_j^2 = \gamma_j,\ \gamma_j\gamma_s = 0$ for $1 \leq j,\ s \leq \ell,\ j \neq s$ and $\sum_j \gamma_j = 1$ modulo $g(v)$ in $\mathcal{R}$.

For $i = 1, 2, \cdots, k, j = 1, 2, ..., \ell$, define η_{ij} as follows

$$\eta_{ij} = \eta_{ij}(u,v) = \epsilon_i(u)\gamma_j(v). \tag{2}$$

Lemma 1: We have $\eta_{ij}^2 = \eta_{ij},\ \eta_{ij}\eta_{rs} = 0$ for $1 \leq i, r \leq k, 1 \leq j, s \leq \ell, (i,j) \neq (r,s)$ and $\sum_{i,j} \eta_{ij} = 1$ in $\mathcal{R}$, i.e., η_{ij}'s are primitive orthogonal idempotents of the ring $\mathcal{R}$.

This is Lemma 2 of [8].

The decomposition theorem of ring theory tells us that $\mathcal{R} = \bigoplus\limits_{i,j} \eta_{ij}\mathcal{R}$.

For a linear code $\mathcal{C}$ of length n over the ring $\mathcal{R}$, let for each pair $(i,j), 1 \leq i \leq k, 1 \leq j \leq \ell$,

$$\mathcal{C}_{ij} = \{x_{ij} \in \mathbb{F}_q^n : \exists\ x_{rs} \in \mathbb{F}_q^n, (r,s) \neq (i,j),\ \text{such that} \sum_{r,s} \eta_{rs}x_{rs} \in \mathcal{C}\}.$$

Then $\mathcal{C}_{ij}$ are linear codes of length n over $\mathbb{F}_q$, $\mathcal{C} = \bigoplus\limits_{i,j} \eta_{ij}\mathcal{C}_{ij}$ and $|\mathcal{C}| = \prod\limits_{i,j} |\mathcal{C}_{ij}|$.

Theorem 1 *Let $\mathcal{C} = \bigoplus\limits_{i,j} \eta_{ij}\mathcal{C}_{ij}$ be a linear code of length n over $\mathcal{R}$. Then*

(i) $\mathcal{C}^\perp = \bigoplus\limits_{i,j} \eta_{ij}\mathcal{C}_{ij}^\perp$,

(ii) *$\mathcal{C}$ is self-dual if and only if $\mathcal{C}_{ij}$ are self-dual,*

(iii) $|\mathcal{C}^{\perp}| = \prod\limits_{i,j} |\mathcal{C}_{ij}^{\perp}|$.

Proof: Let $a = (a_0, a_1, \cdots, a_{n-1}) \in \mathcal{C}^{\perp}$. This gives $a \cdot b = 0$ for all $b = (b_0, b_1, \cdots, b_{n-1}) \in \mathcal{C}$. Let $a_r = \sum\limits_{i,j} \eta_{ij} a_{ijr}$ and $b_r = \sum\limits_{i,j} \eta_{ij} b_{ijr}$ for $0 \leq r \leq n-1$ where $a_{ijr},\ b_{ijr} \in \mathbb{F}_q$. Take $a_{ij} = (a_{ij0}, a_{ij1}, \cdots, a_{ij(n-1)})$ and $b_{ij} = (b_{ij0}, b_{ij1}, \cdots, b_{ij(n-1)})$ so that $a_{ij}, b_{ij} \in \mathbb{F}_q^n$ and $a = \sum\limits_{i,j} \eta_{ij} a_{ij}$, $b = \sum\limits_{i,j} \eta_{ij} b_{ij}$. As $b \in \mathcal{C}$, we find that $b_{ij} \in \mathcal{C}_{ij}$. Now $a \cdot b = 0$ implies

$$0 = (\sum \eta_{ij} a_{ij0})(\sum \eta_{ij} b_{ij0}) + (\sum \eta_{ij} a_{ij1})(\sum \eta_{ij} b_{ij1}) + \cdots + (\sum \eta_{ij} a_{ij(n-1)})(\sum \eta_{ij} b_{ij(n-1)})$$

which gives, using Lemma 1

$$\sum \eta_{ij} a_{ij0} b_{ij0} + \sum \eta_{ij} a_{ij1} b_{ij1} + \cdots + \sum \eta_{ij} a_{ij(n-1)} b_{ij(n-1)} = 0$$

i.e. $\sum \eta_{ij}(a_{ij} \cdot b_{ij}) = 0$. This implies $a_{ij} \cdot b_{ij} = 0$ for all i, j, where $b_{ij} \in \mathcal{C}_{ij}$. Therefore $a_{ij} \in \mathcal{C}_{ij}^{\perp}$. Hence $a \in \bigoplus\limits_{i,j} \eta_{ij} \mathcal{C}_{ij}^{\perp}$, so that $\mathcal{C}^{\perp} \subseteq \bigoplus\limits_{i,j} \eta_{ij} \mathcal{C}_{ij}^{\perp}$. The reverse inclusion can be obtained by reversing the above steps. This proves (i) and (ii), (iii) follow immediately from (i). $\square$

2.2 The Gray map

Every element $r(u, v)$ of the ring $\mathcal{R} = \mathbb{F}_q[u, v]/\langle f(u), g(v), uv - vu\rangle$ can be uniquely expressed as

$$r(u, v) = \sum_{i,j} \eta_{ij} a_{ij},$$

where $a_{ij} \in \mathbb{F}_q$ for $1 \leq i \leq k, 1 \leq j \leq \ell$.

Define a Gray map $\Phi : \mathcal{R} \to \mathbb{F}_q^{k\ell}$ by

$$r(u, v) = \sum_{i,j} \eta_{ij} a_{ij} \longmapsto (a_{11}, a_{12}, \cdots, a_{1\ell}, a_{21}, a_{22}, \cdots, a_{2\ell}, \cdots, a_{k1}, a_{k2}, \cdots, a_{k\ell}). \tag{3}$$

This map can be extended from $\mathcal{R}^n$ to $(\mathbb{F}_q^{k\ell})^n$ component wise i.e. for $r = (r_0, r_1, \cdots, r_{n-1})$, where $r_s = \eta_{11} a_{11}^{(s)} + \eta_{12} a_{12}^{(s)} + \cdots + \eta_{kl} a_{k\ell}^{(s)} \in \mathcal{R}$, define Φ as follows

$$\begin{aligned} \Phi(r_0, r_1, \cdots, r_{n-1}) &= \big(\Phi(r_0), \Phi(r_1), \cdots, \Phi(r_{n-1})\big) \\ &= \Big(a_{11}^{(0)}, a_{12}^{(0)}, \cdots, a_{k\ell}^{(0)}, a_{11}^{(1)}, a_{12}^{(1)}, \cdots, a_{k\ell}^{(1)}, a_{11}^{(n-1)}, \cdots, a_{k\ell}^{(n-1)}\Big). \end{aligned}$$

Let the Gray weight of an element $r \in \mathcal{R}$ be $w_G(r) = w_H(\Phi(r))$, the Hamming weight of $\Phi(r)$. The Gray weight of a codeword $c = (c_0, c_1, \cdots, c_{n-1}) \in \mathcal{R}^n$ is defined as $w_G(c) = \sum_{i=0}^{n-1} w_G(c_i) = \sum_{i=0}^{n-1} w_H(\Phi(c_i)) = w_H(\Phi(c))$. For any two elements $c_1, c_2 \in \mathcal{R}^n$, the Gray distance d_G is given by $d_G(c_1, c_2) = w_G(c_1 - c_2) = w_H(\Phi(c_1) - \Phi(c_2))$. The next theorem is a special case of a result of Goyal and Raka [8].

Theorem 2 *The Gray map Φ is an $\mathbb{F}_q$ - linear, one to one and onto map. It is also distance preserving map from ($\mathcal{R}^n$, Gray distance d_G) to ($\mathbb{F}_q^{k\ell n}$, Hamming distance d_H). Further $\Phi(\mathcal{C}^{\perp}) = (\Phi(\mathcal{C}))^{\perp}$ for any linear code $\mathcal{C}$ over $\mathcal{R}$.*

Sometimes it is more convenient to use a permuted version of the Gray map Φ on $\mathcal{R}^n$. For $r = (r_0, r_1, \cdots, r_{n-1})$, where $r_s = \eta_{11}a_{11}^{(s)} + \eta_{12}a_{12}^{(s)} + \cdots + \eta_{kl}a_{k\ell}^{(s)}$, define $\Phi_\pi : \mathcal{R}^n \to (\mathbb{F}_q^{k\ell})^n$ by

$$\begin{aligned}\Phi_\pi(r_0, r_1, \cdots, r_{n-1}) = \Big(a_{11}^{(0)}, a_{11}^{(1)}, \cdots, a_{11}^{(n-1)}, a_{12}^{(0)}, a_{12}^{(1)}, \cdots, a_{12}^{(n-1)}, \cdots,& \\ a_{k\ell}^{(0)}, a_{k\ell}^{(1)}, \cdots, a_{k\ell}^{(n-1)}\Big).&\end{aligned} \tag{4}$$

Clearly the Gray images $\Phi(\mathcal{C})$ and $\Phi_\pi(\mathcal{C})$ of a linear code $\mathcal{C}$ over $\mathcal{R}$ are equivalent codes.

3 Skew Cyclic codes over the ring $\mathcal{R}$

Let θ be an automorphism of $\mathcal{R}$. The map θ can be extended to $\mathcal{R}^n$ component wise i.e. for $c = (c_0, c_1, \cdots, c_{n-1})$,

$$\theta(c) = \big(\theta(c_0), \theta(c_1), \cdots, \theta(c_{n-1})\big). \tag{5}$$

Let $c = (c_0, c_1, \cdots, c_{n-1}) \in \mathcal{R}^n$. The cyclic shift of $\theta(c)$- called θ-cyclic shift or the skew cyclic shift is defined as

$$\sigma_\theta(c) = (\theta(c_{n-1}), \theta(c_0), \cdots, \theta(c_{n-2})). \tag{6}$$

Let c be divided into m equal parts of length r where $n = mr$, i.e.

$$c = \big(c_{0,0}, c_{0,1}, \cdots, c_{0,r-1}, c_{1,0}, \cdots, c_{1,r-1}, \cdots, c_{m-1,0}, \cdots, c_{m-1,r-1}\big).$$

Write $c = \big(c^{(0)}|c^{(1)}|\cdots|c^{(m-1)}\big)$. The skew quasi-cyclic shift of c of index m is defined as

$$\tau_{\theta,m}(c) = \big(\theta(c^{(m-1)})|\theta(c^{(0)})|\cdots|\theta(c^{(m-2)})\big). \tag{7}$$

A linear code $\mathcal{C}$ of length n over $\mathcal{R}$ is called a skew cyclic code if $\sigma_\theta(\mathcal{C}) = \mathcal{C}$ and a skew quasi-cyclic code of index m if $\tau_{\theta,m}(\mathcal{C}) = \mathcal{C}$.

The set $\mathcal{R}[x, \theta] = \{a_0 + a_1x + a_2x^2 + \cdots + a_sx^s : a_i \in \mathcal{R},\ s \geq 0 \text{ integer}\}$, where the variable x is written on the right of the coefficients, forms a ring under usual addition of polynomials and the multiplication is defined as $ax^i * bx^j = a\theta^i(b)x^{i+j}$ for $a, b \in \mathcal{R}$. The skew polynomial ring $\mathcal{R}[x, \theta]$ is non-commutative unless θ is the identity isomorphism. Let $\mathcal{R}_n = \mathcal{R}[x, \theta]/\langle x^n - 1\rangle$. $\mathcal{R}_n$ is a left $\mathcal{R}[x, \theta]$-module with usual addition and left multiplication defined as $r(x) * (f(x) + \langle x^n - 1\rangle) = r(x) * f(x) + \langle x^n - 1\rangle$ for $r(x) \in \mathcal{R}[x, \theta]$ and $f(x) + \langle x^n - 1\rangle \in \mathcal{R}_n$. In polynomial representation, a linear code of length n over $\mathcal{R}$ is a skew cyclic code if and only if it is a left $\mathcal{R}[x, \theta]$-submodule of $\mathcal{R}[x, \theta]/\langle x^n - 1\rangle$.

In polynomial representation, a skew quasi-cyclic code of length $n = mr$ and index m can be viewed as a left $\mathcal{R}[x, \theta]/\langle x^m - 1\rangle$-submodule of $\big(\mathcal{R}[x, \theta]/\langle x^m - 1\rangle\big)^r$ due to the one-to-one correspondence : $\mathcal{R}^{mr} \to \big(\mathcal{R}[x, \theta]/\langle x^m - 1\rangle\big)^r$ given by

$$\begin{aligned} c = \ & \big(c_{0,0}, c_{0,1}, \cdots, c_{0,r-1}, c_{1,0}, \cdots, c_{1,r-1}, \cdots, c_{m-1,0}, \cdots, c_{m-1,r-1}\big) \\ & \to \big(c_{0,0} + c_{1,0}x + \cdots + c_{m-1,0}x^{m-1}, c_{0,1} + c_{1,1}x + \cdots + c_{m-1,1}x^{m-1}, \\ & \cdots, c_{0,r-1} + c_{1,r-1}x + \cdots + c_{m-1,r-1}x^{m-1}\big) \end{aligned}$$

In this paper, we will consider the following two automorphisms on the ring $\mathcal{R} = \mathbb{F}_q[u, v]/\langle f(u), g(v), uv - vu\rangle$.

1. Without loss of generality, suppose that $\ell \geq 2$. For an $a \in \mathcal{R}$ $a = \sum_{i,j} \eta_{ij} a_{ij} = \sum_{j=1}^{\ell} \eta_{1j} a_{1j} + \sum_{j=1}^{\ell} \eta_{2j} a_{2j} + \cdots + \sum_{j=1}^{\ell} \eta_{kj} a_{kj}$, define

$$\begin{array}{rl} \psi(a) = & (\eta_{1\ell} a_{11} + \eta_{11} a_{12} \cdots + \eta_{1(\ell-1)} a_{1\ell}) + (\eta_{2\ell} a_{21} + \eta_{21} a_{22} \cdots + \eta_{2(\ell-1)} a_{2\ell}) \\ & + \cdots + (\eta_{k\ell} a_{k1} + \eta_{k1} a_{k2} \cdots + \eta_{k(\ell-1)} a_{k\ell}) \end{array}.$$

   Clearly the order of ψ is ℓ.

2. Let $q = p^s$ and t be an integer $1 \leq t \leq s$. Define an automorphism $\theta_t : \mathbb{F}_q \to \mathbb{F}_q$ given by $\theta_t(a) = a^{p^t}$ and extend it to $\theta_t : \mathcal{R} \to \mathcal{R}$ by

$$\theta_t\big(\sum_{i,j} \eta_{ij} a_{ij}\big) = \sum_{i,j} \eta_{ij} a_{ij}^{p^t}.$$

   Note that if $t = s$, θ_t is the identity map and this automorphism is irrelevant if q is a prime.

   Clearly the order of θ_t is $|\theta_t| = s/t$ and the ring $\mathbb{F}_{p^t}[u,v]/\langle f(u), g(v), uv - vu\rangle$ is invariant under θ_t.

3.1 ψ-skew Cyclic codes over the ring $\mathcal{R}$

In this subsection, we discuss skew cyclic codes over $\mathcal{R}$ with respect to automorphisms ψ.

Theorem 3 *The center $Z(\mathcal{R}[x,\psi])$ of $\mathcal{R}[x,\psi]$ is $\mathbb{F}_q[x^\ell]$.*

Proof : Since the order of ψ is ℓ, for any natural number i and $a \in \mathcal{R}$, we have $x^{\ell i} * a = (\psi^\ell)^i(a) x^{\ell i} = a * x^{\ell i}$; so $x^{\ell i}$ is in the center of $\mathcal{R}[x,\psi]$. As the fixed ring of $\mathcal{R}$ by ψ is $\mathbb{F}_q$, any $f \in \mathbb{F}_q[x^\ell]$ is a central element. Conversely for any $f \in Z(\mathcal{R}[x,\psi])$ and $a \in \mathcal{R}$, we have $x * f = f * x$ and $a * f = f * a$ which implies $f \in \mathbb{F}_q[x^\ell]$. $\square$

Corollary 1 *The polynomial $x^n - 1$ is in the center $Z(\mathcal{R}[x,\psi])$ if and only if ℓ divides n.*

Remark 1 If $\ell | n$, then $\mathcal{R}_n = \mathcal{R}[x,\psi]/\langle x^n - 1\rangle$ is a ring and a skew cyclic code $\mathcal{C}$ of length n over $\mathcal{R}$ is a left ideal in $\mathcal{R}_n$.

Theorem 4 *Let $\mathcal{C}$ be a skew cyclic code of length n. If $g(x)$ is a polynomial in $\mathcal{C}$ of minimal degree and leading coefficient of $g(x)$ is a unit in $\mathcal{R}$, then $\mathcal{C} = \langle g(x)\rangle$ where $g(x)$ is a right divisor of $x^n - 1$.*

Proof : Let $c(x) \in \mathcal{C}$. Write $c(x) = q(x)g(x) + r(x)$ where $q(x), r(x) \in \mathcal{R}[x,\psi]$ and deg $r(x)$ <deg $g(x)$. Since $\mathcal{C}$ is a left $\mathcal{R}[x,\psi]$-submodule, $r(x) = c(x) - q(x)g(x) \in \mathcal{C}$. Therefore we must have $r(x) = 0$ and so $\mathcal{C} = \langle g(x)\rangle$. Further if $x^n - 1 = q(x)g(x) + r(x)$ for some skew polynomials $q(x), r(x) \in \mathcal{R}[x,\psi]$ and deg $r(x)$ <deg $g(x)$, then $r(x) = (x^n - 1) - q(x)g(x) \in \mathcal{C}$ and so $r(x) = 0$. Therefore $g(x)$ is a right divisor of $x^n - 1$. $\square$

Theorem 5 *Let $\mathcal{C}$ be a skew cyclic code of length n over $\mathcal{R}$ and let $r = \gcd(n, |\psi|) = \gcd(n, \ell)$. If $r = 1$, then $\mathcal{C}$ is a cyclic code of length n over $\mathcal{R}$; if $r > 1$ then $\mathcal{C}$ is a quasi-cyclic code of index n/r.*

Proof: Let $n = mr$. Find integers a and $b > 0$ such that $a\ell = r + bn$. (As $\gcd(\ell, n) = r$, there exist integers a', b' such that $a'\ell + b'n = r$. If $b' < 0$, we are done. If $b' > 0$, find a positive integer t such that $\ell t - b' > 0$. Then $(a' + nt)\ell = r + (\ell t - b')n$.) Let
$c = \big(c_{0,0}, c_{0,1}, \cdots, c_{0,r-1}, c_{1,0}, \cdots, c_{1,r-1}, \cdots, c_{m-1,0}, \cdots, c_{m-1,r-1}\big)$
be a codeword in $\mathcal{C}$ divided into m equal parts of length r.

Write $c = \big(c^{(0)}|c^{(1)}|\cdots|c^{(m-1)}\big)$. Since $\mathcal{C}$ is a skew cyclic code, $\sigma_\psi(c), \sigma_\psi^2(c), \cdots, \sigma_\psi^r(c), \cdots$ all belong to $\mathcal{C}$. Since $r + bn$ is divisible by $\ell = |\psi|$, we have

$$
\begin{aligned}
\sigma_\psi^r(c) &= \big(\psi^r(c^{(m-1)})|\psi^r(c^{(0)})|\cdots|\psi^r(c^{(m-2)})\big)\\
\sigma_\psi^{2r}(c) &= \big(\psi^{2r}(c^{(m-2)})|\psi^{2r}(c^{(m-1)})|\cdots|\psi^{2r}(c^{(m-3)})\big)\\
\sigma_\psi^{n}(c) &= \big(\psi^{n}(c^{(0)})|\psi^{n}(c^{(1)})|\cdots|\psi^{n}(c^{(m-1)})\big)\\
\sigma_\psi^{bn}(c) &= \big(\psi^{bn}(c^{(0)})|\psi^{bn}(c^{(1)})|\cdots|\psi^{bn}(c^{(m-1)})\big)\\
\sigma_\psi^{r+bn}(c) &= \big(\psi^{r+bn}(c^{(m-1)})|\psi^{r+bn}(c^{(0)})|\cdots|\psi^{r+bn}(c^{(m-2)})\big)\\
&= \big(c^{(m-1)}|c^{(0)}|\cdots|c^{(m-2)}\big) = \tau_{\theta,m}(c),
\end{aligned}
$$

where θ is Identity automorphism. Therefore $\tau_{\theta,m}(c) \in \mathcal{C}$. If $r > 1$, $\mathcal{C}$ is a quasi-cyclic code of index m. If $r = 1$, i.e. $m = n$, then $\mathcal{C}$ is a cyclic code of length n over $\mathcal{R}$. $\square$

Remark 2 The above result holds for any automorphism θ on $\mathcal{R}$.

Example 1 : Let $\mathcal{R} = \mathbb{F}_4[u,v]/\langle u(u-1)(u-\alpha), v^2 - v, uv - vu\rangle$, where $\mathbb{F}_4 = \mathbb{F}_2[\alpha]$ and $\alpha^2 + \alpha + 1 = 0$. Here $\epsilon_1 = \frac{(u-1)(u-\alpha)}{\alpha}$, $\epsilon_2 = \frac{u(u-\alpha)}{1-\alpha}$, $\epsilon_3 = \frac{u(u-1)}{\alpha(\alpha-1)}$, $\gamma_1 = 1 - v$ and $\gamma_2 = v$. We have $\gamma_1 = 1 - v = (1 \cdot \eta_{11} + 0 \cdot \eta_{12}) + (1 \cdot \eta_{21} + 0 \cdot \eta_{22}) + (1 \cdot \eta_{31} + 0 \cdot \eta_{32})$. One finds that $\psi(1-v) = (0 \cdot \eta_{11} + 1 \cdot \eta_{12}) + (0 \cdot \eta_{21} + 1 \cdot \eta_{22}) + (0 \cdot \eta_{31} + 1 \cdot \eta_{32}) = \gamma_2 = v$ and $\psi(v) = 1 - v$. The order of ψ is 2. The polynomial $g(x) = x^6 + vx^5 + x^4 + x^3 + x^2 + (1-v)x + 1$ is a right divisor of $x^{12} - 1$ over the ring $\mathcal{R}[x, \psi]$, therefore it generates a skew cyclic code of length 12 over $\mathcal{R}$. By Theorem 5, this code is a quasi-cyclic code of index 6.

Example 2 : Let $\mathcal{R} = \mathbb{F}_8[u,v]/\langle u(u-1), v(v-1)(v-\beta)(v-\beta^2), uv - vu\rangle$, where $\mathbb{F}_8 = \mathbb{F}_2[\beta]$ and $\beta^3 + \beta + 1 = 0$. Here $\epsilon_1 = 1 - u$ and $\epsilon_2 = u$, $\gamma_1 = \frac{(v-1)(v-\beta)(v-\beta^2)}{\beta+1}$, $\gamma_2 = \frac{v(v-\beta)(v-\beta^2)}{\beta^2}$, $\gamma_3 = \frac{v(v-1)(v-\beta^2)}{\beta}$ and $\gamma_4 = \frac{v(v-1)(v-\beta)}{\beta^2+\beta+1}$. We have $\gamma_1 = (1\cdot\eta_{11} + 0\cdot\eta_{12} + 0\cdot\eta_{13} + 0\cdot\eta_{14}) + (1\cdot\eta_{21} + 0\cdot\eta_{22} + 0\cdot\eta_{23} + 0\cdot\eta_{24})$. One finds that $\psi(\gamma_1) = \gamma_2$, $\psi(\gamma_2) = \gamma_3$, $\psi(\gamma_3) = \gamma_4$, $\psi(\gamma_4) = \gamma_1$ and $\psi(\epsilon_i) = \epsilon_i$ for $i = 1, 2$. The order of ψ is 4. The polynomial $g(x) = x^4 + u(\gamma_1 + \gamma_3)x^3 + u(\gamma_1 + \gamma_3)x + 1$ is a right divisor of $x^8 - 1$ over the ring $\mathcal{R}[x, \psi]$, therefore it generates a skew cyclic code of length 8 over $\mathcal{R}$. By Theorem 5, this code is a quasi-cyclic code of index 2.

Example 3 : Let $\mathcal{R} = \mathbb{F}_5[u,v]/\langle u(u-1), v(v-1), uv - vu\rangle$ and $n = 9$. The polynomial $g(x) = x^6 + x^3 + 1$ generates a skew cyclic code of length 9 over $\mathcal{R}$. This code is equivalent to a cyclic code of length 9, by Theorem 5.

4 θ_t-skew constacyclic codes over the ring $\mathcal{R}$

In this section we will study θ_t-skew α-constacyclic code over $\mathcal{R}$, where α is a unit in $\mathcal{R}$ given by

$$\alpha = \sum_{i,j} \eta_{ij}\alpha_{ij}, \quad \alpha_{ij} \in \mathbb{F}_{p^t} \setminus \{0\}, \tag{8}$$

so that $\theta_t(\alpha_{ij}) = \alpha_{ij}$ and $\theta_t(\alpha) = \alpha$.

Note that $\alpha^2 = 1$ if and only if $\alpha_{ij}^2 = 1$, i.e. if and only if $\alpha_{ij} = \pm 1$.

In the special case when θ_t = identity map, we get all the corresponding results for α-constacyclic codes over $\mathcal{R}$. We shall call θ_t-skew constacyclic code simply as skew constacyclic code.

A linear code $\mathcal{C}$ of length n over $\mathcal{R}$ is said to be skew α-constacyclic code if $\mathcal{C}$ is invariant under the skew α-constacyclic shift ϑ_α, where $\vartheta_\alpha : \mathcal{R}^n \to \mathcal{R}^n$ is defined as

$$\vartheta_\alpha(c_0, c_1, \cdots, c_{n-1}) = \big(\alpha\theta_t(c_{n-1}), \theta_t(c_0), \cdots, \theta_t(c_{n-2})\big), \tag{9}$$

i.e., $\mathcal{C}$ is skew α-constacyclic code if and only if $\vartheta_\alpha(\mathcal{C}) = \mathcal{C}$. Clearly $\mathcal{C}$ is skew cyclic if $\alpha = 1$ and is called skew negacyclic if $\alpha = -1$.

By identifying each codeword by the corresponding polynomial, a linear code $\mathcal{C}$ of length n over $\mathcal{R}$ is skew α-constacyclic code if and only if is left $\mathcal{R}[x, \theta_t]$-submodule of left $\mathcal{R}[x, \theta_t]$-module $\mathcal{R}_{n,\alpha} = \mathcal{R}[x, \theta_t]/\langle x^n - \alpha\rangle$.

Theorem 6 *Let the unit α be as defined in (8). A linear code $\mathcal{C} = \bigoplus_{i,j} \eta_{ij}\mathcal{C}_{ij}$ is a skew α-constacyclic code of length n over $\mathcal{R}$ if and only if $\mathcal{C}_{ij}$ are skew α_{ij}-constacyclic code of length n over $\mathbb{F}_q$.*

Proof: Let $c = (c_0, c_1, \cdots, c_{n-1}) \in \mathcal{C}$, where $c_s = \sum_{i,j} \eta_{ij} a_{ij}^{(s)}$ for each $s, 0 \le s \le n-1$. Let $a_{ij} = (a_{ij}^{(0)}, a_{ij}^{(1)}, \cdots, a_{ij}^{(n-1)})$ so that $c = \sum_{i,j} \eta_{ij} a_{ij}$, $a_{ij} \in \mathcal{C}_{ij}$. Note that, using the properties of idempotents η_{ij} from Lemma 1

$$\alpha c_{n-1} = \Big(\sum_{i,j} \eta_{ij}\alpha_{ij}\Big)\Big(\sum_{i,j} \eta_{ij} a_{ij}^{(n-1)}\Big) = \sum_{i,j} \eta_{ij}\alpha_{ij} a_{ij}^{(n-1)}.$$

Therefore

$$\begin{aligned}
\vartheta_\alpha(c) &= (\alpha\theta_t(c_{n-1}), \theta_t(c_0), \cdots, \theta_t(c_{n-2})\\
&= (\theta_t(\alpha c_{n-1}), \theta_t(c_0), \cdots, \theta_t(c_{n-2})\\
&= \Big(\theta_t(\textstyle\sum_{i,j} \eta_{ij}\alpha_{ij} a_{ij}^{(n-1)}), \theta_t(\sum_{i,j} \eta_{ij} a_{ij}^{(0)}), \cdots, \theta_t(\sum_{i,j} \eta_{ij} a_{ij}^{(n-2)})\Big)\\
&= \Big(\textstyle\sum_{i,j} \eta_{ij}\alpha_{ij}\theta_t(a_{ij}^{(n-1)}), \sum_{i,j} \eta_{ij}\theta_t(a_{ij}^{(0)}), \cdots, \sum_{i,j} \eta_{ij}\theta_t(a_{ij}^{(n-2)})\Big)\\
&= \textstyle\sum_{i,j} \eta_{ij}\Big(\alpha_{ij}\theta_t(a_{ij}^{(n-1)}), \theta_t(a_{ij}^{(0)}), \cdots, \theta_t(a_{ij}^{(n-2)})\Big)\\
&= \textstyle\sum_{i,j} \eta_{ij}\vartheta_{\alpha_{ij}}(a_{ij}).
\end{aligned}$$

Therefore $\vartheta_\alpha(c) \in \mathcal{C}$ if and only if $\vartheta_{\alpha_{ij}}(a_{ij}) \in \mathcal{C}_{ij}$. $\square$

Example 4: Let $f(u) = u^4 - u = u(u-1)(u-\xi)(u-\xi^2)$, where $\xi \in \mathbb{F}_q$, $\xi^3 = 1$ and $q \equiv 1(\mod 3)$. Let $g(v) = v$ so that $\eta_{11} = 1 - u^3$, $\eta_{21} = \frac{1}{3}(u-\xi)(u-\xi^2)$, $\eta_{31} = \frac{1}{3}(u-1)(u-\xi^2)$ and $\eta_{41} = \frac{1}{3}(u-1)(u-\xi)$. Take $\theta_t =$ Identity automorphism and the unit $\alpha = \eta_{11} - \eta_{21} - \eta_{31} - \eta_{41} = 1 - 2u^3$. Then a linear code $\mathcal{C}$ is $(1-2u^3)$-constacyclic code over $\mathcal{R} = \mathbb{F}_q[u]/\langle u^4 - u\rangle$ if and only if $\mathcal{C}_{11}$ is cyclic and $\mathcal{C}_{21}$, $\mathcal{C}_{31}$, $\mathcal{C}_{41}$ are negacyclic codes of length n over $\mathbb{F}_q$. This is Theorem 2 of [15].

Following is Lemma 3.1 of Jitman et al. [14], where R was taken as a finite chain ring, but the result is true for any finite ring.

Lemma 2 Let C be a linear code of length n over a finite ring R. Let θ be an automorphism of R and suppose n is a multiple of the order of θ. Let λ be a unit in R such that $\theta(\lambda) = \lambda$. Then C is skew λ-constacyclic code if and only if $C^\perp$ is skew λ^{-1}-constacyclic code over R.

Theorem 7 *Let the order of θ_t divide n. If the code $\mathcal{C} = \bigoplus_{i,j} \eta_{ij}\mathcal{C}_{ij}$ is a skew α-constacyclic of length n over $\mathcal{R}$, then $\mathcal{C}^\perp$ is skew α^{-1}-constacyclic code over $\mathcal{R}$ and $\mathcal{C}_{ij}^\perp$ are α_{ij}^{-1}-constacyclic codes over $\mathbb{F}_q$, where α is as given in (8). Further for $\mathcal{C}$ to be self-dual it is necessary that $\alpha = \sum_{i,j}(\pm\eta_{ij})$, i.e. $\alpha^2 = 1$.*

Proof: The first statement follows from Lemma 2, as $\theta_t(\alpha) = \alpha$. Also by Theorem 1, we have $\mathcal{C}^\perp = \bigoplus_{i,j} \eta_{ij}\mathcal{C}_{ij}^\perp$, and $\alpha^{-1} = \sum_{i,j}\eta_{ij}\alpha_{ij}^{-1}$. Therefore $\mathcal{C}_{ij}^\perp$ are α_{ij}^{-1}-constacyclic codes over $\mathbb{F}_q$. Further $\mathcal{C}$ is self-dual if and only if $\mathcal{C}_{ij}$ are self-dual. Now for $\mathcal{C}_{ij}$ to be self dual it is necessary that $\alpha_{ij} = \alpha_{ij}^{-1}$ in $\mathbb{F}_q$ i.e. $\alpha_{ij} = \pm 1$. $\square$

Remark 3: It may happen that $\alpha = \sum_{i,j}(\pm\eta_{ij})$, i.e. $\alpha_{ij} = \pm 1$, but $\mathcal{C}_{ij}$ are not self-dual skew α_{ij}-constacyclic codes and so $\mathcal{C}$ may not be self-dual skew α-constacyclic code.

Corollary 2 *Let the order of θ_t divide n. Then the number of units α for which $\mathcal{C}$ can be self-dual skew α-constacyclic of length n over $\mathcal{R}$ is $2^{k\ell}$.*

Gao et al.[5] showed that a skew λ-constacyclic code of length n over $\mathbb{F}_q$ is generated by a monic polynomial $g(x)$ which is a right divisor of $x^n - \lambda$ in $\mathbb{F}_q[x;\theta_t]$. Analogous to this we have the following results for skew constacyclic codes over $\mathcal{R}$.

Theorem 8 *Let $\mathcal{C} = \bigoplus_{i,j} \eta_{ij}\mathcal{C}_{ij}$ be a skew α-constacyclic code of length n over $\mathcal{R}$. Suppose that skew α_{ij}-constacyclic codes $\mathcal{C}_{ij} = \langle g_{ij}(x)\rangle$, where $g_{ij}(x)$ are right divisors of $x^n - \alpha_{ij}$ for $1 \le i \le k, 1 \le j \le \ell$. Then there exists a polynomial $g(x)$ in $\mathcal{R}[x,\theta_t]$ such that*

(i) $\mathcal{C} = \langle g(x)\rangle$

(ii) $g(x)$ *is a right divisor of* $(x^n - \alpha)$.

(iii) $|\mathcal{C}| = q^{k\ell n - \sum_{j=1}^{\ell}\sum_{i=1}^{k} deg(g_{ij})}$.

Proof: First we show that $\mathcal{C} = \langle \eta_{11}g_{11}(x), \cdots, \eta_{1\ell}g_{1\ell}(x), \eta_{21}g_{21}(x), \cdots, \eta_{2\ell}g_{2\ell}(x), \cdots, \eta_{k1}g_{k1}(x), \cdots, \eta_{k\ell}g_{k\ell}(x)\rangle = \mathcal{E}$, say.
Let $c(x) \in \mathcal{C}$. Since $\mathcal{C}_{ij} = \langle g_{ij}\rangle$ and $\mathcal{C} = \bigoplus_{i,j} \eta_{ij}\mathcal{C}_{ij}$, we have $c(x) = \sum_{i,j} \eta_{ij}u_{ij}(x)g_{ij}(x)$ for $u_{ij}(x) \in \mathbb{F}_q[x;\theta_t]$. Therefore $c(x) \in \mathcal{E}$ and so $\mathcal{C} \subseteq \mathcal{E}$.

Conversely let $c(x) = \sum_{i,j} \eta_{ij}f_{ij}(x)g_{ij}(x) \in \mathcal{E}$, where $f_{ij}(x) \in \mathcal{R}[x;\theta_t]$. As $\mathcal{R} = \bigoplus_{r,s} \eta_{rs}\mathbb{F}_q$, each $f_{ij}(x) = \sum_{r,s} \eta_{rs}u_{rs}(x)$ for some $u_{rs}(x) \in \mathbb{F}_q[x;\theta_t]$. Now $\eta_{ij}f_{ij}(x) = \eta_{ij}u_{ij}(x)$ as η_{ij} are primitive orthogonal idempotents, we see find that $c(x) = \sum_{i,j} \eta_{ij}u_{ij}(x)g_{ij}(x) \in \bigoplus_{i,j} \eta_{ij}\langle g_{ij}(x)\rangle = \mathcal{C}$, hence $\mathcal{C} = \mathcal{E}$.

Let $g(x) = \sum_i \sum_j \eta_{ij}g_{ij}(x)$. Then clearly $\langle g(x)\rangle \subseteq \mathcal{E} = \mathcal{C}$. On the other hand $\eta_{ij}g(x) = \eta_{ij}g_{ij}(x)$, so $\mathcal{C} \subseteq \langle g(x)\rangle$.

Let for $1 \leq i \leq k, 1 \leq j \leq \ell$, $x^n - \alpha_{ij} = h_{ij}(x) * g_{ij}(x)$ for some $h_{ij}(x) \in \mathbb{F}_q[x;\theta_t]$. Let $h(x) = \sum_{i,j} \eta_{ij}h_{ij}(x)$, then one finds that $h(x) * g(x) = x^n - \alpha$ so $g(x)$ is a right divisor of $x^n - \alpha$.

Since $|\mathcal{C}| = \prod_{i,j}|\mathcal{C}_{ij}|$ and $|\mathcal{C}_{ij}| = q^{n-deg(g_{ij})}$ we get (iii). $\square$

Next we determine generator polynomial of dual of a skew α-constacyclic codes over $\mathcal{R}$, when the order of θ_t divide n. First we have

Lemma 3: Let order of θ_t divide n and $\mathcal{D} = \langle g(x)\rangle$ be a skew λ-constacyclic code of length n over $\mathbb{F}_q$ then the dual code $\mathcal{D}^\perp$ is a skew λ^{-1}-constacyclic code generated by $h^\perp(x) = h_{n-r} + \theta_t(h_{n-r-1})x + \cdots + \theta_t^{n-r}(h_0)x^{n-r}$, where $x^n - \lambda = h(x) * g(x)$ and $h(x) = \sum_{i=0}^{n-r} h_i x^i$.

The proof is similar to that of Corollary 18 of Boucher et al. [4], where generator of dual of a skew cyclic code was determined.

Theorem 9 *Let the order of θ_t divide n. Let $\mathcal{C} = \bigoplus_{i,j} \eta_{ij}\mathcal{C}_{ij}$ be a skew α-constacyclic code of length n over $\mathcal{R}$. Suppose $\mathcal{C}_{ij} = \langle g_{ij}(x)\rangle$, where $x^n - \alpha_{ij} = h_{ij}(x) * g_{ij}(x)$ for $1 \leq i \leq k, 1 \leq j \leq \ell$. Then*

(i) $\mathcal{C}^\perp = \langle h^\perp(x)\rangle$, *where* $h^\perp(x) = \sum_i \sum_j \eta_{ij}h^\perp_{ij}(x)$,

(ii) $|\mathcal{C}^\perp| = q^{\sum_{j=1}^{\ell}\sum_{i=1}^{k} deg(g_{ij})}$.

Proof : By Theorem 1, we have $\mathcal{C}^\perp = \bigoplus_{i,j} \eta_{ij}\mathcal{C}^\perp_{ij}$. Also, by Lemma 3, $\mathcal{C}^\perp_{ij} = \langle h^\perp_{ij}(x)\rangle$, we get $\mathcal{C}^\perp = \langle h^\perp(x)\rangle$, where $h^\perp(x) = \sum_i \sum_j \eta_{ij}h^\perp_{ij}$. (ii) follows because $|\mathcal{C}||\mathcal{C}^\perp| = q^{k\ell n}$. $\square$

Next we compute idempotent generator of the skew constacyclic code $\mathcal{C}$ over $\mathcal{R}$. First we have

Lemma 4 Let $\mathcal{D}$ be a θ_t-skew λ-constacyclic code of length n over $\mathbb{F}_q$. If $gcd(n,q) = 1$ and $gcd(n, |\theta_t|) = 1$, then there exists an idempotent polynomial $e(x) \in \mathbb{F}_q[x,\theta_t]/\langle x^n - \lambda\rangle$ such that $\mathcal{D} = \langle e(x)\rangle$.

The proof is similar to that of Theorem 6 of Gursoy et al. [10].

Theorem 10 *Let $\mathcal{C} = \bigoplus_{i,j} \eta_{ij}\mathcal{C}_{ij}$ be a θ_t-skew α-constacyclic code of length n over $\mathcal{R}$. If $gcd(n,q) = 1$ and $gcd(n, |\theta_t|) = 1$, then there exists an idempotent polynomial $e(x) \in \mathcal{R}[x,\theta_t]/\langle x^n - \alpha\rangle$ such that $\mathcal{C} = \langle e(x)\rangle$ and $\mathcal{C}^\perp = \langle 1 - e(x^{-1})\rangle$.*

Proof: By Lemma 4, let $e_{ij}(x) \in \mathbb{F}_q[x,\theta_t]/\langle x^n - \alpha_{ij}\rangle$ be idempotent generators of skew α_{ij}-constacyclic codes $\mathcal{C}_{ij}$. Take $e(x) = \sum_i \sum_j \eta_{ij} e_{ij}(x)$. Then $e(x)$ is an idempotent and also a generator of $\mathcal{C}$.

As $\mathcal{C}_{ij}^\perp$ have idempotent generators $1 - e_{ij}(x^{-1})$, $\mathcal{C}^\perp$ has idempotent generator $\sum_i \sum_j \eta_{ij}\big(1 - e_{ij}(x^{-1})\big) = (1 - e(x^{-1}))$. □

Let $c = (c_0, c_1, \cdots, c_{n-1}) = \big(c^{(0)}|c^{(1)}|\cdots|c^{(m-1)}\big)$ be a vector in $\mathcal{R}^n$ divided into m equal parts of length r where $n = mr$. We define two skew α-quasi twisted shifts $\varrho_{\alpha,m}$ and $\rho_{\alpha,m}$ as

$$\varrho_{\alpha,m}(c) = \big(\alpha\theta_t(c^{(m-1)})|\theta_t(c^{(0)})|\cdots|\theta_t(c^{(m-2)})\big). \tag{10}$$

$$\rho_{\alpha,m}(c) = \big(\vartheta_\alpha(c^{(0)})|\vartheta_\alpha(c^{(1)})|\cdots|\vartheta_\alpha(c^{(m-1)})\big), \tag{11}$$

where ϑ_α is as defined in (9).
A linear code C of length n over $\mathcal{R}$ is called a skew α-quasi twisted code of index m if $\varrho_{\alpha,m}(C) = C$ or $\rho_{\alpha,m}(C) = C$.

Theorem 11 *Let $\mathcal{C}$ be a skew α-constacyclic code of length n over $\mathcal{R}$ and let $r = \gcd(n, |\theta_t|)$. If $r = 1$, then $\mathcal{C}$ is α-constacyclic code of length n over $\mathcal{R}$; If $r > 1$ then $\mathcal{C}$ is a α- quasi-twisted code of index n/r.*

Proof: Let $n = mr$. Find integers a and $b > 0$ such that $a|\theta_t| = r + bn$. Let $c = (c_0, c_1, \cdots, c_{n-1}) = \big(c^{(0)}|c^{(1)}|\cdots|c^{(m-1)}\big)$ be a codeword in $\mathcal{C}$ divided into m equal parts of length r. Since $\mathcal{C}$ is a skew α-constacyclic code, $\vartheta_\alpha(c), \vartheta_\alpha^2(c), \cdots, \vartheta_\alpha^r(c), \cdots$ all belong to $\mathcal{C}$. Now

$$\begin{aligned}
\vartheta_\alpha^r(c) &= \big(\alpha\theta_t^r(c^{(m-1)})|\theta_t^r(c^{(0)})|\cdots|\theta_t^r(c^{(m-2)})\big)\\
\vartheta_\alpha^{2r}(c) &= \big(\alpha\theta_t^{2r}(c^{(m-1)})|\alpha\theta_t^{2r}(c^{(0)})|\cdots|\theta_t^{2r}(c^{(m-2)})\big)\\
\vartheta_\alpha^{n}(c) &= \big(\alpha\theta_t^{n}(c^{(0)})|\alpha\theta_t^{n}(c^{(1)})|\cdots|\alpha\theta_t^{n}(c^{(m-1)})\big)\\
\vartheta_\alpha^{bn}(c) &= \big(\alpha^b\theta_t^{bn}(c^{(0)})|\alpha^b\theta_t^{bn}(c^{(1)})|\cdots|\alpha^b\theta_t^{bn}(c^{(m-1)})\big)\\
\vartheta_\alpha^{r+bn}(c) &= \big(\alpha^{b+1}\theta_t^{r+bn}(c^{(m-1)})|\alpha^b\theta_t^{r+bn}(c^{(0)})|\cdots|\alpha^b\theta_t^{r+bn}(c^{(m-2)})\big)\\
&= \big(\alpha^{b+1}c^{(m-1)}|\alpha^b c^{(0)}|\cdots|\alpha^b c^{(m-2)}\big)\\
&= \alpha^b\big(\alpha c^{(m-1)}|c^{(0)}|\cdots|c^{(m-2)}\big) = \alpha^b\varrho_{\alpha,m}(c),
\end{aligned}$$

as $r + bn$ is a multiple of order of θ_t. Therefore $\varrho_{\alpha,m}(c) \in \mathcal{C}$, with θ_t =Identity automorphism. If $r > 1$, $\mathcal{C}$ is a α- quasi-twisted code of index m. If $r = 1$, i.e. $m = n$, then $\mathcal{C}$ is α-constacyclic code of length n over $\mathcal{R}$. □

Example 5 Consider the field $\mathbb{F}_9 = \mathbb{F}_3[\beta]$, where $\beta^2 + \beta - 1 = 0$ and θ be the Frobenius automorphism on $\mathbb{F}_9$ defined by $\theta(\beta) = \beta^3$. Let $f(u) = u^3 - u$, $g(v) = v^2 - 1$ and $\mathcal{R} = \mathbb{F}_9[u,v]/\langle u^3 - u, v^2 - 1, uv - vu\rangle$. Take $\alpha = 1 - u^2 - u^2 v$ a unit in $\mathcal{R}$. The polynomial $h(x) = x^6 + \alpha x^5 + x^4 + \alpha x^3 + x^2 + \alpha x + 1$ is a right divisor of $x^7 - \alpha$ in $\mathcal{R}[x,\theta]$. Also $\gcd(n, |\theta|) = \gcd(7,2) = 1$. Therefore $\mathcal{C} = \langle h(x)\rangle$ is a $(1 - u^2 - u^2 v)$-constacyclic code of length 7 over $\mathcal{R}$.

In fact if α is any unit in $\mathcal{R} = \mathbb{F}_{p^2}[u,v]/\langle f(u), g(v), uv - vu\rangle$ satisfying $\alpha^2 = 1$, i.e. $\alpha = \sum_{i,j}(\pm\eta_{ij})$ and n is odd then $x^n - \alpha = (x-\alpha)(x^{n-1} + \alpha x^{n-2} + x^{n-3} + \cdots + \alpha x^3 + x^2 + \alpha x + 1)$. Therefore the skew α-constacyclic code $\mathcal{C} = \langle x^{n-1} + \alpha x^{n-2} + x^{n-3} + \cdots + \alpha x^3 + x^2 + \alpha x + 1\rangle$ is a α-constacyclic code of length n over $\mathcal{R}$.

Example 6 Consider the field $\mathbb{F}_{25} = \mathbb{F}_5[\beta]$, where $\beta^2 - \beta + 2 = 0$ and θ be the Frobenius automorphism on $\mathbb{F}_{25}$ defined by $\theta(\beta) = \beta^5$. Let $f(u) = u^3 - u$, $g(v) = v^2 - v$ and $\mathcal{R} = \mathbb{F}_{25}[u,v]/\langle u^3 - u, v^2 - 1, uv - vu\rangle$. Now $x^6 - 1 = (x^2-1)(x^2-x+1)(x^2+x+1)$ and $x^6+1 = (x^2+1)(x^2+2x-1)(x^2+3x-1)$. Let $\alpha = \eta_{11} + \eta_{12} - \eta_{21} + \eta_{22} - \eta_{31} + \eta_{32} = 1 - 2u^2 + 2vu^2$, $g_{11} = g_{12} = g_{22} = g_{32} = x^2 + x + 1$ and $g_{21} = g_{31} = x^2 + 3x - 1$. Then $\mathcal{C} = \langle g(x)\rangle$, where $g(x) = \sum_i \sum_j \eta_{ij} g_{ij} = x^2 + (1+2u^2-2u^2v)x + (1-2u^2+2u^2v)$ is a skew $(1-2u^2+2vu^2)$-constacyclic code of length 6 over $\mathcal{R}$. Further as $\gcd(6, |\theta|) = 2$, $\mathcal{C}$ is a $(1-2u^2+2vu^2)$- quasi-twisted code of index 3.

Theorem 12 *Let ϑ_α be the skew α-constacyclic shift defined in (9), $\rho_{\alpha,k\ell}$ be the α-quasi twisted shift as defined in (11) and let Φ_π be the Gray map as defined in (4). Then $\Phi_\pi \sigma_\alpha = \rho_{\alpha,k\ell}\Phi_\pi$.*

Proof : Let $r = (r_0, r_1, \cdots, r_{n-1}) \in \mathcal{R}^n$, where $r_s = \eta_{11}a_{11}^{(s)} + \eta_{12}a_{12}^{(s)} + \cdots + \eta_{kl}a_{k\ell}^{(s)}$. Then

$$\begin{aligned}\vartheta_\alpha(r) &= \big(\alpha\theta_t(r_{n-1}), \theta_t(r_0), \cdots, \theta_t(r_{n-2})\big)\\ &= \Big(\alpha\textstyle\sum_{ij}\eta_{ij}\theta_t(a_{ij}^{(n-1)}), \sum_{ij}\eta_{ij}\theta_t(a_{ij}^{(0)}), \cdots, \sum_{ij}\eta_{ij}\theta_t(a_{ij}^{(n-2)})\Big)\end{aligned}.$$

Applying Φ_π, we get

$$\begin{aligned}\Phi_\pi(\vartheta_\alpha(r)) = \;&\Big(\alpha\theta_t(a_{11}^{(n-1)}), \theta_t(a_{11}^{(0)}), \cdots, \theta_t(a_{11}^{(n-2)}), \alpha\theta_t(a_{12}^{(n-1)}), \theta_t(a_{12}^{(0)}), \cdots, \theta_t(a_{12}^{(n-2)}),\\ &\cdots, \alpha\theta_t(a_{k\ell}^{(n-1)}), \theta_t(a_{k\ell}^{(0)}), \cdots, \theta_t(a_{k\ell}^{(n-2)})\Big)\end{aligned}$$

On the other hand

$$\begin{aligned}\Phi_\pi(r_0, r_1, \cdots, r_{n-1}) = \;&\Big(a_{11}^{(0)}, a_{11}^{(1)}, \cdots, a_{11}^{(n-1)} | a_{12}^{(0)}, a_{12}^{(1)}, \cdots, a_{12}^{(n-1)} |\\ &\cdots | a_{k\ell}^{(0)}, a_{k\ell}^{(1)}, \cdots, a_{k\ell}^{(n-1)}\Big).\end{aligned}$$

Therefore

$$\begin{aligned}\rho_{\alpha,k\ell}(\Phi_\pi(r)) = \;&\Big(\alpha\theta_t(a_{11}^{(n-1)}), \theta_t(a_{11}^{(0)}), \cdots, \theta_t(a_{11}^{(n-2)}) | \alpha\theta_t(a_{12}^{(n-1)}), \theta_t(a_{12}^{(0)}), \cdots, \theta_t(a_{12}^{(n-2)})|\\ &\cdots | \alpha\theta_t(a_{k\ell}^{(n-1)}), \theta_t(a_{k\ell}^{(0)}), \cdots, \theta_t(a_{k\ell}^{(n-2)})\Big)\end{aligned}$$

Hence $\Phi_\pi\vartheta_\alpha = \rho_{\alpha,k\ell}\Phi_\pi$. □

Theorem 13 *A linear code $\mathcal{C}$ of length n over $\mathcal{R}$ is a skew α-constacyclic code if and only if $\Phi_\pi(\mathcal{C})$ is a skew α-quasi-twisted code of length $k\ell n$ over $\mathbb{F}_q$ of index $k\ell$.*

Proof : From Theorem 12, we see that

$$\Phi_\pi(\vartheta_\alpha(\mathcal{C})) = \Phi_\pi\sigma_\alpha(\mathcal{C}) = \rho_{\alpha,k\ell}\Phi_\pi(\mathcal{C}) = \rho_{\alpha,k\ell}(\Phi_\pi(\mathcal{C})).$$

Therefore $\vartheta_\alpha(\mathcal{C}) = \mathcal{C}$ if and only if $\Phi_\pi(\mathcal{C}) = \rho_{\alpha,k\ell}(\Phi_\pi(\mathcal{C}))$. □

Corollary 3 *If a linear code $\mathcal{C}$ of length n over $\mathcal{R}$ is a skew α-constacyclic (a skew cyclic) then $\Phi(\mathcal{C})$ is equivalent to a skew α-quasi-twisted (a skew quasi-cyclic) code of length $k\ell n$ over $\mathbb{F}_q$ of index $k\ell$.*

Example 7 Let $f(u) = u^2 - u$, $g(v) = v(v-1)(v-\beta)$ be polynomials over $\mathbb{F}_4 = \mathbb{F}_2[\beta]$, where $\beta^2+\beta+1 = 0$. Take $\mathcal{R} = \eta_{11}\mathbb{F}_4 \oplus \eta_{12}\mathbb{F}_4 \oplus \eta_{13}\mathbb{F}_4 \oplus \eta_{21}\mathbb{F}_4 \oplus \eta_{22}\mathbb{F}_4 \oplus \eta_{23}\mathbb{F}_4$. Let θ be the Frobenius automorphism on $\mathbb{F}_4$ defined by $\theta(\beta) = \beta^2$. A decomposition of x^6-1 in the skew polynomial ring $\mathbb{F}_4[x,\theta]$ is

$$\begin{aligned} x^6 - 1 &= (x^2-1)(x^4+x^2+1) \\ &= (x^2-\beta)(x^4+\beta x^2+\beta^2) \\ &= (x^2-\beta^2)(x^4+\beta^2x^2+\beta) \\ &= (x^3+\beta x^2+\beta^2 x-\beta^2)(x^3+\beta^2x^2+\beta^2x+\beta). \end{aligned}$$

If we take $\mathcal{C}_{ij} = \langle x^3+\beta^2x^2+\beta^2x+\beta\rangle$ for $i = 1,2$ and $j = 1,2,3$, then $\mathcal{C} = \oplus\eta_{ij}\mathcal{C}_{ij} = \langle x^3+\beta^2x^2+\beta^2x+\beta\rangle$ is a skew cyclic code over $\mathcal{R}$ of length 6. Its Gray image $\Phi(\mathcal{C})$ is a quasi-cyclic code of index 6 with parameters $[36,18,4]$.

If we take $\mathcal{C}_{11} = \mathcal{C}_{13} = \langle x^4+x^2+1\rangle$, $\mathcal{C}_{12} = \mathcal{C}_{21} = \langle x^4+\alpha x^2+\alpha^2\rangle$ and $\mathcal{C}_{22} = \mathcal{C}_{23} = \langle x^4+\beta^2x^2+\beta\rangle$, then $\mathcal{C} = \oplus\eta_{ij}\mathcal{C}_{ij} = \langle x^4+(\beta uv^2+v^2+\beta v+\beta^2u+1)x^2+(\beta^2uv+\beta^2v^2+v+\beta u+1)$ is a skew cyclic code over $\mathcal{R}$. Its Gray image $\Phi(\mathcal{C})$ is a quasi-cyclic code of index 6 with parameters $[36,12,3]$.

Theorem 14 *If n is odd and the unit α satisfies $\alpha^2 = 1$, then a skew α-constacyclic code of length n over $\mathcal{R}$ is equivalent to a skew cyclic code over $\mathcal{R}$.*

Proof: Define a map $\varphi : \mathcal{R}_n = \mathcal{R}[x,\theta_t]/\langle x^n-1\rangle \to \mathcal{R}_{n,\alpha} = \mathcal{R}[x,\theta_t]/\langle x^n-\alpha\rangle$ by $\varphi(f(x)) = f(\alpha x)$. Then φ is $\mathcal{R}[x,\theta_t]$-module isomorphism because

$$\begin{aligned} & f(x) &&= g(x) \text{ in } \mathcal{R}[x,\theta_t]/\langle x^n-1\rangle \\ \Leftrightarrow\quad & f(x)-g(x) &&= h(x)*(x^n-1) \text{ for some } h(x)\in\mathcal{R}[x,\theta_t] \\ \Leftrightarrow\quad & f(\alpha x)-g(\alpha x) &&= h(\alpha x)*(\alpha^n x^n-1) \\ & &&= h(\alpha x)*(\alpha x^n-1) \text{ as } \alpha^n=\alpha \text{ for } n \text{ odd} \\ & &&= \alpha h(\alpha x)*(x^n-\alpha) \text{ as } \alpha^2=1 \text{ and } \theta_t(\alpha)=\alpha \\ \Leftrightarrow\quad & f(\alpha x) &&= g(\alpha x) \text{ in } \mathcal{R}[x,\theta_t]/\langle x^n-\alpha\rangle. \end{aligned}$$

This gives the result. □

Theorem 15 *Let $k\ell \equiv 1(\text{mod } s/t)$. Then for any $r\in\mathcal{R}^n$, $\Phi\sigma_{\theta_t}(r) = \sigma_{\theta_t}^{k\ell}\Phi(r)$.*

Proof : Let $r = (r_0, r_1, \cdots, r_{n-1}) \in \mathcal{R}^n$, where $r_s = \sum_{ij}\eta_{ij}a_{ij}^{(s)}$. Then

$$\begin{aligned} \Phi(\sigma_{\theta_t}(r)) &= \Phi\big(\theta_t(r_{n-1}),\theta_t(r_0),\cdots,\theta_t(r_{n-2})\big) \\ &= \Phi\Big(\sum_{ij}\eta_{ij}\theta_t(a_{ij}^{(n-1)}),\sum_{ij}\eta_{ij}\theta_t(a_{ij}^{(0)}),\cdots,\sum_{ij}\eta_{ij}\theta_t(a_{ij}^{(n-2)})\Big) \\ &= \Big(\Phi\big(\sum_{ij}\eta_{ij}\theta_t(a_{ij}^{(n-1)})\big),\Phi\big(\sum_{ij}\eta_{ij}\theta_t(a_{ij}^{(0)})\big),\cdots,\Phi\big(\sum_{ij}\eta_{ij}\theta_t(a_{ij}^{(n-2)})\big)\Big) \\ &= \Big(\theta_t(a_{11}^{(n-1)}),\theta_t(a_{12}^{(n-1)}),\cdots,\theta_t(a_{k\ell}^{(n-1)}),\theta_t(a_{11}^{(0)}),\theta_t(a_{12}^{(0)})\cdots,\theta_t(a_{k\ell}^{(0)}), \\ &\quad \cdots,\theta_t(a_{11}^{(n-2)}),\theta_t(a_{12}^{(n-2)}),\cdots,\theta_t(a_{k\ell}^{(n-2)})\Big). \end{aligned}$$

On the other hand,

$$\begin{aligned}
\sigma_{\theta_t}(\Phi(r)) &= \sigma_{\theta_t}\big(\Phi(r_0), \Phi(r_1), \cdots, \Phi(r_{n-1})\big)\\
&= \sigma_{\theta_t}\big(a_{11}^{(0)}, a_{12}^{(0)}, \cdots, a_{k\ell}^{(0)}, a_{11}^{(1)}, a_{12}^{(1)}, \cdots, a_{k\ell}^{(1)}, \cdots, a_{11}^{(n-1)}, \cdots, a_{k\ell}^{(n-1)}\big)\\
&= \big(\theta_t(a_{k\ell}^{(n-1)}), \theta_t(a_{11}^{(0)}), \theta_t(a_{12}^{(0)}), \cdots, \theta_t(a_{k\ell}^{(0)}), \theta_t(a_{11}^{(1)}), \cdots, \theta_t(a_{k\ell}^{(1)}),\\
&\qquad \cdots \theta_t(a_{11}^{(n-1)}), \cdots, \theta_t(a_{k\ell-1}^{(n-1)})\big),\\
\sigma_{\theta_t}^2(\Phi(r)) &= \big(\theta_t^2(a_{k\ell-1}^{(n-1)}), \theta_t^2(a_{k\ell}^{(n-1)}), \theta_t^2(a_{11}^{(0)}), \theta_t^2(a_{12}^{(0)}), \cdots, \theta_t^2(a_{k\ell}^{(0)}), \theta_t^2(a_{11}^{(1)}), \cdots,\\
&\qquad \theta_t^2(a_{k\ell}^{(1)}), \cdots, \theta_t^2(a_{11}^{(n-1)}), \cdots, \theta_t^2(a_{k\ell-2}^{(n-1)})\big),\\
&\cdots \qquad \cdots \qquad \cdots\\
\sigma_{\theta_t}^{k\ell}(\Phi(r)) &= \big(\theta_t^{k\ell}(a_{11}^{(n-1)}), \theta_t^{k\ell}(a_{12}^{(n-1)}), \cdots, \theta_t^{k\ell}(a_{k\ell}^{(n-1)}), \theta_t^{k\ell}(a_{11}^{(0)}), \theta_t^{k\ell}(a_{12}^{(0)}), \cdots,\\
&\qquad \theta_t^{k\ell}(a_{k\ell}^{(0)}), \theta_t^{k\ell}(a_{11}^{(1)}), \cdots, \theta_t^{k\ell}(a_{k\ell}^{(1)}), \cdots, \theta_t^{k\ell}(a_{11}^{(n-2)}), \cdots, \theta_t^{k\ell}(a_{k\ell}^{(n-2)})\big).
\end{aligned}$$

Since here $\theta_t^{k\ell} = \theta_t$, we find that $\Phi\sigma_{\theta_t}(r) = \sigma_{\theta_t}^{k\ell}\Phi(r)$. □

Corollary 4 *If $k\ell \equiv 1(\text{mod } |\theta_t|)$, then $\mathcal{C}$ is a skew cyclic code if and only if $\Phi(\mathcal{C})$ is fixed by $\sigma_{\theta_t}^{k\ell}$ skew cyclic shift.*

5 Conclusion

Let $\mathcal{R} = \mathbb{F}_q[u,v]/\langle f(u), g(v), uv - vu\rangle$ be a finite non-chain ring where $f(u)$ and $g(v)$ are two polynomials of degree k and ℓ respectively, which split into distinct linear factors over $\mathbb{F}_q$. We assume that at least one of k and ℓ is ≥ 2. In this paper, we define two automorphisms ψ and θ_t on $\mathcal{R}$ and discuss ψ-skew cyclic and θ_t-skew α-constacyclic codes over $\mathcal{R}$, where α is any unit in $\mathcal{R}$ fixed by the automorphism θ_t, in particular when $\alpha^2 = 1$. We show that a skew α-constacyclic code of length n over $\mathcal{R}$ is either an α-constacyclic code or a α-quasi-twisted code. Some structural properties, specially generator polynomials and idempotent generators for skew constacyclic codes are determined. A Gray map is defined from $\mathcal{R}^n \to \mathbb{F}_q^{k\ell n}$ which preserves duality. It is shown that Gray image of a θ_t-skew α-constacyclic code of length n over $\mathcal{R}$ is a θ_t-skew α-quasi-twisted code of length $k\ell n$ over $\mathbb{F}_q$ of index $k\ell$. Some examples are also given to illustrate the theory.

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References

[1] T. Abualrub and P. Seneviratne, *Skew codes over rings*, in Proc. IMECS, Hong Kong, 2010.

[2] M. Ashraf and G. Mohammad, *Skew-cyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$*, Asian-European Journal of Mathematics, doi.org/10.1142/S1793557118500729.

[3] D. Boucher, W.Geiselmann and F. Ulmer, *Skew cyclic codes*, Appl.Algebra Engrg. Comm. Comput. 18 (4) (2007), 379-389.

[4] D. Boucher and F. Ulmer, *Coding wiyh skew polynomial ring*, J. Symb. Comput. 44 (12) (2009), 1644-1656.

[5] J. Gao, F. Ma and F. Fu Skew constacyclic codes over the ring $\mathbb{F}_q + v\mathbb{F}_q$, Appl. Comput. Math. 6 (3), (2017) 286-295.

[6] M. Goyal and M. Raka, *Duadic and triadic codes over a finite non-chain ring and their Gray images* Int. J. Information and Coding Theory, 5 (1)(2018), 36 -54.

[7] M. Goyal and M. Raka, *Duadic negacyclic codes over a non-chain ring,* Discrete Math. Algorithm Appl., 10 (6) (2018) 1850080, doi.org/10.1142/S1793830918500805.

[8] M. Goyal and M. Raka, *Polyadic cyclic codes over the non-chain ring* $\mathbb{F}_q[u,v]/\langle f(u), g(v), uv - vu\rangle$, submitted for publication, arXiv:1811.01583v1 [cs.IT].

[9] M. Goyal and M. Raka, *Polyadic constacyclic codes over the non-chain ring* $\mathbb{F}_q[u,v]/\langle f(u), g(v), uv - vu\rangle$, under preparation.

[10] F. Gursoy, I. Siap and B. Yildiz, *Construction of skew cyclic codes over* $\mathbb{F}_q + v\mathbb{F}_q$, Adv. Math. Commun. 8 (3),(2014), 313-322.

[11] H. Islam and O. Parkash, *Skew cyclic codes and skew* $(\alpha_1 + u\alpha_2 + v\alpha_3 + uv\alpha_4)$*-constacyclic codes over* $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$, Int. J. Information and Coding Theory, 5 (2), 2018, 101-116.

[12] H. Islam, R. K. Verma and O. Parkash, *A family of constacyclic codes over* $\mathbb{F}_{p^m}[v,w]/\langle v^2-1, w^2-1, vw-wv\rangle$, Int. J. Information and Coding Theory, in press.

[13] Li Jin, *Skew cyclic codes over ring* $\mathbb{F}_p+v\mathbb{F}_p$, Journal of Electronics (China), 31 (3), (2014), 227-231.

[14] S. Jitman, S. Ling and P. Udomkavanich, *Some constacyclic codes over finite chain rings*, Adv. Math. Commun. 6 (1),(2012), 29-63.

[15] M. Raka, L. Kathuria and M. Goyal, $(1-2u^3)$*-constacyclic codes and quadratic residue codes over* $\mathbb{F}_p[u]/\langle u^4 - u\rangle$, Cryptogr. Commun., 9(4) (2017), 459-473.

[16] M. Shi, T. Yao, A. Alahmadi and P. Solé, *Skew Cyclic Codes over* $\mathbb{F}_q+v\mathbb{F}_q+v^2\mathbb{F}_q$, IEICE Transactions on Fundamentals of Electronics Communications and Computer Sciences, E98.A (8)(2015), 1845-1848

[17] M. Shi, T. Yao and P. Solé, *Skew cyclic codes over a non-chain ring,* Chinese ournal of Electronics, 26,(3) (2017), 544-547.

[18] I. Siap, T. Abualrub, N. Aydin and P. Seneviratne, *Skew cyclic codes of arbitary length*, Int. J. Inform. Coding Theory, 2 (1)(2011), 10 -20.

[19] T.Yao, M.Shi, and P. Solé, *Skew cyclic codes over* $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$, J. Algebra Comb. Discrete Appl., 2(3) (2015), 163-168.