On the Index of the Diffie-Hellman Mapping

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Abstract

Let γ be a generator of a cyclic group G of order n. The least index of a self-mapping f of G is the index of the largest subgroup U of G such that $f(x)x^{-r}$ is constant on each coset of U for some positive integer r. We determine the index of the univariate Diffie-Hellman mapping $d(\gamma^a) = \gamma^{a^2}$, $a = 0, 1, \ldots, n-1$, and show that any mapping of small index coincides with d only on a small subset of G. Moreover, we prove similar results for the bivariate Diffie-Hellman mapping $D(\gamma^a, \gamma^b) = \gamma^{ab}$, $a, b = 0, 1, \ldots, n-1$. In the special case that G is a subgroup of the multiplicative group of a finite field we present improvements.

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1 Introduction

Let G be a (multiplicatively written) finite cyclic group of order $n \geq 2$, γ be a generator of G and ℓ be a positive divisor of n. Then the set of nonzero ℓ th powers

$$C_{\ell,0} = \left\{ \gamma^{j\ell} : j = 0, 1, ..., \frac{n}{\ell} - 1 \right\}$$

is a subgroup of G of index ℓ . The elements of the factor group G/C_0 are the cyclotomic cosets

$$C_{\ell,i} = \gamma^i C_{\ell,0}, \quad i = 0, 1, ..., \ell - 1.$$

For any positive integer r and any $a_0, a_1, ..., a_{\ell-1} \in G$, we define the r-th order cyclotomic mapping $f_{a_0, a_1, ..., a_{\ell-1}}^r$ of index ℓ by

$$f_{a_0,a_1,\dots,a_{\ell-1}}^r(x) = a_i x^r \quad \text{if } x \in C_{\ell,i}, \quad i = 0, 1, \dots, \ell - 1.$$
 (1)

For a self-mapping f of G we denote by ind(f) the smallest index ℓ such that f can be represented by a mapping of the form (1).

Any self-mapping of the multiplicative group \mathbb{F}_q^* of a finite field can be uniquely represented by a polynomial over \mathbb{F}_q of degree at most q-1 with f(0)=0. The index of any polynomial over \mathbb{F}_q (with constant term 0) introduced in [1, 19] (which was based on [16]) coincides with our definition. In this special case the index has raised increasing interest, see for example [7], the survey article [20] and references therein. In particular, any mapping of small index is highly predictable and a large index is needed for cryptographic functions.

The security of the Diffie-Hellman key exchange, see for example [17, Chapter 2], for the group G is based on the infeasibility of evaluating the *(bivariate) Diffie-Hellman mapping D*,

$$D(\gamma^a, \gamma^b) = \gamma^{ab}, \quad a, b = 0, \dots, n - 1.$$

The bivariate Diffie-Hellman mapping can be efficiently reduced to the *univariate Diffie-Hellman mapping*,

$$d(\gamma^a) = \gamma^{a^2}, \quad a = 0, \dots, n - 1, \tag{3}$$

since

$$D(\gamma^{a}, \gamma^{b})^{2} = d(\gamma^{a+b})d(\gamma^{a})^{-1}d(\gamma^{b})^{-1}$$

and square roots in G can be calculated efficiently using the *Tonelli-Shanks algorithm*, see for example [2, Chapter 7].

In practice, subgroups of the multiplicative group of a finite field and elliptic curves over finite fields are mainly used. For these groups many results on polynomials representing and interpolating the univariate and bivariate Diffie-Hellman mapping have been obtained, in particular, lower bounds on degree and sparsity, see [3, 4, 5, 6, 8, 9, 12, 13, 14, 21] and the monograph [18].

In this paper, we first study the index of the univariate Diffie-Hellman mapping for a generic cyclic group of order n in Section 2. We show that

ind(d) is n for odd n and n/2 for even n as well as that each mapping of small index coincides with d only on a small subset of G.

In Section 3 we introduce the index pair of a bivariate function over G and obtain similar results for the bivariate Diffie-Hellman mapping, as well. For $G = \mathbb{F}_q^*$ and k-variate polynomials the index k-tuple has already been defined in [15].

In the special case that G is a subgroup of the multiplicative subgroup \mathbb{F}_q^* of the finite field \mathbb{F}_q we obtain some improvements in Section 4.

We will use the notation

$$f(n) = O(g(n))$$
 if $|f(n)| \le cg(n)$

for some constant c > 0 and

$$f(n) = o(g(n))$$
 if $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$.

 $f(n) \ll g(n)$ and $g(n) \gg f(n)$ are both equivalent to f(n) = O(g(n)).

2 Index of the univariate Diffie-Hellman mapping

First we determine the index ind(d) of the univariate Diffie-Hellman mapping.

Theorem 1. Let G be any cyclic group of order n with generator γ . Then the index of the univariate Diffie-Hellman mapping d of G defined by (3) is

$$ind(d) = \begin{cases} n, & n \text{ is odd,} \\ n/2, & n \text{ is even.} \end{cases}$$

Proof. Let ℓ denote the index of d, that is,

$$d = f_{a_0,\dots,a_{\ell-1}}^r$$

for some positive integer r and $a_0, \ldots, a_{\ell-1} \in G$, where $f_{a_0, \ldots, a_{\ell-1}}^r$ is defined by (1). Then we have

$$d(\gamma^{j\ell+i}) = a_i \gamma^{r(j\ell+i)} = \gamma^{(j\ell+i)^2}, \quad j = 0, \dots, \frac{n}{\ell} - 1, \quad i = 0, \dots, \ell - 1.$$
 (4)

Taking j = 0 and j = 1 we get

$$a_i = \gamma^{-ri+i^2} = \gamma^{(\ell+i)^2 - r(\ell+i)}, \quad i = 0, \dots, \ell - 1,$$

which implies

$$r \equiv \ell + 2i \bmod \frac{n}{\ell}, \quad i = 0, \dots, \ell - 1.$$
 (5)

Thus either $\ell = 1$ or n/ℓ divides 2.

If $\ell = 1$, note that $r \equiv \ell \equiv 1 \mod n$ by (5). Then (4) applied with j = 0 and j = n - 1 implies $\gamma^{-1} = \gamma$ and thus $n \in \{1, 2\}$.

If n/ℓ divides 2, we have $\ell = n$ if n is odd and $\ell = n/2$ or $\ell = n$ if n is even. It remains to show that for even n, d can be represented by a mapping of index n/2.

Suppose that n is even and $\ell = n/2$, which means that each coset $C_{\ell,i}$ of G contains only two elements, γ^i and $\gamma^{i+n/2}$, for $i = 0, \ldots, n/2-1$. Choose any r with $r \equiv \frac{n}{2} \mod 2$ and $a_i = \gamma^{i^2-ir}$. Then it is easy to verify that

$$d(\gamma^{i}) = \gamma^{i^{2}} = a_{i}\gamma^{ir}$$
 and $d(\gamma^{i+n/2}) = \gamma^{(i+n/2)^{2}} = a_{i}\gamma^{(i+n/2)r}$

for i = 0, 1, ..., n/2 - 1 and the result follows.

Theorem 1 states only that the univariate Diffie-Hellman mapping d cannot coincide with a mapping of small index in all points. However, by the following result it cannot even coincide in many points.

Theorem 2. The univariate Diffie-Hellman mapping d of the cyclic group G of order n coincides with any mapping of index ℓ in

$$O(\ell n^{1/2})$$

elements of G. If n is prime, we have the better bound 2ℓ .

Proof. For fixed $a \in \{0, 1, ..., n-1\}$ consider the mapping $f_a(y) = \gamma^a y^r$, $y \in G$. We have to estimate the number N of x = 0, 1, ..., n-1 with

$$f_a(\gamma^x) = d(\gamma^x),$$

that is,

$$\gamma^{a+rx} = \gamma^{x^2},$$

or equivalently,

$$x^2 - rx - a \equiv 0 \bmod n.$$

By [10] we have

$$N = O(n^{1/2})$$

for any n. If n is prime, we have obviously $N \leq 2$. Since each function of index ℓ is the combination of at most ℓ different functions of the form f_a and the result follows.

3 Index of the bivariate Diffie-Hellman mapping

Let ℓ_1 and ℓ_2 be divisors of n and G the cyclic group of order n.

For any positive integers r_1 and r_2 and any $a_{0,0}, \ldots, a_{\ell_1-1,\ell_2-1} \in G$, we define the (r_1, r_2) th order cyclotomic mapping $f_{a_{0,0},\ldots,a_{\ell_1-1,\ell_2-1}}^{(r_1,r_2)}$ of index pair (ℓ_1, ℓ_2) by

$$f_{a_0,0,\dots,a_{\ell_1-1},\ell_2-1}^{(r_1,r_2)}(x,y) = a_{k_1,k_2}x^{r_1}y^{r_2} \quad \text{if } (x,y) \in C_{\ell_1,k_1} \times C_{\ell_2,k_2}, \tag{6}$$

for $k_1 = 0, ..., \ell_1 - 1$ and $k_2 = 0, ..., \ell_2 - 1$. For a mapping f over G with the property (6) we call (ℓ_1, ℓ_2) an *index pair of* f.

Theorem 3. Let G be any cyclic group of order n. Then the bivariate Diffie-Hellman mapping D of G defined by (2) has the only index pair (n, n).

Proof. Since otherwise the result is trivial we may assume $n \geq 2$, $\min\{\ell_1, \ell_2\} < n$ and wlog. $\ell_1 \geq \ell_2$.

Let (ℓ_1, ℓ_2) be an index pair of D, that is, D can be represented by a mapping of the form (6). Then

$$D(\gamma^{k_1+j_1\ell_1}, \gamma^{k_2+j_2\ell_2}) = \gamma^{(k_1+j_1\ell_1)(k_2+j_2\ell_2)} = a_{k_1,k_2} \gamma^{r_1(k_1+j_1\ell_1)} \gamma^{r_2(k_2+j_2\ell_2)}$$
(7)

for $j_1 = 0, ..., n/\ell_1 - 1$, $j_2 = 0, ..., n/\ell_2 - 1$, $k_1 = 0, ..., \ell_1 - 1$ and $k_2 = 0, ..., \ell_2 - 1$.

Taking $j_1 = j_2 = 0$ we get

$$a_{k_1,k_2} = \gamma^{k_1 k_2 - r_1 k_1 - r_2 k_2}. (8)$$

Taking $j_1 = 0$ and $j_2 = 1$ gives

$$a_{k_1,k_2} = \gamma^{k_1 k_2 + k_1 \ell_2 - r_1 k_1 - r_2 k_2 - r_2 \ell_2}. (9)$$

Combining (8) and (9) yields

$$r_2 \equiv k_1 \bmod \frac{n}{\ell_2}$$

for $k_1 = 0, \ldots, \ell_1 - 1$. Thus $\ell_1 = 1$ and also $\ell_2 = 1$ by our assumption $\ell_2 \leq \ell_1$. Since $\ell_1 = \ell_2 = 1$, we have $k_1 = k_2 = 0$ and $r_2 \equiv 0 \mod n$. Then (7) becomes

$$D(\gamma^{j_1}, \gamma^{j_2}) = \gamma^{j_1 j_2} = a_{0,0} \gamma^{r_1 j_1}$$

and thus

$$a_{0,0} = \gamma^{j_1 j_2 - r_1 j_1}.$$

Taking $j_1 = 0$ and $j_1 = 1$, respectively, we get

$$a_{0.0} = 1 = \gamma^{j_2 - r_1},$$

that is,

$$j_2 \equiv r_1 \bmod n$$

for $j_2 = 0, ..., n-1$. This is not possible unless n = 1 which contradicts our assumption.

Theorem 4. Any mapping of index pair (ℓ_1, ℓ_2) coincides with the bivariate Diffie-Hellman mapping D of the cyclic group G of order n in at most $n^{1+o(1)}\ell_1\ell_2$ elements of G^2 .

Proof. For each $\gamma^a \in G$ the mapping $f_a(\gamma^x, \gamma^y) = \gamma^a \gamma^{r_1 x} \gamma^{r_2 y}$ coincides with $D(\gamma^x, \gamma^y) = \gamma^{xy}$ if and only if

$$xy \equiv a + r_1 x + r_2 y \bmod n.$$

For fixed y put $t = \gcd(y - r_1, n)$. If t does not divide $a + r_2y$, there is no solution x. Otherwise the equation is equivalent to

$$x\frac{y-r_1}{t} \equiv \frac{a+r_2y}{t} \bmod \frac{n}{t},$$

which has a unique solution x modulo n/t, that is, t solutions modulo n. For each t there are $\varphi(n/t)$ different $y \in \{0, \ldots, n-1\}$ with $\gcd(y-r_1, n) = t$, where φ is Euler's totient function. Hence, we have

$$\sum_{t|n} \varphi(n/t)t = n \sum_{d|n} \frac{\varphi(d)}{d} \le \tau(n)n = n^{1+o(1)}$$

solutions, where $\tau(n) = n^{o(1)}$ is the number of divisors of n. Therefore each mapping of index pair (ℓ_1, ℓ_2) coincides with D in at most $\ell_1 \ell_2 n^{1+o(1)}$ elements of G^2 .

4 Multiplicative subgroups of finite fields

In this section let G be a subgroup of \mathbb{F}_q^* of order n|q-1 and $\gamma \in \mathbb{F}_q^*$ be of order n. First we deal with the univariate case.

Theorem 5. Let f be any self-mapping of \mathbb{F}_q^* satisfying

$$f(\gamma^x) = \gamma^{x^2}, \quad x \in S,$$

for a subset $S \subseteq \{N+1, \ldots, N+H\}$ of cardinality |S| = H-s with $H \le n$. Then we have

 $ind(f) \ge \frac{n}{2(n-H+2s+1)}.$

Proof. For H=n and s=0 the result follows from Theorem 1 and we may restrict ourselves to the case $n-H+2s+1\geq 2$. Since otherwise the result is trivial we may also assume

$$ind(f) \leq n/3.$$

A straightforward extension of [16, Theorem 1] provides that any mapping G of index ℓ can be represented by a polynomial of the form

$$G(X) = X^r \sum_{i=0}^{\ell-1} A_i X^{in/\ell}.$$
 (10)

Now assume that f is of index ℓ and thus h defined by

$$h(\gamma^x) = f(\gamma^x)\gamma^{-rx}, \quad x = 0, \dots, n-1,$$

can be uniquely represented as

$$h(X) = G(X)X^{-r}$$

for some positive integer r and polynomial G(X) of the form (10). In particular, the weight w(h), that is, the number of nonzero coefficients of h(X), is at most ℓ , and the degree of h(X) at most $(\ell-1)n/\ell \leq n-3$. For all but at most s+1 elements x of S we have

$$h(\gamma^{x+1}) = f(\gamma^{x+1})\gamma^{-r(x+1)} = \gamma^{(x+1)^2 - r(x+1)}$$
$$= \gamma^{x^2 - rx}(\gamma^x)^2 \gamma^{1-r} = \gamma^{1-r}(\gamma^x)^2 h(\gamma^x).$$

Hence, the polynomial

$$F(X) = h(\gamma X) - \gamma^{1-r} X^2 h(X)$$

has at least |S| - s - 1 = H - 2s - 1 zeros of the form γ^x , $x \in \{1, ..., n\}$. The weight w(F) of F(X) satisfies

$$w(F) \ge \frac{n}{n - H + 2s + 1}$$

by [11, Lemma 1], which is applicable since $\deg(F) \leq n-1$. On the other hand, $w(F) \leq 2w(h)$ and thus

$$\ell \ge w(h) \ge \frac{n}{2(n-H+2s+1)},$$

which completes the proof.

Remark. Theorem 2 implies

$$ind(f) \gg \frac{|S|}{n^{1/2}}$$

for any S. This lower bound does not exceed $n^{1/2}$. However, Theorem 5 provides a larger lower bound than $n^{1/2}$ for any S satisfying the conditions of Theorem 5 with $n - |S| = o(n^{1/2})$.

Similar ideas can be used to prove an analog of Theorem 5 for the bivariate Diffie-Hellman mapping.

Theorem 6. Let G be a subgroup of \mathbb{F}_q^* of order n|q-1 generated by γ , U be any subset of $\{0,1,\ldots,n-1\}$ and $V=\{N,\ldots,N+H-1\}$ be any set of consecutive integers for some $H \leq n$. Let $f: G \times G \to G$ be any mapping of index pair (ℓ_1,ℓ_2) satisfying

$$f(\gamma^x, \gamma^y) = \gamma^{xy}, \quad (x, y) \in U \times V.$$

Then we have

$$\max\{\ell_1, \ell_2\} \ge \min\{|U|, H\}.$$

Proof. Put $m = \min\{|U|, H\}$.

It is easy to see that any mapping f of index pair (ℓ_1, ℓ_2) and order (r_1, r_2) can be represented by a polynomial f(X, Y) over \mathbb{F}_q of the form

$$f(X,Y) = X^{r_1}Y^{r_2} \sum_{i=0}^{\ell_1-1} \sum_{j=0}^{\ell_2-1} a_{i,j} X^{in/\ell_1} Y^{jn/\ell_2}.$$

Then there is a subset $\{u_0, \ldots, u_{m-1}\}$ of U such that

$$\gamma^{u_x(N+y)-r_1u_x-r_2(N+y)} = \sum_{i=0}^{\ell_1-1} \sum_{j=0}^{\ell_2-1} a_{i,j} \gamma^{inu_x/\ell_1+jn(N+y)/\ell_2}, \quad x, y = 0, \dots, m-1.$$

Assume $\max\{\ell_1, \ell_2\} < m$. Then the coefficient matrix $A = (a_{i,j})_{i,j=0,\dots,m-1}$, with $a_{i,j} = 0$ if $i \geq \ell_1$ or $j \geq \ell_2$, satisfies

$$G = V_1 A V_2$$

where

$$V_1 = (\gamma^{inu_x/\ell_1})_{i,x=0,\dots,m-1}, \quad V_2 = (\gamma^{jn(N+y)/\ell_2})_{y,j=0,\dots,m-1}$$

and

$$G = \left(\gamma^{(u_x - r_2)(N+y) - r_1 u_x}\right)_{x,y=0,\dots,m-1}.$$

 V_1 and V_2 are Vandermonde matrices and G can be reduced to a Vandermonde matrix by multiplying the xth row by the constant $\gamma^{r_1u_x}$. Hence, A is the product of three invertible matrices

$$A = V_1^{-1} G V_2^{-1}$$

and thus invertible itself. In particular, each row and each column of A contains at least one nonzero entry which contradicts our assumption $\max\{\ell_1,\ell_2\} < m$.

Remark. Theorem 4 implies the lower bound

$$\max\{\ell_1, \ell_2\} \ge (\ell_1 \ell_2)^{1/2} \ge \left(\frac{|U|H}{n^{1+o(1)}}\right)^{1/2}.$$

Its right hand side is always smaller than $n^{1/2}$. Theorem 6 provides a lower bound $\geq n^{1/2}$ for any U and H satisfying min $\{|U|, H\} \geq n^{1/2}$.

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