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ORIGINAL PAPER

A discrete SIS-model built on the strictly positive scheme

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Abstract

In this paper we introduce and analyze a discrete *SIS* epidemic model for a homogeneous population. As a discretization method the strictly positive scheme was chosen. The presented model is built from its continuous counterpart known from literature. We firstly present basic properties of the system. Later we discuss local stability of stationary states and global stability for the disease-free stationary state. The results for this state are expressed with the use of the basic reproduction number. The main conclusion from our work is that conditions for stability of the stationary states do not depend on the step size of the discretization method. This fact stays in contrary to other discrete models analyzed in our previous papers. Theoretical results are accomplished with numerical simulations.

Keywords *SIS* model · Discretization of a continuous system · Stability analysis of stationary states

1 Introduction

In mathematical modeling of epidemic the discrete approach is becoming more popular. In most cases, discrete models are built from their continuous counterparts. A discretization can be done for the example with the use of the explicit Euler method (*EEM*), which is one of the simplest methods. The exemplary epidemic model considering the *EEM* can be found in [6]. However, applying the *EEM* can cause dynamical inconsistency of a discrete system with its continuous analogue. Furthermore, negativity of solutions of the discrete system can appear, what is not proper in epidemiological modeling. To avoid these problems, the non-discretization method (*NSDM*) can be applied. This method is introduced and discussed in [7]. Discrete models build with the *NSDM* are dynamically consistent with their continuous analogues [1]. There are different ways of creating non-standard discretized

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models. What is important, using some of non–standard discretization methods can still result in the negativity of solutions of the discretized systems. To ensure that the solutions are always positive, one can apply the strictly positive scheme (*SPS*), which is a version of the *NSDM*. Applying this scheme makes the forms of equations more complicated comparing to other schemes. For this reason the *SPS* is not popular in mathematical modeling. The aim of this paper is to fill the gap in this issue.

Here we introduce a *SIS* (*susceptible–infected–susceptible*) discrete model build on the *SPS*. This model is obtained on the ground of its continuous counterpart. We focus on stability analysis of stationary states appearing in the system. We investigate local stability of these states and global stability of the stationary state for which there is no infection in the population. Our work is a continuation of work presented in [3, 4] and [5], where we introduced the continuous model and its discrete versions built on the *EEM* and the *NSDM*.

This paper is organized as it follows. In Sect. 2 we present the continuous model and its discrete counterparts built on the *EEM* and the *NSDM*. The model built on the *SPS* is introduced in Sect. 3. There we include a mathematical analysis of this model. Section 4 presents numerical simulations, which complement theoretical results. In Sect. 5 we discuss our results.

Here we assume that $\mathbb{N}$ is the set of all natural numbers including zero and $\mathbb{N}_+ := \mathbb{N} \setminus \{0\}$.

2 A continuous model and its discrete counterparts

Firstly we remind the continuous model which was introduced and analyzed in [3]. This model corresponds to dynamics of epidemics in a homogeneous population consisting of susceptible (S) and infected (I) people. The mass–action law was used for proposing an illness transmission function. The analyzed system has the form:

$$\begin{aligned} \dot{S} &= C - \beta SI + \gamma I - \mu S, \\ \dot{I} &= \beta SI - (\gamma + \alpha + \mu)I. \end{aligned} \tag{1}$$

The over dot stands for the time derivation w/time and the follow his entries. A parameter C is a constant inflow into the population, β stands for a transmission coefficient, γ is a recovery rate, μ corresponds to a natural death and α reflects a disease-related death. Every parameter is positive and fixed. Observe that C is a magnitude corresponding to a density of individuals and μ is a ratio for which we have $0 < \mu < 1$. Hence it is reasonable to assume that

$$C \gg \mu. \tag{2}$$

Thanks to a substitution:

$$\tau = \gamma t, \quad x = \beta S, \quad y = \beta I,$$

we reduced the number of parameters in system (1) obtaining

$$
\begin{aligned}
x' &= C - xy + y - \mu x, \\
y' &= xy - ky,
\end{aligned}
\tag{3}
$$

where $k = \alpha + \mu + 1 > 1$ and C is scaled.

Now we recall the basic properties of system (3). Existence, uniqueness and positiveness of solutions of system (3) for positive initial conditions are derived from the form of the right–hand side of the system. The basic reproduction number $\mathcal{R}_0$ related to the system is defined as

$$\mathcal{R}_0 = \frac{C}{\mu k}. \tag{4}$$

We will refer to this value later.

System (3) has two stationary states:

- always existing disease-free: $E_d = (x_d, y_d) = \left(\frac{C}{\mu}, 0\right)$,
- endemic: $E_e = (x_e, y_e) = \left(k, \frac{C-\mu k}{k-1}\right)$ existing for $C > \mu k$ (that is for $\mathcal{R}_0 > 1$).

If $\frac{C}{\mu} < k$, then E_d is locally stable. For $\frac{C}{\mu} > k$ (i.e., when E_e exists) E_d is a saddle point. The state E_e is always locally stable under its existence. Furthermore, the Poincaré–Bendixson theorem [8] implies global stability – either E_d is globally stable whenever E_e does not exist, or E_e is globally stable whenever exists.

Now we recall the discrete version of system (3) for which the *EEM* was used. The paper [4] dealt with analysis of this model. Let $n \in \mathbb{N}$ be an n-th node in a discrete timescale. The analyzed system has the form:

$$x_{n+1} = x_n + h\big(C - x_n y_n + y_n - \mu x_n\big), \tag{5a}$$

$$y_{n+1} = y_n + h\big(x_n y_n - k y_n\big), \tag{5b}$$

where h means a step size of the discretization method. Observe that because of the terms $-\mu x_n$ and $-x_n y_n$ in eq (5a) and the term $-k y_n$ in eq (5b), positivity of the system can be violated.

In order to eliminate this nuance, in [5] we introduced the discrete model built on the *NSDM*. The term $-x_n y_n$ was replaced with an expression $-x_{n+1} y_n$. The modified system has the form

$$x_{n+1} = x_n + h\big(C - x_{n+1} y_n + y_n - \mu x_n\big), \tag{6a}$$

$$y_{n+1} = y_n + h\big(x_n y_n - k y_n\big) = y_n(1 + h(x_n - k)). \tag{6b}$$

The above system can be rewritten as

$$x_{n+1} = \frac{x_n(1 - h\mu) + hC + hy_n}{1 + hy_n}, \tag{7a}$$

$$y_{n+1} = y_n(1 + h(x_n - k)). \tag{7b}$$

Analysis of this system can be found in [5]. In system (7) the x and y variables are positive if $h < \frac{1}{\mu}$ and $h < \frac{1}{k}$, respectively.

3 A discrete model with the use of the *SPS*

Observe that applying the *NSDM* in system (7) allowed to preserve the positivity of the variables in a better way comparing to using the *EEM* for system (5). However, the positivity is still not unconditional, what is not expected in epidemiological models. The variables can be negative if the step size is sufficiently large. Even for $h = 1$ the variables can be negative. Hence there arises a need of proposing a system which is a discrete version of the continuous system (3) preserving the positivity of the variables as in the continuous case.

Let us discretize system (3) with the use of the *SPS*. In Eq. (6a) we replace the term $-\mu x_n$ with an expression $-\mu x_{n+1}$ and in eq (6b) we change $-ky_n$ into $-ky_{n+1}$. We obtain a following system

$$\begin{aligned} x_{n+1} &= x_n + h\big(C - x_{n+1}y_n + y_n - \mu x_{n+1}\big), \\ y_{n+1} &= y_n + h\big(x_n y_n - k y_{n+1}\big). \end{aligned} \tag{8}$$

This system can be rewritten as

$$x_{n+1} = \frac{x_n + hC + hy_n}{1 + h(y_n + \mu)}, \tag{9a}$$

$$y_{n+1} = \frac{y_n(1 + hx_n)}{1 + hk}. \tag{9b}$$

3.1 Basic properties

Firstly we present basic properties of system (9). Following the approach from [2], we compute $\mathcal{R}_0$ for system (9) and obtained (4). From the form of the right–hand side of system (9) we get the positivity of its solutions for any positive initial condition. More generally we write that if

- $x_0 > 0$ and $y_0 = 0$, then $x_n > 0$ and $y_n \equiv 0$ for $n \in \mathbb{N}_+$,
- $x_0 \geq 0$ and $y_0 > 0$, then $x_n, y_n > 0$ for $n \in \mathbb{N}_+$,
- $x_0 = y_0 = 0$, then $x_n \equiv \frac{hC}{1+h\mu}$ and $y_n \equiv 0$ for $n \in \mathbb{N}_+$.

Now we assess the values of the variables of system (9). As it was shown in [3–5], the desired upper bound of the x variable is $\frac{C}{\mu}$. Hence we investigate the inequality $x_{n+1} \leq \frac{C}{\mu}$, from which we have

$$\frac{x_n + hC + hy_n}{1 + h(y_n + \mu)} \leq \frac{C}{\mu},$$

and

$$hy_n(\mu - C) < C - \mu x_n. \tag{10}$$

The left–hand side of Ineq. (10) is negative because of (2). The right–hand side of Ineq. (10) is non–negative if $x_n \leq \frac{C}{\mu}$. We conclude that

Corollary 1 *If $x_0 \leq \frac{C}{\mu}$, then for system (9) we have $x_n \leq \frac{C}{\mu}$ for any $n \in \mathbb{N}_+$.*

See that if $x_0 \leq \frac{C}{\mu}$, then using Cor. 1 for Ineq. (9b) gives

$$y_{n+1} \leq \frac{y_n\left(1 + h\left(\frac{C}{\mu}\right)\right)}{1 + hk} = y_0\left(\frac{1 + h\frac{C}{\mu}}{1 + hk}\right)^{n+1}. \tag{11}$$

Let us introduce a new variable

$$w_n := x_n + y_n,$$

which means a size of the whole population for the n-th point. Now we formulate a lemma stating about the upper bound of the whole population.

Proposition 2 *If*

$$x_n \leq x_{n+1}, \quad y_n \leq y_{n+1}, \quad n \in \mathbb{N}, \tag{12}$$

then

$$w_n \leq w_0 + \frac{C}{\mu} \tag{13}$$

for every $n \in \mathbb{N}_+$.

Proof Adding by sides the equations from system (8) gives

$$w_{n+1} = w_n + h\big(C + (x_n - x_{n+1})y_n + y_n - y_{n+1} - \mu w_{n+1} - \alpha y_{n+1}\big). \tag{14}$$

Using (12) we estimate (14) so that

$$w_{n+1} \leq w_n + h\big(C - \mu w_{n+1} - \alpha y_{n+1}\big),$$

what yields

$$w_{n+1} \le w_n + h\big(C - \mu w_{n+1}\big)$$

and

$$w_{n+1} \le \frac{w_n + hC}{1 + h\mu}.$$

Solving the above inequality gives

$$w_n \le \frac{C}{\mu} + \left(w_0 - \frac{C}{\mu}\right)\left(\frac{1}{1+h\mu}\right)^n, \tag{15}$$

what implies (13). □

Observe that if in Prop. (2) we include an additional assumption $w_0 \le \frac{C}{\mu}$, then Ineq. (13) can be replaced with an inequality $w_n \le \frac{C}{\mu}$.

3.2 Stationary states

Here we investigate the stability of the stationary states of system (9). Obviously their forms are the same as the forms of the stationary states of system (3).

3.2.1 Local stability

Let us express the conditions for local stability of the stationary states.

Theorem 3 *In system* (9) *the state* E_d *is locally stable for* $R_0 < 1$ *and unstable for* $R_0 > 1$.

Proof The Jacobian matrix of system (9) has the form

$$M(x,y) := \begin{pmatrix} \frac{1}{1+h(y_n+\mu)} & \frac{h^2(\mu-C)+h(1-x_n)}{(1+h(y_n+\mu))^2} \\ \frac{hy_n}{1+hk} & \frac{1+hx_n}{1+hk} \end{pmatrix}.$$

For E_d this matrix reads

$$M(E_d) = \begin{pmatrix} \frac{1}{1+h\mu} & \frac{h^2(\mu-C)+h(1-\frac{C}{\mu})}{(1+h\mu)^2} \\ 0 & \frac{1+h\frac{C}{\mu}}{1+hk} \end{pmatrix}.$$

The eigenvalues of $M(E_d)$ equal

$$\lambda_1 = \frac{1}{1+h\mu}, \quad \lambda_2 = \frac{1+h\frac{C}{\mu}}{1+hk}.$$

Conditions for the E_d local stability are $|\lambda_i| < 1$, $i = 1, 2$. See that we always have $0 < \lambda_1 < 1$ and $-1 < \lambda_2$. The inequality $\lambda_2 < 1$ is true for $\frac{C}{\mu} < k$, what is equivalent to $\mathcal{R}_0 < 1$. The last inequality is the condition for the E_d local stability. □

Now let us investigate the E_e local stability. We formulate the following theorem

Theorem 4 *In system (9) the state E_e is always locally stable if it exists.*

Proof The matrix $M(E_e)$ has the form

$$M(E_e) = \begin{pmatrix} \frac{1}{1+h(y_e+\mu)} & \frac{h^2(\mu-C)+h(1-k)}{(1+h(y_e+\mu))^2} \\ \frac{hy_e}{1+hk} & 1 \end{pmatrix}. \tag{16}$$

Let us define

$$y_e^* := y_e + \mu = \frac{C-\mu k}{\alpha+\mu} + \mu = \frac{C-\mu k+\mu(\alpha+\mu)}{\alpha+\mu} = \frac{C-\mu}{\alpha+\mu}. \tag{17}$$

The characteristic polynomial of $M(E_e)$ reads

$$P(\lambda) := \lambda^2 - \left(1 + \frac{1}{1+hy_e^*}\right)\lambda + \frac{1}{1+hy_e^*} + \frac{h^2 y_e}{1+hk} \cdot \frac{(\alpha+\mu)+h(C-\mu)}{(1+hy_e^*)^2}.$$

Let us rewrite the polynomial as $P(\lambda) = \lambda^2 + b\lambda + c$. See that $b < 0$ and $c > 0$ (the second inequality is true from (2)).

Let Δ be a discriminant of $P(\lambda)$. After computations we obtained that discriminator Δ can be positive or negative (and naturally equal to zero – we will omit this case because it is not generic). However, a form of a condition determining a sign of Δ is not needed in the context of the theorem thesis.

• Let us assume that $\Delta > 0$. For this case the eigenvalues of $M(E_e)$ are real and equal to $\lambda_{1,2} = \frac{-b\pm\sqrt{b^2-4c}}{2}$. Sufficient conditions for E_e stability are $\lambda_1 < 1$ and $\lambda_2 > -1$. From the first inequality we get $-b + \sqrt{b^2-4c} < 2$, giving

$$0 < \frac{h^2 y_e}{1+hk} \cdot \frac{(\alpha+\mu)+h(C-\mu)}{(1+hy_e^*)^2}.$$

The last inequality is always true and so does the condition $\lambda_1 < 1$.

The inequality $\lambda_2 > -1$ can be written as $-2 < -b - \sqrt{b^2-4c}$, leading to

$$0 < 8 + \frac{8}{1+hy_e^*} + 4\frac{h^2 y_e}{1+hk} \cdot \frac{(\alpha+\mu)+h(C-\mu)}{(1+hy_e^*)^2}.$$

The above inequality is always true, so we get an unconditional fulfillment of $\lambda_2 > -1$.

• Now assume that $\Delta < 0$. The eigenvalues of $M(E_e)$ are complex with non-zero imaginary part and equal to $\lambda_{1,2} = \frac{-b \pm i\sqrt{4c-b^2}}{2}$, where i is an imaginary unit. The condition $\lambda_1 \lambda_2 = |\lambda| = 1$ guaranties the local stability of E_e. See that

$$|\lambda| = \left(\frac{b}{2}\right)^2 + \frac{4c-b^2}{4} = \frac{4c}{4} = c.$$

Hence it is enough to check the inequality $c < 1$, which can be written as

$$\begin{aligned}
&\frac{1}{1+hy_e^*} + \frac{h^2 y_e}{1+hk} \cdot \frac{(\alpha+\mu)+h(C-\mu)}{(1+hy_e^*)^2} < 1,\\
&1+hy_e^* + \frac{h^2 y_e}{1+hk}((\alpha+\mu)+h(C-\mu)) < \left(1+hy_e^*\right)^2,\\
&\frac{h^2 y_e}{1+hk}((\alpha+\mu)+h(C-\mu)) < hy_e^* + \left(hy_e^*\right)^2,\\
&\frac{hy_e}{1+hk}((\alpha+\mu)+h(C-\mu)) < y_e^* + h\left(y_e^*\right)^2.
\end{aligned}$$

Using (17), we write

$$\begin{aligned}
&\frac{hy_e}{1+hk}\left((\alpha+\mu)+hy_e^*(\alpha+\mu)\right) < y_e^* + h\left(y_e^*\right)^2,\\
&\frac{hy_e(\alpha+\mu)}{1+hk}\left(1+hy_e^*\right) < y_e^*\left(1+hy_e^*\right),\\
&\frac{hy_e(\alpha+\mu)}{1+hk} < y_e^*,\\
&\frac{h(\alpha+\mu)}{1+hk} < \frac{y_e^*}{y_e},\\
&\frac{h(\alpha+\mu)}{1+hk} < \frac{\frac{C-\mu}{\alpha+\mu}}{\frac{C-\mu k}{\alpha+\mu}}
\end{aligned}$$

and

$$\frac{h(\alpha+\mu)}{1+hk} < \frac{C-\mu}{C-\mu k}, \tag{18}$$

From the meaning of variables we have

$$\frac{C-\mu}{C-\mu k} > 1. \tag{19}$$

The condition $k > 1$ yields

$$\frac{h(\alpha+\mu)}{1+hk} < \frac{h(\alpha+\mu)}{1+h}. \tag{20}$$

Meaning of the parameter gives $\alpha + \mu < 1$ and

$$\frac{h(\alpha+\mu)}{1+h} < \frac{h}{1+h} < 1.$$

The above inequality with (20) implies

$$\frac{h(\alpha+\mu)}{1+hk} < 1.$$

If we consider the last inequality and Ineq. (19) together, then we have a fulfillment of (20). We again obtained the unconditional E_e local stability. □

3.2.2 Global stability of E_d

Now let us focus on the global stability of the state E_d.

Theorem 5 *If for system (9) we have $\mathcal{R}_0 < 1$ and $x_0 \leq \frac{C}{\mu}$, then E_d is globally stable.*

Proof From $x_0 \leq \frac{C}{\mu}$ we obtain Cor. 1 and Ineq. (11). The condition $\mathcal{R}_0 < 1$ yields

$$\lim_{n\to\infty} y_n = 0. \tag{21}$$

Now we assess the x variable. From eq (9a) we get

$$x_{n+1} \leq \frac{x_n + hC + hy_n}{1+h\mu}. \tag{22}$$

Let us take $\varepsilon > 0$. From (21) we state that

$$\forall \varepsilon_1 > 0 \quad \exists N_1 \in \mathbb{N} \quad \forall n > N_1 \quad y_n < \varepsilon_1. \tag{23}$$

Combining Ineqs. (22) and (23) yields

$$x_{n+1} < \frac{x_n + hC + h\varepsilon_1}{1+h\mu}. \tag{24}$$

The solution of the above inequality is

$$x_n < \frac{C+\varepsilon_1}{\mu} + \left(x_0 - \frac{C+\varepsilon_1}{\mu}\right)\left(\frac{1}{1+h\mu}\right)^n.$$

Since

$$\lim_{n\to\infty}\left(x_0-\frac{C+\varepsilon_1}{\mu}\right)\left(\frac{1}{1+h\mu}\right)^n=0,$$

we can choose sufficiently large $n(>N_1)$ and we obtain

$$x_n<\frac{C}{\mu}+\varepsilon. \tag{25}$$

Eq. (9a) can be assessed also in a way

$$x_{n+1}\geq\frac{x_n+hC}{1+h(y_n+\mu)}. \tag{26}$$

From (21) for sufficiently large n we have

$$\frac{1}{y_n}>\frac{1}{\varepsilon_1},\quad \frac{1}{1+h(y_n+\mu)}>\frac{1}{1+h(\varepsilon_1+\mu)} \tag{27}$$

and

$$x_{n+1}>\frac{x_n+hC}{1+h(\varepsilon_1+\mu)},$$

what gives

$$x_n>\frac{C}{\varepsilon_1+\mu}+\left(x_0-\frac{C}{\varepsilon_1+\mu}\right)\left(\frac{1}{1+h(\varepsilon_1+\mu)}\right)^n.$$

For sufficiently large n we obtain

$$x_n>\frac{C}{\mu}-\varepsilon. \tag{28}$$

Combining (25) and (28), for sufficiently large n we have

$$\left|x_n-\frac{C}{\mu}\right|<\varepsilon$$

giving

$$\lim_{n\to\infty}x_n=\frac{C}{\mu}.$$

and we reach the global stability of E_d. □

3.2.3 Possibility of a bifurcation

Let us check if there is a possibility of a bifurcation appearance in system (9). Remind that for the matrix $M(E_d)$ the condition $0 < \lambda_1 < 1$ always holds. Observe that we have $\lambda_2 = 1$ for $\mathcal{R}_0 = 1$ and then the bifurcation can occur.

Now look on the polynomial $P(\lambda)$ for the matrix $M(E_e)$. We get

$$P(1) = \frac{h^2 y_e}{1 + hk} \cdot \frac{(\alpha + \mu) + h(C - \mu)}{(1 + hy_e^*)^2} > 0$$

and

$$P(-1) = 2 + \frac{2}{1 + hy_e^*} + \frac{h^2 y_e}{1 + hk} \cdot \frac{(\alpha + \mu) + h(C - \mu)}{(1 + hy_e^*)^2} > 0.$$

Remind that the eigenvalues of the matrix $M(E_e)$ are real. Hence we state that -1 and 1 cannot be the eigenvalues of $M(E_e)$. We conclude that there is no bifurcation for the state E_e.

4 Numerical simulations

In this section we present numerical simulations related to dynamics of system (9). We chose $(x_0, y_0) = (0.5, 0.2)$ as the initial condition and $h = 1$, $\mu_1 = 0.1$, $\alpha = 0.2$ as the values of the parameters.

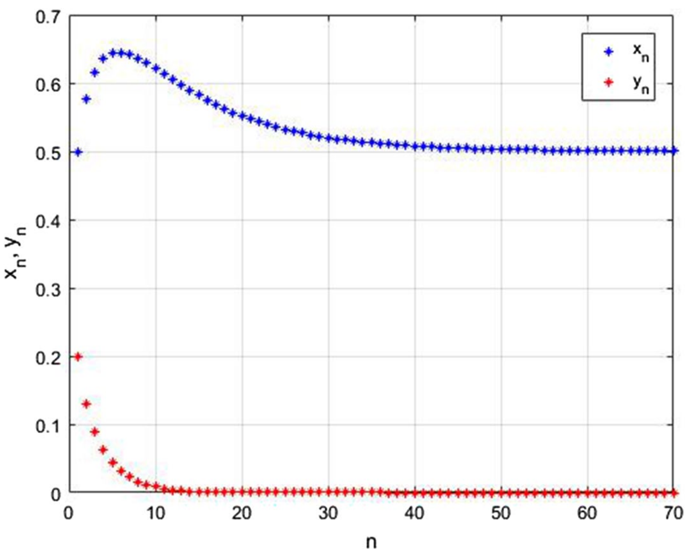

Fig. 1 The next iterations of system (9) for the case $\mathcal{R}_0 < 1$. The iterations for the x and the y variables are depicted with analogically blue and red points

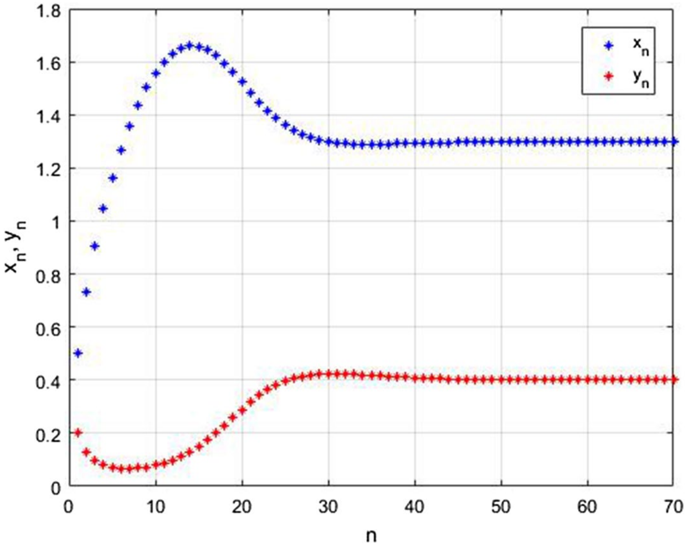

Fig. 2 The next iterations of system (9) for the case $\mathcal{R}_0 > 1$. The iterations for the x and the y variables are depicted with analogically blue and red points

4.1 Stability of the stationary states

Here we illustrate the stability of the stationary states of system (9). Figures 1 and 2 show the dynamics of the system when $\mathcal{R}_0 < 1$ and $\mathcal{R}_0 > 1$, respectively. These figures demonstrate the global stability of the E_d state and the local stability of the

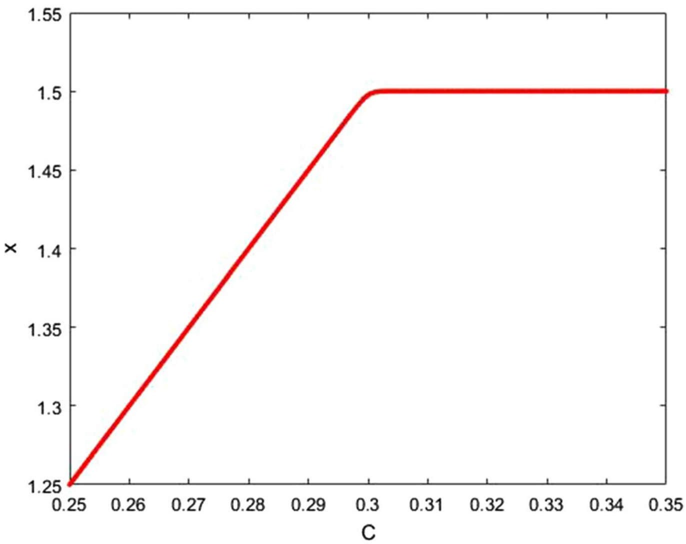

Fig. 3 A bifurcation diagram for the x variable of system (9)

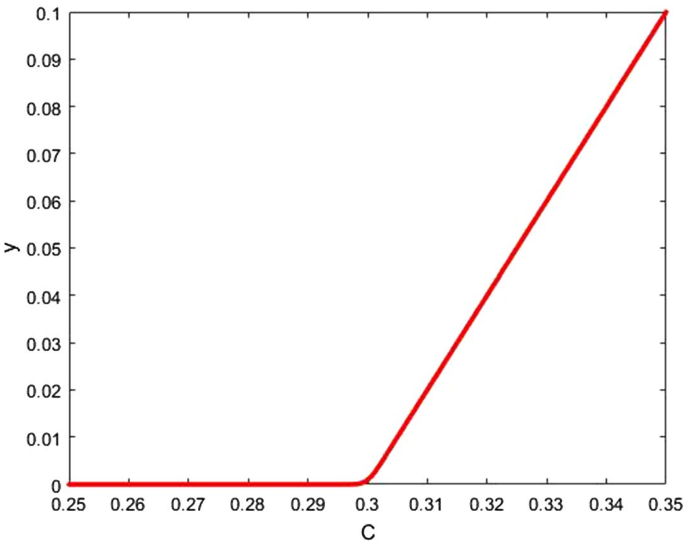

Fig. 4 A bifurcation diagram for the y variable of system (9)

E_e state analogically. For each case $\mathcal{R}_0 < 1$ and $\mathcal{R}_0 > 1$ the values $C = 0.05$ and $C = 0.25$ were taken, respectively. In each figure first 70 iterations of system (9) were presented.

4.2 Appearance of the bifurcation

Now we depict the occurrence of the bifurcation in system (9) for the case $\mathcal{R}_0 = 1$. In Figs. 3 and 4 bifurcation diagrams for the x and y variables are presented. We chose C as the bifurcation parameter. The figures show last ten iterations from initial thousand iterations for values of the x and the y variable, respectively. The values of these last iterations are nearly the same. Hence we observe only one point for each value of C. Observe that for $C = 0.3$ (i.e. $\mathcal{R}_0 = 1$) the state E_d bifurcates into the state E_e.

5 Conclusions

In this paper we introduced and analyzed the strictly positive discretized system of epidemic dynamics. The illness is spread in the population in which we determine two groups: susceptible (healthy) and infected individuals. The global stability of the disease–free stationary state E_d and the local stability of the endemic stationary state E_e were proved. The results are expressed in the terms of the basic reproduction number $\mathcal{R}_0$ of the system.

The conditions obtained for the stability of the stationary states of system (9) are the same as the analogical ones in the case of continuous system (3). What is important, they do not depend on the step size of the discretization method *h*. In our papers [4] and [5] we proved that in systems (5) and (7) conditions for the E_e local stability depend on the step size. Hence we state that system (9) approximates its continuous analog in a better way than systems (5) and (7). We finally conclude that applying the *SPS* is more suitable for modeling the epidemic dynamics comparing to the *EEM* and the *NSDM*.

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