Estimates for polynomial roots.

Maurice Mignotte, Doru Stefanescu

ESTIMATES FOR POLYNOMIAL ROOTS

by

Maurice Mignotte and **Doru Ştefănescu**

Abstract: Given a complex polynomial, we obtain estimates for the lower bounds of the roots outside the unit circle. Our main tool is the method of Dandelin–Graeffe, which can be used directly for polynomials with distinct absolute values of the roots. In the general case the arguments of the powers of the roots must be controlled, and we achieve this by two methods: a theorem of Dirichlet, and an argument on recursive linear sequences.

Introduction

Let $F(X) = \sum_{i=0}^{d} a_i X^{d-i}$ be a nonzero polynomial over $\mathbb{C}$ and suppose that $\alpha_1, \dots, \alpha_d \in \mathbb{C}$ are its roots, with

$$|\alpha_1| \geq \cdots \geq |\alpha_t| > 1 = |\alpha_{t+1}| = \cdots = |\alpha_s| > |\alpha_{s+1}| \geq \cdots \geq |\alpha_d| > 0 .$$

In order to obtain a lower bound strictly larger than 1 for $|\alpha_t|$ we consider the family of polynomials $F_n \in \mathbb{C}[X]$ defined by

$$F_n(X) = \operatorname{Res}\bigl(F(Y), Y^n - X\bigr) = \sum_{i=0}^{d} a_i^{(n)} X^{d-i}.$$

We observe that the roots of F_n are the nth powers of the roots of F, so the polynomials F_n are exactly the polynomials considered in the general case of the Dandelin–Graeffe method [Householder]. We next obtain an asymptotic formula for $|\alpha_1 \cdots \alpha_k|$, for $1 \leq k < d$, as a function of the coefficients of the polynomials F_n. This formula is then used to derive a lower bound greater than 1 for the absolute values of the roots outside the unit circle.

Another estimate is obtained as a function of the measure of the polynomial F. Both estimates allow giving upper bounds for the roots inside the unit circle, and applications include inequalities on the height and the length of polynomial divisors.

1. The first estimate

A1. Application of Dirichlet's theorem

Lemma 1.1. *Let $\beta_1, \dots, \beta_r \in \mathbb{C}$ be such that $|\beta_1| = \cdots = |\beta_r| = \rho > 0$ and let*

$$S_n = \beta_1^n + \cdots + \beta_r^n .$$

There exist infinitely many integers q such that $|S_q| \geq r\rho^q/\sqrt{2}$.

Proof:

Let $\beta_j = \rho\, e^{2\pi i \phi_j}$. We have

$$|S_n| = \rho^n \cdot \left|e^{2n\pi\, i\phi_r}\right| \cdot \Big|\sum_{j=1}^{r} e^{2\pi\, i\theta_j}\Big|, \quad \text{with} \quad \theta_j = \phi_j - \phi_r.$$

Let us first prove that there exists such an integer q. By the theorem of Dirichlet [Schmidt], for a given integer Q, there exists $q \geq 1$ such that

$$q \leq Q^r \quad \text{and} \quad ||q\theta_j|| < \frac{1}{Q} \quad \text{for all} \quad 1 \leq j < r,$$

where $||x|| = \min_{m\in\mathbb{Z}} |x - m|$.

This gives the result: for example, for $Q = 4$ we have $\Re(e^{i\pi\, q\theta_j}) \geq 1/\sqrt{2}$ for all $1 \leq j < r$. So we obtain $S_q \geq r\rho^n/\sqrt{2}$.

Now we choose Q_1 such that $\max ||q\theta_j|| > 1/Q_1$. Our argument gives some integer q_1. Clearly $q_1 > q$. Then we choose Q_2, we get $q_2 > q_1$ and so on. ∎

Proposition 1.2. *Let $T_n = \gamma_1^n + \cdots + \gamma_r^n + \gamma_{r+1}^n + \cdots + \gamma_d^n$, where $\gamma_1, \ldots, \gamma_d$ are complex numbers and*

$$|\gamma_1| = \cdots = |\gamma_r| > |\gamma_{r+1}| \geq \cdots \geq |\gamma_d|.$$

There exist infinitely many $n \in \mathbb{N}$ such that $|T_n| \geq \frac{r}{2\sqrt{2}} \cdot |\gamma_1|^n$.

Proof:
Put $S_n = \gamma_1^n + \cdots + \gamma_r^n$. We have $|\gamma_{r+1}^n + \cdots + \gamma_r^d| \leq (d-r)|\gamma_{r+1}|^n$ and it follows that

$$|T_n| \geq |S_n| - (d-r)|\gamma_{r+1}|^n.$$

For obtaining the result, by Lemma 1.1 it is sufficient to have

$$(*) \qquad (d-r)|\gamma_{r+1}|^n \leq \frac{r|\gamma_1|^n}{2\sqrt{2}}$$

for infinitely many n.

Because $\left|\frac{\gamma_1}{\gamma_{r+1}}\right| > 1$, there exists n_0 such that

$$\left|\frac{\gamma_1}{\gamma_{r+1}}\right|^n \geq \frac{2\sqrt{2}(d-r)}{r} \quad \text{for all} \quad n \geq n_0,$$

so (*) is fulfilled for all $n \geq n_0$. Hence the result. ∎

Theorem 1.3. *With the notation in Proposition 1.2 we have*

$$\limsup_{n\to\infty} |T_n|^{1/n} = |\gamma_1| = \max\{|\gamma_j|;\, 1 \leq j \leq d\}.$$

Proof:
We have

$$|T_n|^{1/n} = |\gamma_1^n + \cdots + \gamma_d^n|^{1/n} \leq d^{1/n} \cdot |\gamma_1|,$$

therefore

$$\limsup_{n\to\infty} |T_n|^{1/n} \le |\gamma_1|. \tag{1}$$

On the other hand, by Proposition 1.2, we have

$$\left|T_n\right|^{1/n} \ge |\gamma_1| \cdot \left(\frac{r}{2\sqrt{2}}\right)^{1/n} \quad \text{for infinitely many } n.$$

Since $\lim_{n\to\infty}(r/2\sqrt{2})^{1/n} = 1$, we have

$$\limsup_{n\to\infty} \left|T_n\right|^{1/n} \ge |\gamma_1|. \tag{2}$$

Inequalities (1) and (2) prove the result. □

Proposition 1.4. *With the notation of the introduction, we have*

$$|a_0\alpha_1\cdots\alpha_k| = \limsup_{n\to\infty} \left|a_k^{(n)}\right|^{1/n} \quad \text{for all } 1 \le k < d.$$

Proof:
We consider the polynomial F_n. Since $\alpha_1^n, \ldots, \alpha_d^n$ are the roots of F_n, we have

$$a_k^{(n)} = (-1)^k\, a_0^{(n)} \sum_{i_1,\ldots,i_k} (\alpha_{i_1}\cdots\alpha_{i_k})^n. \tag{3}$$

Let $\Pi_I = \alpha_{i_1}\cdots\alpha_{i_k}$, where $I = (i_1,\cdots,i_k)$. We consider $U_n = \sum_I \Pi_I^n$. Because $|\alpha_1\cdots\alpha_k| \ge |\Pi_I|$ for all I, by Theorem 1.3 we have

$$\limsup_{n\to\infty} \left|U_n\right|^{1/n} = |\alpha_1\cdots\alpha_k|.$$

By (3) we obtain

$$\limsup_{n\to\infty} \left|a_k^{(n)}/a_0^{(n)}\right|^{1/n} = |\alpha_1\cdots\alpha_k|.$$

Since $\left|a_0^{(n)}\right|^{1/n} = |a_0|$ we have $|a_0\alpha_1\cdots\alpha_k| = \limsup_{n\to\infty}\left|a_k^{(n)}\right|^{1/n}$. □

Theorem 1.5. *For $1 \le k < d$, we have*

$$|\alpha_k| = \frac{\limsup_{n\to\infty}\left|a_k^{(n)}\right|^{1/n}}{\limsup_{n\to\infty}\left|a_{k-1}^{(n)}\right|^{1/n}}.$$

Proof:
By Proposition 1.4 we obtain

$$|\alpha_k| = \frac{|a_0\alpha_1\cdots\alpha_k|}{|a_0\alpha_1\ldots\alpha_{k-1}|} = \frac{\limsup_{n\to\infty}\left|a_k^{(n)}\right|}{\limsup_{n\to\infty}\left|a_{k-1}^{(n)}\right|}.$$

□

Previous approaches for obtaining upper bounds for $|\alpha_1\cdots\alpha_k|$ and $|\alpha_k|$ were derived by W. Specht [Specht] and M. Mignotte [Mignotte]. From the inequalities of W. Specht it is easy to deduce the next two results.

Proposition 1.6. *For all $n \in \mathbb{N}$, $n \geq 1$ and $k \in \mathbb{N}$, $1 \leq k \leq d$, we have*

$$|\alpha_1 \cdots \alpha_k| \;\leq\; \left(|a_0|^n + k \max_{1\leq j\leq d} |a_j^{(n)}|\right)^{\frac{1}{n}}.$$

Proof:

If we apply the inequalities of W. Specht [Specht] to the polynomial F_n, we get

$$|\alpha_1^n \cdots \alpha_k^n| \;\leq\; |a_0^{(n)}| + k\,\mathrm{H}(F_n)\,,$$

where $\mathrm{H}(F_n)$ is the largest absolute value of the coefficients. Hence the statement. ∎

Proposition 1.7. *For all $n \in \mathbb{N}$, $n \geq 1$ and $k \in \mathbb{N}$, $1 \leq k \leq d$, we have*

$$|\alpha_1 \cdots \alpha_k| \;\leq\; (1+k)\,\beta_n^k, \qquad |\alpha_k| \;\leq\; (1+k)^{1/k} \cdot \beta_n\,,$$

where $\beta_n = \max\{|a_1^{(n)}/a_0^n|,\, |a_2^{(n)}/a_0^n|^{1/2}, \ldots,\, |a_d^{(n)}/a_0^n|^{(1/d)}\}$.

Proof:

We observe that the result was proved by W. Specht [Specht] for $n=1$ and $a_0 = 1$ and we apply this case to the polynomial $(1/a_0^n)\,F_n$. ∎

Remarks:

1) The results of W. Specht have no siginificance for those k such that $|\alpha_1 \cdots \alpha_k| < 1$.
2) The coefficients of F_n grow with n, so $\mathrm{H}(F_n)$ and β_n become too large.
3) The smallest index t for which $|\alpha_t| > 1$ can be computed using the Schur–Cohn algorithm which is rather expensive, see [Henrici] and [Marden]. However, for guessing t, the following results will be useful, and this is cheaper.

Corollary 1.8. *There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, for all j with $\max_h |a_h^{(n)}| = |a_j^{(n)}|$, we have*

$$t \leq j \leq s\,.$$

Corollary 1.9. *If F has no roots on the unit circle, then t is the unique j for which $\max_h |a_h^{(n)}|$ is realized for large enough n.*

Proof:

In this case $t = s$ in Corollary 1.8. ∎

We remind that the measure $\mathrm{M}(\alpha)$ of an algebraic number α is the measure of any minimal polynomial P of α over $\mathbb{Z}$, i.e.

$$\mathrm{M}(\alpha) \;=\; \mathrm{M}(P) \;= \mathrm{lc}(P) \prod_{j=1}^{m} \max\{1, |z_j|\},$$

where $\mathrm{lc}\,(P)$ denotes the leading coefficient of P and $z_1, \cdots, z_m$ are the roots of P. The next result allows us to compute the indices t and s.

Proposition 1.10.

i) For all j, $t \le j \le s$, we have

$$\mathrm{M}(F) \;=\; |a_0\alpha_1\cdots\alpha_j| \;>\; |a_0\alpha_1\cdots\alpha_u|$$

if $u < t$ or $u > s$.

ii) We have

$$\mathrm{M}(F) \;=\; \limsup_n \big|a_j^{(n)}\big|^{1/n} \;=\; \limsup_n \big\{\max_j \big|a_j^{(n)}\big|^{1/n}\big\}\,.$$

Proof:

i) By the definition of the measure,

$$\mathrm{M}(F) \;=\; |a_0\alpha_1\cdots\alpha_t| \;=\; |a_0\alpha_1\cdots\alpha_s|\,.$$

We note that

$$|\alpha_1\cdots\alpha_t| \;=\; |\alpha_1\cdots\alpha_j| \;=\; |\alpha_1\cdots\alpha_s| \quad \text{for all} \quad t \le j \le s\,,$$

whereas $|\alpha_1\cdots\alpha_t| > |\alpha_1\cdots\alpha_u|$ for any $u < t$ and any $u > s$.
ii) We observe that $\mathrm{M}(F) = |a_0\alpha_1\cdots\alpha_t|$ and apply Proposition 1.4 for $k = t$. Then we use i. ∎

Remark: Empirically we always got

$$\liminf\{j;\,\max\big|a_h^{(n)}\big| = \big|a_j^{(n)}\big|\} = t, \qquad \limsup\{j;\,\max\big|a_h^{(n)}\big| = \big|a_j^{(n)}\big|\} = s\,,$$

but we are unable so far to prove these relations.

A2. An argument of linear recursive sequences

The drawback of the previous subsection is the occurence of "limsup" in the statements. Using a simple argument of linear algebra, we can obtain similar statements with some "limit". This is an important advantage for the computational approach of the problem.

Proposition 1.11. *Let $P \in \mathbb{C}[X]$, $h = \deg(P)$ and (u_n), (v_n) be linear recursive sequences admitting the polynomial P as characteristic polynomial, where (u_n) has exact order h. Then, there exists a constant $C > 0$ such that*

$$|v_n| \;\le\; C\,\max\{|u_n|, |u_{n+1}|, \ldots, |u_{n+h-1}|\}, \quad \forall n \in \mathbb{N}\,.$$

Proof:

The hypothesis on (u_n) implies that the sequences (u_n), (u_{n+1}), $\ldots$,(u_{n+h-1}) generate the space of the linear recursive sequences with characteristic polynomial P. Hence, there exist c_0, c_1, $\ldots$, $c_{h-1} \in \mathbb{C}$ such that

$$v_n = c_0 u_n + c_1 u_{n+1} + \cdots + c_{h-1} u_{n+h-1}\,.$$

Taking $C = \sum_{i=0}^{h-1} |c_i|$ we get the desired inequality. ∎

Theorem 1.12. *Let $T_n = \gamma_1^n + \gamma_2^n + \cdots + \gamma_d^n$, where $\gamma_1, \gamma_2, \ldots, \gamma_d$ are distinct complex numbers and*

$$|\gamma_1| \geq |\gamma_2| \geq \cdots \geq |\gamma_d|.$$

Then

$$|\gamma_1| = \lim_{n\to\infty} \Big(\max\{|T_n|, |T_{n+1}|, \ldots, |T_{n+d-1}|\}\Big)^{1/n}.$$

Proof:
We apply Proposition 1.11 with $u_n = T_n$, $v_n = \gamma_1^n$, $h = d$ and

$$P = \prod_{j=1}^{d} (X - \gamma_j).$$

The fact that the γ_j's are all distinct implies that T_n is of exact order d. We infer that there exists $C > 0$ with

$$|\gamma_1^n| \leq C \max\{|T_n|, |T_{n+1}|, \ldots, |T_{n+d-1}|\}, \quad \forall n \in \mathbb{N},$$

which gives

$$|\gamma_1| \leq \liminf_{n\to\infty} \Big(\max\{|T_n|, |T_{n+1}|, \ldots, |T_{n+d-1}|\}\Big)^{1/n}.$$

The inequality

$$|\gamma_1| \geq \limsup_{n\to\infty} \Big(\max\{|T_n|, |T_{n+1}|, \ldots, |T_{n+d-1}|\}\Big)^{1/n}$$

follows from the beginning of the proof of Theorem 1.3. Hence the result ∎

Proposition 1.13. *With the notation of the introduction, for all $1 \leq k < d$, we have*

$$|\alpha_1 \cdots \alpha_k| = \lim_{n\to\infty} \big(\max\{V_{k,n}, V_{k,n+1}, \ldots, V_{k,n+u(k)-1}\}\big)^{1/n}$$

where $V_{k,m} = |a_k^{(m)}|/|a_0^{(m)}|$, $u(k) = \binom{d}{k}$.

Proof:
We observe that $U_{k,m} = \sum (\alpha_{i_1} \cdots \alpha_{i_k})^m$ is the sum of the k–products of roots of the polynomial F_m and $U_{k,m} = (-1)^k\, a_k^{(m)}/a_0^{(m)}$. Then we apply Proposition 1.11. ∎

Corollary 1.14. *We have*

$$|\alpha_k| = \frac{\lim_{n\to\infty} \big(\max\{V_{k,n}, V_{k,n+1}, \ldots, V_{k,n+u(k)-1}\}\big)^{1/n}}{\lim_{n\to\infty} \big(\max\{V_{k-1,n}, V_{k-1,n+1}, \ldots, V_{k-1,n+u(k-1)-1}\}\big)^{1/n}}.$$

The following result allows us to obtain another evaluation for the absolute values of the roots.

Lemma 1.15. *Let (u_n) be a sequence of positive real numbers for which there exists a positive integer h such that $\left(\max\{u_n, u_{n+1}, \ldots, u_{n+h}\}\right)^{1/n}$ tends to a nonzero limit γ when n tends to infinity. Then the sequence $\left(\max\{u_n^{\frac{1}{n}}, u_{n+1}^{\frac{1}{n+1}}, \ldots, u_{n+h}^{\frac{1}{n+h}}\}\right)$ tends also to γ when n tends to infinity.*

Proof:
Put $\gamma_1 = \min\{1, \gamma/2\}$, $\gamma_2 = \max\{1, 2\,\gamma\}$ and

$$v_n = \left(\max\{u_n, u_{n+1}, \ldots, u_{n+h}\}\right)^{1/n}, \qquad w_n = \max\{u_n^{\frac{1}{n}}, u_{n+1}^{\frac{1}{n+1}}, \ldots, u_{n+h}^{\frac{1}{n+h}}\}.$$

Then, for $n \geq n_0$ we have

$$\gamma_1 \leq w_n \leq \gamma_2.$$

Thus, there exists $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$ and for any p, with $n \leq p \leq n+h$, we have

$$u_p^{\frac{1}{p}} = u_p^{\frac{1}{n}} \cdot u_p^{-\frac{p-n}{pn}} \leq v_n \cdot \gamma_1^{\frac{-h}{n^2}},$$

which shows immediately that

$$\limsup w_n \leq \gamma.$$

On the other direction, let m be such that $u_m = w_n^n$, then

$$w_n \geq u_m^{\frac{1}{m}} \geq v_n \cdot u_m^{-\frac{m-n}{mn}} \geq v_n \cdot \gamma_2^{-\frac{h}{n^2}},$$

from which we easily deduce that

$$\liminf w_n \geq \gamma.$$

Hence the result. $\square$

Proposition 1.16. *With the notation from Corollary 1.13, we have*

$$|\alpha_1 \cdots \alpha_k| = \lim_{n\to\infty} \left(\max\{V_{k,n}^{1/n}, V_{k,n+1}^{1/(n+1)}, \ldots, V_{k,n+u(k)-1}^{1/(n+u(k)-1)}\}\right)$$

and

$$|\alpha_k| = \frac{\lim_{n\to\infty} \left(\max\{V_{k,n}^{1/n}, V_{k,n+1}^{1/(n+1)}, \ldots, V_{k,n+u(k)-1}^{1/(n+u(k)-1)}\}\right)}{\lim_{n\to\infty} \left(\max\{V_{k-1,n}^{1/n}, V_{k-1,n+1}^{1/(n+1)}, \ldots, V_{k-1,n+u(k-1)-1}^{1/(n+u(k-1)-1)}\}\right)}.$$

Remark: Corollay 1.16 is more convenient than Corollaries 1.13 and 1.14 for the estimation of the absolute values of the roots.

Note that Corollary 1.16 gives also another formula for estimating the measure of a polynomial. We have

$$\mathrm{M}(F) = \lim_{n\to\infty} \left(\max\{V_{t,n}^{1/n}, V_{t,n+1}^{1/(n+1)}, \ldots, V_{t,n+u(t)-1}^{1/(n+u(t)-1)}\}\right).$$

2. The second estimate

Now we will use the measure and an argument à la Liouville.

Proposition 2.1. *If α and β are nonconjugate algebraic numbers, we have*

$$|\alpha-\beta| \geq 2^{1-mn}\,\mathrm{M}(\alpha)^{-n}\,\mathrm{M}(\beta)^{-m},$$

where M *denotes the measure,* $m=\deg(\alpha)$, $n=\deg(\beta)$.

Proof:

We suppose that $\alpha_1,\ldots,\alpha_m$ are the conjugates of α and $\beta_1,\ldots,\beta_n$ are the conjugates of β. We may suppose that $\alpha=\alpha_1$ and $\beta=\beta_1$.

Let $N=a_0^n b_0^m\prod_{i,j}(\alpha_i-\beta_j)$, with $a_0=\mathrm{lc}\,(P)$, $b_0=\mathrm{lc}\,(Q)$, where P is a minimal polynomial of α over $\mathbb{Z}$ and Q is a minimal polynomial of β over $\mathbb{Z}$. Observe that $N\in\mathbb{N}$, $N\neq 0$, so $|N|\geq 1$. We have

$$\begin{aligned}1 &\leq |a_0|^n\cdot|b_0|^m\,|\alpha-\beta|\cdot\prod_{(i,j)\neq(1,1)}|\alpha_i-\alpha_j|\\ &\leq |a_0|^n\cdot|b_0|^m\,|\alpha-\beta|\cdot\prod_{(i,j)\neq(1,1)}\Big(2\max\{|\alpha_i|\}\max\{|\beta_j|\}\Big)\\ &\leq |\alpha-\beta|\,2^{mn-1}\mathrm{M}(\alpha)^n\,\mathrm{M}(\beta)^m,\end{aligned}$$

which gives the desired estimation. ∎

Proposition 2.2. *If α and β are conjugate algebraic numbers, we have*

$$|\alpha-\beta| \geq 2^{1-\frac{m(m-1)}{2}}\,\mathrm{M}(\alpha)^{1-m},$$

where $m=\deg(\alpha)$.

Proof:

Let $N=\mathrm{Discr}(\alpha)$. We have $N\geq 1$ and assuming $\alpha_1=\alpha$ and $\beta=\alpha_2$ we obtain

$$1 \leq |a_0|^{m-1}\cdot|\alpha_1-\alpha_2|\cdot\prod_{\substack{(i,j)\neq(1,2)\\ i<j}}|\alpha_i-\alpha_j| \leq |\alpha-\beta|\cdot 2^{1-\frac{m(m-1)}{2}}\cdot\mathrm{M}(\alpha)^{m-1}$$

which is the desired inequality. ∎

Now we can find a lower bound for $|\alpha_t|>1$ with respect to the measure of F.

Theorem 2.3. *We have*

$$|\alpha_t| \geq \begin{cases} 1 + 2^{1-d} \cdot \mathrm{M}(F)^{-1} & \text{if } \alpha_t \in \mathbb{R}, \\ \sqrt{1 + 2^{1-d(d-1)}\,\mathrm{M}(F)^{-2d}} & \text{if } \alpha_t \notin \mathbb{R}. \end{cases}$$

Proof:
We distinguish two cases: α_t is real or not.

Suppose $\alpha = \alpha_t \in \mathbb{R}$. We have $|\alpha_t| > 1$ and we take $\beta = 1$. Note that α_t is not conjugate with $\beta = 1$, $n = 1$ and $\mathrm{M}(1) = 1$. By Proposition 2.1 we have

$$|\alpha_t - 1| \geq 2^{1-d} \cdot \mathrm{M}(\alpha)^{-1}.$$

Because $\mathrm{M}(\alpha_t) \leq \mathrm{M}(F)$ we obtain

$$|\alpha_t| \geq 1 + 2^{1-d} \cdot \mathrm{M}(F)^{-1}.$$

If $\alpha_t \notin \mathbb{R}$ we consider $\alpha = \alpha_t\overline{\alpha}_t = |\alpha_t|^2$ which is a real number. We observe that $\deg(\alpha) \leq d(d-1)$ and $\mathrm{M}(\alpha) \leq \mathrm{M}(\alpha_t)^{2d} \leq \mathrm{M}(F)^{2d}$. By the previous case we get

$$\Big||\alpha_t|^2 - 1\Big| \geq 2^{1-d(d-1)} \cdot \mathrm{M}(\alpha)^{-1} \geq 2^{1-d(d-1)}\,\mathrm{M}(F)^{-2d}.$$

We deduce that

$$|\alpha_t| \geq \sqrt{1 + 2^{1-d(d-1)}\,\mathrm{M}(F)^{-2d}},$$

which ends the proof. □

Corollary 2.4. *We have*

$$|\alpha_t| \geq \sqrt{1 + 2^{1-d(d-1)} \cdot \mathrm{M}(F)^{-2d}}.$$

Proof:
We observe that

$$\min\Big\{1 + 2^{1-d} \cdot \mathrm{M}(F)^{-1}, \sqrt{1 + 2^{1-d(d-1)} \cdot \mathrm{M}(F)^{-2d}}\Big\} = \sqrt{1 + 2^{1-d(d-1)} \cdot \mathrm{M}(F)^{-2d}}.$$

□

Our results allow us to give also an upper bound for the absolute values of the roots inside the unit circle.

Proposition 2.5. *We have*

$$|\alpha_{s+1}| \leq \left(1 + 2^{1-d(d-1)} \cdot \mathrm{M}(F)^{-2d}\right)^{-1/2},$$

where α_{s+1} is the root of maximal absolute value inside the unit circle. □

3. Applications

Theorem 3.1. *Let $F = PQ$ be a nontrivial factorization of F over $\mathbb{C}$ and suppose that F has no roots on the unit circle. If $M > 1$ is a lower bound for the absolute values of the roots outside the unit circle, $0 < K < 1$ is an upper bound for those inside this circle, $d = \deg(F)$ and t is the number of roots of F outside the unit circle, we have*

$$(M-1)^t\,(1-K)^{d-t}\,\mathrm{H}(P)\,\mathrm{H}(Q)\;\leq\;\mathrm{H}(F)^2\,.$$

Proof:
We suppose that the roots of P are $\{\alpha_i\,;\,i\in I\}$ and those of Q are $\{\alpha_j\,;\,j\in J\}$, where (I,J) is a partition of $\{1,2,\ldots,d\}$. By an inequality of M. Mignotte [Mignotte], we have

$$\mathrm{H}(P)\cdot\prod_{j\in J}\Big||\alpha_j|-1\Big|\;\leq\;\mathrm{H}(F),$$
$$\mathrm{H}(Q)\cdot\prod_{i\in I}\Big||\alpha_i|-1\Big|\;\leq\;\mathrm{H}(F)\,.$$

We obtain

$$\mathrm{H}(P)\,\mathrm{H}(Q)\cdot\prod_{k=1}^{d}\Big||\alpha_k|-1\Big|\;\leq\;\mathrm{H}(F)^2\,.$$

From the hypotheses we know that $s = t$, where s is defined as in the introduction. Then we observe that

$$\prod_{k=1}^{t}\Big||\alpha_k|-1\Big|\;\geq\;(M-1)^t\quad\text{and}\quad\prod_{k=t+1}^{d}\Big||\alpha_k|-1\Big|\;\geq\;(1-K)^{d-t}\,,$$

which gives the result. ☐

Remark: If the polynomial F has no roots on the unit circle (i.e. $s = t$), Theorem 3.1 gives an upper bound for $\min\{\mathrm{H}(P),\mathrm{H}(Q)\}$. By the Schur–Cohn criterion (cf. [Marden], p. 198) there are known sufficient conditions for F not to have roots on the circle $|z| = 1$. These conditions use determinant sequences.
If $F \in \mathbb{R}[X]$ there exist more direct conditions to have $s = t$. For example, if F and its reciprocal F^* are coprime, F has no roots on the unit circle.

Corollary 3.2. *We have*

$$\frac{(B-1)^d}{B^{d-t}}\cdot\mathrm{H}(P)\,\mathrm{H}(Q)\;\leq\;\mathrm{H}(F)^2\,,$$

where $B \in \{B_1, B_2\}$ with

$$B_1\;=\;\frac{\limsup_{n\to\infty}\big|a_t^{(n)}\big|}{\limsup_{n\to\infty}\big|a_{t-1}^{(n)}\big|},\qquad B_2\;=\;\sqrt{1+2^{1-d(d-1)}\cdot\mathrm{M}(F)^{-2d}}.$$

Proof:
We apply Theorems 1.5, 2.3 and Proposition 2.5. Then in Theorem 3.1. we can take $M = B_j$ and $K = B_j^{-1}$, for any $j = 1, 2$. ∎

Further we obtain an evaluation for $|\alpha_t|$ in function of the length and measure of the polynomial F. We remind that the length of F is

$$\mathrm{L}(F) \;=\; |a_0| + |a_1| + \cdots + |a_d|\,.$$

Lemma 3.3. *Suppose that $P \in \mathbb{C}[X] \setminus \mathbb{C}$ and let $Q(X) = (X - \alpha)P(X)$, $\alpha \in \mathbb{C} \setminus \{0\}$. If $|\alpha| > 1$, we have*

$$(|\alpha| - 1)\mathrm{L}(P) \,\leq \mathrm{L}(Q)\,.$$

Proof:
Assume that $m \;=\; \deg(P) \;\geq\; 1$ and let

$$P(X) = \sum_{i=0}^{m} c_i X^i, \qquad Q(X) = \sum_{i=0}^{d+1} b_i X^i\,.$$

We have

$$b_i = a_{i-1} - \alpha a_i \quad \text{for all } i = 0, \ldots, d+1,$$

with $a_{-1} = a_{d+1} = 0$. Therefore

$$\alpha^i\, a_i \;=\; \alpha^{i-1}\, a_{i-1} - \alpha^{i-1}\, b_i$$

and by summation we get

$$\alpha^i\, a_i \;=\; -\sum_{j=0}^{i} \alpha^{j-1} b_j\,.$$

It follows that

$$|\alpha| \cdot |c_i| \;\leq\; \frac{1}{|\alpha|^i} \cdot |b_0| + \frac{1}{|\alpha|^{i-1}} \cdot |b_1| + \ldots + |b_i|\,,$$

hence

$$|\alpha|\,\mathrm{L}(P) \;\leq\; \frac{1 - \frac{1}{|\alpha|^{d+1}}}{1 - \frac{1}{|\alpha|}} \cdot \mathrm{L}(Q) \;<\; \frac{1}{1 - \frac{1}{|\alpha|}} \cdot \mathrm{L}(Q).$$

Therefore

$$(|\alpha| - 1) \cdot \mathrm{L}(P) \;\leq\; \mathrm{L}(Q)\,.$$

∎

Corollary 3.4. *Let F be a polynomial over $\mathbb{C}$ such that $F(0) \neq 0$ and let $\alpha_1, \ldots, \alpha_d$ be its roots, where*

$$|\alpha_1| \;\geq\; \cdots \;\geq\; |\alpha_t| > 1 \;\geq\; |\alpha_{t+1}| \;\geq\; \cdots \;\geq\; |\alpha_d|\,.$$

We have

$$(|\alpha_1|-1)\cdots(|\alpha_t|-1)\cdot \mathrm{L}(P) \;<\; \mathrm{L}(F)\,,$$

where $P(X) \;=\; \dfrac{F(X)}{(X-\alpha_1)\cdots(X-\alpha_t)}$. *Thus*

$$\big(|\alpha_t|-1\big)^t\cdot \mathrm{L}(P) \;<\; \mathrm{L}(F)\,.$$

Proof:

Obvious induction. □

Proposition 3.5. *We have*

$$|\alpha_t| \;\le\; 1+\Big(\frac{\mathrm{L}(F)}{|a_o|\,\big(1+|a_d|\,\mathrm{M}(F)^{-1}\big)}\Big)^{1/t}\,.$$

Proof:

We use the previous notation. Let $b_0=\mathrm{lc}(P)$ and $b_{d-t}=P(0)$. Because

$$F(X)=a_0X^d+\cdots+a_{d-1}X+a_d \;=\; (X-\alpha_1)\cdots(X-\alpha_t)P(X)$$

$$=\big(X^t-(\alpha_1+\ldots+\alpha_t)X^{t-1}+\cdots+(-1)^t\,\alpha_1\cdots\alpha_t\big)\cdot\big(b_0X^{d-t}+\cdots+b_{d-t}\big)\,,$$

we observe that $b_0=a_0$ and $|b_{d-t}|\,\mathrm{M}(F)=|a_0a_d|$. By the previous corollary, we have

$$\big(|\alpha_t|-1\big)\cdot \mathrm{L}(P) \;<\; \mathrm{L}(F)\,.$$

Since $\mathrm{L}(P)\ge |b_0|+|b_{d-t}|=|a_0|+|a_0a_d|\,\mathrm{M}(F)^{-1}$ it follows that

$$\big(|\alpha_t|-1\big)^t \;\le\; \frac{\mathrm{L}(F)}{|a_0|\,\big(1+|a_d|\,\mathrm{M}(F)^{-1}\big)}\,,$$

which proves the statement. □

Remark: Using the reciprocal polynomial of F, Proposition 3.5 gives also a lower bound for $|\alpha_{s+1}|$. Using the polynomials F_n we get:

Corollary 3.6. *For all* $n\ge 1$, *we have*

$$|\alpha_t| \;\le\; \left(1+\Big(\frac{\mathrm{L}(F_n)}{|a_0|^n\,\big(1+|a_d^{(n)}|\,\mathrm{M}(F_n)^{-1}\big)}\Big)^{1/t}\right)^{1/n}\,.$$

4. Examples

We consider the polynomials:

$P_1 = x^3 - 3x + 1$
$P_2 = x^5 - 3x^4 + 6x^3 + 12x^2 - x - 1$
$P_3 = x^7 + x^6 - 6x^5 + x^4 + x^2 - 3x + 2$
$P_4 = 9x^{10} - 12x^7 + 8x + 11$
$P_5 = x^{11} + 2x^3 + 4x^2 + 5x + 2$

4.1. Products of roots

We compute the estimates for the absolute values of the products of k roots using Proposition 1.4, Proposition 1.6 (W. Specht) and Corollary 1.16.

F	k	Prop 1.4	Prop 1.6 (n=8)	Cor 1.16
P_1	1	2.059767	2.879464	2.059767
P_1	2	2.879385	3.140036	2.884499
P_1	3	1.000000	3.303271	1.000000
P_2	1	3.593041	12.003618	3.593041
P_2	2	10.594434	13.090039	10.599896
P_2	3	12.003559	13.770583	12.004757
P_2	4	3.949090	14.274788	3.949090
P_3	1	1.539351	5.831802	3.078702
P_3	2	2.913727	6.359624	5.827455
P_3	3	2.684445	6.690258	5.368890
P_4	1	1.050908	16.933503	1.284443
P_4	2	1.098612	18.458756	1.342748
P_4	3	1.225112	19.415833	1.496186
P_4	4	1.382391	20.125396	1.689589
P_4	5	1.421513	20.693827	1.737405
P_4	6	1.453814	21.170294	1.776884
P_4	7	1.423304	21.581765	1.739594
P_5	1	1.323257	3.463119	1.328016
P_5	2	1.688622	3.776546	1.705553
P_5	3	2.110073	3.972883	2.130057
P_5	4	2.631219	4.118347	2.654690
P_5	5	2.997226	4.234835	2.966073
P_5	6	3.411606	4.332455	3.311505
P_5	7	3.639600	4.416746	3.343063
P_5	8	3.878782	4.491086	3.371435
P_5	9	4.025778	4.557696	3.358011

Note that for P_1, Theorem 1.5 and Corollary 1.16 gives $|\alpha_2| \sim 1.4004$. In this case the real value of $|\alpha_2|$ is close to 1.532.

Remark: The estimates of W. Specht seem to be less precise. In the column from the previous table corresponding to Proposition 1.6 we considered $n = 8$, which is very small. However, we noticed that $n = 12$ (for example) gives almost the same precision. The following table lists the output for polynomials P_1, P_2 and P_4.

F	k	Prop 1.6 (n=8)	Prop 1.6 (n=12)
P_1	1	2.879464	2.879385
P_1	2	3.140036	3.050602
P_1	3	3.303271	3.155439
P_2	1	12.003618	12.003556
P_2	2	13.090039	12.717325
P_2	3	13.770583	13.154369
P_2	4	14.274788	13.473536
P_4	1	16.933503	15.695898
P_4	2	18.458756	16.628349
P_4	3	19.415833	17.199499
P_4	4	20.125396	17.616659
P_4	5	20.693827	17.947216
P_4	6	21.170294	18.221914
P_4	7	21.581765	18.457154

However, if n is too large $\mathrm{H}(F_n)^{1/n}$ may become too big. For example, taking $F = P_4$, we found $\mathrm{H}(F_{16})^{1/16} \sim 17.22$.

The pari function 'polroots' gives the following absolute values of the roots of P_4:

$$\text{The matrix of absolute values of } P_4 = \begin{pmatrix} 0.8520101805895330601809857770 \\ 0.8520101805895330601809857770 \\ 0.97342109417849841536903070 1 \\ 0.973421094178498415369030701 \\ 1.025507677664610779264720220 \\ 1.025507677664610779264720220 \\ 1.138445451646462560098875152 \\ 1.138445451646462560098875152 \\ 1.141769422327605252088522524 \\ 1.141769422327605252088522524 \end{pmatrix}$$

4.2. Bounds for $|\alpha_t|$

We use the estimates given by Theorem 1.5, Corollary 1.16, Corollary 2.4 and Proposition 3.5.

For many polynomials Theorem 2.3 and its corollary give values for $|\alpha_t|$ which are extremely close to 1. In the table we consider $\log(K-1)/\log(10^{-1})$, where K is the estimate given in Corollary 2.4. Working in pari with a realprecision of 72 digits, we obtained:

F	Theorem 2.3
P_1	1.0000021419600373055611078056378338402006294701230459484440013731467839
P_2	1.0000000000000000000749843805588107743091487412004176130365899144697621
P_4	1.0000000000000000000000000000000000000000000000000000000032164769806879

The usual 28 digits pari realprecision gives, for example, the estimate 1.00000000000000000000000000 for $K(P_4)$ which is not satisfactory.

We chose t according to the lower bounds for the absolute values of products of roots given in the column with output by Corollary 1.16.

F	t	Th 1.5	Cor 1.16	Cor 2.4 lo(F)	Prop 3.5 (n=8)
P_1	2	1.400400	1.400400	6.669188	1.704011
P_2	3	1.133006	1.132535	20.125029	2.320128
P_4	6	1.022723	1.022723	57.492619	1.153398
P_3	2	1.892828	1.889433	27.875414	1.892828
P_5	8	1.065716	1.008486	51.670409	1.149588

Remark: Note that for the polynomial P_4 we have $t = 6$. Using the Corollary 1.16 and the Proposition 3.5 we obtain $1.0227 < |\alpha_6| < 1.153$. More precision in the left hand side can be obtained using a larger n. On the other hand the function 'polroots' from the pari package gives $|\alpha_6| \sim 1.025$.

Remark: Theorem 1.5 and Corollary 1.16 give very close results.

The next table compares the results given by Proposition 3.5 for $n = 8$ and $n = 12$.

F	t	Prop 3.5 (n=8)	Prop 3.5 (n=12)
P_1	2	1.704011	1.697580
P_2	3	2.320128	2.302526
P_4	6	1.153398	1.110241

References

[Henrici] P. Henrici — *Computational Complex Analysis*, vol. 1, Wiley, New York (1974).

[Householder] A.S. Householder — Dandelin, Lobachevskii or Graeffe?, *Amer. Math. Monthly*, **66**, 464–466 (1959).

[Landau] E. Landau — Sur quelques théorèmes de M. Petrovič relatifs aux zéros des fonctions analytiques, *Bull. Math. Soc. Math. France*, **33**, 251–261 (1905).

[Marden] M. Marden — *Geometry of Polynomials*, AMS Math. Surv. and Monographs, Providence (1989).

[Mignotte] M. Mignotte — An inequality on the greatest roots of a polynomial, *Elem. d. Math.*, **3**, 86–87 (1991).

[Moussa] O. Moussa — An inequality about the largest roots of a polynomial, *AACC*, **8**, 497–504 (1997).

[Schmidt] W. M. Schmidt — *Diophantine Approximation*, Lect. Notes Math. 785, Springer Verlag, Berlin–Heidelberg (1980).

[Specht] W. Specht — Zur Theorie der algebraischen Gleichungen, *Jahr. DMV*, **48**, 142–145 (1938).

Maurice Mignotte
Université Louis Pasteur
7, Rue Descartes
67084, Strasbourg
France
Email: mignotte@math-u.strasbg.fr

Doru Ştefănescu
University of Bucharest
PO Box 39–D5
Bucharest 39
Romania
Email: stef@irma.u-strasbg.fr