AN ERROR ANALYSIS FOR RADIAL BASIS FUNCTION INTERPOLATION

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Abstract. Radial basis function interpolation refers to a method of interpolation which writes the interpolant to some given data as a linear combination of the translates of a single function ϕ and a low degree polynomial. We develop an error analysis which works well when the Fourier transform of ϕ has a pole of order $2m$ at the origin and a zero at ∞ of order 2κ. In case $0 \le m \le \kappa$, we derive error estimates which fill in some gaps in the known theory; while in case $m > \kappa$ we obtain previously unknown error estimates. In this latter case, we employ dilates of the function ϕ, where the dilation factor corresponds to the fill distance between the data points and the domain.

1. Introduction

Let d be a positive integer. Given a finite set of scattered points $\Xi \subset \mathbb{R}^d$ and data $f_{|\Xi}$, the scattered data interpolation problem refers to the problem of finding a 'nice' function $s : \mathbb{R}^d \to \mathbb{C}$ which satisfies the interpolation conditions

$$s(\xi) = f(\xi) \text{ for all } \xi \in \Xi. \tag{1.1}$$

One standard approach to this problem goes by the name *radial basis function interpolation* which we now describe.

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For integers k, let Π_k denote the space of polynomials (over $\mathbb{R}^d$) whose total degree does not exceed k (note that $\Pi_k = \{0\}$ when $k < 0$). Starting with a continuous function $\phi : \mathbb{R}^d \to \mathbb{R}$ and an integer $k \in \{-1, 0, 1, 2, \dots\}$, the radial basis function approach suggests that the interpolant s be written in the form

$$s = q + \sum_{\xi \in \Xi} \lambda_\xi \phi(\cdot - \xi), \tag{1.2}$$

where $q \in \Pi_k$ and λ satisfies the auxiliary conditions

$$\sum_{\xi \in \Xi} \lambda_\xi q(\xi) = 0 \text{ for all } q \in \Pi_k. \tag{1.3}$$

Let

$$T_{\Xi,\phi,k} f$$

denote the set of all functions of the form (1.2)-(1.3) which satisfy (1.1). In order to guarantee that $T_{\Xi,\phi,k} f$ is nonempty, one must carefully choose the function ϕ and the integer k.

One way of ensuring that $T_{\Xi,\phi,k} f$ contains a unique function is to assume that ϕ is a radially symmetric function which is conditionally positive definite of order k and additionally that Ξ is a finite subset of $\mathbb{R}^d$ which is not contained in the zero set of any nontrivial polynomial in Π_k. This approach has been taken up by Micchellii [10] where such functions ϕ are characterized. However, for the purpose of estimating the interpolation error, the author prefers a construction recently developed by Light and Wayne [8].

Building upon work of Duchon [4] and Madych and Nelson [MN], Light and Wayne give sufficient conditions on a continuous function $\phi : \mathbb{R}^d \to \mathbb{R}$ which ensure that $T_{\Xi,\phi,k} f$ is nonempty. We will not describe these conditions in their full generality, but rather restrict our attention to a special case. Let $\mathcal{R}_0$ denote the space of all tempered distributions f

whose Fourier transform $\widehat{f}$ can be identified on $\mathbb{R}^d \backslash \{0\}$ with a function, denoted $\widehat{f}_|$, which is locally integrable on $\mathbb{R}^d \backslash \{0\}$. For real numbers $m, \kappa \geq 0$, we define $w_{m,\kappa} \in C(\mathbb{R}^d)$ by

$$w_{m,\kappa}(t) := \begin{cases} |t|^m & \text{if } |t| \leq 1 \\ |t|^\kappa & \text{if } |t| > 1 \end{cases}.$$

Definition 1.4. A tempered distribution $\phi \in \mathcal{R}_0$ is *pre-(m, κ, k)-admissible* if the following hold:

1. $m \geq 0$ and $\kappa > d/2$ are real numbers and k is an integer satisfying

$$k \geq \underline{k} := \max\{\lfloor m - d/2 \rfloor, -1\}.$$

2. The function $\widehat{\phi}_|$ is almost everywhere positive and satisfies, for some constant $A \geq 1$,

$$\frac{1}{A} w_{m,\kappa}(t) \leq \frac{1}{\sqrt{\widehat{\phi}_|(t)}} \leq A w_{m,\kappa}(t) \text{ for almost all } t \in \mathbb{R}^d. \tag{1.5}$$

We say that ϕ is (m, κ, k)-*admissible* if ϕ is pre-(m, κ, k)-admissible and if

3. $\langle g, \widehat{\phi} \rangle = \int_{\mathbb{R}^d} g(t) \widehat{\phi}_|(t)\, dt$ for all $g \in C_c^\infty(\mathbb{R}^d)$ satisfying $|g(t)| = O(|t|^{2(k+1)})$ as $|t| \to 0$.

Light and Wayne [8] have shown that if ϕ is pre-(m, κ, k)-admissible, then there exists a polynomial p such that $\phi - p$ is (m, κ, k)-admissible. Moreover, they have proved the following result.

Theorem 1.6. *Let ϕ be (m, κ, k)-admissible and let Ξ be a nonempty, finite subset of $\mathbb{R}^d$ which satisfies*

$$q(\Xi) \neq \{0\} \text{ for all } q \in \Pi_k \backslash \{0\}. \tag{1.7}$$

Then ϕ is continuous and $T_{\Xi,\phi,k} f$ contains exactly one function for all data functions f.

In their construction, the unique interpolant $s \in T_{\Xi,\phi,k} f$ is identified as the unique function in an appropriately defined Hilbert space which minimizes a certain semi-norm subject to the interpolation constraints (1.1). Using a technique previously employed by Bezhaev and Vasilenko [2], we will show in section 3 that even without condition (1.7), $T_{\Xi,\phi,k}f$ is still nonempty.

Example. Let m, κ, k be as in Definition 1.4. An example of a function ϕ which is pre-(m, κ, k)-admissible can be written as

$$\phi(x) = |x|^{\kappa-d/2} K_{\kappa-d/2}(|x|) + c_{m,d} \begin{cases} (1+|x|^2)^{m-d/2}, & \text{if } m - d/2 \notin \mathbb{N}_0, \\ (1+|x|^2)^{m-d/2} \log(1+|x|^2), & \text{if } m - d/2 \in \mathbb{N}_0, \end{cases}$$

where K_ν denotes the modified Bessel function of order ν (see [1]), $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, and
$$c_{m,d} := \begin{cases} m & \text{if } 0 \le m < d/2, \\ (-1)^{1+\lfloor m-d/2 \rfloor} & \text{if } m \ge d/2. \end{cases}$$
It turns out that ϕ is in fact (m, κ, k)-admissible since $|\phi(x)| = o(|x|^{2(k+1)})$ as $|x| \to \infty$ (see Theorem 2.9).

In order to discuss the interpolation error, let Ω be a bounded subset of $\mathbb{R}^d$ having the cone property, and assume that the interpolation points Ξ are contained in Ω. We denote by h the *fill distance* from Ξ to Ω given by

$$h := h(\Xi, \Omega) := \sup_{x \in \Omega} \inf_{\xi \in \Xi} |x - \xi| .$$

For $\gamma > 0$, let W_2^γ denote the Sobolev space of all $f \in L_2 := L_2(\mathbb{R}^d)$ for which

$$\|f\|_{W_2^\gamma} := \left\| (1 + |\cdot|^2)^{\gamma/2} \widehat{f} \right\|_{L_2} < \infty.$$

Employing Light and Wayne's characterization of $T_{\Xi,\phi,k}f$, we are able to prove the following result.

Theorem 1.8. *Let Ω be a bounded subset of $\mathbb{R}^d$ having the cone property, and let ϕ be (m, κ, k)-admissible with $\kappa \ge \lfloor d/2 \rfloor + 1$ and $0 \le m \le \kappa$. There exists $h_0 > 0$ (depending* *only on k, κ, Ω) such that if Ξ is a finite subset of Ω satisfying $h := h(\Xi, \Omega) \le h_0$, then for all $f \in W_2^\kappa$ and $1 \le p \le \infty$,*

$$\|f - s\|_{L_p(\Omega)} \le A^2 const(\kappa, k, \Omega) h^{\kappa - \theta_p} \|f\|_{W_2^\kappa}, \tag{1.9}$$

where $T_{\Xi,\phi,k} f = \{s\}$ and $\theta_p := \max\{d/2 - d/p, 0\}$.

This result was first proved by Duchon [5] for the particular choice of ϕ associated with surface splines. When taken with the construction of Light and Wayne, this amounts to the case when $m = \kappa$ are integers and $k = m - 1$. The case $p = \infty$ has been settled by Wu and Schaback [13], while the case when κ is an integer, $m = 0$, and $k = -1$ has been handled by Wendland [12]. We expect that the theorem remains true in case $d/2 < \kappa < \lfloor d/2 \rfloor + 1$, but our techniques are unable to cope with this case.

Without the restriction $m \le \kappa$, our error analysis breaks down. However, the case $m > \kappa$ can be salvaged if one employs $T_{\Xi,\phi_h,k} f$ instead of $T_{\Xi,\phi,k} f$, where

$$\phi_h := \phi(\cdot/h).$$

Note that the difference between $T_{\Xi,\phi,k} f$ and $T_{\Xi,\phi_h,k} f$ is that the former always employs the function ϕ in (1.2), whereas the latter employs a dilated version of ϕ, where the dilation factor matches the fill distance $h = h(\Xi, \Omega)$. Following the language of 'shift-invariant spaces', we refer to this as a *stationary dilation* of ϕ. In this case our error analysis yields the following result.

Theorem 1.10. *Let Ω be a bounded subset of $\mathbb{R}^d$ having the cone property, and let ϕ be (m, κ, k)-admissible with $\lfloor d/2 \rfloor + 1 \le \kappa \le m$. There exists $h_0 > 0$ (depending only on*

k, m, Ω) such that if Ξ is a finite subset of Ω satisfying $h := h(\Xi, \Omega) \leq h_0$, then for all real numbers γ, with $\kappa \leq \gamma \leq m$, and for all $f \in W_2^\gamma$ and $1 \leq p \leq \infty$,

$$\|f - s\|_{L_p(\Omega)} \leq A^2 const(k, m, \Omega) h^{\gamma - \theta_p} \|f\|_{W_2^\gamma}, \tag{1.11}$$

where $T_{\Xi,\phi_h,k} f = \{s\}$ and $\theta_p := \max\{d/2 - d/p, 0\}$.

We mention that the error estimate in Theorem 1.10 is more robust than that of Theorem 1.8 in the sense that (1.11) holds for all $f \in W_2^\gamma$ with γ in the interval $[\kappa, m]$, whereas (1.9) is asserted only for $f \in W_2^\kappa$ (see [14], [3], and [11] for error estimates when $f \notin W_2^\kappa$).

These theorems are actually special cases of the more general result Theorem 5.4. There it is not assumed that Ω has the cone property, but rather that Ω satisfies a certain condition related to polynomials (see Definition 4.1). Duchon [5] has shown that this condition is satisfied if Ω has the cone property, while Golitschek and Light [6] have established this in case Ω is the sphere $S^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$.

An outline of the paper is as follows. In section 2 we define and examine certain spaces $Y_{m,\kappa} + \Pi_k$. One useful observation made in Theorem 2.9 is that the third condition of Definition 1.4 can be replaced with a certain growth condition on ϕ. The spaces $Y_{m,\kappa} + \Pi_k$ are then identified in section 3 with the Hilbert spaces constructed by Light and Wayne [8] in association with an (m, κ, k)-admissible function ϕ. In section 4 we develop a preliminary result which is then used in section 5 to prove our main result, Theorem 5.4. Following this the case when Ω has the cone property is discussed and it is explained how Theorem 1.8 and Theorem 1.10 can be derived from Theorem 5.4. In section 6, the case when Ω is the sphere S^{d-1} is treated in detail, and finally in section 7, it is explained how Theorem 5.4 can be applied in cases which go beyond those explicitly mentioned in the theorem.

Throughout this paper we use standard multi-index notation: $D^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}$. The natural numbers are denoted $\mathbb{N} := \{1, 2, 3, \ldots\}$, and the non-negative integers are denoted $\mathbb{N}_0$. For $t \in \mathbb{R}$, we employ the notation $\lfloor t \rfloor$ to denote the greatest integer which is less or equal to t, while $\lceil t \rceil$ denotes the least integer which is greater or equal to t. For multi-indices $\alpha \in \mathbb{N}_0^d$, we define $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_d$, while for $x \in \mathbb{R}^d$, we define $|x| := \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$. The monomial $x \mapsto x^\alpha$ is denoted $()^\alpha$. With this notation we can write $\Pi_k = span\{()^\alpha : |\alpha| \le k\}$. The Fourier transform of an integrable function f is defined by $\widehat{f}(w) := \int_{\mathbb{R}^d} e_w(-x) f(x)\, dx$, where $e_w(x) := e^{iw \cdot x}$. The space of compactly supported C^∞ functions whose support is contained in $A \subset \mathbb{R}^d$ is denoted $C_c^\infty(A)$. The open unit ball in $\mathbb{R}^d$ is denoted $B := \{x \in \mathbb{R}^d : |x| < 1\}$. If μ is a distribution and g is a test function, then the application of μ to g is denoted $\langle g, \mu \rangle$. We employ the notation $const$ to denote a generic constant in the range $(0, \infty)$ whose value may change with each occurrence. An important aspect of this notation is that $const$ depends only on its arguments if any, and otherwise depends on nothing.

2. The spaces $Y_{m,\kappa} + \Pi_k$

In this section we assume that $m \ge 0$ and $\kappa > d/2$ are real numbers, and in keeping with Definition 1.4, we define $\underline{k} := \max\{\lfloor m - d/2 \rfloor, -1\}$ and

$$w_{m,\kappa}(x) := \begin{cases} |x|^m & \text{if } |x| \le 1 \\ |x|^\kappa & \text{if } |x| > 1 \end{cases}, \quad x \in \mathbb{R}^d.$$

As in the introduction, let $\mathcal{R}_0$ denote the space of all tempered distributions f whose Fourier transform $\widehat{f}$ can be identified on $\mathbb{R}^d \backslash \{0\}$ with a function, denoted $\widehat{f}_|$, which is locally integrable on $\mathbb{R}^d \backslash \{0\}$. Let $\widetilde{Y}_{m,\kappa}$ be the space of all $f \in \mathcal{R}_0$ for which

$$|f|_{m,\kappa} := \left\| w_{m,\kappa} \widehat{f}_| \right\|_{L_2} < \infty.$$

Note that the kernel of the semi-norm $|\cdot|_{m,\kappa}$ in $\widetilde{Y}_{m,\kappa}$ is the space of polynomials Π. Let $L_2(\mathbb{R}^d, w_{m,\kappa}^2)$ denote the space of (equivalence classes) of measurable functions $\nu : \mathbb{R}^d \to \mathbb{C}$ satisfying

$$\|w_{m,\kappa}\nu\|_{L_2} < \infty,$$

and let $\sigma \in C_c^\infty(2B)$ satisfy $\sigma = 1$ on B. For $\nu \in L_2(\mathbb{R}^d, w_{m,\kappa}^2)$, we define the tempered distribution $f = V_{m,\kappa}(\nu)$ by

$$\langle g, \widehat{f}\rangle := \int_{\mathbb{R}^d} (g - \sigma P_{\underline{k}} g)\nu, \quad g \in C_c^\infty(\mathbb{R}^d), \tag{2.1}$$

where $P_{\underline{k}}g$ denotes the $\underline{k}$-th degree Taylor polynomial of g at 0. The choice of $\underline{k}$ is sufficiently large to ensure that the above integrand is absolutely integrable and that $\widehat{f}$ is a tempered distribution. Hence $f = V_{m,\kappa}(\nu)$ is a tempered which belongs to $\widetilde{Y}_{m,\kappa}$ since $|f|_{m,\kappa} = \|w_{m,\kappa}\nu\|_{L_2} < \infty$. The range of the operator $V_{m,\kappa} : L_2(\mathbb{R}^d, w_{m,\kappa}^2) \to \widetilde{Y}_{m,\kappa}$ apparently depends on the choice of σ. This dependence can be eliminated by simply adding $\Pi_{\underline{k}}$. With this in mind, we define

$$Y_{m,\kappa} := V_{m,\kappa}(L_2(\mathbb{R}^d, w_{m,\kappa}^2)) + \Pi_{\underline{k}}.$$

Proposition 2.2. *The space $Y_{m,\kappa}$ is independent of σ.*

Proof. If $f = V_{m,\kappa}(\nu)$ and $f' = V'_{m,\kappa}(\nu)$, where f' is defined via (2.1) using σ' instead of σ, then for $g \in C_c^\infty(\mathbb{R}^d)$, we have

$$\langle g, \widehat{f} - \widehat{f'}\rangle = \int_{\mathbb{R}^d} (\sigma' - \sigma)(P_{\underline{k}} g)\nu = \sum_{|\alpha| \le \underline{k}} \frac{D^\alpha g(0)}{\alpha_1! \alpha_2! \cdots \alpha_d!} \int_{\mathbb{R}^d \setminus B} (\sigma'(t) - \sigma(t)) t^\alpha \nu(t)\, dt.$$

Hence, $f - f' \in \Pi_{\underline{k}}$. $\square$

We state here some relations which are simple consequences of the definitions. Assume $0 \leq m \leq m'$ and $d/2 < \kappa' \leq \kappa$, and note that $\underline{k}' := \max\{\lfloor m' - d/2 \rfloor, -1\} \geq \underline{k}$. Since $w_{m',\kappa'} \leq w_{m,\kappa}$, it follows that

$$\widetilde{Y}_{m,\kappa} \subset \widetilde{Y}_{m',\kappa'} \text{ and } |f|_{m',\kappa'} \leq |f|_{m,\kappa} \text{ for all } f \in \widetilde{Y}_{m,\kappa}. \tag{2.3}$$

To see that

$$Y_{m,\kappa} \subset Y_{m',\kappa'}, \tag{2.4}$$

let $f \in Y_{m,\kappa}$, say $f = V_{m,\kappa}(\nu) + q$, where $\nu \in L_2(\mathbb{R}^d, w^2_{m,\kappa})$ and $q \in \Pi_{\underline{k}}$. With $f' := V_{m',\kappa'}(\nu)$, we see that

$$\langle g, \widehat{f} - \widehat{f'} \rangle = \langle g, \widehat{q} \rangle + \int_{\mathbb{R}^d} \sigma(P_{\underline{k}'} g - P_{\underline{k}} g)\nu, \quad g \in C_c^\infty(\mathbb{R}^d),$$

whence it follows that $f - f' \in \Pi_{\underline{k}'}$. Hence $f \in Y_{m',\kappa'}$ which proves (2.4). It is easy to see that $W_2^\kappa = Y_{0,\kappa}$. If $d/2 < \kappa \leq \gamma$ and $m \geq 0$, it then follows from the above relations that

$$W_2^\gamma \subset Y_{m,\kappa} \text{ and } |f|_{m,\kappa} \leq \|f\|_{W_2^\gamma} \text{ for all } f \in W_2^\gamma. \tag{2.5}$$

Now let us assume that ϕ is pre-(m, κ, k)-admissible (see Definition 1.4). As mentioned in the introduction, Light and Wayne [8] have shown that there exists a polynomial p such that $\phi - p$ is (m, κ, k)-admissible. However, the technical nature of the third condition in Definition 1.4 makes the task of finding such a polynomial p rather difficult in practice. We will show that the third condition of Definition 1.4 is actually equivalent to a growth condition on ϕ. We begin by examining the growth of $V_{m,\kappa}(\nu)$.

Proposition 2.6. *Let $m \in [0,\infty)$, $\kappa \in (d/2,\infty)$ and put $\mu := \underline{k} + \dfrac{m-d/2}{\underline{k}+1}$ if $\underline{k} > -1$. If $\nu \in L_2(\mathbb{R}^d, w_{m,\kappa}^2)$ and $f = V_{m,\kappa}(\nu)$, then for all $x \in \mathbb{R}^d$,*

$$|f(x)| \le const(d,m,\kappa)\, |f|_{m,\kappa} \begin{cases} (1+|x|)^{\mu}, & \text{if } m > d/2, \\ \sqrt{\log(2+|x|)}, & \text{if } m = d/2, \\ 1, & \text{if } m < d/2. \end{cases}$$

Proof. For the sake of brevity, let us employ the abbreviation $c = const(d,m,\kappa)$. Since $\kappa > d/2$, we have $f(x) = (2\pi)^{-d}\langle e_x, \widehat{f}\rangle$. Hence,

$$(2\pi)^d\,|f(x)| \le \int_{\mathbb{R}^d} \left|e_x - \sigma P_{\underline{k}} e_x\right| |\nu| \le \left\| \frac{e_x - \sigma P_{\underline{k}} e_x}{w_{m,\kappa}} \right\|_{L_2} \|w_{m,\kappa}\nu\|_{L_2},$$

by the Cauchy-Schwarz inequality. We note that $\|w_{m,\kappa}\nu\|_{L_2} = |f|_{m,\kappa}$. For $t \in B$, it follows from Taylor's theorem that

$$\left|e_x(t) - P_{\underline{k}} e_x(t)\right| \le c\,|t|^{\underline{k}+1} \max_{|\alpha|=\underline{k}+1} \|D^\alpha e_x\|_{L_\infty(B)} \le c\,|t|^{\underline{k}+1}(1+|x|)^{\underline{k}+1}.$$

On the other hand, for all $t \in \mathbb{R}^d$ we have the crude estimate

$$\left|e_x(t) - \sigma(t) P_{\underline{k}} e_x(t)\right| \le c(1+|x|)^{\widetilde{k}},$$

where $\widetilde{k} := \max\{\underline{k}, 0\}$. Put $\rho_x := (1+|x|)^{-1/(\widetilde{k}+1)}$. Then

$$\begin{aligned} \left\| \frac{e_x - \sigma P_{\underline{k}} e_x}{w_{m,\kappa}} \right\|_{L_2(\rho_x B)}^2 &\le c(1+|x|)^{2(\underline{k}+1)} \int_{\rho_x B} |t|^{2(\underline{k}+1-m)}\, dt \\ &\le c \begin{cases} (1+|x|)^{2\mu}, & \text{if } m \ge d/2, \\ 1, & \text{if } m < d/2, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \left\| \frac{e_x - \sigma P_{\underline{k}} e_x}{w_{m,\kappa}} \right\|_{L_2(\mathbb{R}^d \setminus \rho_x B)}^2 &\le c(1+|x|)^{2\widetilde{k}} \left(\int_{B\setminus\rho_x B} |t|^{-2m}\, dt + \int_{\mathbb{R}^d\setminus B} |t|^{-2\kappa}\, dt \right) \\ &\le c \begin{cases} (1+|x|)^{2\mu}, & \text{if } m > d/2, \\ \log(2+|x|), & \text{if } m = d/2, \\ 1, & \text{if } m < d/2. \end{cases} \end{aligned}$$

Therefore, $\left\| \dfrac{e_x - \sigma P_{\underline{k}} e_x}{w_{m,\kappa}} \right\|_{L_2} \le c \begin{cases} (1+|x|)^{\mu}, & \text{if } m > d/2, \\ \sqrt{\log(2+|x|)}, & \text{if } m = d/2, \\ 1, & \text{if } m < d/2, \end{cases}$ which completes the proof. □

Corollary 2.7. *With m, κ as in the proposition, if $f \in \widetilde{Y}_{m,\kappa}$ and $k \geq \underline{k}$, then $f \in Y_{m,\kappa}+\Pi_k$ if and only if*

$$|f(x)| = o(|x|^{k+1}) \text{ as } |x| \to \infty. \tag{2.8}$$

Proof. It suffices to show that if $\nu \in L_2(\mathbb{R}^d, w_{m,\kappa}^2)$ and $f = V_{m,\kappa}(\nu)$, then (2.8) holds with $k = \underline{k}$. Since $\mu < \underline{k}+1$ and $\sqrt{\log(2+|x|)} = o(|x|)$, it is clear that (2.8) holds in case $m \geq d/2$. In case $m < d/2$ ($\underline{k} = -1$), then $\nu \in L_1$ and $\widehat{f} = \nu$; hence $|f(x)| = o(1)$ as $|x| \to \infty$ by the Riemann-Lebesgue lemma. □

We can now show that the third condition of Definition 1.4 is equivalent to a certain growth condition on ϕ.

Theorem 2.9. *If ϕ is pre-(m, κ, k)-admissible, then ϕ is (m, κ, k)-admissible if and only if*

$$|\phi(x)| = o(|x|^{2(k+1)}) \text{ as } |x| \to \infty.$$

Proof. We consider first the case $m \geq d/2$ ($\underline{k} \geq 0$). It follows from (1.5) that $\phi \in \widetilde{Y}_{m',\kappa}$ for all $m' > 2m-d/2$. For m' sufficiently close to $2m-d/2$ we have $\underline{k}' := \lfloor m'-d/2 \rfloor = \lfloor 2m-d \rfloor$. With $\psi := V_{m',\kappa}(\widehat{\phi}_|)$, we can write $\phi = \psi + q$ for some polynomial q. Note that if $g \in C_c^\infty(\mathbb{R}^d)$ satisfies $|g(t)| = O(|t|^{\underline{k}'+1})$, then $\langle g, \widehat{\psi} \rangle = \int_{\mathbb{R}^d} g\widehat{\psi}_|$ since $P_{\underline{k}'}g$ would equal 0. But $\underline{k}'+1 \leq 2(k+1)$; hence ψ satisfies the third condition of Definition 1.4. Therefore, ϕ is (m, κ, k)-admissible if and only if $q \in \Pi_{2k+1}$. That is, ϕ is (m, κ, k)-admissible if and only if $\phi \in Y_{m',\kappa} + \Pi_{2k+1}$ which, by Corollary 2.7, happens if and only if $|\phi(x)| = o(|x|^{2(k+1)})$ as $|x| \to \infty$. We consider now the case $m < d/2$ ($\underline{k} = -1$). It follows from (1.5) that $\widehat{\phi}_| \in L_1$, and hence we can write $\phi = \psi + q$, where $\widehat{\psi} = \widehat{\phi}_|$ and q is some polynomial. Since

$\langle g, \widehat{\phi} \rangle = \langle g, \widehat{q} \rangle + \int_{\mathbb{R}^d} g \widehat{\phi}_|$ for all $g \in C_c^\infty(\mathbb{R}^d)$, it follows that ϕ satisfies the third condition of Definition 1.4 if and only if $q \in \Pi_{2k+1}$. Since $\widehat{\psi} \in L_1$, it follows that $|\psi(x)| = o(1)$ as $|x| \to \infty$, and hence $q \in \Pi_{2k+1}$ if and only if $|\phi(x)| = o(|x|^{2(k+1)})$ as $|x| \to \infty$. $\square$

3. The Construction of Light and Wayne

Let ϕ be (m, κ, k)-admissible. Associated with ϕ, we have the semi-norm

$$|f|_\phi := \left\| \frac{\widehat{f}_|}{\sqrt{\widehat{\phi}_|}} \right\|_{L_2}, \quad f \in \widetilde{Y}_{m,\kappa}. \tag{3.1}$$

It is an obvious consequence of (1.5) that

$$\frac{1}{A} |f|_{m,\kappa} \le |f|_\phi \le A\, |f|_{m,\kappa}, \quad f \in \widetilde{Y}_{m,\kappa}. \tag{3.2}$$

We now show that the space $Y_{m,\kappa} + \Pi_k$ is one of the spaces covered by the construction of Light and Wayne. Define $w \in C(\mathbb{R}^d \backslash \{0\})$ by

$$w(t) := \left(\frac{w_{m,\kappa}(t)}{|t|^{k+1}} \right)^2, \quad t \in \mathbb{R}^d \backslash \{0\},$$

and note that $w(t) > 0$ for $t \ne 0$, $1/w(t) = O(|t|^{2(k+1)-2m})$ as $|t| \to 0$, and $1/w(t) = O(|t|^{-2(\kappa-k-1)})$ as $|t| \to \infty$. In particular, since $2(k+1) - 2m > -d$, $1/w$ is locally integrable on $\mathbb{R}^d$. Following Light and Wayne [8, Def. 2.9], we define X to be the space of all tempered distributions f such that $\widehat{D^\alpha f}$ is locally integrable on $\mathbb{R}^d$, for all $|\alpha| = k+1$, and

$$|f|_X := \sqrt{\sum_{|\alpha|=k+1} c_\alpha \int_{\mathbb{R}^d} \left| \widehat{D^\alpha f} \right|^2 w} < \infty,$$

where the positive integers c_α are determined by the equation $|x|^{2(k+1)} = \sum_{|\alpha|=k+1} c_\alpha x^{2\alpha}$, $x \in \mathbb{R}^d$.

Theorem 3.3. *If $m \geq 0$, $\kappa > d/2$ and $k \geq \underline{k}$, then $Y_{m,\kappa} + \Pi_k = X$ and*

$$|f|_X = |f|_{m,\kappa} \text{ for all } f \in X.$$

Proof. Let us say that a distribution is regular if it is locally integrable on $\mathbb{R}^d$. We first show that if $f \in Y_{m,\kappa} + \Pi_k$, then $\widehat{D^\alpha f}$ is regular for all $|\alpha| = k+1$. Let $f = u + q$, where $q \in \Pi_k$ and $u = V_{m,\kappa}(\nu)$ for some $\nu \in L_2(\mathbb{R}^d, w_{m,\kappa}^2)$. Note that $D^\alpha q = 0$ for all $|\alpha| = k+1$. If $g \in C_c^\infty(\mathbb{R}^d)$, then

$$\langle g, \widehat{D^\alpha u}\rangle = i^{|\alpha|}\langle g, ()^\alpha \widehat{u}\rangle = i^{|\alpha|}\int_{\mathbb{R}^d}(()^\alpha g - \sigma P_{\underline{k}}[()^\alpha g])\nu = i^{|\alpha|}\int_{\mathbb{R}^d} g()^\alpha \nu,$$

since $P_{\underline{k}}[()^\alpha g] = 0$. Hence $\widehat{D^\alpha u} = i^{|\alpha|}()^\alpha \nu$ which is regular. Thus $\widehat{D^\alpha f}$ is regular for all $|\alpha| = k+1$. Moreover, since $\displaystyle\sum_{|\alpha|=k+1} c_\alpha \left|\widehat{D^\alpha f}\right|^2 = |\cdot|^{2(k+1)}\left|\widehat{f}_|\right|^2$, we have

$$|f|_X^2 = \int_{\mathbb{R}^d} |\cdot|^{2(k+1)}\left|\widehat{f}_|\right|^2 w = \left\|w_{m,\kappa}\widehat{f}_|\right\|_{L_2}^2 = |f|_{m,\kappa}^2. \tag{3.4}$$

Therefore, $Y_{m,\kappa} + \Pi_k \subset X$. To prove the opposite inclusion, we assume now that $f \in X$. By (3.4), $f \in \widetilde{Y}_{m,\kappa}$, so we can write $f = u+q$ where $u = V_{m,\kappa}(\widehat{f}_|)$ and q is some polynomial. Since $\widehat{D^\alpha f}$ and $\widehat{D^\alpha u}$ are regular for all $|\alpha| = k+1$, it follows that $\widehat{D^\alpha q}$ is regular for all $|\alpha| = k+1$; hence, $q \in \Pi_k$. Therefore $f \in Y_{m,\kappa} + \Pi_k$. $\square$

Light and Wayne [8] prove the following theorem assuming additionally that $\widehat{\phi}_|$ is continuous on $\mathbb{R}^d\backslash\{0\}$; however, this assumption is unnecessary when we assume (1.5).

Theorem 3.5. *Let ϕ be (m,κ,k)-admissible, and let Ξ be a nonempty finite subset of $\mathbb{R}^d$ satisfying*

$$q(\Xi) \neq \{0\} \text{ for all } q \in \Pi_k\backslash\{0\}. \tag{3.6}$$

For every $f \in Y_{m,\kappa} + \Pi_k$*, the set* $T_{\Xi,\phi,k}f$ *contains exactly one function* s *which is the unique function in* $Y_{m,\kappa} + \Pi_k$ *which minimizes* $|s|_\phi$ *subject to the interpolation conditions* $s_{|\Xi} = f_{|\Xi}$*.*

By employing a technique of Bezhaev and Vasilenko, we now adapt Theorem 3.5 to the situation where (3.6) fails. Let $\Xi^\perp$ denote the space of all continuous functions which vanish on Ξ:

$$\Xi^\perp := \{f \in C(\mathbb{R}^d) : f_{|\Xi} = \{0\}\}.$$

Note that condition (3.6) is equivalent to the condition $\Pi_k \cap \Xi^\perp = \{0\}$.

Theorem 3.7. *Let* ϕ *be* (m, κ, k)*-admissible, and let* Ξ *be a nonempty finite subset of* $\mathbb{R}^d$*. For* $f \in Y_{m,\kappa} + \Pi_k$*, let* $\mathcal{T}_{\Xi,\phi,k}f$ *denote the set of all functions* $s \in Y_{m,\kappa} + \Pi_k$ *which minimize* $|s|_\phi$ *subject to the interpolation conditions* $s_{|\Xi} = f_{|\Xi}$*. The following hold:*

(i) $\mathcal{T}_{\Xi,\phi,k}f$ *is nonempty.*

(ii) If $\widetilde{f} \in \mathcal{T}_{\Xi,\phi,k}f$*, then* $\mathcal{T}_{\Xi,\phi,k}f = \widetilde{f} + (\Pi_k \cap \Xi^\perp)$*.*

(iii) $\mathcal{T}_{\Xi,\phi,k}f = T_{\Xi,\phi,k}f$*.*

(iv) If $s_1, s_2 \in T_{\Xi,\phi,k}f$*, then* ${s_1}_{|\Gamma} = {s_2}_{|\Gamma}$*, where* $\Gamma := \{x \in \mathbb{R}^d : q(x) = 0$ *for all* $q \in \Pi_k \cap \Xi^\perp\}$*.*

Proof. Put $Q := \Pi_k \cap \Xi^\perp$ and $\ell := \dim Q$. The case $\ell = 0$ is covered by Theorem 3.5, so assume $\ell > 0$. There exists $\mathcal{N} \subset \mathbb{R}^d \backslash \Gamma$, with $\#\mathcal{N} = \ell$, such that $\mathcal{N}$ is correct for interpolation in Q (ie for any data $g_{|\mathcal{N}}$, there exists a unique $q \in Q$ such that $q_{|\mathcal{N}} = g_{|\mathcal{N}}$). Let s be the unique function in $T_{\Xi\cup\mathcal{N},\phi,k}f$ as described in Theorem 3.5. If $\widetilde{f} \in Y_{m,\kappa} + \Pi_k$ satisfies $\widetilde{f}_{|\Xi} = f_{|\Xi}$, then there exists $q \in Q$ such that $(\widetilde{f} + q)_{|\Xi\cup\mathcal{N}} = f_{|\Xi\cup\mathcal{N}}$; hence, by Theorem 3.5, $|s|_\phi \le \left|\widetilde{f} + q\right|_\phi = \left|\widetilde{f}\right|_\phi$. It follows that $s \in \mathcal{T}_{\Xi,\phi,k}f$ and consequently that (i) holds. Now if $\widetilde{f} \in \mathcal{T}_{\Xi,\phi,k}f$, then again there exists $q \in Q$ such that $(\widetilde{f} + q)_{|\Xi\cup\mathcal{N}} = f_{|\Xi\cup\mathcal{N}}$.

Since $\left|\widetilde{f}+q\right|_\phi = \left|\widetilde{f}\right|_\phi = |s|_\phi$, it follows that $s = \widetilde{f}+q$ which proves (ii). We can write s in the form

$$s = q + \sum_{\xi\in\Xi\cup\mathcal{N}} \lambda_\xi \phi(\cdot - \xi),$$

where $q \in \Pi_k$. Let $\widetilde{\mathcal{N}}$ be another set which is correct for interpolation in Q taken so that $\widetilde{\mathcal{N}} \cap \mathcal{N} = \emptyset$. Using the above arguments, we can write s as

$$s = \widetilde{q} + \sum_{\xi\in\Xi\cup\widetilde{\mathcal{N}}} \widetilde{\lambda}_\xi \phi(\cdot - \xi),$$

where $\widetilde{q} \in \Pi_k$. It follows that

$$\sum_{\xi\in\Xi}(\lambda_\xi - \widetilde{\lambda}_\xi)\phi(\cdot-\xi) + \sum_{\xi\in\mathcal{N}} \lambda_\xi\phi(\cdot-\xi) - \sum_{\xi\in\widetilde{\mathcal{N}}} \widetilde{\lambda}_\xi\phi(\cdot-\xi) = \widetilde{q} - q \in \Pi_k,$$

and hence, in particular, that $\lambda_\xi = 0$ for all $\xi \in \mathcal{N}$. Therefore $s \in T_{\Xi,\phi,k}f$ and consequently, $s + Q \subset T_{\Xi,\phi,k}f$. For the reverse inclusion, let $s' \in T_{\Xi,\phi,k}f$, and let $q' \in Q$ be such that $(s'+q')_{|\Xi\cup\mathcal{N}} = f_{|\Xi\cup\mathcal{N}}$. It then follows from Theorem 3.5 that $s'+q' = s$ which proves (iii). Finally we note that (iv) is an immediate consequence of (ii) and (iii). $\square$

Note that the last statement is significant in the context of interpolation on manifolds. If Ξ is contained in a compact, smooth manifold Ω with sufficient density to ensure that $\Omega \subset \Gamma$, then the trace of $T_{\Xi,\phi,k}f$ on Ω is unique.

4. A Preliminary Error Estimate

In this section we work out an error estimate which serves as the basis for our subsequent error analysis. It works on subsets $\Omega \subset \mathbb{R}^d$ which satisfy a certain technical condition.

Definition 4.1. Let $k \in \mathbb{N}_0$ and let Ω be a nonempty subset of $\mathbb{R}^d$. The space of polynomials Π_k is *locally stable* on Ω if there exist constants $c_\Omega, h_\Omega, r_\Omega > 0$ such that if $\Xi \subset \Omega$ satisfies $h(\Xi, \Omega) \le h_\Omega$, then for all $x \in \Omega$ there exists a finite subset $\mathcal{N} \subset \Xi \cap (x + hr_\Omega B)$ and coefficients b_ξ such that

$$q(x) + \sum_{\xi \in \mathcal{N}} b_\xi q(\xi) = 0 \text{ for all } q \in \Pi_k, \text{ and}$$
$$\sum_{\xi \in \mathcal{N}} |b_\xi| \le c_\Omega.$$

It is obvious that if Π_k is locally stable on Ω, then $\Pi_{k'}$ is locally stable on Ω whenever $0 \le k' \le k$. It is also obvious that Π_0 is locally stable on every nonempty subset $\Omega \subset \mathbb{R}^d$. A crucial item used by Duchon [5] is the fact that if $\Omega \subset \mathbb{R}^d$ has the cone property, then Π_k is locally stable on Ω for all $k \in \mathbb{N}_0$ (see also [7, Lemma 4.1]). Here, $\Omega \subset \mathbb{R}^d$ is said to have the *cone property* if there exist $\varepsilon, r > 0$ such that for all $x \in \Omega$ there exists $y \in \Omega$ such that $|x - y| = \varepsilon$ and

$$(1-t)x + ty + rtB \subset \Omega \quad \forall t \in [0,1].$$

As will be discussed in section 6, if Ω is the sphere $S^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$, then Π_k is locally stable on Ω for all k. This fact is essentially proved by Golitschek and Light [6], where they have additionally demonstrated its relevance to error estimates for zonal basis function interpolation on spheres.

Definition 4.2. Let $C := [-1/2, 1/2]^d$ denote the closed unit cube in $\mathbb{R}^d$. For a function f defined at least on a subset $\Omega \subset \mathbb{R}^d$ and for $h > 0$, we define $M_{\Omega,h}(f) : \mathbb{Z}^d \to [0, \infty)$ by

$$M_{\Omega,h}(f)_j = \begin{cases} \sup_{x \in h(j+C) \cap \Omega} |f(x)| & \text{if } h(j+C) \cap \Omega \ne \emptyset, \\ 0 & \text{otherwise .} \end{cases}$$

Our basic error estimate will actually estimate $\|M_{\Omega,h}(g)\|_{\ell_2}$, where g is meant to equal the interpolation error. When Ω has the cone property, we are really interested in the $L_p(\Omega)$-norm of g with respect to Lebesgue measure in $\mathbb{R}^d$; it is a simple exercise to verify that

$$\|g\|_{L_p(\Omega)} \leq h^{d/p} \|M_{\Omega,h}(g)\|_{\ell_2}, \quad 2 \leq p \leq \infty.$$

On the other hand, when Ω is the sphere S^{d-1}, we are really interested in the $L_p(S^{d-1},\omega)$-norm of g, where ω is the usual measure associated with S^{d-1}. We show in section 6 that

$$\|g\|_{L_p(S^{d-1},\omega)} \leq const(d) h^{(d-1)/p} \|M_{\Omega,h}(g)\|_{\ell_2}, \quad 2 \leq p \leq \infty.$$

Thus any estimate on $\|M_{\Omega,h}(g)\|_{\ell_2}$ can be converted into an estimate on $\|f\|_{L_p(\Omega)}$ or $\|f\|_{L_p(S^{d-1},\omega)}$, as the case may be, simply by multiplying by the appropriate power of h.

For an integer $n > d/2$, let H^n denote the space of tempered distributions f which satisfy $D^\alpha f \in L_2$ for all $|\alpha| = n$, and define $|f|_{H^n}$ by

$$|f|_{H^n}^2 := (2\pi)^d \sum_{|\alpha|=n} c_\alpha \|D^\alpha f\|_{L_2}^2,$$

where the positive integers c_α are determined by the equation $|x|^{2n} = \sum_{|\alpha|=n} c_\alpha x^{2\alpha}$, $x \in \mathbb{R}^d$. We recognize, by Theorem 3.3 and the Plancherel Theorem, that $H^n = Y_{n,n} + \Pi_{n-1}$ and $|f|_{H^n} = |f|_{n,n}$ for all $f \in H^n$. The following result is taken from [7, Lemma 3.2].

Lemma 4.3. *Let $n > d/2$ and $r > 0$. For each $j \in \mathbb{Z}^d$, let $\mathcal{N}_j$ be a finite subset of $j + rB$. If $\{b_{j,\xi}\}_{j\in\mathbb{Z}^d,\xi\in\mathcal{N}_j}$ is such that*

$$\sum_{\xi\in\mathcal{N}_j} b_{j,\xi} q(\xi) = 0 \quad \forall q \in \Pi_{n-1}, j \in \mathbb{Z}^d \quad \textit{and}$$

$$K := \sup_{j\in\mathbb{Z}^d} \sum_{\xi\in\mathcal{N}_j} |b_{j,\xi}| < \infty,$$

then

$$\sum_{j\in\mathbb{Z}^d}\left|\sum_{\xi\in\mathcal{N}_j} b_{j,\xi} f(\xi)\right|^2 \le const(d,n,r)K^2\,|f|_{H^n}^2 \qquad \forall f\in H^n.$$

The following is our preliminary error estimate.

Theorem 4.4. *Let m', κ' be integers satisfying $d/2<\kappa'\le m'$, and let Ω be a subset of $\mathbb{R}^d$ upon which $\Pi_{m'-1}$ is locally stable. There exists $h_0>0$ (depending only on m' and Ω) such that if $\Xi\subset\Omega$ satisfies $h:=h(\Xi,\Omega)\le h_0$, then*

$$\|M_{\Omega,h}(g)\|_{\ell_2}\le const(m',\Omega)\,|g(h\cdot)|_{m',\kappa'}$$

for all $g\in Y_{m',\kappa'}+\Pi_{m'-1}$ which vanish on Ξ.

Proof. Let $c_\Omega, h_\Omega, r_\Omega>0$ be as described in Definition 4.1 with $k=m'-1$ and put $h_0:=h_\Omega$. Let $\Xi\subset\Omega$ satisfy $h:=h(\Xi,\Omega)\le h_0$, and suppose $g\in Y_{m',\kappa'}+\Pi_{m'-1}$ vanishes on Ξ. Let $\mathcal{A}$ denote the set of all $j\in\mathbb{Z}^d$ for which $(j+C)\cap h^{-1}\Omega\ne\emptyset$, where $C:=[-1/2,1/2]^d$, and for each $j\in\mathcal{A}$, let $x_j\in(j+C)\cap h^{-1}\Omega$ be such that $M_{\Omega,h}(g)_j\le 2\,|g(hx_j)|$. Since $\Pi_{m'-1}$ is locally stable on Ω, for each $j\in\mathcal{A}$ there exists a finite $\mathcal{N}_j\subset h^{-1}\Xi\cap(x_j+r_\Omega B)$ and coefficients $b_{j,\xi}$ such that $\sum_{\xi\in\mathcal{N}_j}|b_{j,\xi}|\le c_\Omega$ and

$$q(x_j)+\sum_{\xi\in\mathcal{N}_j} b_{j,\xi}q(\xi)=0 \text{ for all } q\in\Pi_{m'-1}.$$

Put $f:=g(h\cdot)$ and define $f_1\in H^{m'}$ by $\widehat{f_1}:=\chi_B\widehat{f}$. Put $f_2:=f-f_1\in H^{\kappa'}$. Note that

$$|g(h\cdot)|_{m',\kappa'}^2=|f|_{m',\kappa'}^2=|f_1|_{H^{m'}}^2+|f_2|_{H^{\kappa'}}^2\,.$$

Since $f(\xi)=0$ for all $\xi\in\mathcal{N}_j$, we have

$$\begin{aligned}\|M_{\Omega,h}(g)\|_{\ell_2}^2 &\le 4\sum_{j\in\mathcal{A}}|f(x_j)|^2=4\sum_{j\in\mathcal{A}}\left|f(x_j)+\sum_{\xi\in\mathcal{N}_j}f(\xi)\right|^2\\ &\le 8\sum_{j\in\mathcal{A}}\left|f_1(x_j)+\sum_{\xi\in\mathcal{N}_j}f_1(\xi)\right|^2+8\sum_{j\in\mathcal{A}}\left|f_2(x_j)+\sum_{\xi\in\mathcal{N}_j}f_2(\xi)\right|^2\\ &\le const(m',\Omega)(|f_1|_{H^{m'}}^2+|f_2|_{H^{\kappa'}}^2)=const(m',\Omega)\,|g(h\cdot)|_{m',\kappa'}^2\,,\end{aligned}$$

where we have used Lemma 4.3 in the last inequality. □

5. An Error Analysis

Let ϕ be (m,κ,k)-admissible with $\kappa' := \lfloor d/2 \rfloor + 1 \le \kappa$ and put $m' := \max\{\lceil m \rceil, \lceil \kappa \rceil, k+1\}$. Note that for $h>0$, $\phi_h := \phi(\cdot/h)$ is also (m,κ,k)-admissible. Our error analysis will apply to $T_{\Xi,\phi,k}$ when $0 \le m \le \kappa$ and to $T_{\Xi,\phi_h,k}$ when $m \ge \kappa$, where the dilation factor h equals the fill distance between Ξ and Ω.

We will make use of several relations involving semi-norms of the form $|\cdot|_{m,\kappa}$ which we prove in the following two lemmata.

Lemma 5.1. *If $0 < h \le 1$, then*

$$|g(h\cdot)|_{m',\kappa'} \le h^{\kappa-d/2} |g|_{m,\kappa} \text{ for all } g \in \widetilde{Y}_{m,\kappa}.$$

Proof. We first show that

$$w_{m',\kappa'}(ht) \le h^{\kappa} w_{m,\kappa}(t) \text{ for all } t \in \mathbb{R}^d. \tag{5.2}$$

If $|t| < 1$, then

$$w_{m',\kappa'}(ht) = h^{m'} |t|^{m'} \le h^{\kappa} |t|^{m} = h^{\kappa} w_{m,\kappa}(t).$$

While if $1 \le |t| \le h^{-1}$, then

$$w_{m',\kappa'}(ht) = h^{m'} |t|^{m'} = h^{m'} |t|^{m'-\kappa} w_{m,\kappa}(t) \le h^{\kappa} w_{m,\kappa}(t).$$

Finally, if $|t| > h^{-1}$, then

$$w_{m',\kappa'}(ht) = h^{\kappa'} |t|^{\kappa'} = h^{\kappa'} |t|^{\kappa'-\kappa} w_{m,\kappa}(t) \le h^{\kappa} w_{m,\kappa}(t)$$

which establishes (5.2). Now if $g \in \widetilde{Y}_{m,\kappa}$, then

$$|g(h\cdot)|_{m',\kappa'} = h^{-d} \left\| \widehat{g}_{|}(\cdot/h) w_{m',\kappa'} \right\|_{L_2} = h^{-d/2} \left\| \widehat{g}_{|} w_{m',\kappa'}(h\cdot) \right\|_{L_2} \le h^{\kappa-d/2} |g|_{m,\kappa},$$

by (5.2). □

Lemma 5.3. *If $m \ge \kappa$ and $\gamma \in \mathbb{R}$ satisfies $\kappa \le \gamma \le m$, then for $h > 0$ the following hold:*

$$(i) \qquad \frac{1}{A} |f(h\cdot)|_{m,\kappa} \le |f|_{\phi_h} \le A\,|f(h\cdot)|_{m,\kappa} \ \text{for all } f \in \widetilde{Y}_{m,\kappa}.$$

$$(ii) \qquad |f(h\cdot)|_{m,\kappa} \le h^{\gamma - d/2} |f|_{\gamma,\gamma} \ \text{for all } f \in \widetilde{Y}_{\gamma,\gamma}.$$

Proof. For $f \in \widetilde{Y}_{m,\kappa}$, we have $|f(h\cdot)|_{m,\kappa} = h^{-d} \left\| \widehat{f}_{|}(\cdot/h) w_{m,\kappa} \right\|_{L_2}$, while

$$|f|_{\phi_h} = \left\| \frac{\widehat{f}_{|}}{\sqrt{h^d \widehat{\phi}_{|}(h\cdot)}} \right\|_{L_2} = h^{-d} \left\| \frac{\widehat{f}_{|}(\cdot/h)}{\sqrt{\widehat{\phi}_{|}}} \right\|_{L_2}.$$

We then obtain (i) as a consequence of (1.5). For (ii), we first note that $w_{m,\kappa}(ht) \le h^{\gamma} |t|^{\gamma}$ for all $t \in \mathbb{R}^d$. Indeed, if $|t| \le h^{-1}$, then $\dfrac{w_{m,\kappa}(ht)}{|t|^{\gamma}} = h^m |t|^{m-\gamma} \le h^{\gamma}$; while if $|t| > h^{-1}$, then $\dfrac{w_{m,\kappa}(ht)}{|t|^{\gamma}} = h^{\kappa} |t|^{\kappa - \gamma} \le h^{\gamma}$. Thus, for $f \in \widetilde{Y}_{\gamma,\gamma}$,

$$\begin{aligned} |f(h\cdot)|_{m,\kappa} = h^{-d} \left\| \widehat{f}_{|}(\cdot/h) w_{m,\kappa} \right\|_{L_2} &= h^{-d/2} \left\| \widehat{f}_{|} w_{m,\kappa}(h\cdot) \right\|_{L_2} \\ &\le h^{\gamma - d/2} \left\| \widehat{f}_{|}\, |\cdot|^{\gamma} \right\|_{L_2} = h^{\gamma - d/2} |f|_{\gamma,\gamma}. \end{aligned}$$

□

The following is our main error estimate.

Theorem 5.4. *Let ϕ be (m,κ,k)-admissible with $\kappa' := \lfloor d/2 \rfloor + 1 \le \kappa$ and put $m' :=$ $\max\{\lceil m \rceil, \lceil \kappa \rceil, k+1\}$. Let Ω be a bounded subset of $\mathbb{R}^d$ upon which $\Pi_{m'-1}$ is locally stable, and let $h_0 \in (0,1]$ (depending only on Ω and m') be as in Theorem 4.4. Assume that Ξ is a finite subset of $\overline{\Omega}$ satisfying $h := h(\Xi,\Omega) \le h_0$.*

If $0 \le m \le \kappa$ and $f \in Y_{m,\kappa} + \Pi_k$, then

$$\|M_{\Omega,h}(f-s)\|_{\ell_2} \le A^2 \mathit{const}(m',\Omega) h^{\kappa - d/2} |f|_{m,\kappa} \ \text{for all } s \in T_{\Xi,\phi,k} f.$$

If $\kappa \le \gamma \le m$ and $f \in Y_{\gamma,\gamma} + \Pi_k$, then

$$\|M_{\Omega,h}(f-s)\|_{\ell_2} \le A^2 const(m',\Omega) h^{\gamma-d/2} |f|_{\gamma,\gamma} \text{ for all } s \in T_{\Xi,\phi_h,k}f.$$

Proof. For the sake of brevity, let us use the abbreviation $c = const(m',\Omega)$. We consider first the case $0 \le m \le \kappa$. Assume $f \in Y_{m,\kappa} + \Pi_k$ and $s \in T_{\Xi,\phi,k}f$. By Theorem 3.7, $s \in Y_{m,\kappa} + \Pi_k$ and $|s|_\phi \le |f|_\phi$. Since $\kappa' \le \kappa$, $m \le m'$ and $k \le m'-1$, it follows from (2.4) that $g := f - s \in Y_{m',\kappa'} + \Pi_{m'-1}$. Since $g_{|\Xi} = 0$, we have by Theorem 4.4, Lemma 5.1 and (3.2) that

$$\|M_{\Omega,h}(f-s)\|_{\ell_2} \le c\,|g(h\cdot)|_{m',\kappa'} \le c\,h^{\kappa-d/2}\,|g|_{m,\kappa} \le Ac\,h^{\kappa-d/2}\,|g|_\phi\,.$$

Employing the inequality $|g|_\phi \le 2\,|f|_\phi$ and (3.2) yields

$$\|M_{\Omega,h}(f-s)\|_{\ell_2} \le Ac\,h^{\kappa-d/2}\,|f|_\phi \le A^2 c\,h^{\kappa-d/2}\,|f|_{m,\kappa}\,.$$

Next we consider the case $m \ge \kappa$. Assume $\kappa \le \gamma \le m$, $f \in Y_{\gamma,\gamma} + \Pi_k$ and $s \in T_{\Xi,\phi_h,k}f$. By Theorem 3.7, $s \in Y_{m,\kappa} + \Pi_k$ and $|s|_{\phi_h} \le |f|_{\phi_h}$. Since $\kappa' \le \kappa \le \gamma \le m \le m'$ and $k \le m'-1$, it follows from (2.4) that $g := f - s \in Y_{m',\kappa'} + \Pi_{m'-1}$. Since $g_{|\Xi} = 0$, we have by Theorem 4.4, (2.3) and Lemma 5.3 (i) that

$$\|M_{\Omega,h}(f-s)\|_{\ell_2} \le c\,|g(h\cdot)|_{m',\kappa'} \le c\,|g(h\cdot)|_{m,\kappa} \le Ac\,|g|_{\phi_h}\,.$$

Noting that $|g|_{\phi_h} \le 2\,|f|_{\phi_h}$ and employing Lemma 5.3 yields

$$\|M_{\Omega,h}(f-s)\|_{\ell_2} \le Ac\,|f|_{\phi_h} \le A^2 c\,|f(h\cdot)|_{m,\kappa} \le Ac^2\,h^{\gamma-d/2}\,|f|_{\gamma,\gamma}\,.$$

□

We conclude this section by explaining how Theorem 1.8 and Theorem 1.10 follow from Theorem 5.4 in the special case when $\Omega \subset \mathbb{R}^d$ has the cone property. In this case condition (3.6) will be satisfied provided h is sufficiently small, and hence by Theorem 3.5, $T_{\Xi,\phi,k}f$ and $T_{\Xi,\phi_h,k}f$ each contain exactly one element. Moreover, as explained in section 4, Π_ℓ is locally stable on Ω for all ℓ, and we have the inequality

$$\|f - s\|_{L_p(\Omega)} \le h^{d/p} \|M_{\Omega,h}(f-s)\|_{\ell_2}, \quad 2 \le p \le \infty.$$

For $1 \le p < 2$, since Ω is bounded we have

$$\|f - s\|_{L_p(\Omega)} \le const(\Omega) \|f - s\|_{L_2(\Omega)} \le const(\Omega) h^{d/2} \|M_{\Omega,h}(f-s)\|_{\ell_2}.$$

With these observations and noting that W_2^κ is continuously embedded in $Y_{m,\kappa}$ and W_2^γ is continuously embedded in $Y_{\gamma,\gamma}$, one easily deduces Theorem 1.8 and Theorem 1.10 from Theorem 5.4.

6. The case when Ω is a sphere

In this section we consider Theorem 5.4 in the special case when Ω is the sphere $S^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$. We begin with the task of showing that Π_k is locally stable on S^{d-1} for all nonnegative integers k. Put

$$Q_k := \{q \in \Pi_k : q(S^{d-1}) = \{0\}\}, \text{ and}$$

$$P_k := span\{()^\alpha : |\alpha| \le k \text{ and } \alpha_1 \in \{0,1\}\}.$$

The space P_k serves (see [6, p. 23]) as a convenient 'representation' for the restriction of Π_k to the sphere S^{d-1} in the sense that $P_k|_{S^{d-1}} = \Pi_k|_{S^{d-1}}$ and $\dim P_k = \dim \Pi_k|_{S^{d-1}}$. It follows from this that $\Pi_k = Q_k \oplus P_k$. The following result was proved by Golitschek and Light [6, Th. 1.4].

Theorem 6.1. *For each $k \in \mathbb{N}_0$, there exist constants $c, h_0, r > 0$ (depending only on d and k) such that if $\Xi \subset S^{d-1}$ satisfies $h := h(\Xi, S^{d-1}) < h_0$, then for all $x \in S^{d-1}$, there exists $\mathcal{N} \subset \Xi \cap (x + hrB)$ and scalars $\{b_\xi\}_{\xi\in\mathcal{N}}$, with $\sum_{\xi\in\mathcal{N}} |b_\xi| \le c$, such that*

$$p(x) + \sum_{\xi\in\Xi} b_\xi p(\xi) = 0 \text{ for all } p \in P_k. \tag{6.2}$$

It follows immediately from this that Π_k is locally stable on S^{d-1} because the equality in (6.2) in fact holds for all $p \in \Pi_k$. Indeed, if $p \in \Pi_k$, say $p = p_1 + p_2$ with $p_1 \in Q_k$ and $p_2 \in P_k$, then $p(x) + \sum_{\xi\in\Xi} b_\xi p(\xi) = p_2(x) + \sum_{\xi\in\Xi} b_\xi p_2(\xi) = 0$, since p_1 vanishes on S^{d-1}.

Let ω denote the usual measure on S^{d-1}, and let $C := [-1/2, 1/2)^d$ be the unit cube in $\mathbb{R}^d$. It is our desire to estimate the $L_p(S^{d-1}, \omega)$-norm of the interpolation error $f - s$; however, the error estimate in Theorem 5.4 is an estimate on $\|M_{\Omega,h}(f-s)\|_{\ell_2}$. In order to relate these, we note that the ω-measure of $S^{d-1} \cap h(x + C)$ is bounded by a constant multiple of h^{d-1}; that is,

$$\omega(S^{d-1} \cap h(x + C)) \le const(d) h^{d-1} \text{ for all } h > 0, x \in \mathbb{R}^d.$$

Thus, since $f - s \in C(\mathbb{R}^d)$, we have

$$\begin{aligned}
\|f-s\|_{L_p(S^{d-1},\omega)} &= \left\| j \mapsto \|f-s\|_{L_p(S^{d-1}\cap h(j+C),\omega)} \right\|_{\ell_p} \\
&\le const(d) \left\| j \mapsto h^{(d-1)/p} \|f-s\|_{L_\infty(S^{d-1}\cap h(j+C),\omega)} \right\|_{\ell_p} \\
&\le const(d) h^{(d-1)/p} \|M_{\Omega,h}(f-s)\|_{\ell_2}, \qquad \text{for } 2 \le p \le \infty.
\end{aligned}$$

Consequently, Theorem 5.4 specializes to the following result.

Theorem 6.3. *Let ϕ be (m, κ, k)-admissible with $\kappa' := \lfloor d/2 \rfloor + 1 \le \kappa$ and put $m' := \max\{\lceil m \rceil, \lceil \kappa \rceil, k + 1\}$. Let Ω be the sphere S^{d-1}, and let $h_0 \in (0, 1]$ (depending only on d*

and m') be as in Theorem 4.4. Assume that $2 \le p \le \infty$ and that Ξ is a finite subset of S^{d-1} satisfying $h := h(\Xi, S^{d-1}) \le h_0$.

If $0 \le m \le \kappa$ and $f \in Y_{m,\kappa} + \Pi_k$, then

$$\|f - s\|_{L_p(S^{d-1},\omega)} \le A^2 const(d, m') h^{\kappa+(d-1)/p-d/2} |f|_{m,\kappa} \text{ for all } s \in T_{\Xi,\phi,k} f.$$

If $\kappa \le \gamma \le m$ and $f \in Y_{\gamma,\gamma} + \Pi_k$, then

$$\|f - s\|_{L_p(S^{d-1},\omega)} \le A^2 const(d, m') h^{\gamma+(d-1)/p-d/2} |f|_{\gamma,\gamma} \text{ for all } s \in T_{\Xi,\phi_h,k} f.$$

7. Concluding Remarks

The first error estimate in Theorem 5.4 is actually more general than it appears at first glance: If g is any function in $Y_{m,\kappa} + \Pi_k$ which happens to vanish on Ξ, then $s = 0$ belongs to $T_{\Xi,\phi,k} g$ and hence the conclusion of Theorem 5.4 becomes

$$\|M_{\Omega,h}(g)\|_{\ell_2} \le const(m', \Omega) h^{\kappa-d/2} |g|_{m,\kappa}. \tag{7.1}$$

To illustrate the usefulness of this viewpoint we sketch how it can be used to obtain error estimates under weaker assumptions than those of Definition 1.4. Let $k \in \{-1, 0, 1, 2, \dots\}$ and $\phi \in \mathcal{R}_0$, with $\widehat{\phi}_| \in C(\mathbb{R}^d \backslash \{0\})$. In place of the first condition of Definition 1.4, we assume that $|\cdot|^{2(k+1)} \widehat{\phi}_|$ is integrable over the ball B; while in place of the second condition we assume that

$$0 < \widehat{\phi}_|(t) \le A^2 |t|^{-2\kappa}, \quad t \in \mathbb{R}^d \backslash \{0\},$$

for some real constants $A > 0$ and $\kappa \ge \lfloor d/2 \rfloor + 1$. Note that this latter condition is simply the left side of condition (1.5). We assume the third condition of Definition 1.4 without modification. With $|\cdot|_\phi$ as defined in (3.1), the space X which Light and Wayne construct can be equivalently defined as the space of all tempered distributions f such that $\widehat{D^\alpha f}$ is locally integrable on $\mathbb{R}^d$, for all $|\alpha| = k+1$, and $|f|_\phi < \infty$. With $k' := \max\{k, \lfloor \kappa - d/2 \rfloor\}$, it is easy to verify that $X \subset Y_{\kappa,\kappa} + \Pi_{k'}$ and $|f|_{\kappa,\kappa} \le A\,|f|_\phi$ for all $f \in X$. Moreover, Theorem 3.5 and Theorem 3.7 remain valid (with $Y_{m,\kappa} + \Pi_k$ replaced by X), and we note that if $f \in X$ and $s \in T_{\Xi,\phi,k}f$, then $g := f - s$ belongs to X and vanishes on Ξ. Applying estimate (7.1) in conjunction with the inequality $|g|_{\kappa,\kappa} \le A\,|f-s|_\phi \le 2A\,|f|_\phi$ then yields the following result.

Theorem 7.2. *Let κ, k, ϕ, X, and k' be as above, and put $m' := \max\{\lceil \kappa \rceil, k'+1\}$. Let Ω be a bounded subset of $\mathbb{R}^d$ upon which $\Pi_{m'-1}$ is locally stable, and let $h_0 \in (0,1]$ (depending only on Ω and m') be as in Theorem 4.4. Assume Ξ is a finite subset of $\overline{\Omega}$ satisfying $h := h(\Xi,\Omega) \le h_0$. If $f \in X$, then*

$$\|M_{\Omega,h}(f-s)\|_{\ell_2} \le A\,const(m',\Omega)h^{\kappa-d/2}\,|f|_\phi \quad \textit{for all } s \in T_{\Xi,\phi,k}f.$$

Another aspect of the error estimate in Theorem 5.4 which warrants mention is that the estimate's dependence on ϕ is confined to the constant A. Consequently, the error estimate may still be useful when ϕ depends on h provided that one has a good estimate on A. To illustrate this, we consider $T_{\Xi,\phi,k}f$ assuming that ϕ is (m,κ,k)-admissible with $m > \kappa \ge \lfloor d/2 \rfloor + 1$ (which is not a case explicitly addressed in Theorem 5.4). We define

$$\psi^{(h)} := h^{-d-m-\kappa}\phi(h\cdot)$$

and note (for fixed $h > 0$) that $\psi^{(h)}$ is (m,κ,k)-admissible and that $T_{\Xi,\phi,k}f = T_{\Xi,\psi_h^{(h)},k}f$. It is a straightforward matter to deduce from (1.5) the inequalities

$$\frac{1}{A^{(h)}}w_{m,\kappa}(t) \le \frac{1}{\sqrt{\widehat{\psi}^{(h)}_{|}(t)}} \le A^{(h)}w_{m,\kappa}(t) \text{ for almost all } t \in \mathbb{R}^d,$$

where $A^{(h)} := h^{(\kappa-m)/2}A$. Applying Theorem 5.4 to $T_{\Xi,\psi_h^{(h)},k}f$ then yields the following result.

Theorem 7.3. *Under the hypothesis of Theorem 5.4, if $\kappa \le \gamma \le m$ and $f \in Y_{\gamma,\gamma} + \Pi_k$, then*

$$\|M_{\Omega,h}(f-s)\|_{\ell_2} \le A^2 const(m',\Omega) h^{\kappa+\gamma-m-d/2} |f|_{\gamma,\gamma} \ \textit{for all } s \in T_{\Xi,\phi,k}f.$$

Note that the exponent of h, $\kappa+\gamma-m-d/2$, is strictly less than $\kappa - d/2$ and hence the obtained rate of convergence is not 'optimal' relative to the given information $f_{|\Xi}$ and the assumption that $f \in Y_{\gamma,\gamma} + \Pi_k$.

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