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Almost sure exponential stability of numerical solutions for stochastic delay differential equations

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Abstract Using techniques based on the continuous and discrete semimartingale convergence theorems, this paper investigates if numerical methods may reproduce the almost sure exponential stability of the exact solutions to stochastic delay differential equations (SDDEs). The important feature of this technique is that it enables us to study the almost sure exponential stability of numerical solutions of SDDEs directly. This is significantly different from most traditional methods by which the almost sure exponential stability is derived from the moment stability by the Chebyshev inequality and the Borel–Cantelli lemma.

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1 Introduction

Stability theory of numerical solutions is one of central problems in numerical analysis. Stability analysis of numerical methods for stochastic differential equations (SDEs) as well as SDDEs has recently received a great deal of attention. Due to the stochastic nature, the stability concepts of numerical schemes for SDEs and SDDEs include, for example, moment stability (M-stability) and almost sure stability (or

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the trajectory stability (T-stability)). There is an extensive literature concerned with moment stability (for example, [3,4,6–8,21,25] for SDEs and [2,18] for SDDEs). Regarding the almost sure stability of numerical methods for SDEs, it was shown, by the Chebyshev inequality and the Borel–Cantelli lemma, that the moment exponential stability implies almost sure exponential stability under certain conditions (for example, see [7,21]). Higham and his coauthors ([6,7]) directly studied the numerical sequence and obtained almost sure stability by the strong law of large numbers.

Using the technique based on the continuous semimartingale convergence theorem (cf. [9,12]), Mao developed in a series of papers (see e.g. [13–16]) the stochastic versions of the LaSalle theorem, from which follows the almost sure asymptotic stability of SDEs and SDDEs. On the other hand, by the discrete semimartingale convergence theorem (cf. [23,26]), the stability of stochastic difference equations has been examined, for example, by [22]. Noting that there are similar expressions for the continuous and discrete semimartingale convergence theorems, [23] obtained the sufficient conditions for almost sure asymptotic stability of both exact and numerical solutions of *linear* SDEs. To the best knowledge of authors, there is no similar result using martingale techniques for numerical solutions of nonlinear SDEs or SDDEs. This is the first paper that uses the martingale techniques to investigate whether numerical methods may reproduce the almost sure exponential ability of the exact solutions to *nonlinear* SDDEs.

Consider the following n-dimensional nonlinear SDDE

$$dx(t) = f(x(t), x(t-\tau), t)dt + g(x(t), x(t-\tau), t)dw(t), \quad t \geq 0 \tag{1.1}$$

with initial data $x_0 = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, where $x_0 = \{x(\theta) : -\tau \leq \theta \leq 0\}$, $f, g : C(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}^n)$ and $w(t)$ is a scalar Brownian motion. For the purpose of stability, we assume that $f(0, 0, t) = g(0, 0, t) = 0$. As a standing hypothesis, we shall impose the following local Lipschitz condition (cf. [11,12]) on the coefficients f and g.

Assumption 1 Both f and g satisfy the local Lipschitz condition, that is, for each integer $j \geq 0$, there exists a positive constant c_j such that

$$|f(x, y, t) - f(\bar{x}, \bar{y}, t)| \vee |g(x, y, t) - g(\bar{x}, \bar{y}, t)| \leq c_j(|x - \bar{x}| + |y - \bar{y}|) \tag{1.2}$$

for all $t \geq 0$ and those $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq j$.

In this paper, we will address the following question:

– If the SDDE (1.1) is almost surely exponentially stable, will a numerical method be able to reproduce this stability property?

We shall show that the Euler–Maruyama (EM) method (cf. [8,10,12]) will work under an additional linear growth condition but we will demonstrate by a counterexample that it may not work without the linear growth condition. Replacing the linear growth condition with the one-sided Lipschitz condition, we will show that the backward EM method is able to reproduce the stability property.

In the next section, we will give some necessary notations and state the continuous and discrete semimartingale convergence theorems as lemmas for the use of this paper. We will then discuss the almost sure exponential stability of the exact solution to Eq. (1.1) and the almost sure exponential stability of the EM approximations in Sect. 3. In Sect. 4, we will give a counterexample to show that the EM method may not be able to reproduce the almost sure stability property without the linear growth condition. In Sect. 5, we will discuss the almost sure exponential stability of the backward EM approximations under the one-sided Lipschitz condition.

2 Notations and lemmas

Throughout this paper, unless otherwise specified, we use the following notations. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$ while its operator norm is defined by $\|A\|$. If A is a symmetric matrix, its largest eigenvalue is defined by $\lambda_{\max}(A)$. Let $\mathbb{R}_+ = [0, \infty)$, and let $\tau > 0$. Denoted by $C([-\tau, 0], \mathbb{R}^n)$ the family of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-\tau \le \theta \le 0} |\varphi(\theta)|$. Let $C^b_{\mathcal{F}_0}([-\tau, 0], \mathbb{R}^n)$ be the family of all $\mathcal{F}_0$-measurable bounded $C([-\tau, 0], \mathbb{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -\tau \le \theta \le 0\}$. The inner product of $X, Y \in \mathbb{R}^n$ is denoted by $\langle X, Y\rangle$ or $X^T Y$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions, that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $w(t)$ be a scalar Brownian motion defined on this probability space.

The following two lemmas will play important roles in this paper. The first one is the continuous semimartingale convergence theorem (cf. [9,12]). The second one is the corresponding discrete version (cf. [23,26]).

Lemma 1 *Let $A(t), U(t)$ be two $\mathcal{F}_t$-adapted increasing processes on $t \ge 0$ with $A(0) = U(0) = 0$ a.s. Let $M(t)$ be a real-valued local martingale with $M(0) = 0$ a.s. Let ζ be a nonnegative $\mathcal{F}_0$-measurable random variable. Assume that $X(t)$ is nonnegative and*

$$X(t) = \zeta + A(t) - U(t) + M(t) \quad \textit{for } t \ge 0.$$

If $\lim_{t\to\infty} A(t) < \infty$ a.s. then for almost all $\omega \in \Omega$,

$$\lim_{t\to\infty} X(t) < \infty \quad \textit{and} \quad \lim_{t\to\infty} U(t) < \infty,$$

that is, both $X(t)$ and $U(t)$ converge to finite random variables.

Lemma 2 *Let $\{A_i\}, \{U_i\}$ be two sequences of nonnegative random variables such that both A_i and U_i are $\mathcal{F}_{i-1}$-measurable for $i = 1, 2, \ldots,$ and $A_0 = U_0 = 0$ a.s. Let M_i be a real-value local martingale with $M_0 = 0$ a.s. Let ζ be a nonnegative $\mathcal{F}_0$-measurable random variable. Assume that $\{X_i\}$ is a nonnegative semimartingale* *with the Doob–Mayer decomposition*

$$X_i = \zeta + A_i - U_i + M_i.$$

If $\lim_{i\to\infty} A_i < \infty$ *a.s. then for almost all* $\omega \in \Omega$,

$$\lim_{i\to\infty} X_i < \infty \quad \textit{and} \quad \lim_{i\to\infty} U_i < \infty,$$

that is, both X_i *and* U_i *converge to finite random variables.*

In the following sections, we will employ these lemmas to establish the almost sure asymptotic stability theorems for both exact and numerical solutions to Eq. (1.1).

3 Stability of the exact solution and the EM approximation

Applying the EM method (see [1,17]) to Eq. (1.1) yields the following approximation

$$\begin{cases} x_k = \xi(k\triangle) & k = -m, -m+1, \dots, 0, \\ x_{k+1} = x_k + f(x_k, x_{k-m}, k\triangle)\triangle & \\ \quad + g(x_k, x_{k-m}, k\triangle)\triangle w_k, & k = 0, 1, 2, \dots, \end{cases} \tag{3.1}$$

where $\triangle = \tau/m$ (m is an integer) is the stepsize and $\triangle w_k := w((k+1)\triangle) - w(k\triangle)$ is the Brownian increment.

To be precise, let us give the definitions on the almost sure exponential stability of SDDEs and their numerical approximations.

Definition 1 The solution $x(t,\xi)$ to Eq. (1.1) is said to be almost surely exponentially stable if there exists a constant $\eta > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t,\xi)| \le -\eta \quad a.s. \tag{3.2}$$

for any initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$.

Definition 2 The approximate solution x_k to Eq. (3.1) is said to be almost surely exponentially stable if there exists a constant $\bar{\eta} > 0$ such that

$$\limsup_{k\to\infty} \frac{1}{k\triangle} \log |x_k| \le -\bar{\eta} \quad a.s. \tag{3.3}$$

for any bounded variables $\xi(k\triangle)$, $k = -m, -m+1, \dots, 0$.

In this section, our aim is to examine if the EM method can reproduce the almost sure exponential stability of the exact solution of Eq. (1.1). Let us state a theorem which does not only give the existence-and-uniqueness result of the solution but also provides us with a criterion on the almost sure exponential stability of the exact solution (please see [19] for the existence-and-uniqueness result and [14,16] for the almost sure exponential stability).

Theorem 1 *Let Assumption 1 hold. Assume that there are four nonnegative constants* λ_1–λ_4 *such that*

$$2x^T f(x, 0, t) \le -\lambda_1 |x|^2, \tag{3.4}$$

$$|f(x, y, t) - f(x, 0, t)| \le \lambda_2 |y|, \tag{3.5}$$

$$|g(x, y, t)|^2 \le \lambda_3 |x|^2 + \lambda_4 |y|^2 \tag{3.6}$$

for all $x, y \in \mathbb{R}^n$ *and* $t \ge 0$. *If*

$$\lambda_1 > 2\lambda_2 + \lambda_3 + \lambda_4, \tag{3.7}$$

then for any given initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$*, there exists a unique global solution to Eq. (1.1) and this solution, denoted by* $x(t; \xi)$*, has property that*

$$\limsup_{t\to\infty} \frac{1}{t} \log(|x(t; \xi)|) \le -\frac{\gamma}{2} \quad a.s. \tag{3.8}$$

where $\gamma > 0$ *is the unique positive root of*

$$\lambda_1 - \lambda_2 - \lambda_3 - \gamma = (\lambda_2 + \lambda_4) e^{\gamma\tau}. \tag{3.9}$$

This theorem gives a criterion on the robustness of stability. In fact, condition (3.4) guarantees the exponential stability of the ODE

$$\frac{dx(t)}{dt} = f(x(t), 0, t). \tag{3.10}$$

Rewriting Eq. (1.1) as

$$\begin{aligned} dx(t) = f(x(t), 0, t)dt + ([f(x(t), x(t-\tau), t) - f(x(t), 0, t)]dt \\ +g(x(t), x(t-\tau), t)dw(t)), \end{aligned} \tag{3.11}$$

we see that it is a stochastically perturbed system of Eq. (3.10). Theorem 1 gives a criterion on how large the stochastic perturbation which Eq. (3.10) could tolerate so that the perturbed system (3.11), namely Eq. (1.1) remains exponentially stable. One simple example is the linear SDDE

$$dx(t) = [Ax(t) + Bx(t-\tau)]dt + [Cx(t) + Dx(t-\tau)]dw(t),$$

where $A, B, C, D \in \mathbb{R}^{n\times n}$. By Theorem 1, it is easy to show that this linear SDDE is almost surely exponentially stable if

$$-\lambda_{\max}(A + A^T) > 2(\|B\| + \|C\|^2 + \|D\|^2).$$

Let us now discuss the stability of the EM approximate solution (3.1).

Theorem 2 *Let conditions (3.4)–(3.6) and (3.7) hold. Assume also that f satisfies the linear growth condition, namely, there exists a constant $K > 0$ such that*

$$|f(x, y, t)|^2 \leq K(|x|^2 + |y|^2). \tag{3.12}$$

Let $\gamma > 0$ be the number defined by (3.9) and $\varepsilon \in (0, \gamma/2)$ be arbitrary. Then there exists a $\triangle^* > 0$ such that if $\triangle < \triangle^*$, then for any given finite-valued $\mathcal{F}_0$-measurable random variables $\xi(k\triangle)$, $k = -m, -m+1, \ldots, 0$, the EM approximate solution (3.1) obeys

$$\limsup_{k\to\infty} \frac{1}{k\triangle} \log(|x_k|) \leq -\frac{\gamma}{2} + \varepsilon \ \ a.s. \tag{3.13}$$

Proof For any positive constant $C > 1$, we have

$$C^{(k+1)\triangle}|x_{k+1}|^2 - C^{k\triangle}|x_k|^2 = C^{(k+1)\triangle}(|x_{k+1}|^2 - |x_k|^2) + (C^{(k+1)\triangle} - C^{k\triangle})|x_k|^2.$$

Note that

$$\begin{aligned}
|x_{k+1}|^2 &= \langle x_k + f(x_k, x_{k-m}, k\triangle)\triangle + g(x_k, x_{k-m}, k\triangle)\triangle w_k, x_k \\
&\quad + f(x_k, x_{k-m}, k\triangle)\triangle + g(x_k, x_{k-m}, k\triangle)\triangle w_k\rangle \\
&= |x_k|^2 + 2x_k^T f(x_k, x_{k-m}, k\triangle)\triangle + |f(x_k, x_{k-m}, k\triangle)\triangle|^2 \\
&\quad + |g(x_k, x_{k-m}, k\triangle)\triangle w_k|^2 \\
&\quad + 2\langle x_k + f(x_k, x_{k-m}, k\triangle)\triangle, g(x_k, x_{k-m}, k\triangle)\triangle w_k\rangle.
\end{aligned}$$

By conditions (3.4)–(3.6), (3.7) and (3.12), we have

$$\begin{aligned}
&C^{(k+1)\triangle}|x_{k+1}|^2 - C^{k\triangle}|x_k|^2 \\
&\quad \leq C^{(k+1)\triangle}[-\lambda_1\triangle|x_k|^2 + 2\lambda_2 x_k^T x_{k-m}\triangle + K(|x_k|^2 + |x_{k-m}|^2|)\triangle^2 \\
&\qquad + (\lambda_3|x_k|^2 + \lambda_4|x_{k-m}|^2)|\triangle w_k|^2] \\
&\qquad + 2C^{(k+1)\triangle}\langle x_k + f(x_k, x_{k-m}, k\triangle)\triangle, g(x_k, x_{k-m}, k\triangle)\triangle w_k\rangle \\
&\qquad + (C^{(k+1)\triangle} - C^{k\triangle})|x_k|^2 \\
&\quad \leq C^{(k+1)\triangle}[-\lambda_1\triangle + \lambda_2\triangle + (1 - C^{-\triangle}) + K\triangle^2 + \lambda_3|\triangle w_k|^2]|x_k|^2 \\
&\qquad + C^{(k+1)\triangle}(\lambda_2\triangle + K\triangle^2 + \lambda_4|\triangle w_k|^2)|x_{k-m}|^2 \\
&\qquad + 2C^{(k+1)\triangle}\langle x_k + f(x_k, x_{k-m}, k\triangle)\triangle, g(x_k, x_{k-m}, k\triangle)\triangle w_k\rangle,
\end{aligned}$$

which implies that

$$C^{k\triangle}|x_k|^2 \le |x_0|^2 + [-\lambda_1\triangle + \lambda_2\triangle + (1 - C^{-\triangle}) + K\triangle^2]\sum_{i=0}^{k-1} C^{(i+1)\triangle}|x_i|^2$$
$$+\lambda_3\sum_{i=0}^{k-1} C^{(i+1)\triangle}|x_i|^2|\triangle w_i|^2 + (\lambda_2\triangle + K\triangle^2)\sum_{i=0}^{k-1} C^{(i+1)\triangle}|x_{i-m}|^2$$
$$+\lambda_4\sum_{i=0}^{k-1} C^{(i+1)\triangle}|x_{i-m}|^2|\triangle w_i|^2$$
$$+2\sum_{i=0}^{k-1} C^{(i+1)\triangle}\langle x_k + f(x_i, x_{i-m}, i\triangle)\triangle, g(x_i, x_{i-m}, \triangle)\triangle w_i\rangle.$$

Let $m_k = \sum_{i=0}^{k-1} C^{(i+1)\triangle}|x_i|^2(|\triangle w_i|^2 - \triangle)$. Noting that $\mathbb{E}[(|\triangle w_k|^2 - \triangle)|\mathcal{F}_{k\triangle}] = 0$ and x_k is $\mathcal{F}_{k\triangle}$-measurable, then we have

$$\begin{aligned}\mathbb{E}[m_k|\mathcal{F}_{(k-1)\triangle}] &= m_{k-1} + \mathbb{E}[C^{k\triangle}|x_{k-1}|^2(|\triangle w_{k-1}|^2 - \triangle)|\mathcal{F}_{(k-1)\triangle}]\\ &= m_{k-1} + C^{k\triangle}|x_{k-1}|^2\mathbb{E}[(|\triangle w_{k-1}|^2 - \triangle)|\mathcal{F}_{(k-1)\triangle}]\\ &= m_{k-1},\end{aligned}$$

which implies that m_k is a martingale. Similarly, $\hat{m}_k = \sum_{i=0}^{k-1} C^{(i+1)\triangle}|x_{i-m}|^2 (|\triangle w_i|^2 - \triangle)$ is also a martingale. Clearly,

$$\bar{m}_k = 2\sum_{i=0}^{k-1} C^{(i+1)\triangle}\langle x_i + f(x_i, x_{i-m}, i\triangle)\triangle, g(x_i, x_{i-m}, i\triangle)\triangle w_i\rangle$$

is a martingale. These imply that $M_k = \lambda_3 m_k + \lambda_4\hat{m}_k + \bar{m}_k$ is a martingale with $M_0 = 0$. We therefore have that

$$C^{k\triangle}|x_k|^2 \le |x_0|^2 + [-\lambda_1\triangle + \lambda_2\triangle + \lambda_3\triangle + (1 - C^{-\triangle}) + K\triangle^2]\sum_{i=0}^{k-1} C^{(i+1)\triangle}|x_i|^2$$
$$+(\lambda_2\triangle + \lambda_4\triangle + K\triangle^2)\sum_{i=0}^{k-1} C^{(i+1)\triangle}|x_{i-m}|^2 + M_k.$$

Since

$$\sum_{i=0}^{k-1} C^{(i+1)\triangle}|x_{i-m}|^2 = \sum_{i=-m}^{-1} C^{(i+m+1)\triangle}|x_i|^2 + \sum_{k=0}^{k-1} C^{(i+m+1)\triangle}|x_i|^2$$
$$- \sum_{i=k-m}^{k-1} C^{(i+m+1)\triangle}|x_i|^2,$$

we have

$$C^{k\triangle}|x_k|^2 + (\lambda_2\triangle + \lambda_4\triangle + K\triangle^2)\sum_{i=k-m}^{k-1} C^{(i+m+1)\triangle}|x_i|^2 \leq X_k, \tag{3.14}$$

where

$$\begin{aligned}X_k := |x_0|^2 &+ (\lambda_2\triangle + \lambda_4\triangle + K\triangle^2)\sum_{i=-m}^{-1} C^{(i+m+1)\triangle}|x_i|^2 + [-\lambda_1\triangle + \lambda_2\triangle \\ &+\lambda_3\triangle + (1 - C^{-\triangle}) + K\triangle^2 + (\lambda_2\triangle + \lambda_4\triangle + K\triangle^2)C^{m\triangle}] \\ &\times \sum_{i=0}^{k-1} C^{(i+1)\triangle}|x_i|^2 + M_k.\end{aligned}$$

Let us now introduce the function

$$h(C) = (\lambda_2 + \lambda_4 + K\triangle)\triangle C^{(m+1)\triangle} + (1 - \lambda_1\triangle + \lambda_2\triangle + \lambda_3\triangle + K\triangle^2)C^{\triangle} - 1. \tag{3.15}$$

Choose $\triangle_1^* > 0$ such that for any $\triangle < \triangle_1^*$, $1 - \lambda_1\triangle + \lambda_2\triangle + \lambda_3\triangle + K\triangle^2 > 0$. We therefore have $h'(C) > 0$ for any $C \geq 1$. Clearly,

$$h(1) = -(\lambda_1 - 2\lambda_2 - \lambda_3 - \lambda_4 - 2K\triangle)\triangle.$$

Hence, for any $\triangle < \triangle_2^* := (\lambda_1 - 2\lambda_2 - \lambda_3 - \lambda_4)/(2K)$, $h(1) < 0$, which implies that for any $\triangle < \triangle_1^* \wedge \triangle_2^*$, there exists a unique $C_\triangle^* > 1$ such that $h(C_\triangle^*) = 0$. Choosing $C = C_\triangle^*$, we therefore have

$$X_k = |x_0|^2 + (\lambda_2\triangle + \lambda_4\triangle + K\triangle^2)\sum_{i=-m}^{-1} {C_\triangle^*}^{(i+m+1)\triangle}|x_i|^2 + M_k.$$

Noting that the initial sequence $x_i < \infty$ for all $i = -m, \ldots, 0$, by Lemma 2, for $C = C_\triangle^*$, $\lim_{k\to\infty} X_k < \infty$ a.s. By (3.14), we therefore have

$$\begin{aligned}&\limsup_{k\to\infty} {C_\triangle^*}^{k\triangle}|x_k|^2 \\ &\leq \limsup_{k\to\infty}\left[{C_\triangle^*}^{k\triangle}|x_k|^2 + (\lambda_2\triangle + \lambda_4\triangle + K\triangle^2)\sum_{i=k-m}^{k-1} {C_\triangle^*}^{(i+m+1)\triangle}|x_i|^2\right] \\ &\leq \lim_{k\to\infty} X_k < \infty \quad a.s.\end{aligned} \tag{3.16}$$

Noting that $m\triangle = \tau$, by (3.15),

$$(\lambda_2 + \lambda_4 + K\triangle)C_\triangle^{*\,\tau} + \frac{1}{\triangle}(1 - C_\triangle^{*\,-\triangle}) - \lambda_1 + \lambda_2 + \lambda_3 + K\triangle = 0. \quad (3.17)$$

Choose the constant μ such that $C = e^\mu$ and hence $1 - C^{-\triangle} = 1 - e^{-\mu\triangle}$. Define

$$\bar{h}_\triangle(\mu) = (\lambda_2 + \lambda_4 + K\triangle)e^{\mu\tau} + \frac{1}{\triangle}(1 - e^{-\mu\triangle}) - \lambda_1 + \lambda_2 + \lambda_3 + K\triangle.$$

Letting $\mu_\triangle^* = \log C_\triangle^*$, by (3.17), for any $\triangle < \triangle_1^* \wedge \triangle_2^*$, we have

$$\bar{h}_\triangle(\mu_\triangle^*) = 0. \quad (3.18)$$

Noting that $\lim_{\triangle\to 0}(1 - e^{-\mu\triangle})/\triangle = \mu$, we have

$$\lim_{\triangle\to 0} \bar{h}_\triangle(\mu) = (\lambda_2 + \lambda_4)e^{\mu\tau} + \mu - \lambda_1 + \lambda_2 + \lambda_3. \quad (3.19)$$

By the definition of γ, (3.18) and (3.19) yield

$$\lim_{\triangle\to 0} \mu_\triangle^* = \gamma,$$

which implies that for any positive $\varepsilon \in (0, \gamma/2)$, there exists a $\triangle_3^* > 0$ such that for any $\triangle < \triangle_3^*$, we have

$$\mu_\triangle^* > \gamma - 2\varepsilon.$$

Note that (3.16), together with the definition of $\mu_\triangle^*$ shows that

$$\limsup_{k\to\infty} e^{\mu_\triangle^* k\triangle}|x_k|^2 < \infty.$$

We therefore obtain that for any $\triangle < \triangle_1^* \wedge \triangle_2^* \wedge \triangle_3^*$,

$$\limsup_{k\to\infty} \log|x_k| \le -\frac{\gamma}{2} + \varepsilon, \quad a.s.$$

as required. □

Remark 1 There are many results on moment stability for nonlinear SDDEs (see Mao's book [12] and references therein). Using the Halanay inequality, Baker and Buckwar [2] examined the exponential stability in pth ($p \ge 2$) moment of the EM method for the SDDEs. Their results could imply the almost sure exponential stability by the technique using the Chebyshev inequality and the Borel–Cantelli lemma as demonstrated in [6] and [7]. However, we here use martingale techniques to study the almost sure exponential stability of the EM scheme for the SDDEs directly. A further advantage of the martingale techniques is that they will enable us to investigate other types of stochastic stability of the EM scheme e.g. the LaSalle-type stability as Mao did for the true solutions in his series of papers (see e.g. [13–16]), but we will report these results elsewhere due to the page limit here.

Theorem 2 shows that if the coefficient f obeys the linear growth condition, in addition to the conditions imposed in Theorem 1, then the EM approximate solution (3.1) reproduces the almost sure exponential stability of exact solutions of Eq. (1.1) for sufficiently small stepsize $\triangle$. The question is: will the EM method still work without the linear growth condition?

4 A counterexample

To answer the question stated above, let us consider the following scalar stochastic delay differential equation

$$dx(t) = [-3x(t) - x^3(t) + x(t)\sin(x(t-1))]dt + x(t)\sin^3(x(t-1))dw(t) \quad (4.1)$$

for any initial data $\xi \in C^b_{\mathcal{F}_0}([-1, 0]; \mathbb{R})$. Define $f(x, y, t) = -3x - x^3 + x \sin y$ and $g(x, y, t) = x \sin^3 y$. Clearly,

$$\begin{aligned} &2xf(x, 0, t) = 2x(-3x - x^3) \le -6|x|^2, \\ &|f(x, y, t) - f(x, 0, t)| = |x \sin y| \le |x|, \\ &|g(x, y, t)|^2 \le |x \sin^3 y|^2 \le |x|^2. \end{aligned}$$

That is, the coefficients of Eq. (4.1) satisfy conditions (3.4)–(3.6) with $\lambda_1 = 6$, $\lambda_2 = 1$, $\lambda_3 = 1$, $\lambda_4 = 0$ and $\tau = 1$. It is also easy to compute $\gamma = 1.0737$ by (3.9). It follows from Theorem 1 that the solution of Eq. (4.1) obeys

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t, \xi)| \le -0.5, \quad a.s.$$

On the other hands, we observe that the coefficient f does not obey the linear growth condition. We therefore wonder if the EM method will reproduce the almost sure exponential stability of the exact solution?

The EM method (3.1) applied to (4.1) produces

$$x_{k+1} = x_k[1 - 3\triangle - x_k^2\triangle + \sin(x_{k-m})\triangle + \sin^3(x_{k-m})\triangle w_k], \quad (4.2)$$

where $\triangle = 1/m$.

Lemma 3 *Assume that* $\triangle \in (0, 1)$. *If* $|x_0| \ge 8/\sqrt{\triangle}$ *in (4.2), then*

$$\mathbb{P}\left(|x_k| \ge \frac{2^{k+3}}{\sqrt{\triangle}} \ \forall k \ge 1\right) \ge \exp\left(-4e^{-\frac{2}{\sqrt{\triangle}}}\right). \quad (4.3)$$

Proof This proof is motivated by Higham et al. (see Lemma 3.1 in [7]). First, we show that

$$|x_k| \geq \frac{2^{k+3}}{\sqrt{\triangle}} \text{ and } |\triangle w_k| \leq 2^k$$

implies that

$$|x_{k+1}| \geq \frac{2^{k+4}}{\sqrt{\triangle}}. \tag{4.4}$$

To see this, assume that $|x_k| \geq 2^{k+3}/\sqrt{\triangle}$. Then for any $\triangle \in (0, 1)$,

$$\begin{aligned}
|x_{k+1}| &= \left| x_k \left(1 - 3\triangle - x_k^2\triangle + \sin(x_{k-m})\triangle + \sin^3(x_{k-m})\triangle w_k\right) \right| \\
&\geq \left| x_k||x_k|^2\triangle - 1 - 3\triangle - \triangle - |\triangle w_k| \right| \\
&\geq \frac{2^{k+3}}{\sqrt{\triangle}} \left(2^{2k+6} - 1 - 4 - 2^k\right) \\
&\geq \frac{2^{k+4}}{\sqrt{\triangle}} \left(2^{2k+5} - 3 - 2^{k-1}\right) \\
&\geq \frac{2^{k+4}}{\sqrt{\triangle}}
\end{aligned}$$

for all $k \geq 0$.

From (4.4), given that $|x_0| \geq 8/\sqrt{\triangle}$, for any integer $K \geq 0$, the event that $\{|x_k| \geq 2^{k+3}/\sqrt{\triangle},\ \forall 1 \leq k \leq K\}$ contains the event that $\{|\triangle w_k| \leq 2^k,\ \forall 1 \leq k \leq K\}$. Since $\{\triangle w_k\}$ are independent, we have

$$\mathbb{P}\left(|x_k| \geq \frac{2^{k+3}}{\sqrt{\triangle}},\ \forall 1 \leq k \leq K\right) \geq \prod_{k=1}^{K} \mathbb{P}\left(|\triangle w_k| \leq 2^k\right). \tag{4.5}$$

From here we can repeat the proof of Lemma 3.1 in [7] to get the desired result (4.3). □

It should be pointed out that Euler's method for the ODE $dx = (-x - x^3)dt$ is already unstable. Hence it must remain unstable for some nonlinear SDDEs and our counterexample used an SDDE is just to make this more clear.

Now that the EM method may not reproduce the almost sure exponential stability of the exact solution without the linear growth condition, we may ask: are there any other numerical methods that may reproduce the almost sure exponential stability of the exact solution without the linear growth condition? The answer is of course yes. We shall show in the next section that the backward EM method will work.

5 Stability of the backward EM approximations

Applying the backward EM method (cf. [7,8,20]) to Eq. (1.1) yields the approximate solution

$$\begin{cases} x_k = \xi(k\triangle), \quad k = -m, -m+1, \dots, 0, \\ x_{k+1} = x_k + f(x_{k+1}, x_{k-m+1}, (k+1)\triangle)\triangle \\ \quad +g(x_k, x_{k-m}, k\triangle)\triangle w_k, \quad k \geq 0. \end{cases} \tag{5.1}$$

Since the backward EM (5.1) is semi-implicit, we have to ensure that this scheme is well defined. For this purpose, we impose the following one-sided Lipschitz condition on f in x: There exists a positive constant λ such that for any $x_1, x_2, y \in \mathbb{R}^n$ and $t \geq 0$,

$$\langle x_1 - x_2, f(x_1, y, t) - f(x_2, y, t)\rangle \leq \lambda |x_1 - x_2|^2. \tag{5.2}$$

Under this condition, if $\lambda\triangle < 1$, then the backward EM scheme (5.1) is well defined (see e.g. [5,20]). The following theorem shows the almost sure exponential stability of the backward EM numerical solutions.

Theorem 3 *Let conditions* (5.2), (3.4)–(3.6) *and (3.7) hold. Let $\gamma > 0$ be the number defined by (3.9) and $\varepsilon \in (0, \gamma/2)$ be arbitrary. Then there exists a $\triangle^* \in (0, 1/\lambda)$ such that if $\triangle < \triangle^*$, then for any given finite-valued $\mathcal{F}_0$-measurable random variables $\xi(k\triangle)$, $k = -m, -m+1, \dots, 0$, the approximate solution $\{x_k\}$ defined by* (5.1) *has property that*

$$\limsup_{k\to\infty} \frac{1}{k\triangle} \log(|x_k|) \leq -\frac{\gamma}{2} + \varepsilon \quad a.s. \tag{5.3}$$

This theorem shows clearly that the backward EM approximations may reproduce the almost sure exponential stability of Eq. (1.1) without the linear growth condition on f. To highlight this, let us return to the SDDE (4.1) used in the section above. In this case, for any x_1, x_2 and $y \in \mathbb{R}^n$, we have

$$f(x_1, y, t) - f(x_2, y, t) = (x_2 - x_1)(3 + x_1^2 + x_1x_2 + x_2^2 - \sin y),$$

which implies that

$$\begin{aligned} &(x_1 - x_2)[f(x_1, y, t) - f(x_2, y, t)] \\ &\quad = -(x_1 - x_2)^2(3 + x_1^2 + x_1x_2 + x_2^2 - \sin y) \\ &\quad \leq -2|x_1 - x_2|^2 \\ &\quad \leq |x_1 - x_2|^2. \end{aligned}$$

Hence, f satisfies the one-sided Lipschitz condition. By Theorem 3, for any given $\varepsilon \in (0, 0.5)$, there must exist a $\triangle^* > 0$ such that if $\triangle < \triangle^*$, the backward EM approximate solution of the SDDE (4.1) obeys

$$\limsup_{k\to\infty}\frac{1}{k\triangle}\log(|x_k|)\le -0.5+\varepsilon \ \ a.s.$$

Let us now begin to prove Theorem 3.

Proof By conditions (3.4) and (3.5), we have

$$\begin{aligned}
|x_{k+1}|^2 &= \langle x_{k+1}, x_k + f(x_{k+1}, x_{k-m+1}, (k+1)\triangle)\triangle + g(x_k, x_{k-m}, k\triangle)\triangle w_k\rangle\\
&= x_{k+1}^T f(x_{k+1}, x_{k-m+1}, (k+1)\triangle)\triangle + \langle x_{k+1}, x_k + g(x_k, x_{k-m}, k\triangle)\triangle w_k\rangle\\
&\le -\frac{\lambda_1\triangle}{2}|x_{k+1}|^2 + |x_{k+1}||f(x_{k+1}, x_{k-m+1}, (k+1)\triangle)\\
&\quad -f(x_{k+1}, 0, (k+1)\triangle)|\triangle + |x_{k+1}||x_k + g(x_k, x_{k-m}, k\triangle)\triangle w_k|\\
&\le -\frac{\lambda_1\triangle}{2}|x_{k+1}|^2 + \lambda_2|x_{k+1}||x_{k-m+1}|\triangle + \frac{1}{2}|x_{k+1}|^2\\
&\quad +\frac{1}{2}|x_k + g(x_k, x_{k-m}, k\triangle)\triangle w_k|^2\\
&\le -\frac{\lambda_1\triangle}{2}|x_{k+1}|^2 + \frac{\lambda_2\triangle}{2}[|x_{k+1}|^2 + |x_{k-m+1}|^2] + \frac{1}{2}|x_{k+1}|^2\\
&\quad +\frac{1}{2}[|x_k|^2 + |g(x_k, x_{k-m}, k\triangle)\triangle w_k|^2 + 2\langle x_k, g(x_k, x_{k-m}, k\triangle)\triangle w_k\rangle]\\
&= -\frac{1}{2}(\lambda_1\triangle - \lambda_2\triangle - 1)|x_{k+1}|^2 + \frac{\lambda_2\triangle}{2}|x_{k-m+1}|^2 + \frac{1}{2}|x_k|^2\\
&\quad +\frac{1}{2}|g(x_k, x_{k-m}, k\triangle)|^2\triangle + \frac{1}{2}m_k^{\triangle},
\end{aligned}$$

where

$$m_k^{\triangle} = |g(x_k, x_{k-m}, k\triangle)|^2(|\triangle w_k|^2 - \triangle) + 2\langle x_k, g(x_k, x_{k-m}, k\triangle)\triangle w_k\rangle.$$

Note that

$$|g(x_k, x_{k-m}, k\triangle)|^2 \le \lambda_3|x_k|^2 + \lambda_4|x_{k-m}|^2.$$

It therefore follows that

$$(\lambda_1\triangle - \lambda_2\triangle + 1)|x_{k+1}|^2 \le (1+\lambda_3\triangle)|x_k|^2 + \lambda_2\triangle|x_{k-m+1}|^2 + \lambda_4\triangle|x_{k-m}|^2 + m_k^{\triangle}.$$

For any $C > 1$, we therefore have

$$\begin{aligned}
&(\lambda_1\triangle - \lambda_2\triangle + 1)[C^{(k+1)\triangle}|x_{k+1}|^2 - C^{k\triangle}|x_k|^2]\\
&\quad\le [1 - (1+\lambda_1\triangle - \lambda_2\triangle)C^{-\triangle} + \lambda_3\triangle]C^{(k+1)\triangle}|x_k|^2 + \lambda_2\triangle C^{(k+1)\triangle}|x_{k-m+1}|^2\\
&\qquad +\lambda_4\triangle C^{(k+1)\triangle}|x_{k-m}|^2 + C^{(k+1)\triangle}m_k^{\triangle},
\end{aligned}$$

which implies that

$$
\begin{aligned}
&(\lambda_1\triangle-\lambda_2\triangle+1)C^{k\triangle}|x_k|^2\\
&\quad\le(\lambda_1\triangle-\lambda_2\triangle+1)|x_0|^2+[1-(1+\lambda_1\triangle-\lambda_2\triangle)C^{-\triangle}+\lambda_3\triangle]\sum_{i=0}^{k-1}C^{(i+1)\triangle}|x_i|^2\\
&\qquad+\lambda_2\triangle\sum_{i=0}^{k-1}C^{(i+1)\triangle}|x_{i-m+1}|^2+\lambda_4\triangle\sum_{i=0}^{k-1}C^{(i+1)\triangle}|x_{i-m}|^2+M_k,
\end{aligned}
$$

where $M_k=\sum_{i=0}^{k-1}C^{(i+1)\triangle}m_k^{\triangle}$. It is obvious that M_k is a martingale with $M_0=0$. Note that

$$
\begin{aligned}
\sum_{i=0}^{k-1}C^{(i+1)\triangle}|x_{i-m+1}|^2&=\sum_{i=-m+1}^{k-m}C^{(i+m)\triangle}|x_i|^2\\
&=C^{(m-1)\triangle}\sum_{i=-m+1}^{-1}C^{(i+1)\triangle}|x_i|^2+C^{(m-1)\triangle}\sum_{i=0}^{k-1}C^{(i+1)\triangle}|x_i|^2\\
&\quad-C^{(m-1)\triangle}\sum_{i=k-m+1}^{k-1}C^{(i+1)\triangle}|x_i|^2
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{i=0}^{k-1}C^{(i+1)\triangle}|x_{i-m}|^2&=\sum_{i=-m}^{k-m-1}C^{(i+m+1)\triangle}|x_i|^2\\
&=C^{m\triangle}\sum_{i=-m}^{-1}C^{(i+1)\triangle}|x_i|^2+C^{m\triangle}\sum_{i=0}^{k-1}C^{(i+1)\triangle}|x_i|^2\\
&\quad-C^{m\triangle}\sum_{i=k-m}^{k-1}C^{(i+1)\triangle}|x_i|^2.
\end{aligned}
$$

We therefore have

$$
\begin{aligned}
&(\lambda_1\triangle-\lambda_2\triangle+1)C^{k\triangle}|x_k|^2+\lambda_2\triangle C^{(m-1)\triangle}\sum_{i=k-m+1}^{k-1}C^{(i+1)\triangle}|x_i|^2\\
&\quad+\lambda_4\triangle C^{m\triangle}\sum_{i=k-m}^{k-1}C^{(i+1)\triangle}|x_i|^2\le Y_k,
\end{aligned}
\tag{5.4}
$$

where

$$
\begin{aligned}
Y_k := & (\lambda_1\triangle - \lambda_2\triangle + 1)|x_0|^2 + \lambda_2\triangle C^{(m-1)\triangle} \sum_{i=-m+1}^{-1} C^{(i+1)\triangle}|x_i|^2 \\
& +\lambda_4\triangle C^{m\triangle} \sum_{i=-m}^{-1} C^{(i+1)\triangle}|x_i|^2 + [1-(1+\lambda_1\triangle - \lambda_2\triangle)C^{-\triangle} + \lambda_3\triangle \\
& +\lambda_2\triangle C^{(m-1)\triangle} + \lambda_4\triangle C^{m\triangle}]\sum_{i=0}^{k-1} C^{(i+1)\triangle}|x_i|^2 + M_k.
\end{aligned}
$$

We now introduce the function

$$
\varphi(C) = \lambda_4\triangle C^{(m+1)\triangle} + \lambda_2\triangle C^{m\triangle} + (1+\lambda_3\triangle)C^{\triangle} - 1 - \lambda_1\triangle + \lambda_2\triangle. \quad (5.5)
$$

Clearly, $\varphi'(C) > 0$ for any $C > 1$ and

$$
\varphi(1) = -(\lambda_1 - 2\lambda_2 - \lambda_3 - \lambda_4)\triangle < 0,
$$

which implies that there exists a unique $\hat{C}_\triangle > 1$ such that $\varphi(\hat{C}_\triangle) = 0$. Choosing $C = \hat{C}_\triangle$,

$$
\begin{aligned}
Y_k = & (\lambda_1\triangle - \lambda_2\triangle + 1)|x_0|^2 + \lambda_2\triangle \hat{C}_\triangle^{(m-1)\triangle} \sum_{i=-m+1}^{-1} \hat{C}_\triangle^{(i+1)\triangle}|x_i|^2 \\
& +\lambda_4\triangle \hat{C}_\triangle^{m\triangle} \sum_{i=-m}^{-1} \hat{C}_\triangle^{(i+1)\triangle}|x_i|^2 + M_k.
\end{aligned}
$$

Noting that for all $i = -m, \ldots, 0$, the initial data $x_i < \infty$, by Lemma 2, we see that $\lim_{k\to\infty} Y_k < \infty$ a.s. Hence, by (5.4), we have

$$
\begin{aligned}
\limsup_{k\to\infty}(\lambda_1\triangle - \lambda_2\triangle + 1)\hat{C}^{k\triangle}|x_k|^2 \le & \limsup_{k\to\infty}\Big[(\lambda_1\triangle - \lambda_2\triangle + 1)\hat{C}^{k\triangle}|x_k|^2 \\
& +\lambda_2\triangle \hat{C}^{(m-1)\triangle} \sum_{i=k-m+1}^{k-1} \hat{C}^{(i+1)\triangle}|x_i|^2 \\
& +\lambda_4\triangle \hat{C}^{m\triangle} \sum_{i=k-m}^{k-1} \hat{C}^{(i+1)\triangle}|x_i|^2\Big] \\
\le & \lim_{k\to\infty} Y_k < \infty \quad a.s. \qquad (5.6)
\end{aligned}
$$

Noting that $m\triangle = \tau$, by (5.5),

$$
\lambda_4\hat{C}^\tau + \lambda_2\hat{C}^\tau\hat{C}^{-\triangle} + \frac{1}{\triangle}\Big[1-(1+\lambda_1\triangle - \lambda_2\triangle)\hat{C}^{-\triangle}\Big] + \lambda_3 = 0. \quad (5.7)
$$

Introduce a constant η such that $C = e^{\eta}$ and hence $1 - \hat{C}^{-\triangle} = 1 - e^{-\eta\triangle}$. Define

$$\bar{\varphi}_{\triangle}(\eta) = \lambda_4 e^{\eta\tau} + \lambda_2 e^{\eta\tau} e^{-\eta\triangle} + \frac{1}{\triangle}\left[1 - (1 + \lambda_1\triangle - \lambda_2\triangle)e^{-\eta\triangle}\right] + \lambda_3. \quad (5.8)$$

Letting $\eta^*_{\triangle} = \log \hat{C}_{\triangle}$, by (5.7), for any $\triangle < \triangle^*_1 \wedge \triangle^*_2$, we have

$$\bar{\varphi}_{\triangle}(\eta^*_{\triangle}) = 0. \quad (5.9)$$

Noting that $\lim_{\triangle\to 0}(1 - e^{-\eta\triangle})/\triangle = \eta$, we have

$$\lim_{\triangle\to 0} \bar{\varphi}_{\triangle}(\eta) = (\lambda_2 + \lambda_4)e^{\eta\tau} + \eta - \lambda_1 + \lambda_2 + \lambda_3. \quad (5.10)$$

By the definition of γ, (5.9) and (5.10) yield that

$$\lim_{\triangle\to 0} \eta^*_{\triangle} = \gamma, \quad (5.11)$$

which implies that for any positive $\varepsilon \in (0, \gamma/2)$, there exists a $0 < \triangle^* < 1/\lambda$ such that for any $\triangle < \triangle^*$, we have

$$\eta^*_{\triangle} > \gamma - 2\varepsilon.$$

Note that (5.6), together with the definition of $\eta^*_{\triangle}$ shows that

$$\limsup_{k\to\infty}(\lambda_1\triangle - \lambda_2\triangle + 1)e^{\eta^*_{\triangle}k\triangle}|x_k|^2 < \infty.$$

We therefore obtain that for any $\triangle < \triangle^*$,

$$\limsup_{k\to\infty} \frac{1}{k\triangle}\log(|x_k|) \le -\frac{\gamma}{2} + \varepsilon \quad a.s.$$

as required. □

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References

1. Baker, C.T.H., Buckwar, E.: Numerical analysis of explicit one-step methods for stochastic delay differential equations. LMS J. Comput. Math. **3**, 315–335 (2000)
2. Baker, C.T.H., Buckwar, E.: Exponential stability in p-th mean of solutions, and of convergent Euler-type solutions, of stochastic delay differential equations. J. Comput. Appl. Math. **184**, 404–427 (2005)
3. Burrage, K., Burrage, P., Mitsui, T.: Numerical solutions of stochastic differential equations—implematation and stability issues. J. Comput. Appl. Math. **125**, 171–182 (2000)

4. Burrage, K., Tian, T.: A note on the stability propertis of the Euler methods for solving stochastic differential equations. N Z J. Math. **29**, 115–127 (2000)
5. Hairer, E., Wanner, G.: Solving Ordinary Differential Equation II: Stiff and Differential-Algebraic Problems, 2nd edn. Springer, Berlin (1996)
6. Higham, D.J.: Mean-square and asymptotic stability of the stochastic theta methods. SIAM J. Numer. Anal. **38**, 753–769 (2000)
7. Higham, D.J., Mao, X., Yuan, C.: Almost sure and Moment exponential stability in the numerical simulation of stochastic differential equations. SIAM J. Numer. Anal. **45**, 592–607 (2007)
8. Kloeden, P.E., Platen, E.: The Numerical Solution of Stochastic Differential Equations. Springer, Berlin (1992)
9. Liptser, R.Sh., Shiryaev, A.N.: Theory of Martingale. Kluwer Academic Publishers, Dordrecht (1989)
10. Mao, X.: Approximate solutions for a class of stochastic evolution equations with variable delays—part II. Numer. Funct. Anal. Optim. **15**, 65–76 (1994)
11. Mao, X.: Exponential Stability of Stochastic Differential Equation. Marcel Dekker, New York (1994)
12. Mao, X.: Stochastic Differential Equations and their Applications. Horwood, Chichester (1997)
13. Mao, X.: Stochastic versions of the LaSalle theorem. J. Differ. Equ. **153**, 175–195 (1999)
14. Mao, X.: LaSalle-type theorems for stochastic differential delay equations. J. Math. Anal. Appl. **236**, 350–369 (1999)
15. Mao, X.: The LaSalle-type theorems for stochastic differential equations. Nonlinear Stud. **7**, 307–328 (2000)
16. Mao, X.: A note on the LaSalle-type theorems for stochastic differential delay equations. J. Math. Anal. Appl. **268**, 125–142 (2002)
17. Mao, X.: Numerical solutions of stochastic functional differential equations. LMS J. Comput. Math. **6**, 141–161 (2003)
18. Mao, X.: Exponential stability of equidistant Euler-Maruyama approximations of stochastic differential delay equations. J. Comput. Appl. Math. **200**, 297–316 (2007)
19. Mao, X., Rassias, M.J.: Khasminskii-type theorems for stochastic differential delay equations. Stoch. Anal. Appl. **23**, 1045–1069 (2005)
20. Mao, X., Yuan, C.: Stochastic Differential Equations with Markovian Switching. Imperial College Press, London (2006)
21. Pang, S., Deng, F., Mao, X.: Almost sure and moment exponential stability of Euler–Maruyama discretizations for hybrid stochastic differential equations. J. Comput. Appl. Math. **213**, 127–141 (2008)
22. Rodkina, A., Basin, M.: On delay-dependent stability for vector nonlinear stochastic delay-difference equations with Volterra diffusion term. Syst. Control Lett. **56**, 423–430 (2007)
23. Rodkina, A., Schurz, H.: Almost sure asymptotic stability of drift-implicit θ-methods for bilinear ordinary stochastic differential equations in $\mathbb{R}^1$. J. Comput. Appl. Math. **180**, 13–31 (2005)
24. Saito, Y., Mitsui, T.: T-stability of numerical scheme for stochastic differential equations. World Sci. Ser. Appl. Anal. **2**, 333–344 (1993)
25. Saito, Y., Mitsui, T.: Stability analysis of numerical schemes for stochastic differential equations. SIAM J. Numer. Anal. **33**, 2254–2267 (1996)
26. Shiryaev, A.N.: Probability. Springer, Berlin (1996)