# Adaptive FE–BE Coupling for Strongly Nonlinear Transmission Problems with Coulomb Friction

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#### Abstract

We analyze an adaptive finite element/boundary element procedure for scalar elastoplastic interface problems involving friction, where a non-linear uniformly monotone operator such as the p-Laplacian is coupled to the linear Laplace equation on the exterior domain. The problem is reduced to a boundary/domain variational inequality, a discretized saddle point formulation of which is then solved using the Uzawa algorithm and adaptive mesh refinements based on a gradient recovery scheme. The Galerkin approximations are shown to converge to the unique solution of the variational problem in a suitable product of  $L^p$ -and  $L^2$ -Sobolev spaces.

### 1 Introduction

Consider the following transmission problem on a bounded Lipschitz domain  $\Omega \subset \mathbb{R}^n$ :

$$-\operatorname{div}(\varrho(|\nabla u_{1}|)\nabla u_{1}) = f \quad \text{in } \Omega, -\Delta u_{2} = 0 \quad \text{in } \Omega^{c},$$

$$\varrho(|\nabla u_{1}|)\partial_{\nu}u_{1} - \partial_{\nu}u_{2} = t_{0} \quad \text{on } \partial\Omega, u_{1} - u_{2} = u_{0} \quad \text{on } \Gamma_{t},$$

$$(1) \quad -\varrho(|\nabla u_{1}|)\partial_{\nu}u_{1}(u_{0} + u_{2} - u_{1}) + g|(u_{0} + u_{2} - u_{1})| = 0,$$

$$|\varrho(|\nabla u_{1}|)\partial_{\nu}u_{1}| \leq g \quad \text{on } \Gamma_{s}.$$

$$u_{2}(x) = \begin{cases} a + o(1), & n = 2 \\ \mathcal{O}(|x|^{2-n}), & n > 2 \end{cases}.$$

Here  $\varrho(t)$  denotes a function  $\varrho(x,t) \in C(\overline{\Omega} \times (0,\infty))$  satisfying

$$0 \le \varrho(t) \le \varrho^* [t^{\delta} (1+t)^{1-\delta}]^{p-2},$$
$$|\varrho(t)t - \varrho(s)s| \le \varrho^* [(t+s)^{\delta} (1+t+s)^{1-\delta}]^{p-2} |t-s|$$

and

$$\varrho(t)t - \varrho(s)s \ge \varrho_*[(t+s)^{\delta}(1+t+s)^{1-\delta}]^{p-2}(t-s)$$

for all  $t \geq s > 0$  uniformly in  $x \in \Omega$  ( $\delta \in [0,1]$ ,  $\varrho_*, \varrho^* > 0$ ). The interface  $\partial \Omega = \overline{\Gamma_s \cup \Gamma_t}$  is divided into the disjoint components  $\Gamma_s$  and  $\Gamma_t \neq \emptyset$ , and the data belong to the following spaces:

$$f \in L^{p'}(\Omega), \ u_0 \in W^{\frac{1}{2},2}(\partial \Omega), \ t_0 \in W^{-\frac{1}{2},2}(\partial \Omega), \ g \in L^{\infty}(\Gamma_s), \ a \in \mathbb{R}.$$

As usual, the normal derivatives are understood in terms of a Green's formula, and it is convenient to set a = 0 for n > 2. In two dimensions one further condition is required to enforce uniqueness:

(2) 
$$\int_{\Omega} f + \langle t_0, 1 \rangle = 0.$$

We are looking for weak solutions  $(u_1, u_2) \in W^{1,p}(\Omega) \times W^{1,2}_{loc}(\Omega^c)$  when  $p \geq 2$ . A typical example is given by  $\varrho(t) = [t^{\delta}(1+t)^{1-\delta}]^{p-2}$ ,  $\delta \in [0,1]$ , with the p-Laplacian corresponding to the maximally degenerate case  $\delta = 1$ .

In this article we use layer potentials for the Laplace equation on  $\Omega^c$  to reduce the system to a uniquely solvable variational problem on  $W^{1,p}(\Omega) \times W_0^{\frac{1}{2},2}(\Gamma_s)$ . The main idea of our theoretical analysis is simple: Because the traces of  $W^{1,p}(\Omega)$ -functions are continuously embedded into  $W^{\frac{1}{2},2}(\partial\Omega)$  for  $p \geq 2$ , the quadratic form  $\langle Su,u \rangle$  associated to the Steklov-Poincaré operator is accessible to Hilbert space methods whenever it is defined. In this slightly weaker setting, Friedrichs' inequality (Prop. 1) allows to recover control over the  $L^p$ -norms in the interior, and as a consequence the full variational functional associated to the above equations is coercive in  $W^{1,p}(\Omega)$ .

In the numerical part we present a model problem, which shows singularities resulting from the given boundary data, as well as from the change of boundary conditions, leading to a suboptimal convergence rate for uniform mesh refinements. We also present a Uzawa solver to deal with the variational inequality.

With the help of a Korn inequality (Prop. 2), our method easily carries over to transmission problems in nonlinear elasticity, e.g. Hencky materials in  $\Omega$  coupled to the Lamé equation in  $\Omega^c$ . A generalization to certain nonconvex energy functionals will be discussed elsewhere [7].

The outline of the article is as follows: Section 2 recalls some properties of  $L^p$ -Sobolev spaces and introduce a family of quasinorms adapted to the considered class of operators. In the following section 3 we introduce the boundary integral operators and derive our variational formulation. Section 4 is

dedicated to the existence and uniqueness of our model problem. The discretization of our problem is derived in section 5, as well as the a-priori error estimates. In section 6 our a-posteriori error estimator is presented and its reliability proven. Finally, in section 7 we present the Uzawa-solver and two numerical examples, clearly underlining our theoretical results.

### 2 Preliminaries

Let  $\Omega$  be an open subset of  $\mathbb{R}^n$  with Lipschitz boundary  $\partial\Omega$ . Set  $p'=\frac{p}{p-1}$  whenever  $p\in(1,\infty)$ .

**Definition 1.** The Sobolev spaces  $W^{k,p}_{(0)}(\Omega)$ ,  $k \in \mathbb{N}_0$ , are the completion of  $C^{\infty}_{(c)}(\Omega)$  with respect to the norm  $\|u\|_{W^{k,p}(\Omega)} = \|u\|_{k,p} = \|u\|_p + \sum_{|\gamma|=k} \|\partial^{\gamma}u\|_p$ . The second term in the norm will be denoted by  $|u|_{W^{1,p}(\Omega)} = |u|_{k,p}$ . Let  $W^{-k,p'}_0(\Omega) = \left(W^{k,p}(\Omega)\right)'$  and  $W^{-k,p'}(\Omega) = \left(W^{k,p}_0(\Omega)\right)'$ .  $W^{1-\frac{1}{p},p}(\partial\Omega)$  denotes the space of traces of  $W^{1,p}(\Omega)$ -functions on the boundary. It coincides with the Besov space  $B^{1-\frac{1}{p}}_{p,p}(\partial\Omega)$  as obtained by real interpolation of Sobolev spaces [11], and one may define  $W^{s,p}(\partial\Omega) = B^s_{p,p}(\partial\Omega)$  for  $s \in (-1,1)$ .

**Remark 1.** We are going to need the following properties for bounded  $\partial\Omega$  [11]:

- a) All the above spaces are reflexive and  $(W^{s,p}(\partial\Omega))' = W^{-s,p'}(\partial\Omega)$ .
- b) For p=2 they coincide with the Sobolev spaces  $H^s$ .
- c)  $W^{1-\frac{1}{p},p}(\partial\Omega) \hookrightarrow W^{\frac{1}{2},2}(\partial\Omega)$  for  $p \geq 2$ .
- d) If  $\partial\Omega$  is smooth, pseudodifferential operators of order m with symbol in the Hörmander class  $S_{1,0}^m(\partial\Omega)$  map  $W^{s,p}(\partial\Omega)$  continuously to  $W^{s-m,p}(\partial\Omega)$ . For Lipschitz  $\partial\Omega$ , at least the first-order Steklov-Poincaré operator S of the Laplacian on  $\Omega^c$  is continuous between  $W^{\frac{1}{2},2}(\partial\Omega)$  and  $W^{-\frac{1}{2},2}(\partial\Omega)$  [4].
- e) Points a) to d) imply that the quadratic form  $\langle Su, u \rangle$  associated to S is well-defined on  $W^{1-\frac{1}{p},p}(\partial\Omega)$  if  $p \geq 2$ . S being elliptic, the form cannot be defined for p < 2 even if  $\partial\Omega$  is smooth.

Uniform monotony will be shown using a variant of Friedrichs' inequality.

**Proposition 1.** Assume  $\Omega$  is bounded and that  $\Gamma \subset \partial \Omega$  has positive (n-1)-dimensional measure. Then there is a C > 0 such that

$$||u||_p \le C(||\nabla u||_p + ||u|_\Gamma||_{L^1(\Gamma)})$$
 for all  $u \in W^{1,p}(\Omega)$ .

*Proof.* We apply an interpolation argument to the well-known Friedrichs' inequality

$$||u - u_{\Omega}||_p \le C||\nabla u||_p, \qquad u_{\Omega} = \frac{1}{|\Omega|} \int_{\Omega} u,$$

on  $W^{1,p}(\Omega)$  (see e.g. [10]). Let  $L:W^{1,p}(\Omega)\to L^p(\Omega)$  be the rank-1 operator  $Lu=\frac{1}{|\Gamma|}\int_{\Gamma}u|_{\Gamma}$  and I the inclusion of  $W^{1,p}(\Omega)$  into  $L^p(\Omega)$ . Then  $I-L:W^{1,p}(\Omega)\to L^p(\Omega)$  is bounded and

$$||u - Lu||_p = ||(I - L)(u - u_{\Omega})||_p \le ||I - L|| ||u - u_{\Omega}||_{1,p} \le C||\nabla u||_p$$

for all  $u \in W^{1,p}(\Omega)$ . The assertion follows.

Let  $\omega(x,y) = (|x|+|y|)^{\delta}(1+|x|+|y|)^{1-\delta}$ ,  $0 \le \delta \le 1$ . In addition to the above norms, the following family of quasi-norms will prove useful:

**Definition 2.** For  $v, w \in W^{1,p}(\Omega)$  and  $k \in \mathbb{N}_0$ , define

$$|v|_{(k,w,p)} = \left(\int_{\Omega} \omega(\nabla w, D^k v)^{p-2} |D^k v|^2\right)^{\frac{1}{2}},$$

where  $|D^k v|^2 = \sum_{|\gamma|=k} |\partial^{\gamma} v|^2$ .

**Remark 2.** a) If  $p \ge 2$ , the (1, w, p)-quasi-norm can be estimated from above and below by suitable powers of the  $W^{1,p}$ -seminorm [6]:

$$|v|_{1,p}^p \le |v|_{(1,w,p)}^2 \le C(|v|_{1,p},|w|_{1,p})|v|_{1,p}^2.$$

- b) In the nondegenerate case  $\delta = 0$ , we have  $|v|_{1,2}^2 \leq |v|_{(1,w,n)}^2$ .
- c) The following inequality is useful for computations with quasi-norms:

$$\lambda \mu \le \max\{\varepsilon^{-1}, \varepsilon^{1/(1-p)}\} (a^{p-1} + \lambda)^{p'-2} \lambda^2 + \varepsilon (a + \mu)^{p-2} \mu^2$$

for  $\lambda, \mu, a \geq 0$  and  $\varepsilon > 0$ .

The results of this paper easily generalize to the systems of equations describing certain inelastic materials. In this case, Lemma 1 has to be replaced by the following Korn inequality:

**Proposition 2.** Assume  $\Omega \subset \mathbb{R}^n$  is a bounded Lipschitz domain and  $\Gamma \subset \partial \Omega$  has positive (n-1)-dimensional measure. Then there is a C > 0 such that

$$||u||_{1,p} \le C(||\varepsilon(u)||_p + ||u|_{\Gamma}||_{L^1(\Gamma)})$$
 for all  $u \in (W^{1,p}(\Omega))^n$ .

Proof. The  $L^p$ -version  $\|u\|_{1,p} \leq C(\|\varepsilon(u)\|_p + \|u\|_p)$  of Korn's inequality is well-known (see e.g. [5]). Assume the assertion was false. Then  $\|\varepsilon(u_n)\|_p + \|u_n\|_{\Gamma}\|_{L^1(\Gamma)} \leq \frac{1}{n}$  for some sequence in  $W^{1,p}(\Omega)$  normalized to  $\|u_n\|_{1,p} = 1$ . By the compactness of  $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ , we may assume  $u_n$  to converge in  $L^p(\Omega)$ . The cited variant of Korn's inequality shows that  $u_n$  is even Cauchy in  $W^{1,p}(\Omega)$ , hence converges to some  $u_0$  with  $\|\varepsilon(u_0)\|_p = \|u_0\|_{\Gamma}\|_{L^1(\Gamma)} = 0$ . The kernel of  $\varepsilon$  consists of skew-symmetric affine transformations Ax + b,  $A = -A^T$ . As dim ker  $A \equiv n \mod 2$ ,  $u_0$  cannot vanish on all of the (n-1- dimensional)  $\Gamma$  unless  $u_0 = 0$ . Contradiction to  $\|u_0\|_{1,p} = 1$ .

## 3 Variational Formulation and Reduction to $\partial\Omega$

We continue to use the notation from the Introduction and mainly follow [9]. Fix some  $p \geq 2$  and, for  $q(t) = \int_0^t s\varrho(s) \, ds$ , let  $G(u) = \int_\Omega q(|\nabla u|)$  with derivative

$$DG(u,v) = \langle G'u,v \rangle = \int_{\Omega} \varrho(|\nabla u|) \nabla u \nabla v \qquad (u,v \in W^{1,p}(\Omega))$$

and  $j(v) = \int_{\Gamma_s} g|v|, v \in L^1(\Gamma_s)$ . G is known to be strictly convex and G':  $W^{1,p}(\Omega) \to \left(W^{1,p}(\Omega)\right)'$  bounded and uniformly monotone, hence coercive, with respect to the seminorm  $|\cdot|_{1,p}$ : There is some  $\alpha_G > 0$  such that for all  $u, v \in W^{1,p}(\Omega)$ 

$$\langle G'u - G'v, u - v \rangle \ge \alpha_G |u - v|_{1,p}^p$$
 and  $\lim_{|u|_{1,p} \to \infty} \frac{\langle G'u, u \rangle}{|u|_{1,p}} = \infty.$ 

The naive variational formulation of the transmission problem (1) minimizes the functional

$$\Phi(u_1, u_2) = G(u_1) + \frac{1}{2} \int_{\Omega^c} |\nabla u_2|^2 - \int_{\Omega} f u_1 - \langle t_0, u_2 |_{\partial \Omega} \rangle + j((u_2 - u_1 + u_0) |_{\Gamma_s})$$

over a suitable convex set.

**Lemma 1.** Minimizing  $\Phi$  over the nonempty, closed and convex subset

$$C = \{(u_1, u_2) \in W^{1,p}(\Omega) \times W^{1,2}_{loc}(\Omega^c) : (u_1 - u_2)|_{\Gamma_t} = u_0, u_2 \in \mathcal{L}_2\},$$

 $\mathcal{L}_2 = \{v \in W_{loc}^{1,2}(\Omega^c) : \Delta v = 0 \text{ in } W^{-1,2}(\Omega^c) + \text{radiation condition at } \infty \},$  is equivalent to the system (1) in the sense of distributions if  $\varrho \in C^1(\overline{\Omega} \times (0,\infty))$ .

*Proof.* C is apparently convex. A similar argument as in Remarks 2 and 4 of [1] shows that C is closed and nonempty. The proof there almost exclusively involves the exterior problem in  $\mathcal{L}_2$  and only requires basic measure theoretic properties of  $W^{1,2}(\Omega)$ , which also hold for  $W^{1,p}(\Omega)$ . Finally, repeat the computations of [9] to obtain equivalence with (1).

To reduce the exterior problem to the boundary, we are going to need the layer potentials

$$\mathcal{V}\phi(x) = -\frac{1}{\pi} \int_{\partial\Omega} \phi(x') \log|x - x'| dx',$$

$$\mathcal{K}\phi(x) = -\frac{1}{\pi} \int_{\partial\Omega} \phi(x') \partial_{\nu_{x'}} \log|x - x'| dx',$$

$$\mathcal{K}'\phi(x) = -\frac{1}{\pi} \int_{\partial\Omega} \phi(x') \partial_{\nu_{x}} \log|x - x'| dx',$$

$$\mathcal{W}\phi(x) = \frac{1}{\pi} \partial_{\nu_{x}} \int_{\partial\Omega} \phi(x') \partial_{\nu_{x'}} \log|x - x'| dx'$$

associated to the Laplace equation on  $\Omega^c$ . They extend from  $C^{\infty}(\partial\Omega)$  to a bounded map  $\begin{pmatrix} -\mathcal{K} & \mathcal{V} \\ \mathcal{W} & \mathcal{K}' \end{pmatrix}$  on the Sobolev space  $W^{\frac{1}{2},2}(\partial\Omega) \times W^{-\frac{1}{2},2}(\partial\Omega)$ . If the capacity of  $\partial\Omega$  is less than 1, which can always be achieved by scaling,  $\mathcal{V}$  and  $\mathcal{W}$  considered as operators on  $W^{-\frac{1}{2},2}(\partial\Omega)$  are selfadjoint,  $\mathcal{V}$  is positive and  $\mathcal{W}$  non-negative. Similarly, the Steklov-Poincaré operator

$$S = \mathcal{W} + (1 - \mathcal{K}')\mathcal{V}^{-1}(1 - \mathcal{K}) : W^{\frac{1}{2}, 2}(\partial\Omega) \subset W^{-\frac{1}{2}, 2}(\partial\Omega) \to W^{-\frac{1}{2}, 2}(\partial\Omega)$$

defines a positive and selfadjoint operator (pseudodifferential of order 1, if  $\partial\Omega$  is smooth) with the main property

$$\partial_{\nu} u_2|_{\partial\Omega} = -S(u_2|_{\partial\Omega} - a)$$

for solutions  $u_2 \in \mathcal{L}_2$  of the Laplace equation on  $\Omega^c$ . By Remark 1 e), S gives rise to a coercive and symmetric bilinear form  $\langle Su, u \rangle$  on  $W^{\frac{1}{2},2}(\partial\Omega)$  and, in particular, a pairing on the traces of  $W^{1,p}(\Omega)$  if and only if  $p \geq 2$ .

Using the weak definition of  $\partial_{\nu}|_{\partial\Omega}$ , S reduces the integral over  $\Omega^{c}$  in  $\Phi$  to the boundary:

$$\int_{\Omega^c} |\nabla u_2|^2 = -\langle \partial_\nu u_2 |_{\partial\Omega}, u_2 |_{\partial\Omega} \rangle = \langle S(u_2 |_{\partial\Omega} - a), u_2 |_{\partial\Omega} \rangle \quad \text{for } u_2 \in \mathcal{L}_2.$$

Easy manipulations allow to substitute  $u_2$  by a function v on  $\Gamma_s$  (cf. [9]): Let

$$\widetilde{W}^{\frac{1}{2},2}(\Gamma_s) = \{ u \in W^{\frac{1}{2},2}(\partial\Omega) : \text{supp } u \subset \bar{\Gamma}_s \}, \quad X^p = W^{1,p}(\Omega) \times \widetilde{W}^{\frac{1}{2},2}(\Gamma_s)$$

and  $(u, v) = (u_1 - c, u_0 + u_2|_{\partial\Omega} - u_1|_{\partial\Omega}) \in X^p$  for a suitable  $c \in \mathbb{R}$ . Collecting the data-dependent terms in

$$\lambda(u,v) = \langle t_0 + Su_0, u|_{\partial\Omega} + v \rangle + \int_{\Omega} fu$$

leads to

$$\Phi(u_1, u_2) = G(u) + \frac{1}{2} \langle S(u|_{\partial\Omega} + v), u|_{\partial\Omega} + v \rangle - \lambda(u, v) + j(v) + \frac{1}{2} \langle Su_0, u_0 \rangle + \langle t_0, u_0 \rangle.$$

The first three terms on the right hand side will be called J(u, v).

**Lemma 2.** Minimizing  $\Phi$  over C is equivalent to minimizing J+j over the nonempty closed convex set  $D=\{(u,v)\in X^p: \langle S(u|_{\partial\Omega}+v-u_0),1\rangle=0 \text{ if } n=2\}$ 

Proof. As in [9]. The main additional observation here is that the substitution  $v = u_0 + u_2|_{\partial\Omega} - u_1|_{\partial\Omega}$  indeed defines an element of  $\widetilde{W}^{\frac{1}{2},2}(\Gamma_s)$ , because  $u_0, u_2|_{\partial\Omega} \in W^{\frac{1}{2},2}(\partial\Omega), u_1|_{\partial\Omega} \in W^{1-\frac{1}{p},p}(\partial\Omega) \subset W^{\frac{1}{2},2}(\partial\Omega)$  by Remark 1 and  $v|_{\Gamma_t} = 0$ , if  $(u_1, u_2) \in C$ .

## 4 Existence and Uniqueness

Minimization of J+j over D translates into the following variational inequality: Find  $(\hat{u}, \hat{v}) \in X^p$  such that

$$\langle G'\hat{u}, u - \hat{u} \rangle + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (u - \hat{u})|_{\partial\Omega} + v - \hat{v} \rangle + j(v) - j(\hat{v}) \geq \lambda(u - \hat{u}, v - \hat{v})$$

for all  $(u, v) \in X^p$ . Note that D has been replaced by  $X^p$ .

We now prove the crucial monotony estimate:

**Lemma 3.** The operator in the variational inequality is uniformly monotone on  $X^p$ . There exists an  $\alpha = \alpha(C) > 0$  such that for all  $||u,v||_X, ||\hat{u},\hat{v}||_X < C$ 

$$\alpha(\|u-\hat{u}\|_{W^{1,p}(\Omega)}^p + \|v-\hat{v}\|_{\widetilde{W}^{\frac{1}{2},2}(\Gamma_s)}^p) \le \langle G'\hat{u} - G'u, \hat{u} - u \rangle$$
$$+ \langle S((\hat{u}-u)|_{\partial\Omega} + \hat{v} - v), (\hat{u}-u)|_{\partial\Omega} + \hat{v} - v \rangle.$$

*Proof.* Recall the monotony estimate for G' from Section 3:

$$\langle G'\hat{u} - G'u, \hat{u} - u \rangle \ge \alpha_G |\hat{u} - u|_{1,n}^p.$$

The triangle inequality and convexity of  $x^p$  imply

$$\begin{aligned} \|\hat{v} - v\|_{\widetilde{W}^{\frac{1}{2},2}(\Gamma_s)}^p & \leq & (\|(\hat{u} - u)|_{\Gamma_s} + \hat{v} - v\|_{W^{\frac{1}{2},2}(\Gamma_s)} + \|(\hat{u} - u)|_{\Gamma_s}\|_{W^{\frac{1}{2},2}(\Gamma_s)})^p \\ & \leq & 2^{p-1} \left(\|(\hat{u} - u)|_{\Gamma_s} + \hat{v} - v\|_{W^{\frac{1}{2},2}(\Gamma_s)}^p + \|(\hat{u} - u)|_{\Gamma_s}\|_{W^{\frac{1}{2},2}(\Gamma_s)}^p\right). \end{aligned}$$

Using  $W^{1-\frac{1}{p},p}(\Gamma_s) \hookrightarrow W^{\frac{1}{2},2}(\Gamma_s)$  as well as the boundedness of the trace operator,

$$2^{1-p} \|\hat{v} - v\|_{\widetilde{W}^{\frac{1}{2},2}(\Gamma_s)}^p - \beta \|\hat{u} - u\|_{W^{1,p}(\Omega)}^p \le \|(\hat{u} - u)|_{\Gamma_s} + \hat{v} - v\|_{W^{\frac{1}{2},2}(\Gamma_s)}^p$$

follows for some  $\beta \geq 1$ . Let

$$K = \{(u, v, \hat{u}, \hat{v}) \in X^p \times X^p : \|(\hat{u} - u)|_{\partial\Omega} + \hat{v} - v\|_{W^{\frac{1}{2}, 2}(\partial\Omega)} < 2\beta C\}$$

and  $0 < \varepsilon < \beta^{-1}$ . Since S is positive definite on  $W^{\frac{1}{2},2}(\partial\Omega)$ , we obtain from Friedrichs' inequality for  $(u,v,\hat{u},\hat{v}) \in K$  or, in particular, if  $\|u,v\|_X, \|\hat{u},\hat{v}\|_X < C$ :

$$\begin{split} &\langle G'\hat{u} - G'u, \hat{u} - u \rangle + \langle S((\hat{u} - u)|_{\partial\Omega} + \hat{v} - v), (\hat{u} - u)|_{\partial\Omega} + \hat{v} - v \rangle \\ &\gtrsim |\hat{u} - u|_{1,p}^p + \|(\hat{u} - u)|_{\partial\Omega} + \hat{v} - v\|_{W^{\frac{1}{2},2}(\partial\Omega)}^2 \\ &\gtrsim |\hat{u} - u|_{1,p}^p + \|(\hat{u} - u)|_{\partial\Omega} + \hat{v} - v\|_{W^{\frac{1}{2},2}(\partial\Omega)}^p \\ &\gtrsim |\hat{u} - u|_{1,p}^p + \varepsilon \|(\hat{u} - u)|_{\Gamma_s} + \hat{v} - v\|_{W^{\frac{1}{2},2}(\Gamma_s)}^p + \|(\hat{u} - u)|_{\Gamma_t}\|_{W^{\frac{1}{2},2}(\Gamma_t)}^p \\ &\gtrsim \|\hat{u} - u\|_{W^{1,p}(\Omega)}^p + \varepsilon \|(\hat{u} - u)|_{\Gamma_s} + \hat{v} - v\|_{W^{\frac{1}{2},2}(\Gamma_s)}^p \\ &\gtrsim (1 - \varepsilon\beta) \|\hat{u} - u\|_{W^{1,p}(\Omega)}^p + 2^{1-p}\varepsilon \|\hat{v} - v\|_{W^{\frac{1}{2},2}(\Gamma_s)}^p \,. \end{split}$$

Uniform monotony on all of  $X^p$  is shown similarly, but on the unbounded complement  $(X^p \times X^p) \setminus K$  the exponents p on the left hand side have to be replaced by 2.

**Theorem 1.** The variational inequality is equivalent to the transmission problem (1) and has a unique solution.

*Proof.* We repeat the computations in [9] to get the equivalence with the minimization of J+j over D, and hence with (1). Existence and uniqueness follow from Lemma 3, e.g. by applying [12], Proposition 32.36.

## 5 Discretization and Error Analysis

In order to avoid using  $S = W + (1 - K')V^{-1}(1 - K)$  explicitly, the numerical implementation involves a variant of the variational inequality

$$\langle G'\hat{u}, u - \hat{u} \rangle + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (u - \hat{u})|_{\partial\Omega} + v - \hat{v} \rangle + j(v) - j(\hat{v}) \ge \lambda(u - \hat{u}, v - \hat{v})$$

in terms of the layer potentials. Our a posteriori analysis is therefore based on the following equivalent problem: Find  $(\hat{u}, \hat{v}, \hat{\phi}) \in X^p \times W^{-\frac{1}{2}, 2}(\partial\Omega) =: Y^p$ , such that

$$\langle G'\hat{u}, u - \hat{u} \rangle + \langle \mathcal{W}(\hat{u}|_{\partial\Omega} + \hat{v}) + (\mathcal{K}' - 1)\hat{\phi}, (u - \hat{u})|_{\partial\Omega} + v - \hat{v} \rangle$$

$$+ j(v) - j(\hat{v}) \geq \langle t_0 + \mathcal{W}u_0, (u - \hat{u})|_{\partial\Omega} + v - \hat{v} \rangle + \int_{\Omega} f(u - \hat{u}),$$

$$\langle \phi, \mathcal{V}\hat{\phi} + (1 - \mathcal{K})(\hat{u}|_{\partial\Omega} + \hat{v}) \rangle = \langle \phi, (1 - \mathcal{K})u_0 \rangle$$

for all  $(u, v, \phi) \in Y^p$ . More concisely,

$$B(\hat{u},\hat{v},\hat{\phi};u-\hat{u},v-\hat{v},\phi-\hat{\phi})+j(v)-j(\hat{v})\geq\Lambda(u-\hat{u},v-\hat{v},\phi-\hat{\phi})$$

with

$$B(u, v, \phi; \bar{u}, \bar{v}, \bar{\phi}) = \langle G'u, \bar{u} \rangle + \langle \mathcal{W}(u|_{\partial\Omega} + v) + (\mathcal{K}' - 1)\phi, \bar{u}|_{\partial\Omega} + \bar{v} \rangle + \langle \bar{\phi}, \mathcal{V}\phi + (1 - \mathcal{K})(u|_{\partial\Omega} + v) \rangle,$$
  

$$\Lambda(u, v, \phi) = \langle t_0 + \mathcal{W}u_0, u|_{\partial\Omega} + v \rangle + \int_{\Omega} fu + \langle \phi, (1 - \mathcal{K})u_0 \rangle.$$

The more detailed a priori and a posteriori error analysis requires a few basic properties of the quasi–norms [6].

**Remark 3.** a) The continuity and coercivity estimates can be sharpened: For all  $u, v \in W^{1,p}(\Omega)$ 

$$\langle G'u - G'v, u - v \rangle \lesssim |u - v|_{(1,u,p)}^2 \lesssim \langle G'u - G'v, u - v \rangle.$$

b) There is  $\theta > 0$  such that for all  $\varepsilon \in (0, \infty)$  and all  $u, v, w \in W^{1,p}(\Omega)$ 

$$|\langle G'u - G'v, w \rangle| \lesssim \varepsilon |u - v|_{(1,u,p)}^2 + \varepsilon^{-\theta} |w|_{(1,u,p)}^2.$$

**Lemma 4.** For all  $(\hat{u}, \hat{v}, \hat{\phi}), (u, v, \phi) \in Y^p$  we have

$$\begin{split} &|\hat{u}-u|_{(1,\hat{u},p)}^{2}+\|(\hat{u}-u)|_{\partial\Omega}+\hat{v}-v\|_{W^{\frac{1}{2},2}(\partial\Omega)}^{2}+\|\eta\|_{W^{-\frac{1}{2},2}(\partial\Omega)}^{2}\\ &\lesssim |\hat{u}-u|_{(1,\hat{u},p)}^{2}+\|(\hat{u}-u)|_{\partial\Omega}+\hat{v}-v\|_{W^{\frac{1}{2},2}(\partial\Omega)}^{2}+\|\hat{\phi}-\phi\|_{W^{-\frac{1}{2},2}(\partial\Omega)}^{2}\\ &\lesssim B(\hat{u},\hat{v},\hat{\phi};\hat{u}-u,\hat{v}-v,\eta)-B(u,v,\phi;\hat{u}-u,\hat{v}-v,\eta), \end{split}$$

where 
$$2\eta = \hat{\phi} - \phi + V^{-1}(1 - K)((\hat{u} - u)|_{\partial\Omega} + \hat{v} - v)$$
.

*Proof.* The right hand side of the identity

$$B(\hat{u}, \hat{v}, \hat{\phi}; \hat{u} - u, \hat{v} - v, \eta) - B(u, v, \phi; \hat{u} - u, \hat{v} - v, \eta)$$

$$= \langle G'\hat{u} - G'u, \hat{u} - u \rangle + \frac{1}{2} \langle \mathcal{W}((\hat{u} - u)|_{\partial\Omega} + \hat{v} - v), (\hat{u} - u)|_{\partial\Omega} + \hat{v} - v) \rangle$$

$$+ \frac{1}{2} \langle S((\hat{u} - u)|_{\partial\Omega} + \hat{v} - v), (\hat{u} - u)|_{\partial\Omega} + \hat{v} - v) \rangle + \frac{1}{2} \langle \mathcal{V}(\hat{\phi} - \phi), \hat{\phi} - \phi \rangle.$$

is, up to a constant, larger than  $\|\hat{u}-u,\hat{v}-v,\hat{\phi}-\phi\|_{(\hat{u},Y^p)}^2$ . Furthermore,

$$\|\eta\|_{W^{-\frac{1}{2},2}(\partial\Omega)} \lesssim \|\hat{\phi} - \phi\|_{W^{-\frac{1}{2},2}(\partial\Omega)} + \|(\hat{u} - u)|_{\partial\Omega} + \hat{v} - v\|_{W^{\frac{1}{2},2}(\partial\Omega)}.$$

Let  $\{\mathcal{T}_h\}_{h\in I}$  a regular triangulation of  $\Omega$  into disjoint open regular triangles K, so that  $\overline{\Omega}=\bigcup_{K\in\mathcal{T}_h}K$ . Each element has at most one edge on  $\partial\Omega$ , and the closures of any two of them share at most a single vertex or edge. Let  $h_K$  denote the diameter of  $K\in\mathcal{T}_h$  and  $\rho_K$  the diameter of the largest inscribed ball. We assume that  $1\leq \max_{K\in\mathcal{T}_h}\frac{h_K}{\rho_K}\leq R$  independent of h and that  $h=\max_{K\in\mathcal{T}_h}h_K$ .  $\mathcal{E}_h$  is going to be the set of all edges of the triangles in  $\mathcal{T}_h$ , D the set of nodes. Associated to  $\mathcal{T}_h$  is the space  $W_h^{1,p}(\Omega)\subset W^{1,p}(\Omega)$  of functions whose restrictions to any  $K\in\mathcal{T}_h$  are linear.

 $\partial\Omega$  is triangulated by  $\{l\in\mathcal{E}_h:l\subset\partial\Omega\}.\ W_h^{\frac{1}{2},2}(\partial\Omega)$  denotes the corresponding space of piecewise linear functions, and  $\widetilde{W}_h^{\frac{1}{2},2}(\Gamma_s)$  the subspace of those supported on  $\Gamma_s$ . Finally,  $W_h^{-\frac{1}{2},2}(\partial\Omega)\subset W^{-\frac{1}{2},2}(\partial\Omega)$ .

We denote by  $i_h: W_h^{1,p}(\Omega) \hookrightarrow W^{1,p}(\Omega), \ j_h: \widetilde{W}_h^{\frac{1}{2},2}(\Gamma_s) \hookrightarrow \widetilde{W}^{\frac{1}{2},2}(\Gamma_s)$  and  $k_h: W_h^{-\frac{1}{2},2}(\partial\Omega) \hookrightarrow W^{-\frac{1}{2},2}(\partial\Omega)$  the canonical inclusion maps. Set  $X_h^p = W_h^{1,p}(\Omega) \times \widetilde{W}_h^{\frac{1}{2},2}(\Gamma_s)$ , We denote by  $i_h: W_h^{1,p}(\Omega) \hookrightarrow W^{1,p}(\Omega), \ j_h: \widetilde{W}_h^{\frac{1}{2},2}(\Gamma_s) \hookrightarrow \widetilde{W}^{\frac{1}{2},2}(\Gamma_s)$  and  $k_h: W_h^{-\frac{1}{2},2}(\partial\Omega) \hookrightarrow W^{-\frac{1}{2},2}(\partial\Omega)$  the canonical inclusion maps. Set  $X_h^p = W_h^{1,p}(\Omega) \times \widetilde{W}_h^{\frac{1}{2},2}(\Gamma_s)$ ,

$$S_h = \frac{1}{2}(W + (I - K')k_h(k_h^*Vk_h)^{-1}k_h^*(I - K))$$

and

$$\lambda_h(u_h, v_h) = \langle t_0 + S_h u_0, u|_{\partial\Omega} + v \rangle + \int_{\Omega} f u_h.$$

As is well–known, there exists  $h_0 > 0$  such that the approximate Steklov–Poincaré operator  $S_h$  is coercive uniformly in  $h < h_0$ , i.e.  $\langle S_h u_h, u_h \rangle \ge \alpha_S \|u_h\|_{W^{\frac{1}{2},2}(\partial\Omega)}^2$  with  $\alpha_S$  independent of h.

The discretized variational inequality reads as follows: Find  $(\hat{u}_h, \hat{v}_h, \hat{\phi}_h) \in Y_h^p$  such that

$$B(\hat{u}_h, \hat{v}_h, \hat{\phi}_h; u_h - \hat{u}_h, v_h - \hat{v}_h, \phi_h - \hat{\phi}_h) + j(v_h) - j(\hat{v}_h) \ge \Lambda(u_h - \hat{u}_h, v_h - \hat{v}_h, \phi_h - \hat{\phi}_h)$$

for all  $(u_h, v_h, \phi_h) \in Y_h^p$ . Repeating the arguments from the previous section, one obtains a unique solution to the discretized variational inequality.

**Theorem 2.** Let  $(\hat{u}, \hat{v}, \hat{\phi}) \in Y^p$ ,  $(\hat{u}_h, \hat{v}_h, \hat{\phi}_h) \in Y_h^p$  be the solutions of the continuous resp. discretized variational problem. The following a priori bound for the error holds uniformly in  $h < h_0$ :

$$\begin{split} &\|\hat{u} - \hat{u}_{h}, \hat{v} - \hat{v}_{h}, \hat{\phi} - \hat{\phi}_{h}\|_{Y^{p}}^{p} \\ &\lesssim |\hat{u} - \hat{u}_{h}|_{(1,\hat{u},p)}^{2} + \|(\hat{u} - \hat{u}_{h})|_{\partial\Omega} + \hat{v} - \hat{v}_{h}\|_{W^{\frac{1}{2},2}(\partial\Omega)}^{2} + \|\hat{\phi} - \hat{\phi}_{h}\|_{W^{-\frac{1}{2},2}(\partial\Omega)}^{2} \\ &\lesssim \inf_{(u_{h},v_{h},\phi_{h})\in Y_{h}^{p}} \|\hat{u} - u_{h}, \hat{v} - v_{h}, \hat{\phi} - \phi_{h}\|_{Y^{p}}^{2} + \|\hat{v} - v_{h}\|_{L^{2}(\Gamma_{s})}. \end{split}$$

*Proof.* Let  $(u, v, \phi) \in Y^p$ ,  $(u_h, v_h, \phi_h) \in Y_h^p$ . Lemma 4 and the variational inequality imply

$$\begin{split} &|\hat{u}-\hat{u}_{h}|_{(1,\hat{u},p)}^{2}+\|(\hat{u}-\hat{u}_{h})|_{\partial\Omega}+\hat{v}-\hat{v}_{h}\|_{W^{\frac{1}{2},2}(\partial\Omega)}^{2}+\|\hat{\phi}-\hat{\phi}_{h}\|_{W^{-\frac{1}{2},2}(\partial\Omega)}^{2}\\ &\lesssim B(\hat{u},\hat{v},\hat{\phi};\hat{u}-\hat{u}_{h},\hat{v}-\hat{v}_{h},\hat{\phi}-\hat{\phi}_{h})-B(\hat{u}_{h},\hat{v}_{h},\hat{\phi}_{h};\hat{u}-\hat{u}_{h},\hat{v}-\hat{v}_{h},\hat{\phi}-\hat{\phi}_{h})\\ &\lesssim B(\hat{u},\hat{v},\hat{\phi};u,v,\phi)-\Lambda(u-\hat{u},v-\hat{v},\phi-\hat{\phi})+j(v)-j(\hat{v})\\ &+B(\hat{u}_{h},\hat{v}_{h},\hat{\phi}_{h};u_{h},v_{h},\phi_{h})-\Lambda(u_{h}-\hat{u}_{h},v_{h}-\hat{v}_{h},\phi_{h}-\hat{\phi}_{h})+j(v_{h})-j(\hat{v}_{h})\\ &-B(\hat{u}_{h},\hat{v}_{h},\hat{\phi}_{h};\hat{u},\hat{v},\hat{\phi})-B(\hat{u},\hat{v},\hat{\phi};\hat{u}_{h},\hat{v}_{h},\hat{\phi}_{h}) \end{split}$$

Setting  $(u, v, \phi) = (\hat{u}_h, \hat{v}_h, \hat{\phi}_h)$  and adding 0, the right hand side turns into

$$B(\hat{u}, \hat{v}, \hat{\phi}; u_h - \hat{u}, v_h - \hat{v}, \phi_h - \hat{\phi}) - \Lambda(u_h - \hat{u}, v_h - \hat{v}, \phi_h - \hat{\phi}) + j(v_h) - j(\hat{v}) + B(\hat{u}, \hat{v}, \hat{\phi}; \hat{u} - u_h, \hat{v} - v_h, \hat{\phi} - \phi_h) - B(\hat{u}_h, \hat{v}_h, \hat{\phi}_h; \hat{u} - u_h, \hat{v} - v_h, \hat{\phi} - \phi_h).$$

We first consider the friction terms:

$$j(v_h) - j(\hat{v}) = \int_{\Gamma_s} g(|v_h| - |\hat{v}|) \le \int_{\Gamma_s} g(|v_h - \hat{v}|) \le ||g||_{L^2(\Gamma_s)} ||v_h - \hat{v}||_{L^2(\Gamma_s)}.$$

The last two terms are bounded using Remark 3b and Cauchy-Schwarz:

$$\langle G'\hat{u} - G'\hat{u}_{h}, \hat{u} - u_{h} \rangle \lesssim \varepsilon |\hat{u} - \hat{u}_{h}|_{(1,\hat{u},p)}^{2} + \varepsilon^{-\theta} |\hat{u} - u_{h}|_{(1,\hat{u},p)}^{2},$$
  
$$\lesssim \varepsilon |\hat{u}_{h} - \hat{u}|_{(1,\hat{u},p)}^{2} + \varepsilon^{-\theta} C(|\hat{u}|_{1,p}, |u_{h}|_{1,p}) |u_{h} - \hat{u}|_{1,p}^{2},$$

for sufficiently small  $\varepsilon > 0$ . We may replace  $C(|\hat{u}|_{1,p}, |u_h|_{1,p})$  by an honest constant noting that the coercivity of our functional gives an a priori bound on  $\|\hat{u}\|_{W^{1,p}(\Omega)}$  and that we can restrict to those  $u_h$  satisfying  $\|u_h\|_{W^{1,p}(\Omega)} \le 2\|\hat{u}\|_{W^{1,p}(\Omega)}$ . Moreover,

$$\langle \mathcal{W}((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h) + (1 - \mathcal{K}')(\hat{\phi} - \hat{\phi}_h), (\hat{u} - u_h)|_{\partial\Omega} + \hat{v} - v_h \rangle$$

$$\lesssim \varepsilon \|(\hat{u} - \hat{u}_h|_{\partial\Omega} + \hat{v} - \hat{v}_h\|_{W^{\frac{1}{2}, 2}(\partial\Omega)}^2 + \varepsilon \|\hat{\phi} - \hat{\phi}_h\|_{W^{-\frac{1}{2}, 2}(\partial\Omega)}^2 + \varepsilon^{-1} \|\hat{u} - u_h\|_{W^{\frac{1}{2}, 2}(\partial\Omega)}^2 + \varepsilon^{-1} \|\hat{v} - v_h\|_{W^{\frac{1}{2}, 2}(\partial\Omega)}^2,$$

and

$$\langle \hat{\phi} - \phi_h, \mathcal{V}(\hat{\phi} - \hat{\phi}_h) + (1 - \mathcal{K})((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h) \rangle$$

$$\lesssim \varepsilon^{-1} \|\hat{\phi} - \phi_h\|_{W^{-\frac{1}{2},2}(\partial\Omega)}^2 + \varepsilon \|\hat{\phi} - \hat{\phi}_h\|_{W^{-\frac{1}{2},2}(\partial\Omega)}^2 + \varepsilon \|(\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\|_{W^{\frac{1}{2},2}(\partial\Omega)}^2.$$

Substituting  $(u, v, \phi) = (u_h, \hat{v}, 0)$  and  $(u, v, \phi) = (2\hat{u} - u_h, \hat{v}, 0)$  into the variational inequality on  $Y^p$  and using that also the  $\phi$  part is really an equality, the remaining two terms reduce to

$$\begin{split} &\langle -t_0 - \mathcal{W} u_0 + \mathcal{W}(\hat{u}|_{\partial\Omega} + \hat{v}) + (\mathcal{K}' - 1)\hat{\phi}, v_h - \hat{v} \rangle \\ &= -\langle t_0 - S(\hat{u}|_{\partial\Omega} + \hat{v} - u_0), v_h - \hat{v} \rangle \\ &= -\langle \varrho(|\nabla u|)\partial_{\nu}u, v_h - \hat{v} \rangle \leq \|g\|_{L^2(\Gamma_s)} \|v_h - \hat{v}\|_{L^2(\Gamma_s)}. \end{split}$$

Applying these various estimates to the terms of the right hand side, the assertion follows from

$$\begin{split} \|\hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h, \hat{\phi} - \hat{\phi}_h\|_{Y^p}^p &\lesssim |\hat{u} - \hat{u}_h|_{(1,\hat{u},p)}^2 + \|(\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\|_{W^{\frac{1}{2},2}(\partial\Omega)}^2 + \|\hat{\phi} - \hat{\phi}_h\|_{W^{-\frac{1}{2},2}(\partial\Omega)}^2 \\ \text{as in Lemma 3.} & \Box \end{split}$$

In the nondegenerate case  $\delta = 0$ , we essentially recover the estimates for uniformly elliptic operators from [1, 9].

Corollary 1. For  $\delta = 0$ , we obtain

$$\|\hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h, \hat{\phi} - \hat{\phi}_h\|_{Y^2}^2 \lesssim \inf_{(u_h, v_h, \phi_h) \in Y_h^p} \|\hat{u} - u_h, \hat{v} - v_h, \hat{\phi} - \phi_h\|_{Y^p}^2 + \|\hat{v} - v_h\|_{L^2(\Gamma_s)}$$

uniformly in  $h < h_0$ 

*Proof.* Use 2b) to estimate  $|\hat{u}_h - \hat{u}|_{(1,\hat{u},p)}$  in Theorem 2 from below.

## 6 A posteriori error estimate

Denote by

$$(e, \tilde{e}, \epsilon) = (\hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h, \hat{\phi} - \hat{\phi}_h) \in Y^p$$

the error of the Galerkin approximation, and let  $2\nu = \epsilon + \mathcal{V}^{-1}(1-\mathcal{K})(e|_{\partial\Omega} + \tilde{e})$ . Our basic a posteriori estimate is the following.

**Lemma 5.** For all  $(e_h, \tilde{e}_h, \nu_h) \in Y_h^p$ 

$$\begin{split} |e|_{(1,\hat{u},p)}^2 + &\|e|_{\partial\Omega} + \tilde{e}\|_{W^{\frac{1}{2},2}(\partial\Omega)}^2 + \|\epsilon\|_{W^{-\frac{1}{2},2}(\partial\Omega)}^2 \\ \lesssim & \Lambda(e-e_h,\tilde{e}-\tilde{e}_h,\nu-\nu_h) + j(\tilde{e}_h+\hat{v}_h) - j(\hat{v}) \\ & - B(\hat{u}_h,\hat{v}_h,\hat{\phi}_h;e-e_h,\tilde{e}-\tilde{e}_h,\nu-\nu_h) \\ = & \int_{\Omega} f(e-e_h) - \langle G'\hat{u}_h,e-e_h\rangle + \int_{\Gamma_s} g(|\tilde{e}_h+\hat{v}_h|-|\tilde{e}+\hat{v}_h|) \\ & - \langle \nu-\nu_h,\mathcal{V}\hat{\phi}_h + (1-\mathcal{K})(\hat{u}_h|_{\partial\Omega}+\hat{v}_h-u_0)\rangle \\ & + \langle t_0-\mathcal{W}(\hat{u}_h|_{\partial\Omega}+\hat{v}_h-u_0) - (\mathcal{K}'-1)\hat{\phi}_h,(e-e_h)|_{\partial\Omega} + \tilde{e}-\tilde{e}_h\rangle. \end{split}$$

*Proof.* Lemma 4, the continuous and the discretized variational inequality imply

$$\begin{split} &|\hat{u}-u|_{(1,\hat{u},p)}^{2}+\|(\hat{u}-u)|_{\partial\Omega}+\hat{v}-v\|_{W^{\frac{1}{2},2}(\partial\Omega)}^{2}+\|\hat{\phi}-\phi\|_{W^{-\frac{1}{2},2}(\partial\Omega)}^{2}\\ &\lesssim B(\hat{u},\hat{v},\hat{\phi};\hat{u}-\hat{u}_{h},\hat{v}-\hat{v}_{h},\nu)-B(\hat{u}_{h},\hat{v}_{h},\hat{\phi}_{h};\hat{u}-\hat{u}_{h},\hat{v}-\hat{v}_{h},\nu)\\ &\lesssim \Lambda(\hat{u}-\hat{u}_{h},\hat{v}-\hat{v}_{h},\nu)+j(\hat{v}_{h})-j(\hat{v})-B(\hat{u}_{h},\hat{v}_{h},\hat{\phi}_{h};\hat{u}-\hat{u}_{h},\hat{v}-\hat{v}_{h},\nu)\\ &\lesssim \Lambda(\hat{u}-\hat{u}_{h}-(u_{h}-\hat{u}_{h}),\hat{v}-\hat{v}_{h}-(v_{h}-\hat{v}_{h}),\nu-\nu_{h})+j(v_{h})-j(\hat{v})\\ &-B(\hat{u}_{h},\hat{v}_{h},\hat{\phi}_{h};\hat{u}-\hat{u}_{h}-(u_{h}-\hat{u}_{h}),\hat{v}-\hat{v}_{h}-(v_{h}-\hat{v}_{h}),\nu-\nu_{h}). \end{split}$$

Note that the variational inequalities are identities when restricted to the  $\phi$ -variable. The claim follows by setting  $e_h = u_h - \hat{u}_h$  and  $\tilde{e}_h = v_h - \hat{v}_h$ .  $\square$ 

Simplifying the right hand side along the lines of [2] leads to a gradient recovery scheme in the interior with a residual type estimator on the boundary. With a straight forward modification of [8], also a method purely based on residual type estimates could be justified.

For 
$$1 and  $0 \le \delta \le 1$ , define$$

$$G_{p,\delta}(x,y) = |y|^2 \omega(x,y)^{p-2} = |y|^2 [(|x|+|y|)^{\delta} (1+|x|+|y|)^{1-\delta}]^{p-2}$$

whenever |x| + |y| > 0 and 0 otherwise. As in [2], our analysis will be based on the following consequences of the monotony and convexity properties of  $G_{p,\delta}$ .

**Lemma 6.** Assume that  $\Omega$  is connected. Let q be a continuous linear form on  $W^{1,p}(\Omega)$  with  $\mathbb{R} \cap \ker q = \{0\}$ , where  $\mathbb{R}$  is identified with the space of constant functions on  $\Omega$ . Then for any  $1 there exists <math>C_P = C_P(p, q, \Omega) > 0$  such that for all  $a \geq 0$  and  $u \in W^{1,p}(\Omega)$ ,

$$\int_{\Omega} G_{p,\delta}(a,u) \le C_P \left( G_{p,\delta}(a,q(u)) + \int_{\Omega} G_{p,\delta}(a,|\nabla u|) \right).$$

*Proof.* Cf. [2], Lemma 4.1 and its generalization in Remark 4.3.  $\square$ 

**Lemma 7.** For any  $d, k \in \mathbb{N}$  there is  $C_{\Sigma} = C_{\Sigma}(p, d, k) > 0$  such that for all  $a_1, a_2, \ldots, a_k \in \mathbb{R}^d$ 

$$\sum_{j=1}^{k} \sum_{l=1}^{j-1} G_{p,\delta}(a_j, a_j - a_l) \lesssim C_{\Sigma} \sum_{j=1}^{k-1} \min_{1 \le m \le k} G_{p,\delta}(a_m, a_{j+1} - a_j).$$

*Proof.* Cf. [2], Lemma 4.2 and its generalization in Remark 4.3.

Even though Lemma 8 and Lemma 9 hold for any  $1 with minor modifications of the proofs (see [2] for a similar discussion), we will from now on concentrate on the range <math>2 \le p < \infty$  relevant to our transmission problem.

**Definition 3.** Let  $z \in D$  be a node of the triangulation  $\mathcal{T}_h$  and  $\varphi_z \in W_h^{1,p}(\Omega)$  the associated nodal basis function. Let  $\omega_z = \{x \in \Omega : \varphi_z(x) > 0\}$  be the interior of the support of  $\varphi_z$ . The interpolation operator  $\pi : W^{1,p}(\Omega) \to W_h^{1,p}(\Omega)$  is defined as

$$\pi u = \sum_{z \in D} u_z \varphi_z, \qquad u_z = \int_{\Omega} \varphi_z u / \int_{\Omega} \varphi_z.$$

**Lemma 8.** Let  $\mathcal{E}_h^z = \{l \in \mathcal{E}_h : l = \bar{K}_i \cap \bar{K}_j \text{ for some } K_i, K_j \subset \omega_z\}$ . Given  $u_h \in W_h^{1,p}(\Omega)$ , let  $[\partial_{\nu_{\mathcal{E}}} u_h]_l$  denote the jump of the normal derivative across the inner edge l of the triangulation. Then, if  $v \in W^{1,p}(\Omega)$  and  $K \in \mathcal{T}_h$ , the following estimate holds:

$$\int_{K} G_{p,\delta}(\nabla u_{h}, h_{K}^{-1}(v - \pi v)) + \int_{K} G_{p,\delta}(\nabla u_{h}, \nabla(v - \pi v))$$

$$\lesssim \sum_{z \in D \cap \bar{K}} \left( \int_{\omega_{z}} G_{p,\delta}(\nabla u_{h}, \nabla v) + \sum_{l \in \mathcal{E}_{h}^{z}} \min_{\bar{K}' \cap l \neq \emptyset} \int_{\omega_{z}} G_{p,\delta}(\nabla u_{h}|_{K'}, [\partial_{\nu_{\mathcal{E}}} u_{h}]_{l}) \right).$$

*Proof.* The proof is a modification of [2], Lemma 4.3. Concerning the first term on the left hand side, the convexity of  $G_{p,\delta}$  in its second argument (a "triangle inequality") and enlarging the domain of integration leads to

$$\int_{K} G_{p,\delta}(\nabla u_{h}, h_{K}^{-1}(v - \pi v)) = \int_{K} G_{p,\delta}(\nabla u_{h}, \sum_{z \in D \cap \bar{K}} h_{K}^{-1}(v - v_{z})\varphi_{z})$$

$$\lesssim \sum_{z \in D \cap \bar{K}} \int_{K} G_{p,\delta}(\nabla u_{h}, h_{K}^{-1}(v - v_{z})\varphi_{z})$$

$$\leq \sum_{z \in D \cap \bar{K}} \int_{\omega_{z}} G_{p,\delta}(\nabla u_{h}|_{K}, h_{K}^{-1}(v - v_{z})\varphi_{z}).$$

As  $G_{p,\delta}(\nabla u_h|_K,\cdot)$  is increasing and  $|\varphi_z| \leq 1$ , Lemma 6 with  $q(u) = \int_{\omega_z} \varphi_z u$  implies

$$\int_{\omega_{z}} G_{p,\delta}(\nabla u_{h}|_{K}, h_{K}^{-1}(v - v_{z})\varphi_{z}) \leq \int_{\omega_{z}} G_{p,\delta}(\nabla u_{h}|_{K}, h_{K}^{-1}(v - v_{z}))$$

$$\leq C_{P} \int_{\omega_{z}} G_{p,\delta}(\nabla u_{h}|_{K}, \nabla(v - v_{z}))$$

$$= C_{P} \int_{\omega_{z}} G_{p,\delta}(\nabla u_{h}|_{K}, \nabla v)$$

for every term in the sum over  $z \in D \cap \overline{K}$ . To replace the constant  $\nabla u_h|_K$  by  $\nabla u_h$ , we repeatedly apply the usual triangle inequality and the convexity of  $G_{v,\delta}$  to obtain

$$G_{p,\delta}(\nabla u_h|_K, \nabla v)$$

$$\leq G_{p,\delta}(\nabla u_h|_K, |\nabla v| + |\nabla u_h|_K - \nabla u_h|)$$

$$= (|\nabla v| + |\nabla u_h|_K - \nabla u_h|)^2 (|\nabla u_h|_K| + |\nabla v| + |\nabla u_h|_K - \nabla u_h|)^{\delta(p-2)}$$

$$\times (1 + |\nabla u_h|_K| + |\nabla v| + |\nabla u_h|_K - \nabla u_h|)^{(1-\delta)(p-2)}$$

$$\leq (|\nabla v| + |\nabla u_h|_K - \nabla u_h|)^2 (|\nabla v| + 2(|\nabla u_h| + |\nabla u_h|_K - \nabla u_h|))^{\delta(p-2)}$$

$$\times (1 + |\nabla v| + 2(|\nabla u_h| + |\nabla u_h|_K - \nabla u_h|))^{(1-\delta)(p-2)}$$

$$\lesssim G_{p,\delta}(\nabla u_h, |\nabla v| + |\nabla u_h|_K - \nabla u_h|)$$

$$\lesssim G_{p,\delta}(\nabla u_h, \nabla v) + G_{p,\delta}(\nabla u_h, \nabla u_h|_K - \nabla u_h).$$

Altogether

$$\int_K G_{p,\delta}(\nabla u_h|_K, h_K^{-1}(v-\pi v)) \lesssim \sum_{z \in D \cap \bar{K}} \int_{\omega_z} \left\{ G_{p,\delta}(\nabla u_h, \nabla v) + G_{p,\delta}(\nabla u_h, \nabla u_h|_K - \nabla u_h) \right\}.$$

Let  $\overline{\omega}_z = \overline{K}_1 \cup \cdots \cup \overline{K}_k$ . Applying Lemma 7 with  $a_j = \nabla u_h|_{K_j}$ ,  $1 \leq j \leq k$ , leads to the asserted bound for the first term. For the proof, note that the conormal derivatives of the piecewise linear function  $u_h$  are determined by its boundary values on the corresponding edge. But  $u_h \in W_h^{1,p}(\Omega) \subset W^{1,p}(\Omega)$ , so the restrictions from both sides have to coincide, and the conormal derivative does not jump:  $a_j - a_{j-1} = [\partial_{\nu_{\mathcal{E}}} u_h|_{\bar{K}_j \cap \bar{K}_{j-1}}]$ .

As for the second term, let  $c = \frac{1}{|K|} \int_K v$ . Because

$$\int_{K} G_{p,\delta}(\nabla u_h, \nabla(v - \pi v)) \lesssim \int_{K} G_{p,\delta}(\nabla u_h, \nabla v) + \int_{K} G_{p,\delta}(\nabla u_h, \nabla(\pi v - c))$$

by convexity and the triangle inequality, it only remains to consider the second term  $\int_K G_{p,\delta}(\nabla u_h, \nabla(\pi v - c))$ . The inverse estimate

$$|\nabla(\pi v - c)| \lesssim \frac{1}{|K|} \int_K h_K^{-1} |\pi v - c|$$

for the affine function  $\pi v - c$  and Jensen's inequality show

$$\int_{K} G_{p,\delta}(\nabla u_{h}, \nabla(\pi v - c)) \lesssim \int_{K} \frac{1}{|K|} \int_{K} G_{p,\delta}(\nabla u_{h}, h_{K}^{-1}(\pi v - c))$$

$$= \int_{K} G_{p,\delta}(\nabla u_{h}, h_{K}^{-1}(\pi v - c)).$$

However, as before

$$\int_{K} G_{p,\delta}(\nabla u_{h}, h_{K}^{-1}(\pi v - c)) \lesssim \int_{K} G_{p,\delta}(\nabla u_{h}, h_{K}^{-1}(v - \pi v)) + \int_{K} G_{p,\delta}(\nabla u_{h}, h_{K}^{-1}(v - c)),$$

and the first term has been considered in the first step of the proof. Lemma 6 with  $q(u) = \int_K u$  also bounds the final term by  $\int_K G_{p,\delta}(\nabla u_h, \nabla v)$ .

**Lemma 9.** For any  $\varepsilon > 0$ ,  $u_h \in W_h^{1,p}(\Omega)$ ,  $v \in W^{1,p}(\Omega)$  and  $f \in L^{p'}(\Omega)$ ,

$$\int_{\Omega} f(v - \pi v) \leq C\varepsilon \int_{\Omega} G_{p,\delta}(\nabla u_h, \nabla v) 
+ C(\varepsilon) \sum_{z \in D} \sum_{K \subset \overline{\omega}_z} \int_{K} G_{p',1}(|\nabla u_h|^{p-1}, h_K(f - f_K)) 
+ C\varepsilon \sum_{z \in D} \sum_{l \in \varepsilon_z} \min_{K' \cap l \neq \emptyset} \int_{\omega_z} G_{p,\delta}(\nabla u_h|_{K'}, [\partial_{\nu_{\varepsilon}} u_h]_l).$$

Here,  $f_K = \frac{1}{|K|} \int_K f$ . If  $f \in W^{1,p'}(\Omega)$ , the second term may be replaced by

$$C(\varepsilon) \sum_{z \in D} \sum_{K \subset \overline{\omega}_z} \int_K G_{p',1}(|\nabla u_h|^{p-1}, h_K^2 \nabla f).$$

*Proof.* We adapt the proof of [2], Lemma 4.4. Let  $\tilde{K} \subset \overline{\omega}_z$  such that  $|\nabla u_h|_{\tilde{K}}| = \max_{K' \subset \overline{\omega}_z} |\nabla u_h|_{K'}|$ . Applying the inequality from Remark 2c) for some  $\varepsilon > 0$  and  $C(\varepsilon) = C_P \max\{\varepsilon^{-1}, \varepsilon^{1/(1-p)}\}$ ,

$$\int_{\Omega} f(v - \pi v) = \sum_{z \in D} \sum_{K \subset \overline{\omega}_{z}} \int_{K} h_{K}(f - f_{K}) h_{K}^{-1}(v - v_{z}) \varphi_{z}$$

$$\leq C_{P}^{-1} C(\varepsilon) \sum_{z \in D} \sum_{K \subset \overline{\omega}_{z}} \int_{K} (|\nabla u_{h}|_{\tilde{K}}|^{p-1} + h_{K}|f - f_{K}|)^{p'-2} h_{K}^{2}|f - f_{K}|^{2}$$

$$+ \varepsilon \sum_{z \in D} \sum_{K \subset \overline{\omega}_{z}} \int_{K} (|\nabla u_{h}|_{\tilde{K}}| + h_{K}^{-1}|v - v_{z}|\varphi_{z})^{p-2} h_{K}^{-2}|v - v_{z}|^{2} \varphi_{z}^{2}$$

$$\leq C_{P}^{-1} C(\varepsilon) \sum_{z \in D} \sum_{K \subset \overline{\omega}_{z}} \int_{K} G_{p',1}(|\nabla u_{h}|_{\tilde{K}}|^{p-1}, h_{K}(f - f_{K}))$$

$$+ \varepsilon \sum_{z \in D} \sum_{K \subset \overline{\omega}_{z}} \int_{K} G_{p,\delta}(\nabla u_{h}|_{\tilde{K}}, h_{K}^{-1}(v - v_{z})\varphi_{z}),$$

because  $\sum_{K\subset\overline{\omega}_z}\int_K f_K(v-v_z)\varphi_z=0$ . However, by our choice of  $\tilde{K}$  and because  $p'\leq 2$ ,

$$\int_K G_{p',1}(|\nabla u_h|_{\tilde{K}}|^{p-1}, h_K(f-f_K)) \le \int_K G_{p',1}(|\nabla u_h|^{p-1}, h_K(f-f_K)).$$

If  $f \in W^{1,p'}(\Omega)$ , Lemma 6 with  $q(u) = \int_K u$  gives:

$$\int_K G_{p',1}(|\nabla u_h|^{p-1}, h_K(f - f_K)) \le C_P \int_K G_{p',1}(|\nabla u_h|^{p-1}, h_K^2 \nabla f).$$

Concerning the  $G_{p,\delta}$ -term, equation (3) in the proof of Lemma 8 shows that it is dominated by  $\varepsilon \int_{\omega_z} G_{p,\delta}(\nabla u_h|_{\tilde{K}}, \nabla v)$ , which in turn was bounded by

$$\varepsilon \int_{\omega_z} G_{p,\delta}(\nabla u_h, \nabla v) + \varepsilon \sum_{l \in \mathcal{E}_z^z} \min_{\bar{K}' \cap l \neq \emptyset} \int_{\omega_z} G_{p,\delta}(\nabla u_h|_{K'}, [\partial_{\nu_{\mathcal{E}}} u_h]_l).$$

In order to define the a posteriori estimator, we still need to introduce some notation. For any  $z \in D$ , denote by  $K_{j,z} \in \mathcal{T}_h$ ,  $1 \leq j \leq N_z$ , the triangles neighboring z in the sense that  $\overline{\omega}_z = \bigcup_{j=1}^{N_z} \overline{K}_{j,z}$ . To each  $K_{j,z}$  we associate a weight factor  $\alpha_{j,z} \geq 0$  normalized to  $\sum_{j=1}^{N_z} \alpha_{j,z} = 1$ .

**Definition 4.** Given  $u_h \in W_h^{1,p}(\Omega)$ , define the gradient recovery

$$G_h u_h = \sum_{z \in D} (G_h v_h)(z) \ \varphi_z, \quad (G_h v_h)(z) = \sum_{j=1}^{N_z} \alpha_{j,z} \nabla u_h|_{K_{j,z}}.$$

The following theorem states our reliable, but presumably not efficient a posteriori estimate.

**Theorem 3.** Let  $f \in L^{p'}(\Omega)$  and denote by  $(e, \tilde{e}, \epsilon)$  the error between the Galerkin solution  $(\hat{u}_h, \hat{v}_h, \hat{\phi}_h) \in Y_h^p$  and the true solution  $(\hat{u}, \hat{v}, \hat{\phi}) \in Y^p$ . If  $\Gamma_s \neq \emptyset$ , assume that  $\nabla \hat{u}|_{\Gamma_s} \in L^p(\Gamma_s)$ . Then

$$\begin{aligned} \|e, \tilde{e}, \epsilon\|_{Y^p}^p & \lesssim & |e|_{(1, \hat{u}, p)}^2 + \|e|_{\partial \Omega} + \tilde{e}\|_{W^{\frac{1}{2}, 2}(\partial \Omega)}^2 + \|\epsilon\|_{W^{-\frac{1}{2}, 2}(\partial \Omega)}^2 \\ & \lesssim & \eta_{gr}^2 + \eta_f^2 + \eta_S^2 + \eta_\theta^2 + \eta_g^2, \end{aligned}$$

where

$$\begin{split} & \eta_{gr}^{2} = \sum_{K \in \mathcal{T}_{h}} \int_{K} G_{p,\delta}(\nabla \hat{u}_{h}, \nabla \hat{u}_{h} - G_{h} \hat{u}_{h}), \\ & \eta_{f}^{2} = \sum_{K \in \mathcal{T}_{h}} \int_{K} G_{p',1}(|\nabla \hat{u}_{h}|^{p-1}, h_{K}(f - f_{K})), \\ & \eta_{S}^{2} = \sum_{l \subset \partial \Omega} l \ \|\partial_{s} \{ \mathcal{V} \hat{\phi}_{h} + (1 - \mathcal{K})(\hat{u}_{h}|_{\partial \Omega} + \hat{v}_{h} - u_{0}) \} \|_{L^{2}(l)}^{2} \\ & \eta_{\partial}^{2} = \sum_{l \subset \partial \Omega} l \ \| - \varrho(\nabla \hat{u}_{h}) \ \partial_{\nu} \hat{u}_{h} + t_{0} - \mathcal{W}(\hat{u}_{h}|_{\partial \Omega} + \hat{v}_{h} - u_{0}) - (\mathcal{K}' - 1)\hat{\phi}_{h} \|_{L^{2}(l)}^{2} \\ & \eta_{g}^{2} = \sum_{l \subset \Gamma_{S}} l \|\varrho(\nabla \hat{u}_{h}) \ \partial_{\nu} \hat{u}_{h}|_{\Gamma_{S}} \|_{L^{2}(l)}^{2} + \|g\|_{W^{-\frac{1}{2},2}(\Gamma_{S})}^{2} \end{split}$$

If  $f \in W^{1,p'}(\Omega)$ , we may replace  $\eta_f^2$  by  $\sum_{K \in \mathcal{T}_h} \int_K G_{p',1}(|\nabla \hat{u}_h|^{p-1}, h_K^2 \nabla f)$ .

*Proof.* From Lemma 5 we know that for all  $(e_h, \tilde{e}_h, \nu_h) \in Y_h^p$ 

$$\begin{split} &\|e, \tilde{e}, \epsilon\|_{Y^p}^p \lesssim |e|_{(1,\hat{u},p)}^2 + \|e|_{\partial\Omega} + \tilde{e}\|_{W^{\frac{1}{2},2}(\partial\Omega)}^2 + \|\epsilon\|_{W^{-\frac{1}{2},2}(\partial\Omega)}^2 \\ &\lesssim \int_{\Omega} f(e-e_h) - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \varrho(\nabla \hat{u}_h) \; \partial_{\nu} \hat{u}_h|_{\partial K} \; (e-e_h) \\ &+ \int_{\Gamma_s} g(|\tilde{e}_h + \hat{v}_h| - |\hat{v}|) - \langle \nu - \nu_h, \mathcal{V} \hat{\phi}_h + (1-\mathcal{K})(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0) \rangle \\ &+ \langle t_0 - \mathcal{W}(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0) - (\mathcal{K}' - 1)\hat{\phi}_h, (e-e_h)|_{\partial\Omega} + \tilde{e} - \tilde{e}_h \rangle, \end{split}$$

with  $2\nu = \epsilon + \mathcal{V}^{-1}(1 - \mathcal{K})(e|_{\partial\Omega} + \tilde{e})$ . The first two terms are mainly going to give the gradient recovery in the interior, the fourth term the error  $\eta_S$  of constructing the Steklov-Poincaré operator, while the remaining terms add up to  $\eta_{\partial}$ .

Concerning the first term:

$$\int_{\Omega} f(e - e_h) \lesssim \varepsilon \sum_{K \in \mathcal{T}_h} \int_{K} G_{p,\delta}(\nabla \hat{u}_h, \nabla e) 
+ C(\varepsilon) \sum_{K \in \mathcal{T}_h} \int_{K} G_{p',1}(|\nabla \hat{u}_h|^{p-1}, h_z(f - f_z)) 
+ \varepsilon \sum_{z \in D} \sum_{l \in \mathcal{E}_h^z} \min_{\bar{K}' \cap l \neq \emptyset} \int_{\omega_z} G_{p,\delta}(\nabla \hat{u}_h|_{K'}, [\partial_{\nu_{\varepsilon}} u_h]_l) 
\lesssim \varepsilon |e|_{(1,\hat{u},p)}^2 + C(\varepsilon) \eta_f^2 + \varepsilon \sum_{z \in D} \sum_{l \in \mathcal{E}_h^z} \min_{\bar{K}' \cap l \neq \emptyset} \int_{\omega_z} G_{p,\delta}(\nabla \hat{u}_h|_{K'}, [\partial_{\nu_{\varepsilon}} u_h]_l).$$

 $G_h\hat{u}_h$  is continuous across any interior edge l, so that  $[\partial_{\nu}\hat{u}_h]_l = [\partial_{\nu}\hat{u}_h - G_h\hat{u}_h]_l$  and

$$\min_{\bar{K}' \cap l \neq \emptyset} \int_{\omega_z} G_{p,\delta}(\nabla \hat{u}_h|_{K'}, [\partial_{\nu} \hat{u}_h - G_h u_h]_l) \lesssim \int_{\omega_z} G_{p,\delta}(\nabla \hat{u}_h, \nabla \hat{u}_h - G_h \hat{u}_h).$$

Therefore,

$$\int_{\Omega} f(e - e_h) \lesssim \varepsilon |e|_{(1,\hat{u},p)}^2 + C(\varepsilon)\eta_f^2 + \varepsilon \sum_{z \in D} \sum_{l \in \mathcal{E}_h^z} \int_{\omega_z} G_{p,\delta}(\nabla \hat{u}_h, [\partial_{\nu} \hat{u}_h - G_h \hat{u}_h]_l) 
\lesssim \varepsilon |e|_{(1,\hat{u},p)}^2 + C(\varepsilon)\eta_f^2 + \varepsilon \sum_{K \in \mathcal{T}_h} \int_K G_{p,\delta}(\nabla \hat{u}_h, \nabla \hat{u}_h - G_h \hat{u}_h) 
= \varepsilon |e|_{(1,\hat{u},p)}^2 + C(\varepsilon)\eta_f^2 + \varepsilon \eta_{gr}^2.$$

Concerning the second term, let

$$A_l = \varrho(\nabla \hat{u}_h|_{K_{l,1}}) \ \partial_\nu \hat{u}_h|_{K_{l,1}} - \varrho(\nabla \hat{u}_h|_{K_{l,2}}) \ \partial_\nu \hat{u}_h|_{K_{l,2}},$$

where again  $l \subset \bar{K}_{l,1} \cap \bar{K}_{l,2}$ , and the unit normal  $\nu$  points outward of  $K_{l,1}$ . Therefore

$$-\langle G'\hat{u}_h, e - \pi e \rangle = -\sum_{K \in \mathcal{T}_h} \int_{\partial K} \varrho(\nabla \hat{u}_h) \ \partial_{\nu} \hat{u}_h|_{\partial K} \ (e - \pi e)$$
$$= -\sum_{l \notin \partial \Omega} \int_{l} A_l(e - \pi e) - \sum_{l \in \partial \Omega} \int_{l} \varrho(\nabla \hat{u}_h) \ \partial_{\nu} \hat{u}_h|_{l} \ (e - \pi e).$$

Repeating the analysis of [2], Theorem 5.1, with the help of Lemma 8 gives

$$-\sum_{l \notin \partial \Omega} \int_{l} A_{l}(e - \pi e) \lesssim \eta_{gr}^{2} + \varepsilon(|e|_{(1,\hat{u}_{h},p)}^{2} + \eta_{gr}^{2}).$$

Thus

$$\begin{split} \|e, \tilde{e}, \epsilon\|_{Y^{p}}^{p} &\lesssim |e|_{(1, \hat{u}, p)}^{2} + \|e|_{\partial\Omega} + \tilde{e}\|_{W^{\frac{1}{2}, 2}(\partial\Omega)}^{2} + \|\epsilon\|_{W^{-\frac{1}{2}, 2}(\partial\Omega)}^{2} \\ &\lesssim \eta_{f}^{2} + \varepsilon(\eta_{gr}^{2} + |e|_{(1, \hat{u}, p)}^{2}) + \eta_{gr}^{2} + \varepsilon(|e|_{(1, \hat{u}_{h}, p)}^{2} + \eta_{gr}^{2}) \\ &+ \int_{\Gamma_{s}} \left\{ -\varrho(\nabla \hat{u}_{h}) \; \partial_{\nu} \hat{u}_{h}|_{\Gamma_{s}} (\tilde{e}_{h} - \tilde{e}) + g(|\tilde{e}_{h} + \hat{v}_{h}| - |\tilde{e} + \hat{v}_{h}|) \right\} \\ &- \int_{\partial\Omega} \varrho(\nabla \hat{u}_{h}) \; \partial_{\nu} \hat{u}_{h}|_{\partial\Omega} \; ((e - \pi e)|_{\partial\Omega} + \tilde{e} - \tilde{e}_{h}) \\ &+ \langle t_{0} - \mathcal{W}(\hat{u}_{h}|_{\partial\Omega} + \hat{v}_{h} - u_{0}) - (\mathcal{K}' - 1)\hat{\phi}_{h}, (e - \pi e)|_{\partial\Omega} + \tilde{e} - \tilde{e}_{h} \rangle \\ &- \langle \nu - \nu_{h}, \mathcal{V}\hat{\phi}_{h} + (1 - \mathcal{K})(\hat{u}_{h}|_{\partial\Omega} + \hat{v}_{h} - u_{0}) \rangle. \end{split}$$

We bound the second, third + fourth as well as the final line individually. Cauchy-Schwarz and Young's inequality allow to estimate the last term by

$$\varepsilon \|e|_{\partial\Omega} + \tilde{\epsilon}\|_{W^{\frac{1}{2},2}(\partial\Omega)}^2 + \varepsilon \|\epsilon\|_{W^{-\frac{1}{2},2}(\partial\Omega)}^2 + \varepsilon^{-1} \|\mathcal{V}\hat{\phi}_h + (1-\mathcal{K})(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0)\|_{W^{\frac{1}{2},2}(\partial\Omega)}^2,$$

and the latter by  $\eta_S^2$  (cf. [3]). The third and fourth lines are estimated by (cf. [3])

$$\left\|-\varrho(\nabla\hat{u}_h)\,\partial_{\nu}\hat{u}_h+t_0-\mathcal{W}(\hat{u}_h|_{\partial\Omega}+\hat{v}_h-u_0)-(\mathcal{K}'-1)\hat{\phi}_h\right\|_{W^{-\frac{1}{2},2}(\partial\Omega)}\left\|(e-\pi e)|_{\partial\Omega}+\tilde{e}\right\|_{W^{\frac{1}{2},2}(\partial\Omega)}$$

which lead to  $\eta_{\partial}$ , where we have choosen  $\tilde{e}_h = 0$ , i.e.  $v_h = \hat{v}_h$ . Finally, using the triangle inequality, the second line is simplified as follows:

$$\begin{split} &\int_{\Gamma_s} \{ -\varrho(\nabla \hat{u}_h) \ \partial_{\nu} \hat{u}_h|_{\Gamma_s} (\tilde{e}_h - \tilde{e}) + g(|\tilde{e}_h + \hat{v}_h| - |\tilde{e} + \hat{v}_h|) \} \\ &\leq \int_{\Gamma_s} \{ -\varrho(\nabla \hat{u}_h) \ \partial_{\nu} \hat{u}_h|_{\Gamma_s} (\tilde{e}_h - \tilde{e}) + g|\tilde{e}_h - \tilde{e}| \} \\ &= \int_{\Gamma_s} \{ \varrho(\nabla \hat{u}_h) \ \partial_{\nu} \hat{u}_h|_{\Gamma_s} \tilde{e} + g|\tilde{e}| \} \\ &\leq \|\varrho(\nabla \hat{u}_h) \ \partial_{\nu} \hat{u}_h|_{\Gamma_s} \|_{W^{-\frac{1}{2},2}(\Gamma_s)} \|\tilde{e}\|_{W^{\frac{1}{2},2}(\Gamma_s)} + \|g\|_{W^{-\frac{1}{2},2}(\Gamma_s)} \|\tilde{e}\|_{W^{\frac{1}{2},2}(\Gamma_s)}. \end{split}$$

We may use the Cauchy-Schwartz inequality and the inverse inequality, leading to  $\eta_g$ .

## 7 Numerical results

With the subset  $\Lambda_h$  of  $\widetilde{W}_h^{\frac{1}{2},2}(\Gamma_s)$  given by

$$\Lambda_h = \{ \sigma_h \in \widetilde{W}_h^{\frac{1}{2},2}(\Gamma_s) : |\sigma_h(x)| \le 1 \text{ a.e. on } \Gamma_s \},$$

we can define an Uzawa algorithm for solving the variational inequality analogously to [9]. In order to introduce this algorithm, let  $P_{\Lambda}$  be the projection of  $\widetilde{W}_{h}^{\frac{1}{2},2}(\Gamma_{s})$  onto  $\Lambda_{h}$ , i.e. for every nodal point of the mesh  $\mathcal{T}_{h}|_{\Gamma_{s}}$  holds  $\delta \mapsto P_{\Lambda}(\delta) = \sup\{-1,\inf(1,\delta)\}.$ 

Algorithm 1 (Uzawa).

- 1. Choose  $\sigma_h^0 \in \Lambda_h$ .
- 2. For  $n = 0, 1, 2, \ldots$  find  $(u_h^n, v_h^n) \in X_h^p$  such that

$$\langle G'u_h^n, u_h \rangle + \langle S_h(u_h^n|_{\partial\Omega} + v_h^n), u_h|_{\partial\Omega} + v_h \rangle + \int_{\Gamma_s} g\sigma_h^n v_h \, ds = \lambda_h(u_h, v_h)$$

for all  $(u_h, v_h) \in X_h^p$ .

3. Set

$$\sigma_h^{n+1} = P_{\Lambda}(\sigma_h^n + \rho g v_h^n),$$

where  $\rho > 0$  is a sufficiently small parameter that will be specified later.

4. Repeat with 2. until a convergence criterion is satisfied.

In our first example the model problem is defined on the L-shape with  $\Omega = [-\frac{1}{4}, \frac{1}{4}]^2 \backslash [0, \frac{1}{4}]^2, \ \Omega^c = \mathbb{R}^2 \backslash \Omega.$  The friction part of the interface is  $\Gamma_s = (-\frac{1}{4}, -\frac{1}{4})(\frac{1}{4}, -\frac{1}{4}) \cup (-\frac{1}{4}, -\frac{1}{4})(-\frac{1}{4}, \frac{1}{4}),$  see Figure 1. In this example we choose  $\varrho(t) = (\varepsilon + t)^{p-2}$ , with p = 3 and  $\varepsilon = 0.00001$ .

In this example we choose  $\varrho(t) = (\varepsilon + t)^{p-2}$ , with p = 3 and  $\varepsilon = 0.00001$ . Our volume and boundary data are given by f = 0 and  $u_0 = r^{2/3} \sin \frac{2}{3} (\varphi - \frac{\pi}{2})$ ,  $t_0 = \partial_{\nu} u_0|_{\partial\Omega}$ . The friction parameter is g = 0.5, leading to slip conditions on the interface. We have applied the Uzawa algorithm as introduced above with the damping parameter  $\rho = 25$  to solve the variational inequality. The nonlinear variational problem in the Uzawa algorithm is then solved by Newton's method in every Uzawa-iteration step.

In Table 1 we give the degrees of freedom, the value  $J_h(\hat{u}_h, \hat{v}_h)$  and the error measured with the help of J, i.e.  $\delta J = J_h(\hat{u}_h, \hat{v}_h) - J(\hat{u}, \hat{v})$ , where we have obtained the value  $J(\hat{u}, \hat{v})$  by extrapolation of  $J_h(\hat{u}_h, \hat{v}_h)$ . Due to the

slip condition, we need only a few Uzawa steps. But as a consequence of the degeneration of the system matrix, due to the nonlinearity, the iteration numbers for the MINRES solver, applied to the linearized system, are very high, leading to large computation times. The convergence rate  $\alpha_J$  is suboptimal, due to the presence of singularities, in the boundary data as well, as due to the change of boundary conditions.

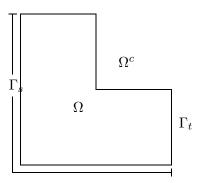


Figure 1: Geometry and interface of the model problem

DOF	$J_h(\hat{u}_h,\hat{v}_h)$	$\delta J$	$lpha_J$	$It_{ m Uzawa}$	$\tau(s)$
28	-0.511609	0.017249	_	2	0.190
80	-0.517938	0.010920	-0.435	2	0.640
256	-0.521857	0.007001	-0.382	2	2.440
896	-0.524293	0.004566	-0.341	2	11.05
3328	-0.525841	0.003017	-0.316	2	61.85
12800	-0.526865	0.001993	-0.308	2	437.5
50176	-0.527571	0.001287	-0.320	2	4218

Table 1: Convergence rates and Uzawa steps for uniform meshes (Example 1)

In our second example we use the same model geometry as before (see Fig. 1). Here we choose the friction boundary  $\Gamma_s = \emptyset$ . Therefore our model problem reduces to a non-linear p-Laplacian FEM-BEM coupling problem, where we can prescribe the solution.

In this example we choose  $\varrho(t)=(\varepsilon+t)^{p-2}$ , with p=3 and  $\varepsilon=0.00001$ . We prescribe the solution by  $u_1=r^{2/3}\sin\frac{2}{3}(\varphi-\frac{\pi}{2})$  and  $u_2=0$ . Then

the boundary data  $u_0, t_0$  and volume data f are given by  $u_0 = u_1|_{\Gamma}$ ,  $t_0 = \varrho(|\nabla u_1|)\partial_{\nu}u_1$  and  $f = -\operatorname{div}(\varrho(|\nabla u_1|)\nabla u_1)$ .

In the following we give errors in the  $\|\cdot\|_{W^{1,p}(\Omega)}$  norm and in the quasinorm  $|u-u_h|_Q = \|u-u_h\|_{(1,u_h,p)}$ .

In Tab. 2 we give the errors, convergence rates, number of Newton iterations  $It_{Newton}$  and the computing time for the uniform h-version with rectangles. We observe that the convergence rate in the quasi-norm  $|\cdot|_Q$  is better than in the  $||\cdot||_{W^{1,3}(\Omega)}$ -norm. The number of Newton iterations appears to be bounded.

In Tab. 3 for the uniform h-version with triangles, we give the errors, convergence rates, error estimator  $\eta$ , efficiency indices  $\delta_u/\eta$  for the  $\|\cdot\|_{W^{1,3}(\Omega)}$ -norm and  $\delta_q/\eta$  for the  $\|\cdot\|_Q$ -norm, number of Newton iterations and the computing time. Again, here we observe that the convergence rate in the quasi-norm  $\|\cdot\|_Q$  is better than in the  $\|\cdot\|_{W^{1,3}(\Omega)}$ -norm and the number of Newton iterations is bounded. The efficiency index  $\delta_u/\eta$  appears to be constant, whereas the efficiency index  $\delta_q/\eta$  appears to be decreasing.

Tab. 4 gives the corresponding numbers for the adaptive version, using a blue-green refining strategy for triangles and refining the 10% elements with the largest indicators. Here we observe that the convergence rates for both norms are very similar and that both efficiency indices are bounded.

Figure 2 give the errors for all methods in the  $\|\cdot\|_{W^{1,3}(\Omega)}$ -norm and the  $|\cdot|_Q$  quasi-norm together with the error indicators for the uniform and adaptive methods.

Figure 3 presents the sequence of meshes generated by the adaptive refinement strategy. We clearly observe the refinement towards the reentrant corner with the singularity of the solution.

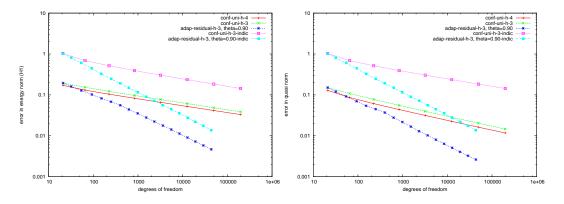


Figure 2:  $||u - u_n||_{W^{1,3}(\Omega)}$  (left) and  $|u - u_n|_Q$  (right).

DOF	$  u-u_h  _{1,3}$	$\alpha$	$ u-u_h _Q$	$\alpha$	$It_{Newton}$	au(s)
21	0.1711499	_	0.1293512	_	22	0.224
65	0.1308635	-0.238	0.0860870	-0.360	22	0.424
225	0.1039326	-0.186	0.0612225	-0.274	23	1.668
833	0.0826578	-0.175	0.0438478	-0.255	23	6.804
3201	0.0657091	-0.170	0.0314280	-0.247	23	27.28
12545	0.0522196	-0.168	0.0225589	-0.243	24	120.8
49665	0.0414910	-0.167	0.0162319	-0.239	24	560.1
197633	0.0329617	-0.167	0.0117169	-0.236	24	2678.

Table 2: Errors, convergence rates (Example 2, uniform mesh with rectangles)  $\,$ 

DOF	$  u-u_h  _{1,3}$ $\alpha$	$ u-u_h _Q$	$\alpha$	$\eta$	$\delta_u/\eta$	$\delta_q/\etaIt_{New}$	$_{w}$ $\tau(s)$
21	0.1945908 —	0.1510064		1.027	0.190	0.147 22	0.620
65	0.1535874 - 0.209	0.1081632	-0.295	0.690	0.223	0.157 22	2.212
225	0.1219287 - 0.186	0.0774765	-0.269	0.516	0.236	0.150 22	8.617
833	0.0969249 - 0.175	0.0555005	-0.255	0.394	0.246	0.141 23	36.00
3201	0.0770270 - 0.171	0.0396882	-0.249	0.304	0.253	0.131 23	144.2
12545	0.0611994 $-0.168$	0.0283778	-0.246	0.236	0.260	0.120 24	608.7
49665	0.0486160 - 0.167	0.0203130	-0.243	0.184	0.265	0.111 24	2530.
197633	0.0386151 - 0.167	0.0145686	-0.241	0.144	0.269	0.102 24	11000

Table 3: Errors, onvergence rates, estimator  $\eta$ , reliability  $\delta_u/\eta$  and  $\delta_q/\eta$  (Example 2, uniform mesh with triangles)

DOF	$  u-u_h  _{1,3}$ $\alpha$	$ u-u_h _Q$	$\alpha$	$\eta$	$\delta_u/\eta$	$\delta_q/\eta~I$	$t_{Nei}$	$_v$ $\tau(s)$
21	0.1945908 —	0.1510064		1.027	0.190	0.147	22	0.196
32	0.1602214 - 0.461	0.1205155	-0.535	0.804	0.199	0.150	22	0.332
54	0.1275298 - 0.436	0.0918131	-0.520	0.603	0.212	0.152	22	0.648
93	0.1019990 -0.411	0.0699054	-0.501	0.442	0.231	0.158	22	1.132
152	0.0821754 - 0.440	0.0540462	-0.524	0.325	0.253	0.166	23	2.000
249	0.0679251 - 0.386	0.0449420	-0.374	0.246	0.276	0.183	23	3.352
400	0.0558447 - 0.413	0.0369614	-0.412	0.190	0.294	0.194	23	5.700
625	0.0439784 - 0.535	0.0277857	-0.639	0.148	0.297	0.188	24	9.896
986	0.0352491 $-0.485$	0.0217361	-0.539	0.116	0.305	0.188	24	17.45
1528	0.0279287 - 0.531	0.0167409	-0.596	0.091	0.308	0.184	25	31.16
2322	0.0222760 - 0.540	0.0129489	-0.614	0.071	0.312	0.181	25	53.98
3620	0.0177640 - 0.510	0.0102552	-0.525	0.056	0.316	0.182	25	106.7
5544	0.0142059 - 0.524	0.0080233	-0.576	0.044	0.320	0.181	25	205.3
8449	0.0112965 - 0.544	0.0063426	-0.558	0.035	0.322	0.181	26	422.4
12810	0.0090396 $-0.536$	0.0050706	-0.538	0.028	0.325	0.183	26	1060.
19222	0.0072288 - 0.551	0.0040370	-0.562	0.022	0.329	0.184	26	2400.
29006	0.0057984 - 0.536	0.0032478	-0.529	0.018	0.333	0.186	27	5460.
43593	0.0046615 - 0.536	0.0026230	-0.524	0.014	0.337	0.190	27	13000

Table 4: p-Laplacian (adaptive), convergence rates, estimator  $\eta$ , reliability  $\delta_u/\eta$  and  $\delta_q/\eta$ 

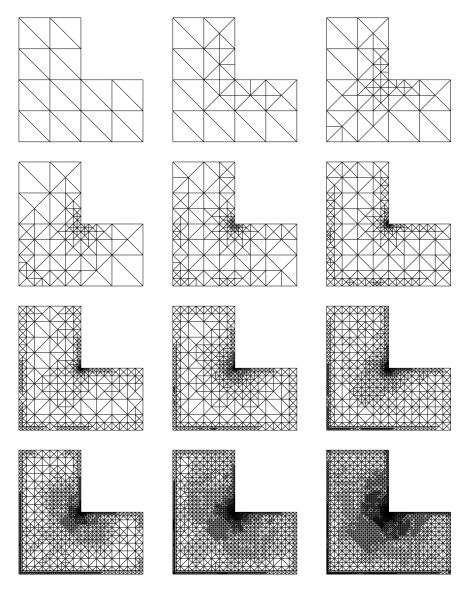


Figure 3: The first 12 meshes generated by the adaptive refinement algorithm

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