NUMERICAL RECOVERY OF LOCATION AND RESIDUE OF POLES OF MEROMORPHIC FUNCTIONS

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ABSTRACT. We present a method able to recover location and residue of poles of functions meromorphic in a half-plane from samples of the function on the real positive semi-axis. The function is assumed to satisfy appropriate asymptotic conditions including, in particular, that required by Carlson's theorem. The peculiar features of the present procedure are: (i) it does not make use of the approximation of meromorphic functions by rational functions; (ii) it does not use the standard methods of regularization of ill-posed problems. The data required for the determination of the pole parameters (i.e., location and residue) are the approximate values of the meromorphic function on a finite set of equidistant points on the real positive semi-axis. We show that this method is numerically stable by proving that the algorithm is convergent as the number of data points tends to infinity and the noise on the input data goes to zero. Moreover, we can also evaluate the degree of approximation of the estimates of pole location and residue which we obtain from the knowledge of a finite number of noisy samples.

1. INTRODUCTION

A classical problem of numerical complex analysis consists in recovering location and residue of poles of meromorphic functions. The classical approach to this problem is based on the approximation by rational functions and, in this framework, the Padé approximants play a particularly significant role [2, 9, 12, 15].

In this paper we present a completely different method, whose origin goes back to a much earlier paper written by one of the authors, in collaboration with Tullio Regge, in connection with the interpolation problem in the complex angular momentum plane [11]. Work on the method continued in [1]. However, soon, we ran up against difficulties related with the ill–posedness of the numerical analytic continuation. Now, after more than forty years of experience regarding the regularization of ill–posed problems [6, 14], we can reconsider the method originated at that time, and present a regularized algorithm which is safe from the pathologies of ill–posedness.

First, we consider a function f(z), analytic in the half-plane $\operatorname{Re} z > 0$, and satisfying appropriate asymptotic conditions (detailed below in the article) which, in particular, include that required by Carlson's theorem [4]. First, we suppose that the data set consists of an infinite number of samples of the function f(z), taken on a regular grid on the real positive semi-axis, and moreover, f(z) is assumed to be known exactly: i.e., the input data are noiseless. More precisely, denoting by $\{f_N\}_{N=0}^{\infty}$ $(f_N \doteq f(N + \frac{1}{2}); N \in \mathbb{N})$ the set of input samples and assuming the series $\sum_{N=0}^{\infty} f_N$ to be absolutely convergent, as a first result we find an interpolation formula for f(x) $(x \doteq \operatorname{Re} z)$ along with a relation which allows every sample f_N to be expressed in terms of all the other samples. The analysis performed for the analytic functions is then generalized to the case of a meromorphic function f(z) with one first order pole in the half-plane Re z > 0. Still assuming to work with a data set made of an infinite number of noiseless samples of f(x), and supposing, initially, that position and residue of the pole are known, we obtain a generalization of the previous formula in which each datum f_N can be reconstructed from all the other samples and from the pole parameters (i.e., position and residue). Stated in other words, this formula provides us with a set of consistency relations, which mutually constrain the values of all the samples and of the pole parameters. It is exactly this overall consistency which is exploited in order to construct the algorithm for recovering pole location and residue from the function samples taken on the real positive semi-axis.

The successive step is to consider as input a more realistic data set $\{f_N^{(\varepsilon)}\}_{N=0}^{N_0}$ $(\varepsilon > 0, N_0 < \infty)$, made of a finite number of function samples $f_N^{(\varepsilon)}$ perturbed by noise. The algorithm for pole recovery, defined previously for the case of an infinite set of noiseless input data, can be suitably adapted to this different situation. More precisely, we can prove that the limit for $N_0 \to +\infty$ and $\varepsilon \to 0$ of appropriately defined estimates $z_p^{(\varepsilon,N_0)}(n)$ $(n \in \mathbb{N} \text{ fixed})$ of the pole location z_p tends to z_p when n tends to infinity, i.e.: $\lim_{n\to+\infty} \lim_{\varepsilon\to 0} z_p^{(\varepsilon,N_0)}(n) = z_p$. But, in practice, since N_0 is necessarily finite and ε is non-null, we are faced with the problem that the two limits in the above formula cannot be interchanged. This delicate point will be discussed in detail in Section 5, where it is shown that a proper estimate $z_p^{(\varepsilon,N_0)}$, close to the true pole position z_p , can be obtained for ε sufficiently small and N_0 sufficiently large. In the same section we will also show how to evaluate the degree of approximation to z_p by $z_p^{(\varepsilon,N_0)}$. Analogous arguments (leading to similar results) are also developed for the problem of finding a suitable estimate $R_p^{(\varepsilon,N_0)}$ of the residue R_p .

The paper is organized as follows. In Section 2 we derive the interpolation formula for functions analytic in Re z > 0, by taking a data set $\{f_N\}_{N=0}^{\infty}$ of noise-less samples. In Section 3 we obtain the interpolation formula for a meromorphic function in the half-plane Re z > 0, which has one first order pole at $z = z_p$ (Re $z_p > 0$), continuing to assume an input made of an infinite number of noiseless data. In Section 4 the consistency relations mentioned previously are derived, and the algorithm for recovering location and residue of the pole from an infinite set of noiseless function samples is presented.

In Section 5 the algorithm for recovering the pole parameters with a finite number of noisy input samples is given. In the same section we show how to evaluate the degree of approximation of the estimate $z_p^{(\varepsilon,N_0)}$ and $R_p^{(\varepsilon,N_0)}$ to z_p and R_p . Section 6 is devoted to the numerical examples, which illustrate the various steps of the algorithm. Finally, some conclusions will be drawn, and possible extensions of the present method will be outlined. In the Appendix we briefly recall some properties of the Meixner–Pollaczek polynomials, which are extensively used in the paper.

2. Interpolation formula for a class of functions analytic in the half–plane ${\rm Re}\,z>0$

Most of the results we present in this article rely on a celebrated theorem by Carlson, which states the growth properties that a function of a specified class must enjoy in order to be determined by its values on a certain set of points. Some preliminary notions are necessary. The entire function f(z), $z = re^{i\theta}$, is of exponential type (or, of order 1) if

(1)
$$\limsup_{r \to \infty} \frac{\log \log M(r)}{\log r} = 1$$

where M(r) denotes the maximum modulus of f(z) for |z| = r. In order to specify the rate of growth of a function of exponential type in different directions use can be made of the Phragmén–Lindelöf indicator function:

(2)
$$h_f(\theta) \doteq \limsup_{r \to \infty} \frac{\log |f(re^{i\theta})|}{r}$$

Then, we have the following theorem.

Theorem A (Carlson [4]). If f(z) is regular and of exponential type in the halfplane Re $z \ge 0$ and $h_f(\frac{\pi}{2}) + h_f(-\frac{\pi}{2}) < 2\pi$, then $f(z) \equiv 0$ if f(N) = 0 for N = 0, 1, 2, ...

For our purposes we shall consider a subset of the functions which fulfill Carlson's theorem, that is, the functions satisfying the following bound, which can be named *Carlson's bound*:

(3)
$$h_f(\theta) \leq b |\sin \theta| \qquad \left(b < \pi; -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right).$$

Evidently, the functions satisfying condition (3) satisfy also the assumptions of Carlson's theorem.

Let $\mathbb{N} = \{0, 1, 2, ...\}$ denote the set of all natural numbers. Throughout the paper we shall use the notation $f_N \doteq f\left(N + \frac{1}{2}\right), N \in \mathbb{N}$, to denote the samples of the function f(z) at the equidistant interpolation nodes $\left(N + \frac{1}{2}\right)$; we shall refer to the set $\{f_N\}_{N=0}^{\infty}$ as the "data set". We can prove the following interpolation theorem.

Theorem 1. Assume that the function f(z) ($z \in \mathbb{C}$; z = x + iy; $x, y \in \mathbb{R}$) enjoys the following properties:

- (i) f(z) is holomorphic in $\operatorname{Re} z > 0$, continuous at $\operatorname{Re} z = 0$;
- (ii) f(z) satisfies Carlson's bound (3);
- (iii) $f(iy) \in L^2(-\infty, +\infty);$ (iv) $\sum_{N=1}^{\infty} \frac{|f_N|}{N} < \infty.$

Then the following equality holds for $x > -\frac{1}{2}$:

(4)
$$f\left(x+\frac{1}{2}\right) = \sum_{N=0}^{\infty} f_N \operatorname{sinc}(x-N) - \frac{\sin \pi x}{2\pi} \int_{-\infty}^{+\infty} \frac{\sum_{n=0}^{\infty} c_n \psi_n(y)}{(x+\frac{1}{2}-\mathrm{i}y) \cosh \pi y} \,\mathrm{d}y$$

where $\operatorname{sinc}(t) \doteq \frac{\sin \pi t}{\pi t}$ for $t \neq 0$ and $\operatorname{sinc}(0) \doteq 1$. The coefficients c_n are given by:

(5)
$$c_n = 2\sqrt{\pi} \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} f_N P_n \left[-i\left(N + \frac{1}{2}\right) \right] \qquad (n \in \mathbb{N}),$$

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FIGURE 1. Integration path C for formula (7).

and the set of functions $\{\psi_n\}_{n=0}^{\infty}$ is defined by

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(6)
$$\psi_n(y) \doteq \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} + iy\right) P_n(y) \qquad (n \in \mathbb{N}; y \in \mathbb{R}),$$

where Γ is the Euler Gamma function, and P_n denotes the Meixner–Pollaczek polynomials $P_n^{(\alpha)}$ with $\alpha = \frac{1}{2}$.

Proof. In view of conditions (i) and (ii), by Cauchy's integral formula we have, for $R \to \infty$ (see Fig. 1):

(7)
$$\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)\Gamma(\frac{1}{2}+z)\Gamma(\frac{1}{2}-z)}{z-z'} \,\mathrm{d}z = f(z')\Gamma\left(\frac{1}{2}+z'\right)\Gamma\left(\frac{1}{2}-z'\right),$$

where \mathcal{C} is the path shown in Fig. 1, and z' belongs to the half-plane Re z > 0, but $z' \notin \mathcal{C}$. Next, the integral along the path \mathcal{C} on the left-hand side (l.h.s.) of (7) can be evaluated; noting that the integrand has simple poles at $z_N = N + \frac{1}{2}, N \in \mathbb{N}$, which are brought by $\Gamma(\frac{1}{2} - z)$, we have:

(8)
$$\frac{\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)\Gamma(\frac{1}{2}+z)\Gamma(\frac{1}{2}-z)}{z-z'} dz}{= \sum_{N=0}^{\infty} f_N \frac{(-1)^N}{N+\frac{1}{2}-z'} - \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{f(iy)\Gamma(\frac{1}{2}+iy)\Gamma(\frac{1}{2}-iy)}{iy-z'} dy.$$

Then we put $z' = x + \frac{1}{2} + i\epsilon$ $(x > -\frac{1}{2}, x \notin \mathbb{N}; \epsilon > 0)$. Next, deforming appropriately the integration path, we push the point z' up to the real axis computing the limit for $\epsilon \to 0$. By exploiting the relations $\Gamma(-x)\Gamma(x+1) = -\pi(\sin \pi x)^{-1}$ and $\Gamma(\frac{1}{2} + iy)\Gamma(\frac{1}{2} - iy) = \pi(\cosh \pi y)^{-1}$, from (7) and (8) we obtain for $x > -\frac{1}{2}$:

(9)
$$f\left(x+\frac{1}{2}\right) = \sum_{N=0}^{\infty} f_N \operatorname{sinc}(x-N) - \frac{\sin \pi x}{2\pi} \int_{-\infty}^{+\infty} \frac{f(\mathrm{i}y)}{[(x+\frac{1}{2})-\mathrm{i}y]\cosh \pi y} \,\mathrm{d}y.$$

The integral on the right-hand side (r.h.s.) of (9) converges, as can be shown by using assumption (iii) and Schwarz's inequality (see also the inequalities which will be given at the end of the proof). Formula (9) has been obtained under the hypothesis $x \notin \mathbb{N}$; however, it is easy to see that the limit for $x \to N$ ($N \in \mathbb{N}$) of both sides of (9) leads to the identity $f(N + \frac{1}{2}) = f(N + \frac{1}{2})$, so that formula (9) actually holds for every $x > -\frac{1}{2}$. Let us now introduce the Meixner–Pollaczek polynomials $P_n^{(1/2)}(y)$ [7, 10, 13], which are orthonormal with respect to the weight function $w(y)^1$:

(10)
$$w(y) \doteq \frac{1}{\pi} \left| \Gamma\left(\frac{1}{2} + \mathrm{i}y\right) \right|^2 = \frac{1}{\cosh \pi y}.$$

Then we consider the set of functions $\{\psi_n\}_{n=0}^{\infty}$ defined in (6), which form an orthonormal basis in $L^2(-\infty, +\infty)$ [8]. Now, in view of property (iii), f(iy) may be expanded in the basis $\{\psi_n\}_{n=0}^{\infty}$:

(11)
$$f(\mathbf{i}y) = \sum_{n=0}^{\infty} c_n \psi_n(y),$$

the convergence being in the L^2 -norm. By the orthonormal property of the basis $\{\psi_n\}_{n=0}^{\infty}$, the coefficients are given by:

(12)
$$c_n = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} f(\mathrm{i}y) \Gamma\left(\frac{1}{2} - \mathrm{i}y\right) P_n(y) \,\mathrm{d}y \qquad (n \in \mathbb{N}).$$

Next, putting iy = z, $(z \in \mathbb{C})$, and evaluating the integral in (12) by the complex integration method along the path \mathcal{C} shown in Fig. 1, we obtain formula (5). Now, inserting expansion (11) into the integral on the r.h.s. of (9) we have the term:

(13)
$$\int_{-\infty}^{+\infty} \frac{\sum_{n=0}^{\infty} c_n \,\psi_n(y)}{(x+\frac{1}{2}-\mathrm{i}y)\cosh\pi y} \,\mathrm{d}y,$$

whose convergence is easily proved by using the Schwarz inequality:

(14)
$$\begin{pmatrix} \int_{-\infty}^{+\infty} \frac{\sum_{n=0}^{\infty} c_n \psi_n(y)}{(x + \frac{1}{2} - \mathrm{i}y) \cosh \pi y} \, \mathrm{d}y \end{pmatrix}^2 \leqslant \sum_{n=0}^{\infty} |c_n|^2 \int_{-\infty}^{+\infty} \frac{1}{|x + \frac{1}{2} - \mathrm{i}y|^2 \cosh^2 \pi y} \, \mathrm{d}y \\ = \|f(\mathrm{i}y)\|^2 \int_{-\infty}^{+\infty} \frac{1}{|x + \frac{1}{2} - \mathrm{i}y|^2 \cosh^2 \pi y} \, \mathrm{d}y < \infty.$$

Finally, plugging integral (13) in formula (9), we obtain formula (4).

Remark 1. In the numerical analysis (see Section 6) sums of the type $\sum_{n=0}^{\infty} c_n \psi_n$ (or similar) are repeatedly used but (obviously) truncated at a suitable finite value of n, say $n = \hat{n}$. In these cases, sum and integral in (13) may be interchanged, yielding:

(15)
$$\frac{1}{2} \int_{-\infty}^{+\infty} \frac{\sum_{n=0}^{\widehat{n}} c_n \psi_n(y)}{[(x+\frac{1}{2}) - \mathrm{i}y] \cosh \pi y} \,\mathrm{d}y = \sum_{n=0}^{\widehat{n}} c_n \,Q_n \left[-\mathrm{i}\left(x+\frac{1}{2}\right) \right],$$

¹Hereafter the superscript (1/2) in $P_n^{(1/2)}(y)$ will be omitted for simplicity (see also the Appendix).

where the function Q_n is defined by

(16)
$$Q_n\left[-\mathrm{i}\left(x+\frac{1}{2}\right)\right] \doteq \frac{\mathrm{i}}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{P_n(y)\,\Gamma(\frac{1}{2}+\mathrm{i}y)}{[\mathrm{i}(x+\frac{1}{2})+y]\cosh\pi y}\,\mathrm{d}y \qquad (n\in\mathbb{N}).$$

3. Interpolation formula for a function meromorphic in the half–plane ${\rm Re}\,z>0$

Consider now the case of meromorphic functions. For simplicity, we consider a function f(z), which has only one singularity in the half-plane Re z > 0, and we assume that this singularity is a first order pole, whose residue is $R_{\rm p}$ ($R_{\rm p} \neq 0$). The extension to the case of several first order poles is straightforward. We now prove the following theorem.

Theorem 2. Assume that the meromorphic function f(z) has a first order pole at $z = z_p$ with $\operatorname{Re} z_p > 0$, whose residue is R_p ($R_p \neq 0$). Suppose that f(z)is holomorphic in the half-plane $\operatorname{Re} z > \operatorname{Re} z_p$, continuous at $\operatorname{Re} z = 0$, satisfies Carlson's bound (3) and conditions (iii) and (iv) of Theorem 1. Then, the following interpolation formula holds:

(17)
$$f\left(x+\frac{1}{2}\right) = \sum_{N=0}^{\infty} f_N \operatorname{sinc}(x-N) - \frac{R_{\rm p}}{\cos(\pi z_{\rm p})} \frac{\sin \pi x}{(x+\frac{1}{2}-z_{\rm p})} - \frac{\sin \pi x}{2\pi} \int_{-\infty}^{+\infty} \frac{\sum_{n=0}^{\infty} c_n^{({\rm p})} \psi_n(y)}{[(x+\frac{1}{2})-\mathrm{i}y] \cosh \pi y} \,\mathrm{d}y \qquad \left(x > -\frac{1}{2}\right),$$

where, for $n \in \mathbb{N}$: (18)

$$c_n^{(p)} = 2\sqrt{\pi} \left\{ \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} f_N P_n \left[-i\left(N + \frac{1}{2}\right) \right] - R_p \Gamma \left(\frac{1}{2} - z_p\right) P_n \left(-iz_p\right) \right\}.$$

Proof. Proceeding similarly to what we have done in the proof of Theorem 1, and recalling that the function f(z) has a first order pole in $z = z_p$ with residue R_p , we evaluate the following integral by using the residue theorem:

(19)
$$\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(z) \Gamma(\frac{1}{2} + z) \Gamma(\frac{1}{2} - z)}{z - z'} dz = \pi \left[\frac{f(z')}{\cos \pi z'} + \frac{R_{\rm p}}{(z_{\rm p} - z') \cos \pi z_{\rm p}} \right]$$

where \mathcal{C} is the path shown in Fig. 1 which encloses z' and z_p , but $z', z_p \notin \mathcal{C}$. Then, proceeding as in Theorem 1, we obtain: (20)

$$\frac{f(z')}{\cos \pi z'} = \frac{1}{\pi} \sum_{N=0}^{\infty} \frac{(-1)^N f_N}{N + \frac{1}{2} - z'} - \frac{R_p}{(z_p - z') \cos \pi z_p} - \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{f(iy)}{(iy - z') \cosh \pi y} \, \mathrm{d}y.$$

Putting $z' = x + \frac{1}{2} + i\epsilon$, $(x > -\frac{1}{2}, x \notin \mathbb{N}; \epsilon > 0)$, and proceeding as in Theorem 1, from (20) we obtain:

(21)
$$f\left(x+\frac{1}{2}\right) = \frac{\sin \pi x}{\pi} \left\{ \sum_{N=0}^{\infty} (-1)^N \frac{f_N}{x-N} - \frac{\pi R_p}{\left(x+\frac{1}{2}-z_p\right)\cos \pi z_p} -\frac{1}{2} \int_{-\infty}^{+\infty} \frac{f(\mathrm{i}y)}{\left[\left(x+\frac{1}{2}\right)-\mathrm{i}y\right]\cosh \pi y} \,\mathrm{d}y \right\}.$$

We then expand f(iy) on the orthonormal basis $\{\psi_n\}_{n=0}^{\infty}$: i.e.,

(22)
$$f(iy) = \sum_{n=0}^{\infty} c_n^{(p)} \psi_n(y),$$

where the convergence is in the L^2 -norm (the superscript '(p)' in $c_n^{(p)}$ is to recall that these coefficients refer to a function with a pole). In view of the orthonormality property of the basis $\{\psi_n\}_{n=0}^{\infty}$ we have:

(23)
$$c_n^{(p)} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} f(\mathrm{i}y) \,\Gamma\left(\frac{1}{2} - \mathrm{i}y\right) P_n(y) \,\mathrm{d}y.$$

Next, putting iy = z, $(z \in \mathbb{C})$, and evaluating this integral by the use of the complex integration method along the path \mathcal{C} , we obtain formula (18). Then, plugging (22) into (21) we obtain the interpolation formula (17).

4. Consistency relations and the algorithm for pole recovery: case of input data made of an infinite number of noiseless samples

4.1. Consistency relations. We now continue to consider a meromorphic function f(z) with only one first order pole in Re z > 0. Let us introduce the following function associated with f(z): i.e., $h(k;z) \doteq \left[z - \left(k + \frac{1}{2}\right)\right] f(z), (z \in \mathbb{C}, k \in \mathbb{N})$. We can prove the following theorem.

Theorem 3. Assume that the meromorphic function f(z) has a first order pole at $z = z_p$ with $\operatorname{Re} z_p > 0$, whose residue is R_p ($R_p \neq 0$). Suppose that f(z)is holomorphic in the half-plane $\operatorname{Re} z > \operatorname{Re} z_p$, continuous at $\operatorname{Re} z = 0$, satisfies Carlson's bound (3), and

(24) (i')
$$\sum_{N=0}^{\infty} |f_N| < \infty.$$

Moreover, assume that for every $k \in \mathbb{N}$:

(25) (ii')
$$h(k; iy) \in L^2(-\infty, +\infty)$$
 $(k \in \mathbb{N}).$

Then, for every $k \in \mathbb{N}$ the following equalities hold:

(26)
$$f_{k} = (-1)^{k+1} \left\{ \sum_{N=0}^{\infty} (-1)^{N} f_{N} (1 - \delta_{Nk}) - \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\sum_{n=0}^{\infty} c_{n,k}^{(p)} \psi_{n}(y)}{[(x + \frac{1}{2}) - \mathrm{i}y] \cosh \pi y} \, \mathrm{d}y - \frac{\pi R_{\mathrm{p}}}{\cos \pi z_{\mathrm{p}}} \right\} \qquad (k \in \mathbb{N}),$$

where:

(27)
$$c_{n,k}^{(p)} = 2\sqrt{\pi} \left\{ \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} (N-k) f_N P_n \left[-i\left(N + \frac{1}{2}\right) \right] - \left(z_p - k - \frac{1}{2}\right) R_p \Gamma \left(\frac{1}{2} - z_p\right) P_n \left(-iz_p\right) \right\} \quad (n,k \in \mathbb{N}),$$

the functions $\psi_n(y)$ being defined in (6).

Proof. In view of the conditions listed above, the results of Theorem 2 can be applied to the function h(k; z), which has a first order pole in $z = z_p$ with residue $(z_p - k - \frac{1}{2})R_p$, and whose samples are $h(k; N + \frac{1}{2}) = (N - k)f_N$; then, for every $k \in \mathbb{N}$ and $x > -\frac{1}{2}$ the following interpolation formula holds (see (17)):

(28)
$$h\left(k;x+\frac{1}{2}\right) = (x-k)f\left(x+\frac{1}{2}\right) = \sum_{N=0}^{\infty} (N-k) f_N \operatorname{sinc}(x-N)$$
$$-\frac{\sin \pi x}{2\pi} \int_{-\infty}^{+\infty} \frac{h(k;iy)}{[(x+\frac{1}{2})-iy]\cosh \pi y} \,\mathrm{d}y - \frac{(z_{\mathrm{p}}-k-\frac{1}{2}) R_{\mathrm{p}}}{\cos \pi z_{\mathrm{p}}} \frac{\sin \pi x}{x+\frac{1}{2}-z_{\mathrm{p}}}$$

Next, for $k \in \mathbb{N}$ we compute the following derivative:

(29)
$$\left[\frac{\mathrm{d}}{\mathrm{d}x}(x-k)f\left(x+\frac{1}{2}\right)\right]_{x=k} = \left[f\left(x+\frac{1}{2}\right) + (x-k)f'\left(x+\frac{1}{2}\right)\right]_{x=k} = f_k,$$

so that we can equate f_k to the derivative with respect to x, computed at x = k, of the r.h.s. of formula (28). Therefore, we can formally write:

$$f_k = \sum_{N=0}^{\infty} (N-k) f_N \left[\frac{\mathrm{d}\operatorname{sinc}(x-N)}{\mathrm{d}x} \right]_{x=k}$$
$$- \frac{1}{2\pi} \left(\frac{\mathrm{d}}{\mathrm{d}x} \sin \pi x \right)_{x=k} \int_{-\infty}^{+\infty} \frac{h(k; \mathrm{i}y)}{[(x+\frac{1}{2}) - \mathrm{i}y] \cosh \pi y} \, \mathrm{d}y - \left(\frac{\sin \pi x}{2\pi} \right)_{x=k}$$
$$\times \frac{\mathrm{d}}{\mathrm{d}x} \int_{-\infty}^{+\infty} \frac{h(k; \mathrm{i}y)}{[(x+\frac{1}{2}) - \mathrm{i}y] \cosh \pi y} \, \mathrm{d}y - \frac{(z_{\mathrm{p}} - k - \frac{1}{2}) R_{\mathrm{p}}}{\cos \pi z_{\mathrm{p}}} \left[\frac{\mathrm{d}}{\mathrm{d}x} \frac{\sin \pi x}{x + \frac{1}{2} - z_{\mathrm{p}}} \right]_{x=k}$$

We have: $\frac{\mathrm{d}}{\mathrm{d}x} \sin \pi x |_{x=k} = \pi (-1)^k$, and

(31)
$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{sinc}(x-N)\Big|_{x\to k} = \frac{(-1)^{N+k+1}}{N-k}(1-\delta_{Nk}),$$

(32)
$$\frac{\mathrm{d}}{\mathrm{d}x} \left. \frac{\sin \pi x}{x + \frac{1}{2} - z_{\mathrm{p}}} \right|_{x=k} = \frac{\pi \, (-1)^k}{k + \frac{1}{2} - z_{\mathrm{p}}},$$

which, substituted in (30), yield formally:

(33)
$$f_{k} = (-1)^{k+1} \left\{ \sum_{N=0}^{\infty} (-1)^{N} f_{N} (1 - \delta_{Nk}) + \frac{1}{2} \int_{-\infty}^{+\infty} \frac{h(k; iy)}{[(x + \frac{1}{2}) - iy] \cosh \pi y} \, \mathrm{d}y - \frac{\pi R_{\mathrm{p}}}{\cos \pi z_{\mathrm{p}}} \right\} \qquad (k \in \mathbb{N})$$

It should be observed that the term by term differentiation of the series $\sum_{N=0}^{\infty} (N-k)f_N \operatorname{sinc}(x-N)$ is legitimate in view of condition (i'). We can thus conclude that equalities (33) are proved. Next, since $h(k; iy) \in L^2(-\infty, +\infty)$ $(k \in \mathbb{N}, y \in \mathbb{R})$, we can expand h(k; iy) on the basis $\{\psi_n\}_{n=0}^{\infty}$:

(34)
$$h(k; \mathbf{i}y) = \sum_{n=0}^{\infty} c_{n,k}^{(\mathbf{p})} \psi_n(y) \qquad (k \in \mathbb{N}, y \in \mathbb{R}),$$

which converges in the L^2 -norm. Following procedures closely analogous to those used in Theorems 1 and 2, the explicit expression of the coefficients $c_{n,k}^{(p)}$, which is given in (27), is easily obtained. Finally, inserting the expansion (34) into the integral on the r.h.s. of (33), formula (26) follows. \Box

For every $k \in \mathbb{N}$, formula (26) gives the value of the sample f_k of the function f(z) in terms of the value of all the other samples f_N , with $N \neq k$ (notice that also in (27) the contribution of the sample f_N to the coefficient $c_{n,k}^{(p)}$ is null for N = k). Therefore, equations (26) can be regarded as an (infinite) set of consistency relations, which make explicit the mutual constraints among the samples of f(z) and the pole parameters z_p and R_p . However, for this purpose, first we need to extend (in part) the results of the previous theorem, and consider the expansion of the function h(k; iy) on the orthonormal basis $\{\psi_n\}_{n=0}^{\infty}$ for $k \in \mathbb{R}$ (not only for integral values of k). What we need is stated in the following corollary.

Corollary 1. Assume for the meromorphic function f(z) all the conditions of Theorem 3. Assume that the function $h(k; z) \doteq (z - k - \frac{1}{2})f(z)$ ($z \in \mathbb{C}; k \in \mathbb{R}$) satisfies the following condition (which substitutes condition (ii') of Theorem 3):

(35) (ii'')
$$h(k; iy) \in L^2(-\infty, +\infty) \quad (k \in \mathbb{R}).$$

Then the following equality holds for $x > -\frac{1}{2}$ and for any $k \in \mathbb{R}$:

(36)
$$h\left(k;x+\frac{1}{2}\right) = (x-k)f\left(x+\frac{1}{2}\right) = \sum_{N=0}^{\infty} (N-k)f_N \operatorname{sinc}(x-N)$$
$$-\frac{\sin \pi x}{2\pi} \int_{-\infty}^{+\infty} \frac{\sum_{n=0}^{\infty} c_n^{(\mathrm{p})}(k)\psi_n(y)}{[(x+\frac{1}{2})-\mathrm{i}y]\cosh \pi y} \,\mathrm{d}y - \frac{(z_{\mathrm{p}}-k-\frac{1}{2})R_{\mathrm{p}}}{\cos \pi z_{\mathrm{p}}} \frac{\sin \pi x}{x+\frac{1}{2}-z_{\mathrm{p}}},$$

where the coefficients $c_n^{(p)}(k)$ are given by:

(37)
$$c_n^{(p)}(k) = 2\sqrt{\pi} \left\{ \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} (N-k) f_N P_n \left[-i\left(N + \frac{1}{2}\right) \right] - \left(z_p - k - \frac{1}{2} \right) R_p \Gamma \left(\frac{1}{2} - z_p \right) P_n \left(-iz_p \right) \right\} \quad (n \in \mathbb{N}, k \in \mathbb{R}),$$

and

(38)
$$\lim_{n \to \infty} c_n^{(p)}(k) = 0 \qquad (k \in \mathbb{R}).$$

Proof. Applying the results of Theorem 2 to the function h(k; z) with $k \in \mathbb{R}$, formulae (36) and (37) follow immediately from the interpolation formula (17) and from (18), respectively. For any $k \in \mathbb{R}$ the $c_n^{(p)}(k)$ represent the coefficients of the expansion of h(k; iy) in terms of the basis $\{\psi_n\}_{n=0}^{\infty}$, i.e.:

(39)
$$h(k; \mathbf{i}y) = \sum_{n=0}^{\infty} c_n^{(\mathbf{p})}(k)\psi_n(y) \qquad (k \in \mathbb{R}),$$

the convergence being in the sense of the L^2 -norm. Finally, from expansion (39) we have for $k \in \mathbb{R}$: $||h(k; iy)||^2 = \sum_{n=0}^{\infty} |c_n^{(p)}(k)|^2$, which implies (38).

4.2. The algorithm for recovering pole location and residue. Let us continue to consider a meromorphic function f(z) with one first order pole in $z = z_{\rm p}$ with $\operatorname{Re} z_{\rm p} > 0$, whose residue is $R_{\rm p} \neq 0$. Moreover, the conditions required by Corollary 1 are assumed to be satisfied by f(z) and its associated function h(k; z) $(k \in \mathbb{R})$. Now, it is convenient to rewrite the coefficients $c_n^{(p)}(k)$, given in (37), as follows:

(40)
$$c_n^{(p)}(k) = \mathfrak{c}_n(k) - (\zeta_p - k) \tau_n \qquad (n \in \mathbb{N}, k \in \mathbb{R}),$$

where

(41)
$$\mathbf{c}_n(k) \doteq 2\sqrt{\pi} \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} (N-k) f_N P_n\left[-i\left(N+\frac{1}{2}\right)\right],$$

(42)
$$\tau_n \doteq 2\sqrt{\pi} R_p \Gamma\left(\frac{1}{2} - z_p\right) P_n \left(-iz_p\right),$$

(43)
$$\zeta_{\rm p} \doteq z_{\rm p} - \frac{1}{2}.$$

Note that for every $k \in \mathbb{R}$ the coefficients $\mathfrak{c}_n(k)$ can be computed from the input data set $\{f_N\}_{N=0}^{\infty}$, and, consequently, can be regarded as known, whereas the explicit dependence of the coefficients $c_n^{(p)}(k)$ on the unknown pole is contained only in the second term on the r.h.s. of (40).

Now, Eq. (38) allows us to connect the unknowns $\zeta_{\rm p}$ and $R_{\rm p}$ to the input data $\{f_N\}$ through the function $\mathfrak{c}_n(k)$. In fact, from (38) and (40) we have:

(44)
$$\lim_{n \to +\infty} \mathfrak{c}_n(k) = \lim_{n \to +\infty} \left(-\tau_n \, k + \zeta_p \tau_n \right) \qquad (k \in \mathbb{R}),$$

which shows that, in the limit for n tending to infinity, the coefficients of the form $\mathfrak{c}_n(k)$, which is linear in k, are related to the unknown pole parameters. Now, in order to obtain $\mathfrak{c}_n(k)$ $(n \in \mathbb{N}, k \in \mathbb{R})$ from the input data, it is sufficient to compute the coefficients $\mathfrak{c}_{n,k}$ for any two integer values of k, say k_1 and k_2 , and successively for every $n \in \mathbb{N}$ interpolate linearly \mathfrak{c}_{n,k_1} and \mathfrak{c}_{n,k_2} to yield

(45)
$$\mathbf{c}_n(k) = m_n k + q_n \qquad (n \in \mathbb{N}, k \in \mathbb{R}).$$

In this way, for any $n \in \mathbb{N}$ we can associate the coefficients m_n and q_n with the function samples $\{f_N\}$, i.e., for any $n \in \mathbb{N}$:

(46)
$$\{f_N\}_{N=0}^{\infty} \xrightarrow[\text{Formula (41)}]{} \{\mathfrak{c}_{n,k_1},\mathfrak{c}_{n,k_2}\} \xrightarrow[\text{Linear interpolation in } k]{} (m_n,q_n).$$

It should be recalled that in the current case we are assuming to know an infinite number of noiseless input samples $\{f_N\}_{N=0}^{\infty}$, which amounts to saying that the calculated coefficients $\mathfrak{c}_{n,k}$ are exact. As will be discussed in the next section, in practice, when only a finite number of noisy function samples is available and, consequently, only an approximation of the coefficients $\mathfrak{c}_{n,k}$ is computable, the scheme in (46) needs to be generalized.

Comparing (44) and (45), it can be seen that, for finite values of n, the computed coefficients m_n and q_n can be considered estimates of $(-\tau_n)$ and $(\zeta_p \tau_n)$, respectively (i.e., for $n \gg 1$, $m_n \sim -\tau_n$ and $q_n \sim \zeta_p \tau_n$), which Eq. (44) guarantees to be such

that:

(47a)
$$\lim_{n \to +\infty} m_n = -\lim_{n \to +\infty} \tau_n,$$

(47b)
$$\lim_{n \to +\infty} q_n = \zeta_p \lim_{n \to +\infty} \tau_n.$$

Now, Eqs. (47) guide us to define, for every $n \in \mathbb{N}$, the approximate pole position $\zeta_{\mathbf{p}}(n)$ as

(48)
$$\zeta_{\mathbf{p}}(n) \doteq -\frac{q_n}{m_n} \qquad (n \in \mathbb{N}),$$

(in order to avoid proliferation of symbols, we denote the approximate pole position computed at a certain value of n by $\zeta_{\rm p}(n)$, making explicit the dependence on n; instead, the true pole position is simply denoted by $\zeta_{\rm p}$. Moreover, for simplicity, we will refer interchangeably to $z_{\rm p}$ and $\zeta_{\rm p}$ as the pole position). Finally, Eqs. (47) and (48) guarantee that

(49)
$$\lim_{n \to +\infty} \zeta_{\mathbf{p}}(n) = \zeta_{\mathbf{p}},$$

which, explicitly, reads:

(50a)
$$\operatorname{Re} \zeta_{\mathbf{p}} = -\lim_{n \to +\infty} \frac{\operatorname{Re} q_n \operatorname{Re} m_n + \operatorname{Im} q_n \operatorname{Im} m_n}{|m_n|^2}$$

(50b)
$$\operatorname{Im} \zeta_{\mathrm{p}} = -\lim_{n \to +\infty} \frac{\operatorname{Im} q_n \operatorname{Re} m_n - \operatorname{Re} q_n \operatorname{Im} m_n}{|m_n|^2}$$

Once $\zeta_{\rm p}$ has been recovered (and, accordingly, also $z_{\rm p}$ by formula (43)), also the residue can be readily recovered from the data. In fact, for every $n \in \mathbb{N}$ we can define the approximate residue $R_{\rm p}(n)$ as

(51)
$$R_{\mathbf{p}}(n) \doteq -\frac{m_n}{2\sqrt{\pi} \Gamma\left(\frac{1}{2} - z_{\mathbf{p}}\right) P_n\left(-\mathrm{i}z_{\mathbf{p}}\right)} \qquad (n \in \mathbb{N}).$$

Finally, Eqs. (42), (47a) and (51) allow us to state

(52)
$$\lim_{n \to +\infty} R_{\rm p}(n) = -\frac{1}{2\sqrt{\pi} \,\Gamma(\frac{1}{2} - z_{\rm p})} \lim_{n \to +\infty} \frac{m_n}{P_n(-{\rm i}z_{\rm p})} = R_{\rm p}.$$

5. Consistency relations and the algorithm for pole recovery: case of input data made of a finite number of noisy samples

In practice, actual data handling requires the analysis of more realistic situations in which the input data set is made of a finite number of noisy data: the data set now is $\{f_N^{(\varepsilon)}\}_{N=0}^{N_0}$, where ε characterizes a bound on the noise that will be specified below. Various models of noise are actually possible. Since in our case the data f_N are required to vanish as $N \to +\infty$, we assume a noise model such that the relative error remains bounded, namely, we write: $f_N^{(\varepsilon)} = (1 + \nu_N^{(\varepsilon)})f_N$, where $\nu_N^{(\varepsilon)}$ denotes a noise term such that $\left|\nu_N^{(\varepsilon)}\right| \leq \varepsilon$. It follows that: $\left|(f_N^{(\varepsilon)} - f_N)/f_N\right| \leq \varepsilon$, $f_N \neq 0$, $\varepsilon > 0$ constant; evidently, if $f_N = 0$ the relative error becomes meaningless, so in this particular case we simply assume that $\lim_{\varepsilon \to 0} \left|f_N^{(\varepsilon)}\right| = 0$. 5.1. Algorithm for recovering pole location and residue. When $\varepsilon > 0$ and $N_0 < \infty$, the coefficients $\mathfrak{c}_n(k)$ in formula (41) can be computed only approximately. Then, for fixed values of ε and N_0 , we can define for any $n \in \mathbb{N}$ and $k \in \mathbb{R}$ the following approximate coefficients: (53)

$$\mathfrak{c}_n^{(\varepsilon,N_0)}(k) \doteq 2\sqrt{\pi} \sum_{N=0}^{N_0} \frac{(-1)^N}{N!} (N-k) f_N^{(\varepsilon)} P_n\left[-\mathrm{i}\left(N+\frac{1}{2}\right)\right] \quad (\varepsilon > 0, N_0 < \infty).$$

Evidently, $\mathfrak{c}_n^{(0,\infty)}(k)\equiv\mathfrak{c}_n(k).$ We can prove the following lemma.

Lemma 4. For every $n \in \mathbb{N}$ and $k \in \mathbb{R}$, the following statement holds:

(54)
$$\lim_{\substack{N_0 \to +\infty\\\varepsilon \to 0}} \mathfrak{c}_n^{(\varepsilon,N_0)}(k) = \mathfrak{c}_n^{(0,\infty)}(k) = \mathfrak{c}_n(k).$$

Proof. Consider

(55)

$$\frac{\mathfrak{c}_{n}^{(0,\infty)}(k) - \mathfrak{c}_{n}^{(\varepsilon,N_{0})}(k)}{2\sqrt{\pi}} = \left\{ \sum_{N=0}^{N_{0}} \frac{(-1)^{N}}{N!} (N-k) \left[f_{N} - f_{N}^{(\varepsilon)} \right] P_{n} \left[-i \left(N + \frac{1}{2} \right) \right] + \sum_{N=N_{0}+1}^{\infty} \frac{(-1)^{N}}{N!} (N-k) f_{N} P_{n} \left[-i \left(N + \frac{1}{2} \right) \right] \right\}.$$

We know that the series $2\sqrt{\pi} \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} (N-k) f_N P_n \left[-i\left(N+\frac{1}{2}\right)\right]$ converges to $\mathfrak{c}_n^{(0,\infty)}(k)$, which is finite for every finite $n \in \mathbb{N}$ and $k \in \mathbb{R}$. The latter statement follows from formula (40): in fact, $\left|c_n^{(p)}(k)\right| < \infty$ since they are the coefficients of the expansion of h(k; iy), and $|(z_p - k - \frac{1}{2})R_p\Gamma(\frac{1}{2} - z_p)P_n(-iz_p)| < \infty$ for $n \in \mathbb{N}$, $k \in \mathbb{R}$ (of course, $z_p \neq N + \frac{1}{2}$, which merely means that the pole cannot be located on the input datum). It follows that the second sum on the r.h.s. of (55) vanishes as $N_0 \to +\infty$. Concerning the first term, we may write the inequality:

(56)
$$\left| \sum_{N=0}^{N_0} \frac{(-1)^N}{N!} (N-k) \left[f_N - f_N^{(\varepsilon)} \right] P_n \left[-i \left(N + \frac{1}{2} \right) \right] \right|$$
$$\leqslant \varepsilon \sum_{N=0}^{N_0} \frac{|N-k|}{N!} |f_N| \left| P_n \left[-i \left(N + \frac{1}{2} \right) \right] \right|,$$

where the assumption made on the noise has been used. Next, by rewriting the Pollaczek polynomials $P_n\left[-i\left(N+\frac{1}{2}\right)\right]$ as

(57)
$$P_n\left[-i\left(N+\frac{1}{2}\right)\right] = \sum_{j=0}^n p_j^{(n)}\left(N+\frac{1}{2}\right)^j,$$

and substituting this expression in the r.h.s. of inequality (56), we obtain

(58)
$$\varepsilon \sum_{N=0}^{N_0} \frac{|N-k|}{N!} |f_N| \left[\sum_{j=0}^n \left| p_j^{(n)} \right| \left(N + \frac{1}{2} \right)^j \right].$$

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Next, we compute the limit for $N_0 \to +\infty$. Since the sum $\sum_{j=0}^{n} p_j^{(n)} \left(N + \frac{1}{2}\right)^j$ is finite, the order of the sums in (58) may be exchanged:

(59)
$$\varepsilon \sum_{j=0}^{n} \left| p_{j}^{(n)} \right| \sum_{N=0}^{\infty} \frac{|N-k|}{N!} \left| f_{N} \right| \left(N + \frac{1}{2} \right)^{j}$$

The inner series $\sum_{N=0}^{\infty} \frac{|N-k|}{N!} |f_N| (N+\frac{1}{2})^j$ is evidently convergent in view of assumption (i') of Theorem 3, and therefore, the expression in (59) vanishes for $\varepsilon \to 0$. Statement (55) is thus proved.

Let us now tackle the problem of recovering, in practice, the position of the pole. For this purpose we follow a procedure analogous to that described in Subsection 4.2, using now the computable coefficients $\mathfrak{c}_n^{(\varepsilon,N_0)}(k)$ instead of the (exact but unknown) coefficients $\mathfrak{c}_n(k)$. Then, for given fixed values of $\varepsilon > 0$ and $N_0 < \infty$, the actual implementation is realized by the following procedure:

- **1.** For every $n \in \mathbb{N}$ compute by means of formula (53) the coefficients $\mathfrak{c}_n^{(\varepsilon,N_0)}(k)$ for some integral values of k, say $k = 0, \ldots, k_*$.
- **2.** Since $\mathfrak{c}_n^{(\varepsilon,N_0)}(k)$ is a linear function of k (see (53)), we associate by a linear regression procedure (in k) the set of computed "noisy" coefficients $\{\mathfrak{c}_{n,k}^{(\varepsilon,N_0)}\}_{k=0}^{k_*}$ with the linear form

(60)
$$\mathbf{c}_n^{(\varepsilon,N_0)}(k) = m_n^{(\varepsilon,N_0)} k + q_n^{(\varepsilon,N_0)} \qquad (n \in \mathbb{N}, k \in \mathbb{R}).$$

For every $n \in \mathbb{N}$, we therefore link the coefficients $m_n^{(\varepsilon,N_0)}$ and $q_n^{(\varepsilon,N_0)}$ to the noisy input data $\{f_N^{(\varepsilon)}\}$ according to the scheme (see also (46)):

$$\left\{f_N^{(\varepsilon)}\right\}_{N=0}^{N_0} \xrightarrow[\text{Formula (53)}]{} \left\{\mathfrak{c}_{n,k}^{(\varepsilon,N_0)}\right\}_{k=0}^{k_*} \xrightarrow[\text{Linear regression}]{} \left(m_n^{(\varepsilon,N_0)}, q_n^{(\varepsilon,N_0)}\right).$$

3. For every $n \in \mathbb{N}$, compute the function $\zeta_{p}^{(\varepsilon,N_{0})}(n)$ as (see also (48))

(61)
$$\zeta_{\mathbf{p}}^{(\varepsilon,N_0)}(n) \doteq -\frac{q_n^{(\varepsilon,N_0)}}{m_n^{(\varepsilon,N_0)}}$$

Now, Lemma 4 informs us that:

(62)
$$\mathfrak{c}_{n}^{(\varepsilon,N_{0})}(k) \xrightarrow[N_{0} \to +\infty]{N_{0} \to +\infty}{} \mathfrak{c}_{n}^{(0,\infty)}(k) \equiv \mathfrak{c}_{n}(k) \qquad (n \in \mathbb{N}, k \in \mathbb{R}),$$

and, consequently, we have for every $n \in \mathbb{N}$ (see also (45)):

(63a)
$$m_n^{(\varepsilon,N_0)} \xrightarrow[\varepsilon \to 0]{N_0 \to +\infty} m_n^{(0,\infty)} \equiv m_n,$$

(63b)
$$q_n^{(\varepsilon,N_0)} \xrightarrow[\varepsilon \to 0]{N_0 \to +\infty} q_n^{(0,\infty)} \equiv q_n$$

Accordingly, from formulae (48) and (61) it follows

(64)
$$\lim_{\substack{N_0 \to +\infty\\\varepsilon \to 0}} \zeta_{\mathbf{p}}^{(\varepsilon,N_0)}(n) = \zeta_{\mathbf{p}}^{(0,\infty)}(n) = \zeta_{\mathbf{p}}(n) \qquad (n \in \mathbb{N}).$$

4. Finally, in view of formula (49), we obtain the formula for recovering the position of the pole:

(65)
$$\lim_{n \to +\infty} \left(\lim_{\substack{N_0 \to +\infty \\ \varepsilon \to 0}} \zeta_{\mathbf{p}}^{(\varepsilon, N_0)}(n) \right) = \zeta_{\mathbf{p}},$$

or, equivalently, defining $z_{p}^{(\varepsilon,N_{0})}(n) \doteq \zeta_{p}^{(\varepsilon,N_{0})}(n) + \frac{1}{2}$, and in view of (43):

(66)
$$\lim_{n \to +\infty} \left(\lim_{\substack{N_0 \to +\infty \\ \varepsilon \to 0}} z_{\mathbf{p}}^{(\varepsilon, N_0)}(n) \right) = z_{\mathbf{p}}.$$

For its actual implementation, formula (66) deserves a deeper analysis. To begin with, assume (unrealistically) that we can perform the inner limit for $N_0 \to +\infty$ and $\varepsilon \to 0$ to get the function $z_p^{(0,\infty)}(n)$. Now, the outer limit in (66), which is a direct consequence of limit (38), tells us that $z_p^{(0,\infty)}(n)$ is expected to become close to z_p from a certain value of n on (say, $n > n_{\min}$), in correspondence of the values of n for which $c_n^{(p)}(k)$ becomes nearly zero (see (38)). This means that, in practice, in the plot of $z_p^{(0,\infty)}(n)$ against n we should be able to identify a "range of convergence", that is, a set of n-values where $z_p^{(0,\infty)}(n)$ is nearly constant (actually, since in general $z_p \in \mathbb{C}$, two "ranges of convergence", one for the real and one for the imaginary part, separately). More precisely, for an arbitrary constant $\eta > 0$, we expect to find an integer $n_{\min} = n_{\min}(\varepsilon, N_0; \eta)$ such that:

(67)
$$\left| z_{\mathbf{p}}^{(0,\infty)}(n) - z_{\mathbf{p}} \right| < \eta \quad \text{for } n \ge n_{\min}(0,\infty;\eta).$$

Notice that, in this case with $\varepsilon = 0$ and $N_0 = \infty$ and in view of (38), the range of convergence is expected to be superiorly unlimited.

Now, in a realistic situation ε cannot be null, N_0 is necessarily finite, and both must be regarded as fixed. Therefore the inner limit in (66) cannot be actually performed. This fact has consequences on the algorithm in view of the fact that the two limits in (66) cannot be interchanged. In order to see this, let us define, in close analogy with formula (40) (see also (37)), the following approximate coefficients:

(68)
$$c_n^{(\mathbf{p};\varepsilon,N_0)}(k) \doteq \mathfrak{c}_n^{(\varepsilon,N_0)}(k) - (\zeta_{\mathbf{p}} - k)\tau_n \qquad (n \in \mathbb{N}, k \in \mathbb{R}; \varepsilon > 0, N_0 < \infty),$$

where $\mathfrak{c}_n^{(\varepsilon,N_0)}(k)$, τ_n and ζ_p are given by (53), (42) and (43), respectively. Comparing (68) with (37), and by Lemma 4, it follows: $c_n^{(p;0,\infty)}(k) = c_n^{(p)}(k)$. Now, we have:

(69)
$$\lim_{n \to +\infty} \left(\lim_{\substack{N_0 \to +\infty \\ \varepsilon \to 0}} c_n^{(\mathbf{p};\varepsilon,N_0)}(k) \right) \neq \lim_{\substack{N_0 \to +\infty \\ \varepsilon \to 0}} \left(\lim_{n \to +\infty} c_n^{(\mathbf{p};\varepsilon,N_0)}(k) \right).$$

In fact, the l.h.s. of (69) is null since $c_n^{(p;0,\infty)}(k)$ are the coefficients of expansion (39). Instead, for what concerns the r.h.s. of (69) we have, by using the asymptotic formulae (A.5) and (A.6) for the Pollaczek polynomials, with $N_0 < \infty$, $\varepsilon > 0$:

(70)

$$\begin{aligned} & \left| c_n^{(\mathbf{p};\varepsilon,N_0)}(k) \right| \\ & \\ & \\ & \\ & \\ & n \gg 1} 2\sqrt{\pi} \left| (-1)^{N_0} \frac{(N_0-k) f_{N_0}^{(\varepsilon)}}{(N_0!)^2} (2n)^{N_0} - (\zeta_{\mathbf{p}}-k) R_{\mathbf{p}} \frac{\Gamma(\frac{1}{2}-z_{\mathbf{p}})}{\Gamma(\frac{1}{2}+z_{\mathbf{p}})} (2n)^{z_{\mathbf{p}}-1/2} \right|, \end{aligned}$$

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which tends to infinity as $n \to +\infty$. Now, since Eq. (66) is a direct consequence of the fact that the l.h.s. of (69) is null, then formula (69) does not allow the limits in (66) to be switched.

Assume now (more realistically) that ε and N_0 take on the fixed values $\overline{\varepsilon}$ and \overline{N}_0 , respectively: i.e., $\varepsilon \equiv \overline{\varepsilon}$ and $N_0 \equiv \overline{N}_0$. In view of (66) we have therefore to deal with the following limit: $\lim_{n \to +\infty} z_p^{(\overline{\varepsilon}, \overline{N}_0)}(n)$.

Immediate consequence of the divergence of $c_n^{(\overline{p};\overline{\epsilon},\overline{N}_0)}(k)$ as $n \to +\infty$ (with $\overline{\epsilon} > 0$ and $\overline{N}_0 < \infty$) is the divergence of $z_p^{(\overline{\epsilon},\overline{N}_0)}(n)$ from z_p as $n \to +\infty$ (see (66)). Therefore, in the actual analysis of $z_p^{(\overline{\epsilon},\overline{N}_0)}(n)$, n cannot be pushed to infinity, but must be stopped before this divergence sets in. However, if $\overline{\epsilon}$ is "sufficiently small" and \overline{N}_0 is "sufficiently large", then, according to formula (64), $z_p^{(\overline{\epsilon},\overline{N}_0)}(n)$ (at fixed n) is expected to be close to $z_p^{(0,\infty)}(n)$, and consequently, for not too large values of n, say $n < n_{\max}$ (and with $n > n_{\min}$), we will have also $z_p^{(\overline{\epsilon},\overline{N}_0)}(n) \simeq z_p$. Therefore, in the plot of $z_p^{(\overline{\epsilon},\overline{N}_0)}(n)$ against n we aim at identifying a range of nvalues (the "range of convergence"), now limited superiorly, where $z_p^{(\overline{\epsilon},\overline{N}_0)}(n)$ is nearly constant. More precisely, given an arbitrary constant $\eta > 0$ (whose value determines the allowed range of variability of the estimate), our goal is to find two integers $n_{\min}(\varepsilon, N_0; \eta)$ and $n_{\max}(\varepsilon, N_0; \eta)$ and a value $z_p^{(\overline{\epsilon},\overline{N}_0)}$, which represents the estimate of the pole position at the given values $\varepsilon = \overline{\varepsilon}$ and $N_0 = \overline{N}_0$, such that:

(71)
$$\left| z_{\mathbf{p}}^{(\overline{\varepsilon},\overline{N}_{0})}(n) - z_{\mathbf{p}}^{(\overline{\varepsilon},\overline{N}_{0})} \right| < \eta \quad \text{for } n_{\min}(\varepsilon,N_{0};\eta) \leq n \leq n_{\max}(\varepsilon,N_{0};\eta).$$

Since $z_{p}^{(\overline{e},N_{0})}(n)$ may vary significantly within the 2η -wide interval defined in (71), in the actual numerical implementation of the algorithm (see Section 6), once the range $[n_{\min}, n_{\max}]$ has been detected (if any), it can be taken as estimate $z_{p}^{(\overline{e},\overline{N}_{0})}$ of the pole position the sample mean of the values of $z_{p}^{(\overline{e},\overline{N}_{0})}(n)$ within this range, while the sample standard deviation can be used as an estimate of the uncertainty ². Finally, in view of the arguments discussed above (and comparing (71) with (67)) it is worth observing that $\lim_{\substack{N_{0} \to +\infty \\ \overline{e} \to 0}} n_{\max}(\overline{e}, N_{0}; \eta) = +\infty$.

We can now move on to consider the problem of evaluating the residue $R_{\rm p}$. Inspired by (51) and (61) (and recalling that $z_{\rm p}^{(\varepsilon,N_0)}(n) = \zeta_{\rm p}^{(\varepsilon,N_0)}(n) + \frac{1}{2}$), we compute, for every $n \in \mathbb{N}$, the function

(72)
$$R_{\mathbf{p}}^{(\varepsilon,N_0)}(n) \doteq -\frac{m_n^{(\varepsilon,N_0)}}{2\sqrt{\pi} \Gamma\left(\frac{1}{2} - z_{\mathbf{p}}^{(\varepsilon,N_0)}\right) P_n\left(-\mathrm{i}z_{\mathbf{p}}^{(\varepsilon,N_0)}\right)} \qquad (n \in \mathbb{N}).$$

Then, from (52), (63a), and (66) we have:

(73)
$$\lim_{n \to +\infty} \left(\lim_{\substack{N_0 \to +\infty\\\varepsilon \to 0}} R_{\mathbf{p}}^{(\varepsilon, N_0)}(n) \right) = R_{\mathbf{p}}.$$

The structure of Eq. (73) is equal to that of Eq. (66). Then, the arguments used earlier for estimating the pole position by the analysis of Eq. (66) can be

²For simplicity, we cease henceforth to use the notation $\overline{\varepsilon}$, \overline{N}_0 that we adopted in this subsection to emphasize the case when ε , N_0 take on fixed values.

used similarly for estimating the residue by means of Eq. (73). Therefore, if ε is "sufficiently small" and N_0 is "sufficiently large" $R_{\rm p}^{(\varepsilon,N_0)}(n)$ is expected to show, as a function of n, a "range of convergence" within which $R_{\rm p}^{(\varepsilon,N_0)}(n)$ is nearly constant. The estimate $R_{\rm p}^{(\varepsilon,N_0)}$ of the residue at the given fixed values of ε and N_0 can then be obtained as the sample mean of $R_{\rm p}^{(\varepsilon,N_0)}(n)$ within this range of n-values, in a way completely similar to what done in the case of the position of the pole.

Now, the arguments given sofar are mainly qualitative, and therefore, the following problem emerges.

Problem. How can the degree of approximation to $z_{\rm p}$ and $R_{\rm p}$ by the estimates $z_{\rm p}^{(\varepsilon,N_0)}$ and $R_{\rm p}^{(\varepsilon,N_0)}$ be evaluated?

The discussion of this problem is given in the next subsection.

5.2. Consistency relations for a meromorphic function, and measure of the degree of approximation of the estimates to pole position and residue. Referring to definition (68) of the approximate coefficients $c_{n,k}^{(p;\varepsilon,N_0)}$, we can state the following auxiliary lemma.

Lemma 5. For every fixed $k \in \mathbb{N}$, the following statements hold:

(74) (i)
$$\sum_{n=0}^{\infty} \left| c_{n,k}^{(p;0,\infty)} \right|^2 = \|h_k(\mathbf{i}y)\|_{L^2(-\infty,+\infty)}^2 \doteq C_k \quad (C_k = \text{const.}).$$

(75) (ii)
$$\sum_{n=0}^{\infty} \left| c_{n,k}^{(\mathbf{p};\varepsilon,N_0)} \right|^2 = +\infty \qquad (\varepsilon > 0, N_0 < \infty).$$

(76) (iii)
$$\lim_{\substack{N_0 \to +\infty\\\varepsilon \to 0}} c_{n,k}^{(\mathbf{p};\varepsilon,N_0)} = c_{n,k}^{(\mathbf{p};0,\infty)} = c_{n,k}^{(\mathbf{p})} \qquad (n \in \mathbb{N}).$$

(iv) Let $m^{(p)}(\varepsilon, N_0; k)$ be defined as

(77)
$$m^{(\mathbf{p})}(\varepsilon, N_0; k) \doteq \max\left\{m \in \mathbb{N} : \sum_{n=0}^m \left|c_{n,k}^{(\mathbf{p};\varepsilon, N_0)}\right|^2 \leqslant C_k\right\},$$

then

(78)
$$\lim_{\substack{N_0 \to +\infty\\\varepsilon \to 0}} m^{(p)}(\varepsilon, N_0; k) = +\infty \qquad (k \in \mathbb{N}).$$

(v) The sum

(79)
$$M_k^{(\mathbf{p};\varepsilon,N_0)}(m) \doteq \sum_{n=0}^m \left| c_{n,k}^{(\mathbf{p};\varepsilon,N_0)} \right|^2 \qquad (k \in \mathbb{N})$$

satisfies the following properties:

- (v.a) it does not decrease for increasing values of m;
- (v.b) for every $k \in \mathbb{N}$ the following asymptotic relationship holds:

(80)
$$M_k^{(\mathbf{p};\varepsilon,N_0)}(m) \ge \left| c_{m,k}^{(\mathbf{p};\varepsilon,N_0)} \right|^2 \underset{m \gg 1}{\sim} A_k^{(\varepsilon,N_0)} \cdot (2m)^{2\max\left\{N_0,\operatorname{Re} z_{\mathbf{p}} - \frac{1}{2}\right\}}$$

 $(\varepsilon > 0 \text{ and } N_0 < \infty \text{ fixed}),$

 $A_k^{(\varepsilon,N_0)}$ being a quantity independent of m.

Proof. (i) Since $c_{n,k}^{(p;0,\infty)} = c_{n,k}^{(p)}$, the statement follows from expansion (39) and Parseval's theorem. (ii) From the asymptotic expression in (70) it follows:

(81)
$$\left| c_{n,k}^{(\mathbf{p};\varepsilon,N_0)} \right| \sim A_k^{(\varepsilon,N_0)} \cdot (2n)^{\max\{N_0,\operatorname{Re} z_{\mathbf{p}}-1/2\}} \xrightarrow[n \to +\infty]{} +\infty,$$

since $A_k^{(\varepsilon,N_0)} > 0$, $N_0 > 0$. Then, statement (ii) follows. (iii) Consider the difference

(82)
$$c_{n,k}^{(\mathbf{p};0,\infty)} - c_{n,k}^{(\mathbf{p};\varepsilon,N_0)} = \left(\mathfrak{c}_{n,k}^{(0,\infty)} - \mathfrak{c}_{n,k}^{(\varepsilon,N_0)}\right).$$

For $N_0 \to \infty$ and $\varepsilon \to 0$, the r.h.s. of (82) vanishes by Lemma 4, and statement (iii) follows. (iv) Define $m_1(\varepsilon, N_0; k) \doteq m^{(p)}(\varepsilon, N_0; k) + 1$. From definition (77) it follows that $\sum_{\substack{n=0\\ \varepsilon\to 0}}^{m_1} |c_{n,k}^{(p;\varepsilon,N_0)}|^2 > C_k$. For our purpose it is sufficient to prove that $\lim_{\substack{N_0\to+\infty\\\varepsilon\to 0}} m_1(\varepsilon, N_0; k) = +\infty$. Suppose, instead, that such a limit is finite. Then, there exists a finite number $m_*(k)$ (independent of ε and N_0) such that $\lim_{\substack{N_0\to+\infty\\\varepsilon\to 0}} m_1(\varepsilon, N_0; k) \leqslant m_*(k)$. Then, we would have

(83)
$$C_k < \sum_{n=0}^{m_1(\varepsilon, N_0; k)} \left| c_{n,k}^{(p;\varepsilon, N_0)} \right|^2 \leq \sum_{n=0}^{m_*(k)} \left| c_{n,k}^{(p;\varepsilon, N_0)} \right|^2.$$

But, as $N_0 \to +\infty$ and $\varepsilon \to 0$ we have (see also (76)):

(84)
$$C_k < \sum_{n=0}^{m_*(k)} \left| c_{n,k}^{(p;0,\infty)} \right|^2 < \sum_{n=0}^{\infty} \left| c_{n,k}^{(p;0,\infty)} \right|^2 = C_k,$$

which is a contradiction. Then, statement (iv) is proved. Statement (v.a) is obvious. Statement (v.b) follows from the asymptotic behavior of the polynomials $P_m(-iz)$ for large values of m and z fixed, given in the Appendix (see formula (A.5) and statement (ii)).

Corollary 2. From statements (iv) and (v) of Lemma 5 it follows that, for N_0 sufficiently large, ε sufficiently small and for every $k \in \mathbb{N}$, the sum $M_k^{(\mathbf{p};\varepsilon,N_0)}(m)$ exhibits (as a function of m) a plateau: i.e., a range of m-values where it is nearly constant. The upper limit of this range is given by $m^{(\mathbf{p})}(\varepsilon, N_0; k)$.

Remark 2. The plateau mentioned in Corollary 2 refers to the evaluation of $M_k^{(\mathrm{p};\varepsilon,N_0)}(m)$ against $m \ (k \in \mathbb{N})$ and should not be confused with the range of convergence mentioned at the end of the previous subsection, which refers to the evaluation of $z_{\mathrm{p}}^{(\varepsilon,N_0)}$ and $R_{\mathrm{p}}^{(\varepsilon,N_0)}$.

Next, we introduce the sum defined by

(85)
$$\mathfrak{M}_{k}^{(\varepsilon,N_{0})}(m) \doteq \sum_{n=0}^{m} \left|\mathfrak{c}_{n,k}^{(\varepsilon,N_{0})}\right|^{2} \qquad (k \in \mathbb{N}; \varepsilon > 0, N_{0} < \infty),$$

where the coefficients $\mathfrak{c}_{n,k}^{(\varepsilon,N_0)}$ are given in formula (53) (restricted to $k \in \mathbb{N}$). The following two cases are worth being discussed:

(1) Suppose that the function f(z) being analyzed is analytic in Re z > 0. In this case the $\mathfrak{c}_{n,k}$ (see (41)) represent the expansion coefficients of the function $h_k(iy)$ (analytic in Re z > 0) on the basis $\{\psi_n\}$ (see also formula (27), where the sum on

the r.h.s. coincides with $\mathfrak{c}_{n,k}$). Therefore the terms $\mathfrak{c}_{n,k}$ and $\mathfrak{c}_{n,k}^{(\varepsilon,N_0)}$ enjoy properties completely analogous to those established in Lemma 5 for the coefficients $c_{n,k}^{(p;\varepsilon,N_0)}$. In particular, the sum $\mathfrak{M}_k^{(\varepsilon,N_0)}(m)$ is expected to exhibit (for ε sufficiently small and N_0 sufficiently large) a *plateau* whose upper limit will be denoted by $\mathfrak{m}(\varepsilon, N_0; k)$. The properties of this plateau are strictly analogous to those stated by Corollary 2.

(2) If f(z) is meromorphic in Re z > 0, the coefficients $\mathfrak{c}_{n,k}$ are not the expansion coefficients of the (meromorphic) function $h_k(iy)$ on the basis $\{\psi_n\}$, the actual coefficients being given instead by formula (27). Consequently, the sum $\mathfrak{M}_k^{(\varepsilon,N_0)}(m)$ no longer satisfies the properties mentioned in the previous point (1), in view of the absence of the pole term in the $\mathfrak{c}_{n,k}$. Then, $\mathfrak{M}_k^{(\varepsilon,N_0)}(m)$ (as a function of m) ought not to exhibit a *plateau* in the neighborhood of a certain value of m.

Summarizing, the analysis, as a function of m, of the sum $\mathfrak{M}_{k}^{(\varepsilon,N_{0})}(m)$ (which can be computed from the input data set) can be exploited as an initial test of analyticity for the function under consideration.

We can now proceed to give an answer to the problem posed in the previous subsection. The main idea is inspired by the consistency relations (26) (along with formula (27) for the coefficients), which make explicit the mutual relations among the pole parameters and the function samples. Equations (26) suggest to compare the samples of the input data set $\{f_k^{(\varepsilon)}\}_{k=0}^{N_0}$ with the corresponding values, denoted $\{\hat{f}_k^{(p;\varepsilon,N_0)}\}_{k=0}^{N_0}$, which can be actually computed when the estimates $z_p^{(\varepsilon,N_0)}$ and $R_p^{(\varepsilon,N_0)}$ of pole position and residue have been evaluated. Therefore, we are led to define the following approximate coefficients:

$$\widehat{f}_{k}^{(\mathbf{p};\varepsilon,N_{0})} \doteq (-1)^{k+1} \left\{ \sum_{N=0}^{N_{0}} (-1)^{N} f_{N}^{(\varepsilon)} \left(1-\delta_{Nk}\right) - \frac{\pi R_{\mathbf{p}}^{(\varepsilon,N_{0})}}{\cos \pi z_{\mathbf{p}}^{(\varepsilon,N_{0})}} - \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\sum_{n=0}^{\widehat{m}} \widehat{c}_{n,k}^{(\mathbf{p};\varepsilon,N_{0})} \psi_{n}(y)}{\left[\left(x+\frac{1}{2}\right)-iy\right] \cosh \pi y} \,\mathrm{d}y \right\}$$

$$= (-1)^{k+1} \left\{ \sum_{N=0}^{N_{0}} (-1)^{N} f_{N}^{(\varepsilon)} \left(1-\delta_{Nk}\right) - \frac{\pi R_{\mathbf{p}}^{(\varepsilon,N_{0})}}{\cos \pi z_{\mathbf{p}}^{(\varepsilon,N_{0})}} + \sum_{n=0}^{\widehat{m}^{(\mathbf{p})}} \widehat{c}_{n,k}^{(\mathbf{p};\varepsilon,N_{0})} Q_{n} \left[-i\left(k+\frac{1}{2}\right) \right] \right\} \quad (k=0,\ldots,N_{0}),$$

where:

$$\widehat{c}_{n,k}^{(\mathbf{p};\varepsilon,N_{0})} \doteq 2\sqrt{\pi} \left\{ \sum_{N=0}^{N_{0}} \frac{(-1)^{N}}{N!} (N-k) f_{N}^{(\varepsilon)} P_{n} \left[-i\left(N+\frac{1}{2}\right) \right] - \left(z_{\mathbf{p}}^{(\varepsilon,N_{0})} - k - \frac{1}{2} \right) R_{\mathbf{p}}^{(\varepsilon,N_{0})} \Gamma \left(\frac{1}{2} - z_{\mathbf{p}}^{(\varepsilon,N_{0})} \right) P_{n} \left(-iz_{\mathbf{p}}^{(\varepsilon,N_{0})} \right) \right\} \\
= \mathfrak{c}_{n,k}^{(\varepsilon,N_{0})} - \left(\zeta_{\mathbf{p}}^{(\varepsilon,N_{0})} - k \right) \tau_{n}^{(\varepsilon,N_{0})} \qquad (n = 0, 1, 2, \ldots),$$

 $\psi_n(y)$ being defined in (6), $Q_n\left[-i\left(k+\frac{1}{2}\right)\right]$ in (16), while the truncation number $\hat{m}^{(p)} = \hat{m}^{(p)}(\varepsilon, N_0; k)$ will be defined in what follows. Notice that the coefficients $\hat{c}_{n,k}^{(p;\varepsilon,N_0)}$ differ from the corresponding coefficients $c_{n,k}^{(p;\varepsilon,N_0)}$ (see (68)) for the fact that in the expression of $\hat{c}_{n,k}^{(p;\varepsilon,N_0)}$ use is made of the computed estimates $z_p^{(\varepsilon,N_0)}$ and $R_p^{(\varepsilon,N_0)}$ instead of their exact values, which are unknown as long as $\varepsilon > 0$ and $N_0 < \infty$. However, if the computed estimates $z_p^{(\varepsilon,N_0)}$ and $R_p^{(\varepsilon,N_0)}$ are "close" to the exact values z_p and R_p , then correspondingly the terms $\hat{c}_{n,k}^{(p;\varepsilon,N_0)}$ are "close" to the coefficients $c_{n,k}^{(p;\varepsilon,N_0)}$. Hence, in strict analogy with the definitions given in Lemma 5 concerning the coefficients $c_{n,k}^{(p;\varepsilon,N_0)}$, we can introduce the following sum (see (79)) :

(88)
$$\widehat{M}_{k}^{(\mathbf{p};\varepsilon,N_{0})}(m) \doteq \sum_{n=0}^{m} \left|\widehat{c}_{n,k}^{(\mathbf{p};\varepsilon,N_{0})}\right|^{2} \qquad (k \in \mathbb{N}).$$

Notice that $\widehat{M}_{k}^{(\mathrm{p};0,\infty)}(m) = M_{k}^{(\mathrm{p};0,\infty)}(m)$ since $\widehat{c}_{n,k}^{(\mathrm{p};0,\infty)} = c_{n,k}^{(\mathrm{p};0,\infty)}$.

The analysis of $\widehat{M}_{k}^{(\mathbf{p};\varepsilon,N_{0})}(m)$ may provide us with a first qualitative test of the degree of approximation of $z_{\mathbf{p}}^{(\varepsilon,N_{0})}$ and $R_{\mathbf{p}}^{(\varepsilon,N_{0})}$. In fact, if $\widehat{M}_{k}^{(\mathbf{p};\varepsilon,N_{0})}(m)$ exhibits a plateau as a function of m, that is, if it manifests a behavior analogous to that expected from the analysis of $M_{k}^{(\mathbf{p};\varepsilon,N_{0})}(m)$ against m (the existence of a plateau in $M_{k}^{(\mathbf{p};\varepsilon,N_{0})}(m)$ is proved by Corollary 2), then the coefficients $\widehat{c}_{n,k}^{(\mathbf{p};\varepsilon,N_{0})}$ are likely to be close to the corresponding values $c_{n,k}^{(\mathbf{p};\varepsilon,N_{0})}$, and, accordingly, the estimates $z_{\mathbf{p}}^{(\varepsilon,N_{0})}$ and $R_{\mathbf{p}}^{(\varepsilon,N_{0})}$ are close to their exact values $z_{\mathbf{p}}$ and $R_{\mathbf{p}}$. In this case, the upper limit of this plateau, which we denote by $\widehat{m}^{(\mathbf{p})}(\varepsilon, N_{0}; k)$, enjoys properties strictly analogous to those of $m^{(\mathbf{p})}(\varepsilon, N_{0}; k)$, which have been stated in Lemma 5, and therefore, can be used as the truncation number for the rightmost sum in formula (86).

Now, once $\widehat{m}^{(p)}(\varepsilon, N_0; k)$ has been set, formula (86) allows us to compute the approximate samples $\widehat{f}_k^{(p;\varepsilon,N_0)}$, representing the set of samples which are compatible with the computed estimates of pole position $z_p^{(\varepsilon,N_0)}$ and residue $R_p^{(\varepsilon,N_0)}$. At this point we are naturally brought to compare these approximate samples $\widehat{f}_k^{(p;\varepsilon,N_0)}$ with the input data, i.e. the noisy samples $f_k^{(\varepsilon)}$. Then, as a measure of the accuracy of the computation of the samples, and hence also of the pole parameters, use can be made of the relative root mean squared error:

(89)
$$\delta^{(\varepsilon,N_0)} \doteq \left(\frac{1}{(N_0+1)} \sum_{k=0}^{N_0} \frac{\left|f_k^{(\varepsilon)} - \widehat{f}_k^{(\mathbf{p};\varepsilon,N_0)}\right|^2}{\left|f_k^{(\varepsilon)}\right|^2}\right)^{1/2},$$

which gives a quantitative numerical evaluation of the degree of approximation to $z_{\rm p}$ and $R_{\rm p}$ by the estimates $z_{\rm p}^{(\varepsilon,N_0)}$ and $R_{\rm p}^{(\varepsilon,N_0)}$, as initially required by the problem in Section 5.1.

Remark 3. In the case of functions f(z) analytic in Re z > 0, the formula for computing the *approximate samples* is obviously different for the absence of the pole term. Then, in formula (89), instead of $\hat{f}_k^{(p;\varepsilon,N_0)}$ given in (86), there must be

inserted the terms $\widehat{f}_k^{(\varepsilon,N_0)}$ given by:

(90)
$$\widehat{f}_{k}^{(\varepsilon,N_{0})} \doteq (-1)^{k+1} \left\{ \sum_{N=0}^{N_{0}} (-1)^{N} f_{N}^{(\varepsilon)} \left(1 - \delta_{Nk}\right) + \sum_{n=0}^{\mathfrak{m}(\varepsilon,N_{0};k)} \widehat{c}_{n,k}^{(\varepsilon,N_{0})} Q_{n} \left[-\mathrm{i} \left(k + \frac{1}{2}\right) \right] \right\} \quad (k = 0, \dots, N_{0}),$$

where, for n = 0, 1, 2, ...

(91)
$$\widehat{c}_{n,k}^{(\varepsilon,N_0)} \doteq 2\sqrt{\pi} \sum_{N=0}^{N_0} \frac{(-1)^N}{N!} (N-k) f_N^{(\varepsilon)} P_n\left[-\mathrm{i}\left(N+\frac{1}{2}\right)\right] \equiv \mathfrak{c}_{n,k}^{(\varepsilon,N_0)},$$

 $\mathfrak{m}(\varepsilon, N_0; k)$ being the upper limit of the plateau exhibited by $\mathfrak{M}_k^{(\varepsilon, N_0)}(m)$, regarded as a function of m.

Remark 4. In the case f(z) is a meromorphic function, for all values $m \lesssim \widehat{m}^{(\mathrm{p})}(\varepsilon, N_0; k)$ such that $\widehat{M}_k^{(\mathrm{p};\varepsilon,N_0)}(m)$ exhibits a *plateau* we have $\widehat{c}_{n,k}^{(\mathrm{p};\varepsilon,N_0)} \sim 0$. It then follows that the actual value of the truncation number in the rightmost sum on the r.h.s. of formula (86) is not critical, and therefore any such $m \lesssim \widehat{m}^{(\mathrm{p})}(\varepsilon, N_0; k)$ may be selected as an acceptable value where to stop the sum. In the next section, devoted to numerical examples, we will see that, in the actual numerical implementation, it can be convenient to truncate the sum at a m-value slightly different from $\widehat{m}^{(\mathrm{p})}(\varepsilon, N_0; k)$. The practical evaluation of this truncation point, denoted $\widehat{m}_{\mathrm{t}}^{(\mathrm{p})}$, will be specified in the next section.

Similar arguments hold *mutatis mutandis* in the case f(z) is an analytic function, the role of $\widehat{M}_{k}^{(\mathbf{p};\varepsilon,N_{0})}(m)$, $\widehat{m}^{(\mathbf{p})}(\varepsilon,N_{0};k)$, $\widehat{c}_{n,k}^{(\mathbf{p};\varepsilon,N_{0})}$, and $\widehat{m}_{\mathbf{t}}^{(\mathbf{p})}$ being now played by $\mathfrak{M}_{k}^{(\varepsilon,N_{0})}(m)$, $\mathfrak{m}(\varepsilon,N_{0};k)$, $\mathfrak{c}_{n,k}^{(\varepsilon,N_{0})}$, and $\mathfrak{m}_{\mathbf{t}}$, respectively.

Remark 5. In the case f(z) is a meromorphic function, and in view of what has been discussed above, the formulae for the actual numerical implementation of the interpolation formula (17) (along with (18)) read:

(92)
$$\hat{f}^{(\varepsilon,N_0)}\left(x+\frac{1}{2}\right) = \sum_{N=0}^{N_0} f_N^{(\varepsilon)} \operatorname{sinc}(x-N) - \frac{R_{\mathrm{p}}^{(\varepsilon,N_0)}}{\cos\left(\pi z_{\mathrm{p}}^{(\varepsilon,N_0)}\right)} \frac{\sin\pi x}{\left(x+\frac{1}{2}-z_{\mathrm{p}}^{(\varepsilon,N_0)}\right)} - \frac{\sin\pi x}{\pi} \sum_{n=0}^{\widehat{m}^{(\mathrm{p})}} \widehat{c}_n^{(\mathrm{p};\varepsilon,N_0)} Q_n \left[-\mathrm{i}\left(x+\frac{1}{2}\right)\right] \qquad \left(x > -\frac{1}{2}\right),$$

where, for n = 0, 1, 2, ...

(93)
$$\widehat{c}_{n}^{(\mathbf{p};\varepsilon,N_{0})} = 2\sqrt{\pi} \left\{ \sum_{N=0}^{N_{0}} \frac{(-1)^{N}}{N!} f_{N}^{(\varepsilon)} P_{n} \left[-i\left(N + \frac{1}{2}\right) \right] -R_{\mathbf{p}}^{(\varepsilon,N_{0})} \Gamma\left(\frac{1}{2} - z_{\mathbf{p}}^{(\varepsilon,N_{0})}\right) P_{n} \left(-iz_{\mathbf{p}}^{(\varepsilon,N_{0})}\right) \right\}$$

and, for the given values of ε and N_0 and for every $x > -\frac{1}{2}$, $\widehat{f}^{(\varepsilon,N_0)}(x+\frac{1}{2})$ represents the approximation to the function $f(x+\frac{1}{2})$.

6. NUMERICAL EXAMPLES

The purpose of this section is to illustrate through numerical examples the main steps of the theory. To begin with, we consider as a preparatory example the function $f_1(z) = C/(z+5)^5$, (C constant), which satisfies the conditions assumed in Theorem 1 and, in particular, it is analytic in $\operatorname{Re} z > 0$; the analysis is summarized in Figs. 2 and 3. In Fig. 2a the plot of the sum $\mathfrak{M}_{k}^{(\varepsilon,N_{0})}(m)$ (defined in (85)), computed for various values of k (see the figure legend for numerical details), displays clearly the presence of the plateaux, which manifest the analyticity of the function $f_1(z)$ in Re z > 0 (see also Corollary 2). In this example, as well as in all those that follow (with the exception of that referring to Fig. 8), we set $\varepsilon \equiv \varepsilon_{\rm R}$, which means that no noise has been added to the input samples f_N , and that the only source of (inevitable) noise is given by the numerical roundoff error. The plateaux range approximately from $n_{\min}(\varepsilon, N_0; k) \sim 40$ through $\mathfrak{m}(\varepsilon, N_0; k) \sim 240$; for $m \gtrsim \mathfrak{m}(\varepsilon, N_0; k)$ we see that $\mathfrak{M}_k^{(\varepsilon, N_0)}(m)$ starts to diverge as a power of m (see (80)) for the presence of the roundoff noise and the finiteness of the number of input samples (in this case $N_0 = 60$). From the inspection of these plateaux we can determine the truncation number \mathfrak{m}_t , which must lie within the plateaux, and which is necessary for the reconstruction of the data samples (see (90)). The choice of \mathfrak{m}_{t} within the plateaux is not critical for the accuracy of the final result (see Fig. 3a); the actual value of \mathfrak{m}_{t} (within the plateau) can be conveniently set by exploiting formula (90) for the sample reconstruction. In fact, by (90) we can compute the approximate samples $\widehat{f}_{k}^{(\varepsilon,N_{0})}$ (which depend on the truncation number \mathfrak{m}_{t}) and, correspondingly, also the mean error $\delta^{(\varepsilon,N_{0})}$ (see (89)). The strategy is then to set \mathfrak{m}_{t} as the integral number which minimizes $\delta^{(\varepsilon,N_{0})}$ (ε and N_{0} being fixed). In this example, the value of \mathfrak{m}_{t} which minimizes $\delta^{(\varepsilon,N_{0})}$ (with $\varepsilon = \varepsilon_{\mathrm{R}}$ and $N_{0} = 60$) is $\mathfrak{m}_{t}(\varepsilon, N_{0}; k) = 122$; the reconstructed samples, computed through (90), are shown in Fig. 2b (filled dots) superimposed to the function $f_1(x)$ (solid line); the high quality of the reconstruction ($\delta^{(\varepsilon, N_0)} = 2.74 \times 10^{-5}$) is evident.

The role played by the various parameters intervening in the algorithm is summarized in Fig. 3. In Fig. 3a we show the behavior of the reconstruction error $\delta^{(\varepsilon,N_0)}$ as a function of the truncation number \mathfrak{m}_t (see (90)). We see that the error becomes tiny when \mathfrak{m}_t enters the plateaux ($\mathfrak{m}_t \sim 50$, see Fig. 2a), and does not vary appreciably as long as \mathfrak{m}_t remains within it. Figure 3b shows the sum $\mathfrak{M}_k^{(\varepsilon,N_0)}(m)$ against m (k = 10) for various values of the number of input samples N_0 . It can be seen that the length of the plateaux increases as the number of input data increases, reflecting the increase of *information* available for the computation. At $N_0 = 60$ the effect of the roundoff noise appears, and it is this noise indeed which limits superiorly the length of the plateau; in fact, for this function, the length of the plateau no longer increases for $N_0 \gtrsim 60$. In Fig. 3c we investigate how the analysis depends on the asymptotic behavior (for $z \to \infty$) of the function f(z). For this purpose we consider the function $f_1^{(q)}(z) = C/(z+5)^q$, and plot in Fig. 3c the sum $\mathfrak{M}_{k}^{(\varepsilon,N_{0})}(m)$ $(k = 10, \varepsilon = \varepsilon_{\mathrm{R}}, \text{ and } N_{0} = 60 \text{ fixed})$ at different values of q. Even in this case the length of the plateaux increases evidently with q. It should be remarked that, as expected, for q = 1 no plateau appears since the function $h_k(iy) = (iy - k - \frac{1}{2})f_1^{(1)}(iy) \notin L_2(-\infty, +\infty)$ (see Theorem 3). Parallelly, in



FIGURE 2. Analysis of a function analytic in $\operatorname{Re} z > 0$: $f_1(z) = C(z+5)^{-5}$. $C = 10^3$; $N_0 = 60$; $\varepsilon \equiv \varepsilon_{\mathrm{R}}$: i.e., no noise is added to the function samples but only numerical roundoff noise is present. (a): Plot of the sum $\mathfrak{M}_k^{(\varepsilon,N_0)}(m)$ vs. *m* for some values of *k* (see formula (85)). (b): Reconstructed samples $\widehat{f}_j^{(\varepsilon,N_0)}$ (filled dots) of the function $f_1(x)$ (solid line) at 20 half-integers values $x_j \doteq j + \frac{1}{2}, j = 0, \ldots, 19$, computed by using Eq. (90). For every *k*, the truncation number has been set to the value $\mathfrak{m}_t(\varepsilon, N_0; k) = 122$, which represents the value that minimizes the relative root mean squared error $\delta^{(\varepsilon,N_0)}$ (see Eq. (89)): in this case, $\delta^{(\varepsilon,N_0)} = 2.74 \times 10^{-5}$.

Fig. 3d, where the plots of the reconstructed samples $\hat{f}_k^{(\varepsilon,N_0)}$ are shown for various values of q, we see that for q = 1 the reconstruction clearly deteriorates.

We can now move on to analyze the case of meromorphic functions. In order to elucidate the algorithm presented in Sections 4.2 and 5.1, we begin by considering a simple, though canonical, meromorphic function: $f_2(z) = \frac{C}{(z+3)^3} \frac{1}{(z-z_p)}$, which has a first order pole in the half-plane Re z > 0 at $z_p = 6.2 + 0.15$; the factor $(z+3)^{-3}$ in $f_2(z)$ guarantees the necessary asymptotic behavior for Re $z \to +\infty$, without introducing poles in the half-plane Re z > 0. The various steps, which lead to the recovery of pole location and residue, are summarized in Figs. 4 and 5.

From the input data $(f_2)_N^{(\varepsilon_R)}$ $(N = 0, ..., N_0)$, which are the numerical approximation of the function samples $f_2(N + \frac{1}{2})$, we can compute, for integral values of k, the sum $\mathfrak{M}_k^{(\varepsilon,N_0)}(m)$ (see formulae (85) and (53)), whose plot versus m is shown in Fig. 4a. The absence of any plateau in this plot indicates the lack of analyticity of the function $f_2(z)$ in Re z > 0; in fact, as discussed in Section 5.2 (see, in particular, Corollary 2), it is instead the sum $M_k^{(p;\varepsilon,N_0)}(m)$ (and its approximate version $\widehat{M}_k^{(p;\varepsilon,N_0)}(m)$, see (79) and (88)), containing the terms $(\zeta_p - k)\tau_n$ related to the (unknown) pole, which is expected to exhibit a plateau. Thus, as discussed in Section 5.2, the study of $\mathfrak{M}_k^{(\varepsilon,N_0)}(m)$ can be used as a preliminary tool for testing the analyticity for f(z). It is worth remarking that this test is not limited to functions with only polar singularities, but works also in the case of multivalued functions with branch cuts in Re z > 0; moreover, this procedure works also as a test when the function f(z) has a pole (of order > 1), whose residue is null (for the sake of brevity, the numerical examples are not reported here).



FIGURE 3. Analysis of a function analytic in $\operatorname{Re} z > 0$: $f_1^{(q)}(z) = C(z+5)^{-q}$; $C = 10^3$; $N_0 = 60$; $\varepsilon \equiv \varepsilon_{\mathrm{R}}$. (a): q = 5. Relative root mean squared error $\delta^{(\varepsilon,N_0)}$ vs. the truncation number $\mathfrak{m}_{\mathrm{t}}$ (see (89), (90), and the legend of Fig. 2b). (b): q = 5. Plot of the sum $\mathfrak{M}_k^{(\varepsilon,N_0)}(m)$ vs. m, at various values of the number N_0 of input function samples: $N_0 = 10, 20, 30, 40, 50, 60$ (see (85), (53)); k = 10. (c): Plot of $\mathfrak{M}_k^{(\varepsilon,N_0)}(m)$ vs. m for different values of the exponent $q: q = 1, \ldots, 10$; k = 10. (d): Reconstructed samples $\widehat{f}_j^{(\varepsilon,N_0)}$ (filled dots) of the function $f_1^{(q)}(x)$ (solid line) at the half-integers x_j , computed by Eq. (90) for various values of the exponent q. For each q, the value of the truncation number $\mathfrak{m}_{\mathrm{t}}(\varepsilon,N_0;k)$, used in formula (90) for reconstructing the samples, is the one which minimizes $\delta^{(\varepsilon,N_0)}; q = 1: \mathfrak{m}_{\mathrm{t}}(\varepsilon,N_0;k) = 41, \delta^{(\varepsilon,N_0)} = 2.08 \times 10^{-2}; q = 2: \mathfrak{m}_{\mathrm{t}}(\varepsilon,N_0;k) = 72, \delta^{(\varepsilon,N_0)} = 1.38 \times 10^{-2}; q = 3: \mathfrak{m}_{\mathrm{t}}(\varepsilon,N_0;k) = 29, \delta^{(\varepsilon,N_0)} = 1.08 \times 10^{-3}; q = 4: \mathfrak{m}_{\mathrm{t}}(\varepsilon,N_0;k) = 138, \delta^{(\varepsilon,N_0)} = 4.87 \times 10^{-4}; q = 5: \mathfrak{m}_{\mathrm{t}}(\varepsilon,N_0;k) = 122, \delta^{(\varepsilon,N_0)} = 2.74 \times 10^{-5}.$

Figure 4b shows the plots of $\operatorname{Re} z_{p}^{(\varepsilon,N_{0})}(n)$ and $\operatorname{Im} z_{p}^{(\varepsilon,N_{0})}(n)$ (computed by means of Eq. (61) and recalling that $z_{p}^{(\varepsilon,N_{0})}(n) = \zeta_{p}^{(\varepsilon,N_{0})}(n) + \frac{1}{2}$) versus n. In this example both ε and N_{0} must be considered as fixed, and take on the values $\varepsilon = \varepsilon_{R}$ (which means that only numerical roundoff error is present) and $N_{0} = 60$. In Fig. 4b wild oscillations can be observed in both plots for $n < n_{\min} \simeq 10$, whereas, for



FIGURE 4. Analysis of a function with one first order pole in Re z > 0: $f_2(z) = C(z + 5)^{-3}(z - z_p)^{-1}$. $z_p = 6.2 + 0.15i$, $R_p = 7.110146 - 0.2858122i$. $C = 10^4$, $N_0 = 60$, $\varepsilon \equiv \varepsilon_R$. (a): Analyticity test for $f_2(z)$ in Re z > 0: plot of the sum $\mathfrak{M}_k^{(\varepsilon,N_0)}(m)$ against m, at different values of k (notice the absence of the pole term in (53)). (b): Recovery of pole location. Real (left scale) and imaginary (right scale) part of the function $z_p^{(\varepsilon,N_0)}(n)$ vs. n: for every value of n, $z_p^{(\varepsilon,N_0)}(n)$ has been computed by means of formula (61) (see the algorithm described in Subsection 5.1). Length of the ranges of convergence, obtained with bin–width $W_p = 10^{-3}\%$; Real part: $L_p^R = 416$; Imaginary part: $L_p^I = 418$. The position of the pole, computed as the sample mean of Re $z_p^{(\varepsilon,N_0)}(n)$ for n within the range of convergence, is: $z_p^{(\varepsilon,N_0)} = (6.20000005 \pm 1.5 \times 10^{-7}) + (1.499996 \times 10^{-1} \pm 1.9 \times 10^{-7})$ i. (c): Recovery of the pole residue. Real (left scale) and imaginary (right scale) part of the function $R_p^{(\varepsilon,N_0)}(n)$ vs. n: for every n, $R_p^{(\varepsilon,N_0)}(n)$ has been computed by Eq. (72), using the value of $z_p^{(\varepsilon,N_0)}$ estimated from the data shown in panel (b). Length of the ranges of convergence, obtained with bin–width $W_p = 10^{-3}\%$: $L_p^R = 509$, $L_p^I = 454$. The residue, computed as the sample mean of $R_p^{(\varepsilon,N_0)}(n)$ for n within the range of convergence is: $R_p^{(\varepsilon,N_0)} = (7.1101458 \pm 7.4 \times 10^{-6}) - (2.8581197 \times 10^{-1} \pm 5.6 \times 10^{-7})$ i. (d): Plateaux after pole recovery. Plot of the sum $\widehat{M}_k^{(p;\varepsilon,N_0)}(m)$ vs. m for various k (see formula (88), and cf. panel (a)). The value of the plateau of $\widehat{M}_k^{(p;\varepsilon,N_0)}$ ins in correspondence of the squared norm $||h_k(iy)||^2$ (see (74)). The values of $z_p^{(\varepsilon,N_0)}$ and $R_p^{(\varepsilon,N_0)}$ inserted in the coefficients in (68) (and used to calculate $\widehat{M}_{\nu}^{(p;\varepsilon,N_0)}(m)$), are those obtained from the analysis of the data shown in panels (b) and (c).

 $n > n_{\min}$ the value of $z_{\rm p}^{(\varepsilon, N_0)}(n)$ (that is, of its real and imaginary parts) remains nearly constant over a wide range of *n*-values; the upper limit of this interval (not visible in this figure) is $n_{\max} = 426$ (see Section 5.1).



FIGURE 5. Analysis of the function $f_2(z)$ with one first order pole in $\operatorname{Re} z > 0$ (see also the legend of Fig. 4). $N_0 = 60$, $\varepsilon \equiv \varepsilon_{\mathrm{R}}$. (a) and (b): Reconstructed samples $\widehat{f}_j^{(\mathrm{p};\varepsilon,N_0)}$ (filled dots) of $f_2(x)$ (solid line) at the half-integers x_j , computed by Eq. (86). The truncation number used in (86) is $\widehat{m}_{\mathrm{t}}^{(\mathrm{p})}(\varepsilon, N_0; k) = 40$, which lies within the range of convergence of $z_{\mathrm{P}}^{(\varepsilon,N_0)}(n)$ shown in Fig. 4b, and minimizes the relative root mean squared error: $\delta^{(\varepsilon,N_0)} = 1.19 \times 10^{-3}$. (c): Interpolation of the function $h_{10}(x) \doteq (x - \frac{21}{2})f_2(x)$ (solid line) associated with $f_2(x)$. The interpolated values (filled dots) have been computed through formula (92) and, for better visualization, are shown only for values of x on a uniform grid with step 0.2. The crosses represent the samples $h_{10}(N + \frac{1}{2})$ ($N \in [0, N_0]$) used as input data. (d): Relative error in the evaluation of pole location and residue versus the number N_0 of input function samples $f_N, N \in [0, N_0]$.

The occurrence of this extended plateau (its length is $L_{\rm p} \doteq (n_{\rm max} - n_{\rm min}) = 416)^3$ guarantees that the coefficients $c_n^{({\rm p};\varepsilon,N_0)}(k)$ (see (68)), from which the values of $z_{\rm p}^{(\varepsilon,N_0)}(n)$ follow, and which are expected to vanish within a certain interval

³ Actually, we have two lengths $L_{\rm p}^{\rm R}$ and $L_{\rm p}^{\rm I}$ associated with the range of convergence of the real and of the imaginary part, respectively. Since they are usually very similar, for simplicity we will frequently refer only to $L_{\rm p} = L_{\rm p}^{\rm R} \simeq L_{\rm p}^{\rm I}$.

 $n_{\min} < n < n_{\max}$ (see Section 5.1), are nearly constant indeed in this interval of n-values. The actual value of this constant, along with the extension of the plateau (or range of convergence), is determined by the values of ε and N_0 . If ε is "sufficiently small" and N_0 is "sufficiently large" we have that within this range of convergence $c_n^{(p;\varepsilon,N_0)}(k) \simeq 0$ (see also (38)), which implies $z_p^{(\varepsilon,N_0)}(n) \simeq z_p^{(0,\infty)}(n) \simeq z_p$ (see (64) and (66)). That this is actually the case, and that the estimate $z_p^{(\varepsilon,N_0)}$ obtained from the set of values of $z_p^{(\varepsilon,N_0)}(n)$ within the range of convergence is indeed close to the true value z_p , will be confirmed (or refuted) a posteriori by the study of the sum $\widehat{M}_k^{(p;\varepsilon,N_0)}(m)$ against m (see (88) and Fig. 4d), and by evaluating the mean error $\delta^{(\varepsilon,N_0)}$ defined in (89).

Patterns of the type shown in Fig. 4b, in which a range of convergence (or a plateau) is present and has to be localized, are ubiquitous in our analysis; in practice, the plateau is selected as the most extended range of consecutive *n*-values for which all the values of the plotted quantity lie within a band, whose width $W_{\rm p}$ is a small percentage of its central value (with only numerical roundoff error, i.e, $\varepsilon = \varepsilon_{\rm R}$, the bin-width we used is $W_{\rm p} = 10^{-3}$ % of the central value of the band); for instance, referring to Fig. 4b, we have that $\operatorname{Re} z_{\rm p}^{(\varepsilon,N_0)}(n)$ lies within the range [6.199969 : 6.200031] for $n \in [10 : 426]$, i.e., the length of the range of convergence is $L_{\rm p} = 416$ (only values up to n = 100 are shown). Once the range of convergence has been located, the value of the quantity being considered is chosen to be the sample mean of the values belonging to the range of convergence, while the sample standard deviation is used as an estimate of the uncertainty; in the example of Fig. 4b we obtain $\operatorname{Re} z_{\rm p}^{(\varepsilon,N_0)} = 6.2000005 \pm 1.5 \times 10^{-7}$, which is in excellent agreement with the true value $\operatorname{Re} z_{\rm p} = 6.2$.

A situation very similar to that just discussed for the position of the pole arises from the analysis of the residue $R_{\rm p}$ of $f_2(z)$ in $z_{\rm p}$. After that the location $z_{\rm p}^{(\varepsilon,N_0)}$ of the pole has been computed, use can be made of formula (72) to obtain $R_{\rm p}^{(\varepsilon,N_0)}(n)$ (see Fig. 4c). Even in this case, after some wild oscillations for small values of n, the value of $R_{\rm p}^{(\varepsilon,N_0)}(n)$ stabilizes on a value $R_{\rm p}^{(\varepsilon,N_0)}$, which is very close to the true value $R_{\rm p}$ of the residue (see the figure legend for numerical details).

We can now proceed to test the reliability of our results. The estimates just computed of pole location $z_{\rm p}^{(\varepsilon,N_0)}$ and residue $R_{\rm p}^{(\varepsilon,N_0)}$ can be inserted into the terms $\widehat{c}_{n,k}^{({\rm p};\varepsilon,N_0)}$ (see (87)) in order to compute the sum $\widehat{M}_k^{({\rm p};\varepsilon,N_0)}(m)$ given by formula (88). The plot of the latter, as a function of m, is shown in Fig. 4d for various values of k; the clear presence of the plateaux validates (qualitatively) the correctness of the values of $z_{\rm p}^{(\varepsilon,N_0)}$ and $R_{\rm p}^{(\varepsilon,N_0)}$ (cf. Fig. 4a). It is worth observing how the roundoff noise and the finiteness of N_0 limit superiorly the length of the plateaux, causing, for $m > \widehat{m}^{({\rm p})}(\varepsilon, N_0; k) \simeq 230$, the divergence of $\widehat{M}_k^{({\rm p};\varepsilon,N_0)}(m)$ as a power of m (see (80)).

The values of $z_{\rm p}^{(\varepsilon,N_0)}$ and $R_{\rm p}^{(\varepsilon,N_0)}$ can be further validated by comparing the input samples $(f_2)_N^{(\varepsilon)}$ with the approximate samples $\widehat{f}_k^{({\rm p};\varepsilon,N_0)}$, which can be calculated by means of (86). These reconstructed samples $\widehat{f}_k^{({\rm p};\varepsilon,N_0)}$, shown in Figs. 5a and 5b (filled dots), reproduce extremely well the input data $(f_2)_N^{(\varepsilon_{\rm R})}$, and the small value of the root mean squared error $\delta^{(\varepsilon,N_0)} = 1.19 \times 10^{-3}$ (see Eq. (89)) confirms the great accuracy of the values of pole location and residue just recovered.



FIGURE 6. Analysis of a function with one first order pole in Re z > 0. $N_0 = 60$; $\varepsilon \equiv \varepsilon_{\rm R}$. $f_3(z) = [\cosh(z-z_{\rm p})-1]\{(z+5)^2[\sinh(z-z_{\rm p})-(z-z_{\rm p})]\}^{-1}$. $z_{\rm p} = 9.45\pm0.37i$, $R_{\rm p} = 1.433941 \times 10^{-2} - 7.348186 \times 10^{-4}i$. (a): Recovery of pole location. $z_{\rm p}^{(\varepsilon,N_0)}(n)$ vs. n (see (61)). Length of the ranges of convergence with $W_{\rm p} = 10^{-3}\%$: $L_{\rm p}^{\rm R} = 552$, $L_{\rm p}^{\rm I} = 489$. The computed pole position is: $z_{\rm p}^{(\varepsilon,N_0)} = (9.4500018 \pm 4.7 \times 10^{-6}) +$ $(3.700014 \times 10^{-1} \pm 4.8 \times 10^{-6})i$. (b): Recovery of pole residue. Upper part of the panel: Re $R_{\rm p}^{(\varepsilon,N_0)}(n)$ (multiplied by 10^2) vs. n (see (72)). Lower part of the panel: Im $R_{\rm p}^{(\varepsilon,N_0)}(n)$ (multiplied by 10^4) vs. n. Length of the ranges of convergence with $W_{\rm p} = 10^{-3}\%$: $L_{\rm p}^{\rm R} = 553$, $L_{\rm p}^{\rm I} = 524$. The computed residue is: $R_{\rm p}^{(\varepsilon,N_0)} =$ $(1.4339431 \times 10^{-2} \pm 1.4 \times 10^{-8}) - (7.34952 \times 10^{-4} \pm 1.2 \times 10^{-8})i$. (c) and (d): Real and imaginary parts of the reconstructed samples $\hat{f}_j^{({\rm p};\varepsilon,N_0)}$ (filled dots) of $f_3(x)$ (solid line) at the half-integers x_j , computed by Eq. (86): $\hat{m}_{\rm t}^{({\rm p})}(\varepsilon,N_0;k) = 36$ for every k; $\delta^{(\varepsilon,N_0)} = 1.17 \times 10^{-2}$.

Now that position and residue of the pole are known, we can use the interpolation formula (92) (see also the related formula (17), which holds in the case of an infinite number of noiseless input data) to obtain the values of the meromorphic function for every x > 0. An example of such a calculation is given in Fig. 5c, where the (approximate) values of the meromorphic function $h_{10}(x) \doteq (x - \frac{21}{2})f_2(x)$ (which has a first order pole at $z = z_p$ with residue $(z_p - \frac{21}{2})R_p$), have been computed by



FIGURE 7. Analysis of a function with a first order pole close to a zero. $f_4(z) = (z+10)^{-3} + \eta(z-z_p)^{-1}$. $\eta = 10^{-4}$; $z_p = 5.0$, $R_p = 10^{-4}$; $N_0 = 60$, $\varepsilon \equiv \varepsilon_R$. $W_p = 10^{-2}\%$. The position of the zero is $z_0 = 4.68343$. (a): Upper part of the panel: $z_p^{(\varepsilon,N_0)}(n)$ vs. $n; L_p = 318$. Computed pole position: $z_p^{(\varepsilon,N_0)} = 5.000003 \pm 2.1 \times 10^{-5}$. Lower part of the panel: $R_p^{(\varepsilon,N_0)}(n)$ (multiplied by 10^4) vs. $n; L_p = 295$. Computed residue: $R_p^{(\varepsilon,N_0)} = 9.999946 \times 10^{-5} \pm 6.4 \times 10^{-10}$. (b): Reconstructed samples $\hat{f}_j^{(p;\varepsilon,N_0)}$ (filled dots) of $f_4(x)$ (solid line) at the half-integers x_j , computed by Eq. (86): $\hat{m}_t^{(p)}(\varepsilon, N_0; k) = 82$ for every $k; \delta^{(\varepsilon,N_0)} = 9.91 \times 10^{-3}$.

means of Eq. (92) (indicated by the filled dots only on a regular grid for better visualization) and compared with the true $h_{10}(x)$ (solid line).

Finally, Fig. 5d illustrates the accuracy of the computation of pole location and residue as a function of the number of input data N_0 . It can be seen that the relative error on $z_{\rm p}^{(\varepsilon,N_0)}$ and $R_{\rm p}^{(\varepsilon,N_0)}$ rapidly decreases when the number of input samples increases, being quite small even in the case of a rather limited number of (accurate) input data.

The example discussed so far is typical of an extensive numerical experimentation performed with numerous test functions, which, for brevity, are not reported here. The analysis of an other example is shown in Fig. 6, where we study the function $f_3(z) = \frac{1}{(z+5)^2} \frac{\cosh(z-z_p)-1}{\sinh(z-z_p)-(z-z_p)}$, which features a *non-trivial* first order pole at $z = z_p$, which is what is left by the cancellation of a second order zero in the numerator with a third order zero in the denominator. The analysis of this function follows faithfully the one made previously for the function $f_2(z)$; it is worth remarking in the plots of $z_p^{(\varepsilon,N_0)}(n)$ and $R_p^{(\varepsilon,N_0)}(n)$ versus n (see Figs. 6a,b) the large extent of the interval in which these functions are practically equal to their corresponding true values.

A well-known "defect" affecting the methods for pole recovery which are based on rational approximants, notably the Padé approximant method [3], is the difficulty in handling effectively situations when a pole is very close to a zero of the function. A typical example of such a situation is the case of a function of the type [2, p. 57]: $f(z) = g(z) + \eta(z - z_p)^{-1}$, where g(z) is analytic: if the real parameter η is small, the pole z_p can get close to a zero z_0 of f(z). Instead, the method we propose is, as expected, practically insensitive to this kind of problem. To exemplify our analysis

TABLE 1. Analysis of a function with a first order pole close to a zero. $f_4(z) = (z+10)^{-3} + \eta(z-z_p)^{-1}$. $z_p = 5.0$; $N_0 = 60$, $\varepsilon = 0$. The second column gives the location of the real zero of $f_4(z)$ in Re z > 0, which is close to z_p . The fourth column gives the range of n values within which $z_p^{(\varepsilon,N_0)}(n)$, computed by Eq. (61), differs from the true pole position by less than $W_p = 0.01\%$: i.e., it belongs to [4.99975 : 5.00025] (see also Fig. 7a).

η	Zero location	Zero–pole distance	Range of <i>n</i> values where $z_{p}^{(\varepsilon, N_{0})}(n) \in [4.99975: 5.00025]$	Plateau length $L_{\rm p}$ with $W_{\rm p} = 10^{-2} \%$
10^{-4}	4.68343	0.31657	$n \in [23:364]$	342
10^{-5}	4.96637	0.03363	$n \in [36:327]$	292
10^{-6}	4.99662	0.00338	$n \in [55:296]$	242
10^{-7}	4.99966	0.00034	$n \in [93:257]$	165

we have taken $g(z) = (z + 10)^{-3}$, which is analytic in Re z > 0, and $z_p = 5.0$; the results are summarized in Fig. 7 and Table 1.

In Fig. 7a we see, for $\eta = 10^{-4}$, that both pole position and residue are correctly identified, and that the range over which $z_{\rm p}^{(\varepsilon,N_0)}(n)$ and $R_{\rm p}^{(\varepsilon,N_0)}(n)$ are nearly constant is rather extended $(z_{\rm p}^{(\varepsilon,N_0)}(n)$ is within a band centered around the true value, with $W_{\rm p} = 10^{-2}$ %, for 23 $\leq n \leq 350$). The accuracy of the values of $z_{\rm p}^{(\varepsilon,N_0)}$ and $R_{\rm p}^{(\varepsilon,N_0)}$ obtained from the data shown in Fig. 7a, is then verified in Fig. 7b, which exhibits the excellent reconstruction of the data samples $\hat{f}_k^{({\rm p};\varepsilon,N_0)}$ ($\delta^{(\varepsilon,N_0)} \sim 10^{-2}$) in spite of the great closeness between pole and zero of $f_4(x)$. Table 1 shows that these results hold even for much smaller values of η . For instance, with $\eta = 10^{-7}$ the zero–pole distance becomes tiny ($\sim 3.4 \times 10^{-4}$) but, nevertheless, the location of the pole is recovered correctly (within a 0.01% error) over a very extended range of n–values ($n_{\min}(\varepsilon, N_0; k) = 74$ and $n_{\max}(\varepsilon, N_0; k) = 274$).

Finally, in Fig. 8 we summarize the analysis in the case of noisy input samples $f_N^{(\varepsilon)}$. The input data have been obtained from the noiseless samples f_N by adding white noise uniformly distributed in the interval $[-\varepsilon, \varepsilon]$, in such a way that for all N we have: $|(f_N - f_N^{(\varepsilon)})/f_N| \leq \varepsilon$. For this analysis we used the test function: $f_5(z) = \frac{1}{(z+5)^5} \frac{1}{(z-z_p)}$. Figure 8a shows $\operatorname{Re} z_p^{(\varepsilon,N_0)}(n)$ as a function of n (see Eq. (61)), computed for various values of the noise bound ε . These plots show how the range of n-values over which the value of $\operatorname{Re} z_p^{(\varepsilon,N_0)}(n)$ remains constant (and almost equal to the true value $\operatorname{Re} z_p = 5.2$) gets shorter as the level of noise increases. When ε is varied we have that $n_{\min}(\varepsilon, N_0; k)$ remains nearly constant, whereas (as expected) $n_{\max}(\varepsilon, N_0; k)$ changes quite considerably (e.g., $n_{\max}(10^{-4}, N_0; k) = 65$ and $n_{\max}(10^{-2}, N_0; k) = 20$). This behavior exemplifies the discussion of formula (66) made in Section 5.1: the limits in (66) cannot be interchanged and, as a consequence, when $\varepsilon > 0$ and $N_0 < \infty$ are kept fixed, $z_p^{(\varepsilon,N_0)}(n)$ diverges as $n \to \infty$. The value of n where such a divergence "approximately" sets in, which we denoted by n_{\max} , depends on ε and N_0 : it is finite when $\varepsilon > 0$ and $N_0 < \infty$, and it is such that $\lim_{N_0 \to +\infty} n_{\max}(\varepsilon, N_0) = +\infty$.

However, even in the presence of quite noisy input data, this interval remains extended enough to allow a reliable recovery of $\operatorname{Re} z_{p}$ (see the figure legend for



FIGURE 8. Analysis of a function with one first order pole in Re z > 0 with noisy input samples $f_N^{(\varepsilon)}$: $f_5(z) = (z + 5)^{-5}(z - z_p)^{-1}$. $z_p = 5.2 + 0.2i$, $R_p = 9.005168 \times 10^{-6} - 8.855849 \times 10^{-7}i$; $N_0 = 60$. The noisy function samples $f_N^{(\varepsilon)}$ have been computed as $f_N^{(\varepsilon)} = (1 + \nu_N^{(\varepsilon)})f_N$, $\nu_N^{(\varepsilon)}$ being random variables uniformly distributed in the interval $[-\varepsilon, \varepsilon]$, and f_N are the noiseless function samples. (a): Function Re $z_p^{(\varepsilon,N_0)}(n)$ vs. n, computed by Eq. (61) with various values of noise bound ε . Bin-width $W_p = 10^{-2}\%$. Length of the range of convergence and computed value of Re $z_p^{(\varepsilon,N_0)}$ are: $\varepsilon = \varepsilon_R$ (i.e., only numerical roundoff error): $L_p^R = 399$, Re $z_p^{(\varepsilon,N_0)} = 5.199999 \pm 1.7 \times 10^{-5}$; $\varepsilon = 10^{-4}$: $L_p^R = 70$, Re $z_p^{(\varepsilon,N_0)} = 5.20006 \pm 1.4 \times 10^{-4}$; $\varepsilon = 10^{-3}$: $L_p^R = 41$, Re $z_p^{(\varepsilon,N_0)} = 5.19984 \pm 1.7 \times 10^{-4}$; $\varepsilon = 5 \times 10^{-3}$: $L_p^R = 27$, Re $z_p^{(\varepsilon,N_0)} = 5.20168 \pm 6.3 \times 10^{-4}$; $\varepsilon = 10^{-2}$: $L_p^R = 24$, Re $z_p^{(\varepsilon,N_0)} = 5.19519 \pm 9.7 \times 10^{-4}$; $\varepsilon = 5 \times 10^{-2}$: $L_p^R = 15$, Re $z_p^{(\varepsilon,N_0)} = 5.1951 \pm 5.4 \times 10^{-3}$. (b): Plot of $\widehat{M}_k^{(p;\varepsilon,N_0)}(m)$ vs. m (see (88)), computed with different values of ε ; k = 13. (c): Relative error in the evaluation of pole location and residue versus the noise bound ε . (d): Reconstructed samples $\widehat{f}_j^{(p;\varepsilon,N_0)}$ (filled dots) of $f_5(x)$ (multiplied by 10^4, and plotted with solid line) at the half-integers x_j , computed by Eq. (86); $\varepsilon = 10^{-1}$. $\widehat{m}_t^{(p)}(\varepsilon,N_0;k) = 11$ for every k, $\delta^{(\varepsilon,N_0)} = 1.105$.

numerical details). Correspondingly, in panel (b), the sum $\widehat{M}_{k}^{(\mathbf{p};\varepsilon,N_{0})}(m)$, computed with pole location $z_{\mathbf{p}}^{(\varepsilon,N_{0})}$ and residue $R_{\mathbf{p}}^{(\varepsilon,N_{0})}$ obtained from the analysis of the data in (a), displays the same behavior, that is, shorter plateaux for higher levels of noise.

The accuracy achieved in the computation of the pole parameters is given in Fig. 8c. We see that the relative error remains always quite satisfactory: for instance, the real part of the pole Re $z_{\rm p}$ is computed within nearly 1% when the input data suffer of (at most) a 10% error. Also $R_{\rm p}$ exhibits a similar behavior, though its estimate always results less accurate than that of $z_{\rm p}$. Finally, in Fig. 8d an example of reconstruction of the data samples $\hat{f}_k^{({\rm p};\varepsilon,N_0)}$ (with $\varepsilon = 0.1$) is shown.

7. Concluding remarks and extensions

The method we propose is able to compute estimates of location and residue of a single first order pole of a function meromorphic in Re z > 0 from a finite set of noisy samples taken on a uniform grid of points spaced one unit apart on the real positive semi-axis. Moreover, the degree of approximation of these estimates to the true values $z_{\rm p}$ and $R_{\rm p}$ can be evaluated by computing the relative mean squared error $\delta^{(\varepsilon,N_0)}$.

In conclusion, the following comments and remarks are in order.

(1) A limit of the method we have proposed consists in the fact that the pole parameters cannot be determined if the pole is located outside the range of the data set: e.g., when the pole lies in the half–plane $\operatorname{Re} z < 0$, while the input data $\{f_N^{(\varepsilon)}\}_{N=0}^{N_0}$ are given in the half–plane $\operatorname{Re} z > 0$, or if we have a pole at z_p while the data are given only up to $z = N_0 + \frac{1}{2} < \operatorname{Re} z_p$. In other words, the singularities should lie well inside a region where the function is partially known through the data set $\{f_N^{(\varepsilon)}\}_{N=0}^{N_0}$.

(2) As a typical example of a physical problem which can be properly tackled by our method, it can be considered the following one: suppose that, for various values of the angular momentum ℓ , a finite set of partial–waves a_{ℓ} , at fixed energy, has been determined in a scattering process: i.e., the data set is given by $\{a_{\ell}\}_{\ell=0}^{L_0}$. By means of our method, we can explore whether these partial–waves are the restriction to the integers of a function which is analytic in a certain domain: e.g., in the half– plane Re $\lambda > -\frac{1}{2}$ ($\lambda \in \mathbb{C}; \lambda = \ell + i\nu$). If, instead, some resonances are present in the collision process, location and residue of the pole which represents these resonances can be determined. This analysis is particularly relevant in the inverse scattering problem at fixed energy, especially in the case of Yukawian potentials, whose partial–wave amplitudes are known to satisfy Carlson's bound.

(3) The sampling rate of the input data can be generalized taking as input data the set $\{f_N^{(\alpha)}\}_{N=0}^{\infty}$ made of samples $f_N^{(\alpha)} \doteq f[\alpha(N+\frac{1}{2})]$ $(N \in \mathbb{N}, \alpha > 0)$ of a meromorphic function f(z) taken at equidistant real points, α units apart.

(4) Functions f(z) with a pole of any order higher than unity can be analyzed, and an algorithmic procedure capable to return the pole parameters, i.e., location and Laurent coefficients, can also be given. The case of function f(z) with more than one pole in Re z > 0 can also be considered, and even in this case an algorithmic procedure for recovering location and residue of each pole can be presented. These latter extensions will be the argument of a forthcoming paper.

Appendix

(A) For the reader's convenience, some properties of the Pollaczek polynomials $P_n(y)$ are here briefly summarized [7].

The definition of $P_n(y)$ in terms of Gauss hypergeometric series reads:

(A.1)
$$P_n(y) \doteq P_n^{(1/2)}(y) = i^n {}_2F_1\left(-n, \frac{1}{2} + iy; 1; 2\right).$$

The polynomials $P_n(y)$ satisfy the following recurrence relation:

(A.2)
$$(n+1)P_{n+1}(y) - 2yP_n(y) + nP_{n-1}(y) = 0, P_{-1}(y) = 0, \quad P_0(y) = 1.$$

The polynomials $P_n(y)$ are orthonormal with respect to the weight function $w(y) = \frac{1}{\pi} |\Gamma(\frac{1}{2} + iy)|^2 = (\cosh \pi y)^{-1}$, i.e.:

(A.3)
$$\int_{-\infty}^{+\infty} P_m(y) P_n(y) w(y) dy = \delta_{m,n} \qquad (m, n \in \mathbb{N}).$$

(B) In Ref. [5] (see formula (75)) we have proved the following asymptotic formula for $P_n(z)$, for large values of n (at fixed $z \in \mathbb{C}$):

(A.4)
$$P_n(z) \sim_{n \gg 1} i^n \left[\frac{(2n)^{-iz-1/2}}{\Gamma(\frac{1}{2} - iz)} + (-1)^n \frac{(2n)^{iz-1/2}}{\Gamma(\frac{1}{2} + iz)} \right] \qquad (z \in \mathbb{C} \text{ fixed}).$$

If $\operatorname{Re} z > 0$, it follows that

(A.5)
$$P_n(-iz) \sim_{n \gg 1} \frac{(-i)^n}{\Gamma\left(\frac{1}{2} + z\right)} (2n)^{z-1/2} \qquad (z \in \mathbb{C} \text{ fixed}, \operatorname{Re} z > 0).$$

If in formula (A.5) we put $z = (N + \frac{1}{2})$ $(N \in \mathbb{N})$, we have:

(A.6)
$$P_n\left[-i\left(N+\frac{1}{2}\right)\right] \underset{n\gg 1}{\sim} \frac{(-i)^n}{N!} (2n)^N \qquad (N \in \mathbb{N}).$$

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