# On the order optimality of the regularization via inexact Newton iterations 

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#### Abstract

Inexact Newton regularization methods have been proposed by Hanke and Rieder for solving nonlinear ill-posed inverse problems. Every such a method consists of two components: an outer Newton iteration and an inner scheme providing increments by regularizing local linearized equations. The method is terminated by a discrepancy principle. In this paper we consider the inexact Newton regularization methods with the inner scheme defined by Landweber iteration, the implicit iteration, the asymptotic regularization and Tikhonov regularization. Under certain conditions we obtain the order optimal convergence rate result which improves the suboptimal one of Rieder. We in fact obtain a more general order optimality result by considering these inexact Newton methods in Hilbert scales.


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## 1 Introduction

Inverse problems arise whenever one searches for unknown causes based on observation of their effects. Driven by the requirements from huge amount of practical applications, the field of inverse problems has undergone a tremendous growth. Such problems are usually ill-posed in the sense that their solutions do not depend continuously on the data. In practical applications, one never has exact data, instead only noisy data are available due to errors in the measurements. Even if the deviation is very small, algorithms developed for well-posed problems may fail, since noise could be amplified by an arbitrarily large factor. Therefore, the development of stable methods for solving inverse problems is a central topic.

[^0]In this paper we consider the stable resolution of nonlinear inverse problems which mathematically can be formulated as the nonlinear equations

$$
\begin{equation*}
F(x)=y, \tag{1.1}
\end{equation*}
$$

where $F: \mathcal{D}(F) \subset \mathcal{X} \mapsto \mathcal{Y}$ is a nonlinear Fréchet differentiable operator between two Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$ whose norms and inner products are denoted as $\|\cdot\|$ and $(\cdot, \cdot)$ respectively. We use $F^{\prime}(x)$ to denote the Fréchet derivative of $F$ at $x \in \mathcal{D}(F)$ and use $F^{\prime}(x)^{*}$ to denote the adjoint of $F^{\prime}(x)$. We assume that (1.1) has a solution $x^{\dagger}$ in the domain $\mathcal{D}(F)$ of $F$, i.e. $F\left(x^{\dagger}\right)=y$. Let $y^{\delta}$ be the only available noisy data of $y$ satisfying

$$
\begin{equation*}
\left\|y^{\delta}-y\right\| \leq \delta \tag{1.2}
\end{equation*}
$$

with a given small noise level $\delta>0$. Due to the intrinsic ill-posedness, regularization methods should be employed to produce from $y^{\delta}$ a stable approximate solution of (1.1).

Many regularization methods have been considered in the last two decades. Due to their straightforward implementation and fast convergence property, Newton type regularization methods are attractive for solving nonlinear inverse problems. In [8 we considered a general class of Newton type methods of the form

$$
\begin{equation*}
x_{n+1}=x_{n}+g_{t_{n}}\left(F^{\prime}\left(x_{n}\right)^{*} F^{\prime}\left(x_{n}\right)\right) F^{\prime}\left(x_{n}\right)^{*}\left(y^{\delta}-F\left(x_{n}\right)\right), \tag{1.3}
\end{equation*}
$$

where $x_{0}$ is an initial guess of $x^{\dagger},\left\{t_{n}\right\}$ is a sequence of positive numbers, and $\left\{g_{t}\right\}$ is a family of spectral filter functions. The scheme (1.3) can be derived by applying the linear regularization method defined by $\left\{g_{t}\right\}$ to the linearized equation

$$
F^{\prime}\left(x_{n}\right)\left(x-x_{n}\right)=y^{\delta}-F\left(x_{n}\right)
$$

which follows from (1.1) by replacing $y$ by $y^{\delta}$ and $F(x)$ by its linearization $F\left(x_{n}\right)+F^{\prime}\left(x_{n}\right)\left(x-x_{n}\right)$ at $x_{n}$. When the sequence $\left\{t_{n}\right\}$ is given a priori with suitable property, we showed in [8] that, under the discrepancy principle, the methods are convergent and order optimal. We also considered in [9] the methods in Hilbert scales and obtained the order optimal convergence rates.

In the definition of the Newton type methods (1.3), one may determine the sequence $\left\{t_{n}\right\}$ adaptively during computation. Motivated by the inexact Newton methods in [1 for well-posed problems, Hanke proposed in 4] his regularizing Levenberg-Marquardt scheme for solving nonlinear inverse problems with $\left\{t_{n}\right\}$ chosen to satisfy

$$
\left\|y^{\delta}-F\left(x_{n}\right)-F^{\prime}\left(x_{n}\right)\left(x_{n+1}-x_{n}\right)\right\|=\eta\left\|y^{\delta}-F\left(x_{n}\right)\right\|
$$

at each step for some preassigned number $\eta \in(0,1)$ and with the discrepancy principle used to terminate the iteration. Rieder generalized the idea in [4] and proposed in [12] (see also [10]) a general class of inexact Newton methods; every such a method consists of two components: an outer Newton iteration and an inner scheme providing increment by regularizing local linearized equations. When the inner scheme is defined by an iterative method, the number
of iterations is determined adaptively which has the advantage to avoid the over-solving of the linearized equation that may occur when the inner scheme is terminated a priori. The convergence rates of inexact Newton regularization methods were considered in [13] but only suboptimal ones were derived. It is a longstanding question whether the inexact Newton methods are order optimal. Important progress has been made recently in [5] where the regularizing Levenberg-Marquardt scheme is shown to be order optimal. In this paper we consider the inexact Newton regularization methods in which the inner schemes are defined by applying various linear regularization methods, including Landweber iteration, the implicit iteration, the asymptotic regularization and Tikhonov regularization, to the local linearized equations and show that these methods are indeed order optimal by exploiting ideas developed in [5) 9, 10. We even consider these methods in Hilbert scales and derive the order optimal convergence rates. Our theoretical results confirm those numerical illustrations in 12, 13.

This paper is organized as follows. In Section 2 we formulate the methods precisely and state the main results on the order optimal convergence rates. In Section 3 we show that these methods are well-defined, and prove that the error decays monotonically. In Section 4 we complete the proof of the the main result by deriving the order optimal convergence rates.

## 2 Main results

The inexact Newton regularization methods are a family of methods for solving nonlinear ill-posed inverse problems. Every such a method consists of two components, an outer Newton iteration and an inner scheme providing increments by regularizing local linearized equations. An approximate solution is output by a discrepancy principle.

To be more precise, the method starts with an initial guess $x_{0} \in \mathcal{D}(F)$. Assume that $x_{n}$ is a current iterate, one may apply any regularization scheme to the linearized equation

$$
\begin{equation*}
F^{\prime}\left(x_{n}\right) u=y^{\delta}-F\left(x_{n}\right) \tag{2.1}
\end{equation*}
$$

to produce a family of regularized approximations $\left\{u_{n}(t)\right\}$. One may choose $t_{n}$ to be the smallest number $t_{n}>0$ such that

$$
\begin{equation*}
\left\|y^{\delta}-F\left(x_{n}\right)-F^{\prime}\left(x_{n}\right) u_{n}\left(t_{n}\right)\right\| \leq \eta\left\|y^{\delta}-F\left(x_{n}\right)\right\| \tag{2.2}
\end{equation*}
$$

for some preassigned value $0<\eta<1$. The next iterate is then updated as $x_{n+1}=x_{n}+u_{n}\left(t_{n}\right)$. The outer Newton iteration is terminated by the discrepancy principle

$$
\begin{equation*}
\left\|y^{\delta}-F\left(x_{n_{\delta}}\right)\right\| \leq \tau \delta<\left\|y^{\delta}-F\left(x_{n}\right)\right\|, \quad 0 \leq n<n_{\delta} \tag{2.3}
\end{equation*}
$$

for some given number $\tau>1$. This outputs an integer $n_{\delta}$ and hence $x_{n_{\delta}}$ which is used to approximate the exact solution $x^{\dagger}$.

The convergence rates of the inexact Newton regularization methods have been considered in 12,13. It has been shown that if

$$
x_{0}-x^{\dagger} \in \mathcal{R}\left(\left(F^{\prime}\left(x^{\dagger}\right)^{*} F^{\prime}\left(x^{\dagger}\right)\right)^{\mu}\right)
$$

for some $0<\mu \leq 1 / 2$, then there is a number $0<\mu_{0}<\mu$ such that

$$
\left\|x_{n_{\delta}}-x^{\dagger}\right\|=O\left(\delta^{2\left(\mu-\mu_{0}\right) /(1+2 \mu)}\right)
$$

which is only suboptimal. It is a long-standing question whether the inexact Newton regularization methods are order optimal. Important progress has been made recently in [5] where the regularizing Levenberg-Marquardt scheme is proved to be order optimal.

In this paper we will consider the inexact Newton regularization methods in which the inner schemes are defined by applying Landweber iteration, the implicit iteration, the asymptotic regularization, or Tikhonov regularization to the linearized equation (2.1) and show that these methods are indeed order optimal. For these four methods, $u_{n}(t)$ are defined by

$$
u_{n}(t)=g_{t}\left(F^{\prime}\left(x_{n}\right)^{*} F^{\prime}\left(x_{n}\right)\right) F^{\prime}\left(x_{n}\right)^{*}\left(y^{\delta}-F\left(x_{n}\right)\right)
$$

with the spectral filter functions $\left\{g_{t}\right\}$ given by

$$
\begin{equation*}
g_{t}(\lambda)=\sum_{j=0}^{[t]-1}(1-\lambda)^{j}, \quad \sum_{j=1}^{[t]}(1+\lambda)^{-j}, \quad \frac{1}{\lambda}\left(1-e^{-t \lambda}\right), \quad\left(\frac{1}{t}+\lambda\right)^{-1} \tag{2.4}
\end{equation*}
$$

respectively, where $[t]$ denotes the largest integer not greater than $t$.
We need the following standard condition which is known as the NewtonMysovskii condition (see [2]).

Assumption 2.1 (a) There exists $K_{0} \geq 0$ such that

$$
\left\|\left[F^{\prime}(x)-F^{\prime}(z)\right] h\right\| \leq K_{0}\|x-z\|\left\|F^{\prime}(z) h\right\|, \quad \forall h \in \mathcal{X}
$$

for all $x, z \in B_{\rho}\left(x^{\dagger}\right) \subset \mathcal{D}(F)$, where $B_{\rho}\left(x^{\dagger}\right)$ denotes the ball of radius $\rho>0$ with center at $x^{\dagger}$.
(b) $F$ is properly scaled so that $\left\|F^{\prime}(x)\right\| \leq \Theta<1$ for all $x \in B_{\rho}\left(x^{\dagger}\right)$.

The order optimality of these four inexact Newton regularization methods is contained in the following result.

Theorem 2.1 Let $F$ satisfy Assumption [2.1, let $\tau>2$ and $0<\eta<1$ be such that $\tau \eta>2$, and let $x_{0} \in B_{\rho}\left(x^{\dagger}\right)$. If $K_{0}\left\|x_{0}-x^{\dagger}\right\|$ is sufficiently small, then the inexact Newton regularization methods with the inner scheme defined by Landweber iteration, the implicit iteration, the asymptotic regularization, or Tikhonov regularization are well-defined and terminate after $n_{\delta}=O(1+$ $|\log \delta|)$ iterations. If, in addition, $x_{0}-x^{\dagger}=\left(F^{\prime}\left(x^{\dagger}\right)^{*} F^{\prime}\left(x^{\dagger}\right)\right)^{\mu} \omega$ for some $\omega \in$ $\mathcal{N}\left(F^{\prime}\left(x^{\dagger}\right)\right)^{\perp} \subset \mathcal{X}$ and $0<\mu \leq 1 / 2$ and if $K_{0}\|\omega\|$ is sufficiently small, then there holds

$$
\left\|x_{n_{\delta}}-x^{\dagger}\right\| \leq C\|\omega\|^{\frac{1}{1+2 \mu}} \delta^{\frac{2 \mu}{1+2 \mu}}
$$

for some constant $C$ independent of $\delta$ and $\|\omega\|$.

We will not give the proof of Theorem 2.1 directly. Instead, we will prove a more general result by considering these four inexact Newton regularization methods in Hilbert scales. Let $L$ be a densely defined self-adjoint strictly positive linear operator in $\mathcal{X}$ satisfying

$$
\|x\|^{2} \leq \gamma(L x, x), \quad x \in \mathcal{D}(L)
$$

for some constant $\gamma>0$, where $\mathcal{D}(L)$ denotes the domain of $L$. For each $t \in \mathbb{R}$, we define $\mathcal{X}_{t}$ to be the completion of $\cap_{k=0}^{\infty} \mathcal{D}\left(L^{k}\right)$ with respect to the Hilbert space norm

$$
\|x\|_{t}:=\left\|L^{t} x\right\| .
$$

This family of Hilbert spaces $\left\{\mathcal{X}_{t}\right\}_{t \in \mathbb{R}}$ is called the Hilbert scales generated by $L$. The following are fundamental properties (see [3]):
(a) For any $-\infty<q<r<\infty, \mathcal{X}_{r}$ is densely and continuously embedded into $\mathcal{X}_{q}$ with

$$
\begin{equation*}
\|x\|_{q} \leq \gamma^{r-q}\|x\|_{r}, \quad x \in \mathcal{X}_{r} \tag{2.5}
\end{equation*}
$$

(b) For any $-\infty<p<q<r<\infty$ there holds the interpolation inequality

$$
\begin{equation*}
\|x\|_{q} \leq\|x\|_{p}^{\frac{r-q}{r-p}}\|x\|_{r}^{\frac{q-p}{r-p}}, \quad x \in \mathcal{X}_{r} . \tag{2.6}
\end{equation*}
$$

(c) If $T: \mathcal{X} \mapsto \mathcal{Y}$ is a bounded linear operator satisfying

$$
m\|h\|_{-a} \leq\|T h\| \leq M\|h\|_{-a}, \quad h \in \mathcal{X}
$$

for some constants $M \geq m>0$ and $a \geq 0$, then for the operator $A:=T L^{-s}$ : $\mathcal{X} \mapsto \mathcal{Y}$ with $s \geq-a$ there holds for any $|\nu| \leq 1$ that

$$
\begin{equation*}
\underline{c}(\nu)\|h\|_{-\nu(a+s)} \leq\left\|\left(A^{*} A\right)^{\nu / 2} h\right\| \leq \bar{c}(\nu)\|h\|_{-\nu(a+s)} \tag{2.7}
\end{equation*}
$$

on $\mathcal{D}\left(\left(A^{*} A\right)^{\nu / 2}\right)$, where $A^{*}:=L^{-s} T^{*}: \mathcal{Y} \rightarrow \mathcal{X}$ is the adjoint of $A$ and

$$
\underline{c}(\nu):=\min \left\{m^{\nu}, M^{\nu}\right\} \quad \text { and } \quad \bar{c}(\nu)=\max \left\{m^{\nu}, M^{\nu}\right\}
$$

We will consider the inexact Newton regularization methods in which the inner schemes are defined by applying Landweber iteration, the implicit iteration, the asymptotic regularization, or Tikhonov regularization in Hilbert scales to the linearized equation (2.1). Now we have

$$
\begin{equation*}
u_{n}(t)=g_{t}\left(L^{-2 s} F^{\prime}\left(x_{n}\right)^{*} F^{\prime}\left(x_{n}\right)\right) L^{-2 s} F^{\prime}\left(x_{n}\right)^{*}\left(y^{\delta}-F\left(x_{n}\right)\right) \tag{2.8}
\end{equation*}
$$

with $g_{t}$ defined by (2.4), where $s \in \mathbb{R}$ is a suitable chosen number. The iterative solutions are defined by $x_{n+1}=x_{n}+u_{n}\left(t_{n}\right)$ with $t_{n}>0$ chosen to be the smallest number satisfying (2.2). The iteration is then terminated by the discrepancy principle (2.3) to output an approximate solution $x_{n_{\delta}}$.

We will use $x_{n_{\delta}}$, constructed from these four inexact Newton regularization methods in Hilbert scales, to approximate the true solution $x^{\dagger}$ of (1.1) and derive the order optimal convergence rate when $x_{0}-x^{\dagger} \in \mathcal{X}_{\mu}$ with $s<\mu \leq$ $b+2 s$. We need the following condition on the nonlinear operator $F$.

Assumption 2.2 (a) There exist constants $a \geq 0$ and $0<m \leq M<\infty$ such that

$$
m\|h\|_{-a} \leq\left\|F^{\prime}(x) h\right\| \leq M\|h\|_{-a}, \quad h \in \mathcal{X}
$$

for all $x \in B_{\rho}\left(x^{\dagger}\right)$.
(b) $F$ is properly scaled so that $\left\|F^{\prime}(x) L^{-s}\right\|_{\mathcal{X} \rightarrow \mathcal{Y}} \leq \Theta<1$ for all $x \in$ $B_{\rho}\left(x^{\dagger}\right)$, where $s \geq-a$.
(c) There exist $0<\beta \leq 1,0 \leq b \leq a$ and $K_{0} \geq 0$ such that

$$
\left\|F^{\prime}(x)-F^{\prime}(z)\right\|_{\mathcal{X}_{-b} \rightarrow \mathcal{Y}} \leq K_{0}\|x-z\|^{\beta}
$$

for all $x, z \in B_{\rho}\left(x^{\dagger}\right)$.
This condition was first used in [11 for the convergence analysis of the nonlinear Landweber iteration in Hilbert scales. It was then used recently in [7] and [9] for nonlinear Tikhonov regularization and some Newton-type regularization methods in Hilbert scales respectively. One can consult [11,7 for several examples satisfying Assumption 2.2

Theorem 2.2 Let $F$ satisfy Assumption 2.2 with $s \geq(a-b) / \beta$, let $\tau>2$ and $0<\eta<1$ be such that $\tau \eta>2$, and let $x_{0} \in \mathcal{D}(F)$ be such that $\gamma^{s}\left\|x_{0}-x^{\dagger}\right\|_{s} \leq \rho$. If $K_{0}\left\|x_{0}-x^{\dagger}\right\|_{s}^{\beta}$ is sufficiently small, then the inexact Newton regularization methods with the inner scheme defined by Landweber iteration, the implicit iteration, the asymptotic regularization, or Tikhonov regularization in Hilbert scales are well-defined and terminate after $n_{\delta}=O(1+|\log \delta|)$ iterations. If, in addition, $x_{0}-x^{\dagger} \in \mathcal{X}_{\mu}$ for some $s<\mu \leq b+2 s$ and $K_{0}\left\|x_{0}-x^{\dagger}\right\|_{\mu}^{\beta}$ is sufficiently small, then there holds

$$
\left\|x_{n_{\delta}}-x^{\dagger}\right\|_{r} \leq C\left\|x_{0}-x^{\dagger}\right\|_{\mu}^{\frac{a+r}{a+\mu}} \delta^{\frac{\mu-r}{a+\mu}}
$$

for all $r \in[-a, s]$, where $C$ is a constant independent of $\delta$ and $\left\|x_{0}-x^{\dagger}\right\|_{\mu}$.
The proof of Theorem 2.2 will be given in the next two sections. Here some remarks are in order.

Remark 2.1 When the inner scheme is defined by the asymptotic regularization or Tikhonov regularization, there is flexibility to choose $t_{n}$ to satisfy

$$
\eta_{1}\left\|y^{\delta}-F\left(x_{n}\right)\right\| \leq\left\|y^{\delta}-F\left(x_{n}\right)-F^{\prime}\left(x_{n}\right) u_{n}\left(t_{n}\right)\right\| \leq \eta_{2}\left\|y^{\delta}-F\left(x_{n}\right)\right\|
$$

with some numbers $0<\eta_{1} \leq \eta_{2}<1$. Furthermore, we only need $\tau>2$ and $\tau \eta_{1}>1$ in the convergence analysis.
Remark 2.2 When $s>(a-b) / \beta$, the same order optimal convergence rate in Theorem 2.2 holds for $x_{0}-x^{\dagger} \in \mathcal{X}_{\mu}$ with $s \leq \mu \leq b+2 s$ which can be seen from the proof of Lemma 4.4 in Section 4.

Remark 2.3 If the Fréchet derivative $F^{\prime}(x)$ satisfies the Lipschitz condition

$$
\left\|F^{\prime}(x)-F^{\prime}(z)\right\| \leq K_{0}\|x-z\|, \quad x, z \in B_{\rho}\left(x^{\dagger}\right)
$$

then Assumption 2.2 (c) holds with $b=0$ and $\beta=1$, and thus, for these inexact Newton regularization methods in Hilbert scales with $s \geq a$, the order optimal convergence rates hold for $x_{0}-x^{\dagger} \in \mathcal{X}_{\mu}$ with $s<\mu \leq 2 s$.

Remark 2.4 We indicate how Theorem 2.1 can be derived from Theorem 2.2 First, we note that Assumption 2.1 (a) implies

$$
\left\|F(x)-F(z)-F^{\prime}(z)(x-z)\right\| \leq \frac{1}{2} K_{0}\|x-z\|\left\|F^{\prime}(z)(x-z)\right\|
$$

for all $x, z \in B_{\rho}\left(x^{\dagger}\right)$. One can then follow the proofs in Section 3 to show that, if $x_{0} \in B_{\rho}\left(x^{\dagger}\right)$ and $K_{0}\left\|x_{0}-x^{\dagger}\right\|$ is sufficiently small, then these inexact Newton regularization methods are well-defined and

$$
\left\|x_{n+1}-x^{\dagger}\right\| \leq\left\|x_{n}-x^{\dagger}\right\|, \quad n=0, \cdots, n_{\delta}-1
$$

which implies $x_{n} \in B_{\rho}\left(x^{\dagger}\right)$ for $0 \leq n \leq n_{\delta}$. By shrinking the ball $B_{\rho}\left(x^{\dagger}\right)$ if necessary, we can derive from Assumption 2.1 (a) that there exist two constants $0<C_{0} \leq C_{1}<\infty$ such that

$$
\begin{equation*}
C_{0}\left\|F^{\prime}(z) h\right\| \leq\left\|F^{\prime}(x) h\right\| \leq C_{1}\left\|F^{\prime}(z) h\right\|, \quad h \in \mathcal{X} \tag{2.9}
\end{equation*}
$$

for all $x, z \in B_{\rho}\left(x^{\dagger}\right)$. This implies that all the operators $F^{\prime}(x)$ have the same null space $\mathcal{N}$ as long as $x \in B_{\rho}\left(x^{\dagger}\right)$. By the condition of Theorem 2.1] we have $x_{0}-x^{\dagger} \in \mathcal{N}^{\perp}$. By the definition of $\left\{x_{n}\right\}$ we also have $x_{n+1}-x_{n} \in \mathcal{R}\left(F^{\prime}\left(x_{n}\right)^{*}\right) \subset$ $\mathcal{N}^{\perp}$ for $n=0, \cdots, n_{\delta}-1$. By considering the operator $G(z):=F\left(z+x_{0}\right)$ if necessary, we may assume $x_{0}=0$. Therefore $x^{\dagger}, x_{n} \in \mathcal{N}^{\perp}$ for $n=0, \cdots, n_{\delta}$, and we may consider the equation (1.1) on $\mathcal{N}^{\perp}$. Consequently we may assume $\mathcal{N}=\{0\}$, i.e. each $F^{\prime}(x)$ is injective for $x \in B_{\rho}\left(x^{\dagger}\right)$.

Now we introduce the operator $L:=\left(F^{\prime}\left(x^{\dagger}\right)^{*} F^{\prime}\left(x^{\dagger}\right)\right)^{-1 / 2}$ which is clearly densely defined self-adjoint strictly positive linear operator in $\mathcal{X}$ satisfying

$$
\|x\|^{2} \leq \Theta(L x, x), \quad x \in \mathcal{D}(L) .
$$

From (2.9) it follows that $C_{0}\|h\|_{-1} \leq\left\|F^{\prime}(x) h\right\| \leq C_{1}\|h\|_{-1}$ which implies Assumption 2.2 (a) with $a=1$. Moreover, from Assumption 2.1 (b) it follows for $x, z \in B_{\rho}\left(x^{\dagger}\right)$ that

$$
\left\|\left[F^{\prime}(x)-F^{\prime}(z)\right]\right\|_{\mathcal{X}_{-1} \rightarrow \mathcal{Y}}=\left\|\left[F^{\prime}(x)-F^{\prime}(z)\right] L\right\|_{\mathcal{X} \rightarrow \mathcal{Y}} \leq K_{0}\|x-z\|\left\|F^{\prime}(z) L\right\|_{\mathcal{X} \rightarrow \mathcal{Y}} .
$$

Since (2.9) implies $\left\|F^{\prime}(z) L\right\|_{\mathcal{X} \rightarrow \mathcal{Y}} \leq C_{1}$, Assumption 2.2 (c) holds with $b=1$ and $\beta=1$. Since $\mathcal{R}\left(\left(F^{\prime}\left(x^{\dagger}\right) F^{\prime}\left(x^{\dagger}\right)\right)^{\mu}\right)=\mathcal{X}_{2 \mu}$, Theorem[2.1ffollows immediately from Theorem 2.2 with $s=0$.

## 3 Monotonicity of the error

We start with a simple consequence of Assumption 2.2 which will be used frequently.
Lemma 3.1 Let $F$ satisfy Assumption 2.2 and let $x, z \in B_{\rho}\left(x^{\dagger}\right)$. If $t \geq 0$ then

$$
\begin{equation*}
\left\|F(x)-F(z)-F^{\prime}(z)(x-z)\right\| \leq \frac{1}{1+\beta} K_{0}\|x-z\|_{t}^{\frac{a(1+\beta)-b}{a+t}}\|x-z\|_{-a^{\frac{t(1+\beta)+b}{a+t}}} \tag{3.1}
\end{equation*}
$$

If, in addition, $t \geq(a-b) / \beta$, then

$$
\begin{equation*}
\left\|F(x)-F(z)-F^{\prime}(z)(x-z)\right\| \leq \frac{1}{1+\beta} \gamma^{t \beta+b-a} K_{0}\|x-z\|_{t}^{\beta}\|x-z\|_{-a} \tag{3.2}
\end{equation*}
$$

Proof From Assumption 2.2 (c) and the identity

$$
F(x)-F(z)-F^{\prime}(z)(x-z)=\int_{0}^{1}\left[F^{\prime}(z+t(x-z))-F^{\prime}(z)\right](x-z) d t
$$

it follows immediately that

$$
\begin{equation*}
\left\|F(x)-F(z)-F^{\prime}(z)(x-z)\right\| \leq \frac{1}{1+\beta} K_{0}\|x-z\|^{\beta}\|x-z\|_{-b} \tag{3.3}
\end{equation*}
$$

With the help of the interpolation inequality (2.6) we have

$$
\|x-z\| \leq\|x-z\|_{t}^{\frac{a}{a+t}}\|x-z\|_{-a}^{\frac{t}{a+t}} \quad \text { and } \quad\|x-z\|_{-b} \leq\|x-z\|_{t}^{\frac{a-b}{a+t}}\|x-z\|_{-a}^{\frac{t+b}{a+t}}
$$

This together with (3.3) gives (3.1). If, in addition, $t \geq(a-b) / \beta$, then we have $[t(1+\beta)+b] /(a+t) \geq 1$. Thus, by using $\|x-z\|_{-a} \leq \gamma^{a+t}\|x-z\|_{t}$ which follows from the embedding (2.5), we can derive (3.2) immediately from (3.1).

In this section we will use the ideas from [4,6,10 to show that the four inexact Newton regularization methods in Hilbert scales stated in Theorem [2.2 are well-defined and for the error term

$$
e_{n}:=x_{n}-x^{\dagger}
$$

there holds $\left\|e_{n+1}\right\|_{s} \leq\left\|e_{n}\right\|_{s}$ for $n=0, \cdots, n_{\delta}-1$. We will use the notation

$$
T:=F^{\prime}\left(x^{\dagger}\right), \quad T_{n}:=F^{\prime}\left(x_{n}\right), \quad A:=T L^{-s} \quad \text { and } \quad A_{n}:=T_{n} L^{-s}
$$

It follows easily from the definition (2.8) of $\left\{u_{n}(t)\right\}$ that

$$
\begin{equation*}
u_{n}(t)=L^{-s} g_{t}\left(A_{n}^{*} A_{n}\right) A_{n}^{*}\left(y^{\delta}-F\left(x_{n}\right)\right) \tag{3.4}
\end{equation*}
$$

and

$$
\begin{equation*}
y^{\delta}-F\left(x_{n}\right)-T_{n} u_{n}(t)=r_{t}\left(A_{n} A_{n}^{*}\right)\left(y^{\delta}-F\left(x_{n}\right)\right), \tag{3.5}
\end{equation*}
$$

where $r_{t}(\lambda):=1-\lambda g_{t}(\lambda)$ denotes the residual function associated with $g_{t}$. For the spectral filter functions given in (2.4), it is easy to see that $\lim _{t \rightarrow \infty} r_{t}(\lambda)=$ 0 for each $\lambda>0$. This implies that

$$
\begin{equation*}
\lim _{t \rightarrow \infty}\left\|y^{\delta}-F\left(x_{n}\right)-T_{n} u_{n}(t)\right\|=\left\|P_{\mathcal{R}\left(A_{n}\right)^{\perp}}\left(y^{\delta}-F\left(x_{n}\right)\right)\right\|, \tag{3.6}
\end{equation*}
$$

where $P_{\mathcal{R}\left(A_{n}\right) \perp}$ denotes the orthogonal projection of $\mathcal{Y}$ onto $\mathcal{R}\left(A_{n}\right)^{\perp}$, the orthogonal complement of the range $\mathcal{R}\left(A_{n}\right)$ of $A_{n}$.

Lemma 3.2 Let $F$ satsify Assumption 2.2 with $s \geq(a-b) / \beta$, let $\tau>1$ and $0<\eta<1$ satisfy $\tau \eta>1$, and let $x_{0} \in \mathcal{D}(F)$ be such that $\gamma^{s}\left\|e_{0}\right\|_{s} \leq \rho$. Assume that $K_{0}\left\|e_{0}\right\|_{s}^{\beta}$ is sufficiently small. If $\left\|y^{\delta}-F\left(x_{n}\right)\right\|>\tau \delta$ and $\left\|e_{n}\right\|_{s} \leq\left\|e_{0}\right\|_{s}$, then $t_{n}$ is well-defined and $t_{n} \geq c_{0}$ for some constant $c_{0}>0$ independent of $n$ and $\delta$.

Proof From (2.5) and the given conditions it follows that $\left\|e_{n}\right\| \leq \gamma^{s}\left\|e_{n}\right\|_{s} \leq$ $\gamma^{s}\left\|e_{0}\right\|_{s} \leq \rho$ which implies $x_{n} \in B_{\rho}\left(x^{\dagger}\right)$. Since $\left\|e_{n}\right\|_{s} \leq\left\|e_{0}\right\|_{s}<\infty$ implies $L^{s} e_{n} \in \mathcal{X}$, we have

$$
\left\|P_{\mathcal{R}\left(A_{n}\right)^{\perp}}\left(y^{\delta}-F\left(x_{n}\right)\right)\right\| \leq\left\|y^{\delta}-F\left(x_{n}\right)+A_{n} L^{s} e_{n}\right\|=\left\|y^{\delta}-F\left(x_{n}\right)+T_{n} e_{n}\right\| .
$$

In order to show that $t_{n}$ is well-defined, in view of (3.6) it suffices to show

$$
\begin{equation*}
\left\|y^{\delta}-F\left(x_{n}\right)+T_{n} e_{n}\right\|<\eta\left\|y^{\delta}-F\left(x_{n}\right)\right\| \tag{3.7}
\end{equation*}
$$

Since $s \geq(a-b) / \beta$, we can use (1.2) and (3.2) in Lemma 3.1 to derive

$$
\left\|y^{\delta}-F\left(x_{n}\right)+T_{n} e_{n}\right\| \leq \delta+\frac{1}{1+\beta} \gamma^{s \beta+b-a} K_{0}\left\|e_{n}\right\|_{s}^{\beta}\left\|e_{n}\right\|_{-a}
$$

Now by using Assumption 2.2 (a), $\left\|e_{n}\right\|_{s} \leq\left\|e_{0}\right\|_{s}$ and $\tau \delta<\left\|y^{\delta}-F\left(x_{n}\right)\right\|$, we obtain with $C=\gamma^{s \beta+b-a} /[(1+\beta) m]$ that

$$
\begin{aligned}
\left\|y^{\delta}-F\left(x_{n}\right)+T_{n} e_{n}\right\| \leq & \frac{1}{\tau}\left\|y^{\delta}-F\left(x_{n}\right)\right\|+C K_{0}\left\|e_{0}\right\|_{s}^{\beta}\left\|T_{n} e_{n}\right\| \\
\leq & \left(\frac{1}{\tau}+C K_{0}\left\|e_{0}\right\|_{s}^{\beta}\right)\left\|y^{\delta}-F\left(x_{n}\right)\right\| \\
& +C K_{0}\left\|e_{0}\right\|_{s}^{\beta}\left\|y^{\delta}-F\left(x_{n}\right)+T_{n} e_{n}\right\| .
\end{aligned}
$$

Since $\tau \eta>1$, we therefore obtain (3.7) if $K_{0}\left\|e_{0}\right\|_{s}$ is sufficiently small.
For the inner scheme defined by Landweber iteration or the implicit iteration in Hilbert scales, it is obvious that $t_{n}$ is an integer with $t_{n} \geq 1$. For the inner scheme defined by the asymptotic regularization or Tikhonov regularization in Hilbert scales, we have

$$
\eta\left\|y^{\delta}-F\left(x_{n}\right)\right\|=\left\|y^{\delta}-F\left(x_{n}\right)-T_{n} u_{n}\left(t_{n}\right)\right\|=\left\|r_{t_{n}}\left(A_{n} A_{n}^{*}\right)\left(y^{\delta}-F\left(x_{n}\right)\right)\right\|
$$

where $r_{t}(\lambda)=e^{-t \lambda}$ or $r_{t}(\lambda)=(1+t \lambda)^{-1}$. Since $\left\|A_{n}\right\| \leq 1$, we can obtain either $e^{-t_{n}} \leq \eta$ or $\left(1+t_{n}\right)^{-1} \leq \eta$. Therefore $t_{n} \geq \log (1 / \eta)$ or $t_{n} \geq 1 / \eta-1$.

Lemma 3.3 Let $F$ satisfy Assumption 2.2 with $s \geq(a-b) / \beta$, let $\tau>2$ and $0<\eta<1$ be such that $\tau \eta>2$, and let $x_{0} \in \mathcal{D}(F)$ be such that $\gamma^{s}\left\|e_{0}\right\|_{s} \leq \rho$. If $K_{0}\left\|e_{0}\right\|_{s}^{\beta}$ is sufficiently small, then the four inexact Newton regularization methods in Hilbert scales stated in Theorem 2.2 are well-defined and terminate after $n_{\delta}<\infty$ iterations, and

$$
\begin{equation*}
\sum_{n=0}^{n_{\delta}-1} t_{n}\left\|y^{\delta}-F\left(x_{n}\right)\right\|^{2} \leq C_{2}\left\|e_{0}\right\|_{s}^{2} \tag{3.8}
\end{equation*}
$$

for some constant $C_{2}>0$. Moreover

$$
\begin{equation*}
\left\|x_{n+1}-x^{\dagger}\right\|_{s} \leq\left\|x_{n}-x^{\dagger}\right\|_{s} \tag{3.9}
\end{equation*}
$$

for $n=0, \cdots, n_{\delta}-1$.

Proof We will prove this result for the four inexact Newton methods case by case.
(a) We first consider the inexact Newton method with inner scheme defined by Landweber iteration in Hilber scales. We first show the monotonicity (3.9). We may assume $n_{\delta} \geq 1$. Let $0 \leq n<n_{\delta}$ and assume that $\left\|e_{n}\right\|_{s} \leq\left\|e_{0}\right\|_{s}$. By the definition of $n_{\delta}$ we have $\left\|y^{\delta}-F\left(x_{n}\right)\right\|>\tau \delta$. It follows from Lemma 3.2 that $t_{n}$ is a well-defined positive integer. Let $u_{n, k}:=u_{n}(k)$ for each integer $k$. Then $u_{n, 0}=0$ and

$$
u_{n, k}=u_{n, k-1}+L^{-2 s} T_{n}^{*}\left(y^{\delta}-F\left(x_{n}\right)-T_{n} u_{n, k-1}\right)
$$

for $k=1, \cdots, t_{n}$. Recall that $x_{n+1}=x_{n}+u_{n, t_{n}}$. Therefore, in order to show $\left\|e_{n+1}\right\|_{s} \leq\left\|e_{n}\right\|_{s}$, it suffices to show

$$
\begin{equation*}
\left\|e_{n}+u_{n, k}\right\|_{s} \leq\left\|e_{n}+u_{n, k-1}\right\|_{s}, \quad k=1, \cdots, t_{n} \tag{3.10}
\end{equation*}
$$

We set $z_{n, k}=y^{\delta}-F\left(x_{n}\right)-T_{n} u_{n, k}$. Then $u_{n, k}-u_{n, k-1}=L^{-2 s} T_{n}^{*} z_{n, k-1}$ and thus

$$
\begin{aligned}
\| e_{n}+u_{n, k} & \left\|_{s}^{2}-\right\| e_{n}+u_{n, k-1} \|_{s}^{2} \\
& =2\left(e_{n}+u_{n, k-1}, u_{n, k}-u_{n, k-1}\right)_{s}+\left\|u_{n, k}-u_{n, k-1}\right\|_{s}^{2} \\
& =\left(u_{n, k}-u_{n, k-1}, u_{n, k}+u_{n, k-1}+2 e_{n}\right)_{s} \\
& =\left(z_{n, k-1}, T_{n}\left(u_{n, k}+u_{n, k-1}+2 e_{n}\right)\right) .
\end{aligned}
$$

According to the definition of $z_{n, k}$ one can see

$$
T_{n}\left(u_{n, k}+u_{n, k-1}+2 e_{n}\right)=-z_{n, k}-z_{n, k-1}+2\left(y^{\delta}-F\left(x_{n}\right)+T_{n} e_{n}\right) .
$$

Therefore

$$
\begin{aligned}
& \left\|e_{n}+u_{n, k}\right\|_{s}^{2}-\left\|e_{n}+u_{n, k-1}\right\|_{s}^{2} \\
& \quad=-\left(z_{n, k-1}, z_{n, k}\right)-\left\|z_{n, k-1}\right\|^{2}+2\left(z_{n, k-1}, y^{\delta}-F\left(x_{n}\right)+T_{n} e_{n}\right)
\end{aligned}
$$

Observing that (3.5) and $r_{t}(\lambda)=(1-\lambda)^{[t]}$ imply $z_{n, k}=\left(I-A_{n} A_{n}^{*}\right)^{k}\left(y^{\delta}-\right.$ $F\left(x_{n}\right)$ ), we have $\left(z_{n, k-1}, z_{n, k}\right) \geq 0$. Hence

$$
\begin{aligned}
& \left\|e_{n}+u_{n, k}\right\|_{s}^{2}-\left\|e_{n}+u_{n, k-1}\right\|_{s}^{2} \\
& \quad \leq-\left\|z_{n, k-1}\right\|\left(\left\|z_{n, k-1}\right\|-2\left\|y^{\delta}-F\left(x_{n}\right)+T_{n} e_{n}\right\|\right)
\end{aligned}
$$

Since $\tau \eta>2$, we can pick $0<\eta_{0}<\eta / 2$ with $\tau \eta_{0}>1$. By using Assumption [2.2, $\tau \delta<\left\|y^{\delta}-F\left(x_{n}\right)\right\|$ and $\left\|e_{n}\right\|_{s} \leq\left\|e_{0}\right\|_{s}$, we can derive as in the proof of Lemma 3.2 that if $K_{0}\left\|e_{0}\right\|_{s}^{\beta}$ is sufficiently small then

$$
\left\|y^{\delta}-F\left(x_{n}\right)+T_{n} e_{n}\right\| \leq \eta_{0}\left\|y^{\delta}-F\left(x_{n}\right)\right\|
$$

On the other hand, by the definition of $t_{n}$ we have $\left\|z_{n, k-1}\right\|>\eta\left\|y^{\delta}-F\left(x_{n}\right)\right\|$. Therefore

$$
\begin{equation*}
\left\|e_{n}+u_{n, k}\right\|_{s}^{2}-\left\|e_{n}+u_{n, k-1}\right\|_{s}^{2} \leq-\varepsilon_{0}\left\|y^{\delta}-F\left(x_{n}\right)\right\|^{2} \tag{3.11}
\end{equation*}
$$

where $\varepsilon_{0}:=\eta\left(\eta-2 \eta_{0}\right)>0$. This in particular implies (3.10) and hence $\left\|e_{n+1}\right\|_{s} \leq\left\|e_{n}\right\|_{s}$. An induction argument then shows the monotonicity result (3.9).

Moreover, it follows from (3.11) that

$$
\begin{aligned}
\left\|e_{n+1}\right\|_{s}^{2}-\left\|e_{n}\right\|_{s}^{2} & =\sum_{k=1}^{t_{n}}\left(\left\|e_{n}+u_{n, k}\right\|_{s}^{2}-\left\|e_{n}+u_{n, k-1}\right\|_{s}^{2}\right) \\
& \leq-\varepsilon_{0} t_{n}\left\|y^{\delta}-F\left(x_{n}\right)\right\|^{2} .
\end{aligned}
$$

Consequently

$$
\varepsilon_{0} \sum_{n=0}^{n_{\delta}-1} t_{n}\left\|y^{\delta}-F\left(x_{n}\right)\right\|^{2} \leq\left\|e_{0}\right\|_{s}^{2}-\left\|e_{n_{\delta}}\right\|_{s}^{2} \leq\left\|e_{0}\right\|_{s}^{2}<\infty
$$

which shows (3.8). Since $t_{n} \geq 1$ and $\left\|y^{\delta}-F\left(x_{n}\right)\right\|>\tau \delta$ for $0 \leq n<n_{\delta}$, one can see that $n_{\delta}$ must be finite.
(b) For the inexact Newton method with inner scheme defined by the implicit iteration in Hilbert scales, all $t_{n}$ must be positive integer and with the notation $u_{n, k}:=u_{n}(k)$ we have $u_{n, 0}=0$ and

$$
u_{n, k}=u_{n, k-1}+\left(L^{2 s}+T_{n}^{*} T_{n}\right)^{-1} T_{n}^{*}\left(y^{\delta}-F\left(x_{n}\right)-T_{n} u_{n, k-1}\right)
$$

Let $z_{n, k}:=y^{\delta}-F\left(x_{n}\right)-T_{n} u_{n, k}$. We have from (3.5) and $r_{t}(\lambda)=(1+\lambda)^{-[t]}$ that $z_{n, k}=\left(I+A_{n} A_{n}^{*}\right)^{-1} z_{n, k-1}$ and $u_{n, k}-u_{n, k-1}=L^{-2 s} T_{n}^{*} z_{n, k}$. Thus

$$
\begin{aligned}
&\left\|e_{n}+u_{n, k}\right\|_{s}^{2}-\left\|e_{n}+u_{n, k-1}\right\|_{s}^{2} \\
&=\left(u_{n, k}-u_{n, k-1}, u_{n, k}+u_{n, k-1}+2 e_{n}\right)_{s} \\
&=\left(z_{n, k}, T_{n}\left(u_{n, k}+u_{n, k-1}+2 e_{n}\right)\right) \\
&=\left(z_{n, k},-z_{n, k}-z_{n, k-1}+2\left(y^{\delta}-F\left(x_{n}\right)+T_{n} e_{n}\right)\right) .
\end{aligned}
$$

Note that $\left(z_{n, k}, z_{n, k-1}\right) \geq\left\|z_{n, k}\right\|^{2}$. We then obtain

$$
\left\|e_{n}+u_{n, k}\right\|_{s}^{2}-\left\|e_{n}+u_{n, k-1}\right\|_{s}^{2} \leq-2\left\|z_{n, k}\right\|\left(\left\|z_{n, k}\right\|-\left\|y^{\delta}-F\left(x_{n}\right)+T_{n} e_{n}\right\|\right)
$$

By using $\left\|A_{n}\right\| \leq 1$ and the definition of $t_{n}$, we have

$$
\left\|z_{n, k}\right\| \geq \frac{1}{2}\left\|z_{n, k-1}\right\| \geq \frac{1}{2} \eta\left\|y^{\delta}-F\left(x_{n}\right)\right\|, \quad k=1, \cdots, t_{n}
$$

Since $\tau \eta>2$, we can obtain

$$
\left\|e_{n}+u_{n, k}\right\|_{s}^{2}-\left\|e_{n}+u_{n, k-1}\right\|_{s}^{2} \leq-\frac{1}{2} \eta\left(\eta-2 \eta_{0}\right)\left\|y^{\delta}-F\left(x_{n}\right)\right\|^{2}
$$

for $k=1, \cdots, t_{n}$ when $K_{0}\left\|e_{0}\right\|_{s}$ is sufficiently small, where $0<\eta_{0}<\eta / 2$ is such that $\tau \eta_{0}>1$. This together with an induction argument implies (3.8) and (3.9).
(c) For the inexact Newton method with inner scheme defined by the asymptotic regularization in Hilbert scales, $u_{n}(t)$ is the solution of the initial value problem

$$
\begin{aligned}
\frac{d}{d t} u_{n}(t) & =L^{-2 s} T_{n}^{*}\left(y^{\delta}-F\left(x_{n}\right)-T_{n} u_{n}(t)\right), \quad t>0 \\
u_{n}(0) & =0
\end{aligned}
$$

Therefore, with $z_{n}(t):=y^{\delta}-F\left(x_{n}\right)-T_{n} u_{n}(t)$ we have

$$
\begin{aligned}
\frac{d}{d t}\left\|e_{n}+u_{n}(t)\right\|_{s}^{2} & =2\left(\frac{d}{d t} u_{n}(t), e_{n}+u_{n}(t)\right)_{s}=2\left(z_{n}(t), T_{n}\left(e_{n}+u_{n}(t)\right)\right) \\
& =2\left(z_{n}(t),-z_{n}(t)+y^{\delta}-F\left(x_{n}\right)+T_{n} e_{n}\right) \\
& \leq-2\left\|z_{n}(t)\right\|\left(\left\|z_{n}(t)\right\|-\left\|y^{\delta}-F\left(x_{n}\right)+T_{n} e_{n}\right\|\right)
\end{aligned}
$$

According to the definition of $t_{n}$ we have $\left\|z_{n}\left(t_{n}\right)\right\|=\eta\left\|y^{\delta}-F\left(x_{n}\right)\right\|$ and $\left\|z_{n}(t)\right\|>\eta\left\|y^{\delta}-F\left(x_{n}\right)\right\|$ for $0 \leq t \leq t_{n}$. Since $\tau \eta>1$, we therefore obtain

$$
\frac{d}{d t}\left\|e_{n}+u_{n}(t)\right\|_{s}^{2} \leq-2 \eta\left(\eta-\eta_{0}\right)\left\|y^{\delta}-F\left(x_{n}\right)\right\|^{2}, \quad 0<t \leq t_{n}
$$

if $K_{0}\left\|e_{0}\right\|_{s}^{\beta}$ is sufficiently small, where $0<\eta_{0}<\eta$ is such that $\tau \eta_{0}>1$. In view of $u_{n}(0)=0$ and $x_{n+1}=x_{n}+u_{n}\left(t_{n}\right)$, we obtain

$$
\left\|e_{n+1}\right\|_{s}^{2}-\left\|e_{n}\right\|_{s}^{2} \leq-2 \eta\left(\eta-\eta_{0}\right) t_{n}\left\|y^{\delta}-F\left(x_{n}\right)\right\|^{2}
$$

This implies (3.8) and (3.9) immediately.
(d) For the inexact Newton method with inner scheme defined by Tikhonov regularization, we have

$$
u_{n}(t)=\left(t^{-1} L^{2 s}+T_{n}^{*} T_{n}\right)^{-1} T_{n}^{*}\left(y^{\delta}-F\left(x_{n}\right)\right)
$$

We first observe that

$$
\begin{aligned}
\left\|e_{n+1}\right\|_{s}^{2}-\left\|e_{n}\right\|_{s}^{2} & \leq 2\left\|x_{n+1}-x_{n}\right\|_{s}^{2}+2\left(x_{n+1}-x_{n}, e_{n}\right)_{s} \\
& =2\left(x_{n+1}-x_{n}, x_{n+1}-x_{n}+e_{n}\right)_{s} .
\end{aligned}
$$

Let $z_{n}=y^{\delta}-F\left(x_{n}\right)-T_{n}\left(x_{n+1}-x_{n}\right)$. We have from (3.5) and $r_{t}(\lambda)=(1+t \lambda)^{-1}$ that $u_{n}\left(t_{n}\right)=t_{n} L^{-2 s} T_{n}^{*} z_{n}$ and hence $x_{n+1}-x_{n}=t_{n} L^{-2 s} T_{n}^{*} z_{n}$. Therefore

$$
\begin{aligned}
\left\|e_{n+1}\right\|_{s}^{2}-\left\|e_{n}\right\|_{s}^{2} & \leq 2 t_{n}\left(z_{n}, T_{n}\left(x_{n+1}-x_{n}+e_{n}\right)\right) \\
& =2 t_{n}\left(z_{n},-z_{n}+\left(y^{\delta}-F\left(x_{n}\right)+T_{n} e_{n}\right)\right) \\
& \leq-2 t_{n}\left\|z_{n}\right\|\left(\left\|z_{n}\right\|-\left\|y^{\delta}-F\left(x_{n}\right)+T_{n} e_{n}\right\|\right)
\end{aligned}
$$

By the definition of $t_{n}$ we have $\left\|z_{n}\right\|=\eta\left\|y^{\delta}-F\left(x_{n}\right)\right\|$. Since $\tau \eta>1$, we can obtain

$$
\left\|e_{n+1}\right\|_{s}^{2}-\left\|e_{n}\right\|_{s}^{2} \leq-2 \eta\left(\eta-\eta_{0}\right) t_{n}\left\|y^{\delta}-F\left(x_{n}\right)\right\|^{2}
$$

if $K_{0}\left\|e_{0}\right\|_{s}^{\beta}$ is sufficiently small, where $0<\eta_{0}<\eta$ is such that $\tau \eta_{0}>1$. This implies (3.8) and (3.9).

Remark 3.1 The inequality (3.8) will find its use in the proof of Lemma 4.4 From (3.8), $t_{n} \geq c_{0}>0$, and the fact $\left\|y^{\delta}-F\left(x_{n}\right)\right\| \geq \tau \delta$ for $0 \leq n<n_{\delta}$, it follows easily that $n_{\delta}=O\left(\delta^{-2}\right)$ which gives only a rough estimate on the number of outer iterations. However, we should point out that the inexact Newton iterations in Hilbert scales in fact terminate after $n_{\delta}=O(1+|\log \delta|)$ outer iterations. This can be confirmed by using the fact

$$
\begin{equation*}
\eta\left\|y^{\delta}-F\left(x_{n}\right)\right\| \geq\left\|y^{\delta}-F\left(x_{n}\right)-T_{n}\left(x_{n+1}-x_{n}\right)\right\|, \quad 0 \leq n<n_{\delta} \tag{3.12}
\end{equation*}
$$

which follows from the definition of $t_{n}$ and $x_{n+1}=x_{n}+u_{n}\left(t_{n}\right)$. To see this, by using (3.2) in Lemma 3.1 we have

$$
\left\|F\left(x_{n+1}\right)-F\left(x_{n}\right)-T_{n}\left(x_{n+1}-x_{n}\right)\right\| \leq \frac{\gamma^{s \beta+b-a}}{1+\beta} K_{0}\left\|x_{n+1}-x_{n}\right\|_{s}^{\beta}\left\|x_{n+1}-x_{n}\right\|_{-a} .
$$

Since (3.9) implies $\left\|x_{n+1}-x_{n}\right\|_{s} \leq\left\|e_{n+1}\right\|_{s}+\left\|e_{n}\right\|_{s} \leq 2\left\|e_{0}\right\|_{s}$, from Assumption 2.2 (a) we have with $C:=2^{\beta} \gamma^{s \beta+b-a} /[(1+\beta) m]$ that

$$
\left\|F\left(x_{n+1}\right)-F\left(x_{n}\right)-T_{n}\left(x_{n+1}-x_{n}\right)\right\| \leq C K_{0}\left\|e_{0}\right\|_{s}^{\beta}\left\|T_{n}\left(x_{n+1}-x_{n}\right)\right\|
$$

Therefore, if $K_{0}\left\|e_{0}\right\|_{s}^{\beta}$ is sufficiently small, then there holds $\left\|T_{n}\left(x_{n+1}-x_{n}\right)\right\| \leq$ $2\left\|F\left(x_{n+1}\right)-F\left(x_{n}\right)\right\|$ and consequently

$$
\begin{equation*}
\left\|F\left(x_{n+1}\right)-F\left(x_{n}\right)-T_{n}\left(x_{n+1}-x_{n}\right)\right\| \leq 2 C K_{0}\left\|e_{0}\right\|_{s}^{\beta}\left\|F\left(x_{n+1}\right)-F\left(x_{n}\right)\right\| . \tag{3.13}
\end{equation*}
$$

Combining this with (3.12) yields

$$
\eta\left\|y^{\delta}-F\left(x_{n}\right)\right\| \geq\left\|y^{\delta}-F\left(x_{n+1}\right)\right\|-2 C K_{0}\left\|e_{0}\right\|_{s}^{\beta}\left\|F\left(x_{n+1}\right)-F\left(x_{n}\right)\right\| .
$$

Considering $\eta<1$, this in particular implies that if $K_{0}\left\|e_{0}\right\|_{s}^{\beta}$ is sufficiently small then

$$
\frac{\left\|y^{\delta}-F\left(x_{n+1}\right)\right\|}{\left\|y^{\delta}-F\left(x_{n}\right)\right\|} \leq \frac{\eta+2 C K_{0}\left\|e_{0}\right\|_{s}^{\beta}}{1-2 C K_{0}\left\|e_{0}\right\|_{s}^{\beta}} \leq \frac{1+\eta}{2}<1
$$

Therefore for all $n=0, \cdots, n_{\delta}$ there holds

$$
\left\|y^{\delta}-F\left(x_{n}\right)\right\| \leq\left(\frac{1+\eta}{2}\right)^{n}\left\|y^{\delta}-F\left(x_{0}\right)\right\|
$$

By taking $n=n_{\delta}-1$ and using $\left\|y^{\delta}-F\left(x_{n_{\delta}-1}\right)\right\| \geq \tau \delta$ we obtain $\tau \delta \leq$ $\left(\frac{1+\eta}{2}\right)^{n_{\delta}-1}\left\|y^{\delta}-F\left(x_{0}\right)\right\|$ which shows that $n_{\delta}=O(1+|\log \delta|)$.

## 4 Proof of Theorem 2.2

In this section we will show the order optimality of the four inexact Newton method in Hilbert scales stated in Theorem 2.2, For simplicity of further exposition, we will always use $C$ to denote a generic constant independent of $\delta$ and $n$, we will also use the convention $\Phi \lesssim \Psi$ to mean that $\Phi \leq C \Psi$ for some generic constant $C$ when the explicit expression of $C$ is not important. Furthermore, we will use $\Phi \sim \Psi$ to mean that $\Phi \lesssim \Psi$ and $\Psi \lesssim \Phi$.

Lemma 4.1 Under the same conditions in Lemma 3.3, there holds

$$
\left\|y^{\delta}-F\left(x_{n}\right)\right\| \lesssim\left\|y^{\delta}-F\left(x_{n+1}\right)\right\|, \quad n=0, \cdots, n_{\delta}-1
$$

Proof We first claim that there is a constant $c_{1}>0$ such that

$$
\begin{equation*}
c_{1}\left\|y^{\delta}-F\left(x_{n}\right)\right\| \leq\left\|y^{\delta}-F\left(x_{n}\right)-T_{n}\left(x_{n+1}-x_{n}\right)\right\| . \tag{4.1}
\end{equation*}
$$

This is clear from the definition of $t_{n}$ when the inner scheme is defined by Tikhonov regularization or the asymptotic regularization. When the inner scheme is defined by Landweber iteration, we have $r_{t}(\lambda)=(1-\lambda)^{[t]}$. According to the definition of $t_{n}$ and (3.5), we have

$$
\begin{aligned}
\eta\left\|y^{\delta}-F\left(x_{n}\right)\right\| & \leq\left\|y^{\delta}-F\left(x_{n}\right)-T_{n} u_{n}\left(t_{n}-1\right)\right\| \\
& =\left\|\left(I-A_{n} A_{n}^{*}\right)^{t_{n}-1}\left(y^{\delta}-F\left(x_{n}\right)\right)\right\| .
\end{aligned}
$$

Since $\left\|A_{n}\right\| \leq \Theta<1$, we have $\left\|\left(I-A_{n} A_{n}^{*}\right)^{-1}\right\| \leq\left(1-\Theta^{2}\right)^{-1}$. Therefore, using (3.5) again it follows

$$
\begin{aligned}
\left(1-\Theta^{2}\right) \eta\left\|y^{\delta}-F\left(x_{n}\right)\right\| & \leq\left\|\left(I-A_{n} A_{n}^{*}\right)^{t_{n}}\left(y^{\delta}-F\left(x_{n}\right)\right)\right\| \\
& =\left\|y^{\delta}-F\left(x_{n}\right)-T_{n}\left(x_{n+1}-x_{n}\right)\right\|
\end{aligned}
$$

which shows (4.1) with $c_{1}=\left(1-\Theta^{2}\right) \eta$. When the inner scheme is defined by the implicit iteration, we have $r_{t}(\lambda)=(1+\lambda)^{-[t]}$. Thus it follows from (3.5) and $\left\|A_{n}\right\| \leq 1$ that

$$
\begin{aligned}
\eta\left\|y^{\delta}-F\left(x_{n}\right)\right\| & \leq\left\|\left(I+A_{n} A_{n}^{*}\right)^{-t_{n}+1}\left(y^{\delta}-F\left(x_{n}\right)\right)\right\| \\
& \leq 2\left\|\left(I+A_{n} A_{n}^{*}\right)^{-t_{n}}\left(y^{\delta}-F\left(x_{n}\right)\right)\right\| \\
& =2\left\|y^{\delta}-F\left(x_{n}\right)-T_{n}\left(x_{n+1}-x_{n}\right)\right\|
\end{aligned}
$$

which shows (4.1) with $c_{1}=\eta / 2$.
The combination of (4.1) and (3.13) gives

$$
\begin{aligned}
& c_{1}\left\|y^{\delta}-F\left(x_{n}\right)\right\| \\
& \quad \leq\left\|y^{\delta}-F\left(x_{n+1}\right)\right\|+C K_{0}\left\|e_{0}\right\|_{s}^{\beta}\left\|F\left(x_{n+1}\right)-F\left(x_{n}\right)\right\| \\
& \quad \leq\left\|y^{\delta}-F\left(x_{n+1}\right)\right\|+C K_{0}\left\|e_{0}\right\|_{s}^{\beta}\left(\left\|y^{\delta}-F\left(x_{n+1}\right)\right\|+\left\|y^{\delta}-F\left(x_{n}\right)\right\|\right) .
\end{aligned}
$$

This shows the result if $K_{0}\left\|e_{0}\right\|_{s}^{\beta}$ is sufficiently small.

For the spectral filter functions defined by (2.4), we have shown in 9 that for any sequence of positive numbers $\left\{t_{n}\right\}$ there hold

$$
\begin{array}{r}
0 \leq \lambda^{\nu} \prod_{k=j}^{n-1} r_{t_{k}}(\lambda) \leq\left(s_{n}-s_{j}\right)^{-\nu} \\
0 \leq \lambda^{\nu} g_{t_{j}}(\lambda) \prod_{k=j+1}^{n-1} r_{t_{k}}(\lambda) \leq t_{j}\left(s_{n}-s_{j}\right)^{-\nu} \tag{4.3}
\end{array}
$$

and

$$
\begin{equation*}
0 \leq \lambda^{\nu} \sum_{i=0}^{n-1} g_{t_{i}}(\lambda) \prod_{k=i+1}^{n-1} r_{t_{k}}(\lambda) \leq s_{n}^{1-\nu} \tag{4.4}
\end{equation*}
$$

for $0 \leq \nu \leq 1,0 \leq \lambda \leq 1$ and $j=0,1, \cdots, n-1$, where $\left\{s_{n}\right\}$ is defined by

$$
\begin{equation*}
s_{0}=0 \quad \text { and } \quad s_{n}=\sum_{j=0}^{n-1} t_{j} \quad \text { for } n=1,2, \cdots \tag{4.5}
\end{equation*}
$$

Moreover, we have the following crucial estimate.
Lemma 4.2 Let $F$ satisfy Assumption 2.2, let $\left\{g_{t}\right\}$ be defined by 2.4 and $r_{t}(\lambda)=1-\lambda g_{t}(\lambda)$, and let $\left\{t_{n}\right\}$ be a sequence of positive numbers with $\left\{s_{n}\right\}$ defined by (4.5). Let $A=F^{\prime}\left(x^{\dagger}\right) L^{-s}$ and for any $x \in B_{\rho}\left(x^{\dagger}\right)$ let $A_{x}=F^{\prime}(x) L^{-s}$. Then for $-\frac{b+s}{2(a+s)} \leq \nu \leq 1 / 2$ there holds

$$
\left\|\left(A^{*} A\right)^{\nu} \prod_{k=j+1}^{n-1} r_{t_{k}}\left(A^{*} A\right)\left[g_{t_{j}}\left(A^{*} A\right) A^{*}-g_{t_{j}}\left(A_{x}^{*} A_{x}\right) A_{x}^{*}\right]\right\| \text { } \quad \begin{aligned}
& \quad t_{j}\left(s_{n}-s_{j}\right)^{-\nu-\frac{b+s}{2(a+s)}} K_{0}\left\|x-x^{\dagger}\right\|^{\beta}
\end{aligned}
$$

for $j=0,1, \cdots, n-1$.
Proof We refer to [9, Lemma 2] in which similar estimates have been derived for a general class of spectral filter functions.

We also need the following estimate concerning the sums of suitable types which will occur in the convergence analysis.

Lemma 4.3 Let $\left\{t_{n}\right\}$ be a sequence of numbers satisfying $t_{n} \geq c_{2}>0$, and let $s_{n}$ be defined by 4.5). Let $p \geq 0$ and $q \geq 0$ be two numbers. Then

$$
\sum_{j=0}^{n-1} t_{j}\left(s_{n}-s_{j}\right)^{-p} s_{j+1}^{-q} \leq C_{3} s_{n}^{1-p-q} \begin{cases}1, & \max \{p, q\}<1 \\ \log \left(1+s_{n}\right), & \max \{p, q\}=1 \\ s_{n}^{\max \{p, q\}-1}, & \max \{p, q\}>1\end{cases}
$$

where $C_{3}$ is a constant depending only on $p, q$ and $c_{2}$.
Proof This is essentially contained in [5, Lemma 4.3] and its proof. A simplified proof can be found in [9, Lemma 3].

Now we are ready to give the crucial estimates on $\left\|e_{n}\right\|_{\mu}$ and $\left\|T e_{n}\right\|$ for $0 \leq n<n_{\delta}$. We will exploit the ideas developed in [5] 8, 9 .

Lemma 4.4 Let $F$ satisfy Assumption 2.2 with $s \geq(a-b) / \beta$, let $\tau>2$ and $0<\eta<1$ be such that $\tau \eta>2$, let $x_{0} \in \mathcal{D}(F)$ satisfy $\gamma^{s}\left\|e_{0}\right\|_{s} \leq \rho$. If $e_{0} \in \mathcal{X}_{\mu}$ for some $s<\mu \leq b+2 s$ and if $K_{0}\left\|e_{0}\right\|_{\mu}^{\beta}$ is sufficiently small, then there exists a constant $C_{*}>0$ such that

$$
\left\|e_{n}\right\|_{\mu} \leq C_{*}\left\|e_{0}\right\|_{\mu} \quad \text { and } \quad\left\|T e_{n}\right\| \leq C_{*}\left\|e_{0}\right\|_{\mu}\left(1+s_{n}\right)^{-\frac{a+\mu}{2(a+s)}}
$$

for all $n=0, \cdots, n_{\delta}-1$.
Proof Since $s<\mu \leq b+2 s$, from (2.7) we have $\left\|e_{n}\right\|_{\mu} \sim\left\|\left(A^{*} A\right)^{\frac{s-\mu}{2(a+s)}} L^{s} e_{n}\right\|$. Therefore, it suffices to show that there exists a constant $C_{*}>0$ such that

$$
\begin{equation*}
\left\|\left(A^{*} A\right)^{\frac{s-\mu}{2(a+s)}} L^{s} e_{n}\right\| \leq C_{*}\left\|e_{0}\right\|_{\mu} \quad \text { and } \quad\left\|T e_{n}\right\| \leq C_{*}\left\|e_{0}\right\|_{\mu}\left(1+s_{n}\right)^{-\frac{a+\mu}{2(a+s)}} \tag{4.6}
\end{equation*}
$$

for all $n=0, \cdots, n_{\delta}-1$. We will show (4.6) by induction. By using (2.7) and Assumption 2.2 (b) we have

$$
\left\|\left(A^{*} A\right)^{\frac{s-\mu}{2(a+s)}} L^{s} e_{0}\right\| \leq \bar{c}\left(\frac{s-\mu}{a+s}\right)\left\|e_{0}\right\|_{\mu}
$$

and

$$
\left\|T e_{0}\right\|=\left\|\left(A^{*} A\right)^{1 / 2} L^{s} e_{0}\right\| \leq\left\|\left(A^{*} A\right)^{\frac{s-\mu}{2(a+s)}} L^{s} e_{0}\right\| \leq \bar{c}\left(\frac{s-\mu}{a+s}\right)\left\|e_{0}\right\|_{\mu}
$$

Therefore (4.6) with $n=0$ holds for $C_{*} \geq \bar{c}\left(\frac{s-\mu}{a+s}\right)$. Now we assume that (4.6) is true for all $0 \leq n<l$ for some $0<l<n_{\delta}$ and want to show that it is also true for $n=l$.

From the equation (3.4) and $x_{n+1}=x_{n}+u_{n}\left(t_{n}\right)$ it follows that

$$
\begin{aligned}
e_{n+1}= & e_{n}+L^{-s} g_{t_{n}}\left(A_{n}^{*} A_{n}\right) A_{n}^{*}\left(y^{\delta}-F\left(x_{n}\right)\right) \\
= & L^{-s} r_{t_{n}}\left(A^{*} A\right) L^{s} e_{n}+L^{-s} g_{t_{n}}\left(A^{*} A\right) A^{*}\left(y^{\delta}-F\left(x_{n}\right)+T e_{n}\right) \\
& +L^{-s}\left[g_{t_{n}}\left(A_{n}^{*} A_{n}\right) A_{n}^{*}-g_{t_{n}}\left(A^{*} A\right) A^{*}\right]\left(y^{\delta}-F\left(x_{n}\right)\right) .
\end{aligned}
$$

By induction on this equation we obtain

$$
\begin{align*}
e_{l}= & L^{-s} \prod_{j=0}^{l-1} r_{t_{j}}\left(A^{*} A\right) L^{s} e_{0}+L^{-s} \sum_{j=0}^{l-1} \prod_{k=j+1}^{l-1} r_{t_{k}}\left(A^{*} A\right) g_{t_{j}}\left(A^{*} A\right) A^{*}\left(y^{\delta}-y\right) \\
& +L^{-s} \sum_{j=0}^{l-1} \prod_{k=j+1}^{l-1} r_{t_{k}}\left(A^{*} A\right) g_{t_{j}}\left(A^{*} A\right) A^{*}\left(y-F\left(x_{j}\right)+T e_{j}\right) \\
& +L^{-s} \sum_{j=0}^{l-1} \prod_{k=j+1}^{l-1} r_{t_{k}}\left(A^{*} A\right)\left[g_{t_{j}}\left(A_{j}^{*} A_{j}\right) A_{j}^{*}-g_{t_{j}}\left(A^{*} A\right) A^{*}\right]\left(y^{\delta}-F\left(x_{j}\right)\right) . \tag{4.7}
\end{align*}
$$

By multiplying (4.7) by $T:=F^{\prime}\left(x^{\dagger}\right)$, noting that $A=T L^{-s}$, and using the identity

$$
1-\lambda \sum_{j=0}^{l-1} g_{t_{j}}(\lambda) \prod_{k=j+1}^{l-1} r_{t_{k}}(\lambda)=\prod_{j=0}^{l-1} r_{t_{j}}(\lambda)
$$

which follows from the relation $r_{t}(\lambda)=1-\lambda g_{t}(\lambda)$, we can obtain

$$
\begin{align*}
T e_{l}= & A \prod_{j=0}^{l-1} r_{t_{j}}\left(A^{*} A\right) L^{s} e_{0}+\left[I-\prod_{j=0}^{l-1} r_{t_{j}}\left(A A^{*}\right)\right]\left(y^{\delta}-y\right) \\
& +\sum_{j=0}^{l-1} \prod_{k=j+1}^{l-1} r_{t_{k}}\left(A A^{*}\right) g_{t_{j}}\left(A A^{*}\right) A A^{*}\left(y-F\left(x_{j}\right)+T e_{j}\right) \\
& +\sum_{j=0}^{l-1} A \prod_{k=j+1}^{l-1} r_{t_{k}}\left(A^{*} A\right)\left[g_{t_{j}}\left(A_{j}^{*} A_{j}\right) A_{j}^{*}-g_{t_{j}}\left(A^{*} A\right) A^{*}\right]\left(y^{\delta}-F\left(x_{j}\right)\right) . \tag{4.8}
\end{align*}
$$

Since $e_{0} \in \mathcal{X}_{\mu}$ with $s<\mu \leq b+2 s$, by using (2.7), (4.2), (4.3), (4.4) and Lemma 4.2 we can derive from (4.7) that

$$
\begin{align*}
& \left\|\left(A^{*} A\right)^{\frac{s-\mu}{2(a+s)}} L^{s} e_{l}\right\| \\
& \quad \leq c_{3}\left\|e_{0}\right\|_{\mu}+s_{l}^{\frac{a+\mu}{2(a+s)}} \delta+\sum_{j=0}^{l-1} t_{j}\left(s_{l}-s_{j}\right)^{-\frac{a+2 s-\mu}{2(a+s)}}\left\|y-F\left(x_{j}\right)+T e_{j}\right\| \\
& \quad+C \sum_{j=0}^{l-1} t_{j}\left(s_{l}-s_{j}\right)^{-\frac{b+2 s-\mu}{2(a+s)}} K_{0}\left\|e_{j}\right\|^{\beta}\left\|y^{\delta}-F\left(x_{j}\right)\right\| \tag{4.9}
\end{align*}
$$

where $c_{3}=\bar{c}\left(\frac{\mu-s}{a+s}\right)$ and $C$ is a generic constant independent of $l$ and $\delta$.
Next by using again $e_{0} \in \mathcal{X}_{\mu}$ with $s<\mu \leq b+2 s$, (2.7) and (4.2), we can obtain

$$
\begin{aligned}
\left\|A \prod_{j=0}^{l-1} r_{t_{j}}\left(A^{*} A\right) L^{s} e_{0}\right\| & \leq\left\|A \prod_{j=0}^{l-1} r_{t_{j}}\left(A^{*} A\right)\left(A^{*} A\right)^{\frac{\mu-s}{2(a+s)}}\right\|\left\|\left(A^{*} A\right)^{-\frac{\mu-s}{2(a+s)}} L^{s} e_{0}\right\| \\
& \leq c_{3} \sup _{0 \leq \lambda \leq 1}\left(\lambda^{\frac{a+\mu}{2(a+s)}} \prod_{j=0}^{l-1} r_{t_{j}}(\lambda)\right)\left\|e_{0}\right\|_{\mu} \\
& \leq c_{3} s_{l}^{-\frac{a+\mu}{2(a+s)}}\left\|e_{0}\right\|_{\mu} .
\end{aligned}
$$

Therefore, it follows from (4.8), (4.3) and Lemma 4.2 that

$$
\begin{align*}
\left\|T e_{l}\right\| \leq & c_{3} s_{l} \frac{-\frac{a+\mu}{2(a+s)}}{\left\|e_{0}\right\|_{\mu}+\delta+\sum_{j=0}^{l-1} t_{j}\left(s_{l}-s_{j}\right)^{-1}\left\|y-F\left(x_{j}\right)+T e_{j}\right\|} \\
& +C \sum_{j=0}^{l-1} t_{j}\left(s_{l}-s_{j}\right)^{-\frac{b+a+2 s}{2(a+s)}} K_{0}\left\|e_{j}\right\|^{\beta}\left\|y^{\delta}-F\left(x_{j}\right)\right\| . \tag{4.10}
\end{align*}
$$

We first use (4.10) to derive the desired estimate for $\left\|T e_{l}\right\|$. According to the relation $\left\|e_{j}\right\|_{\mu} \sim\left\|\left(A^{*} A\right)^{\frac{s-\mu}{2(a+s)}} L^{s} e_{j}\right\|$, we have from the induction hypotheses that

$$
\begin{equation*}
\left\|e_{j}\right\|_{\mu} \lesssim\left\|e_{0}\right\|_{\mu} \quad \text { and } \quad\left\|T e_{j}\right\| \lesssim\left\|e_{0}\right\|_{\mu}\left(1+s_{j}\right)^{-\frac{a+\mu}{2(a+s)}}, \quad 0 \leq j \leq l-1 \tag{4.11}
\end{equation*}
$$

We need to estimate the terms

$$
\left\|e_{j}\right\|, \quad\left\|y^{\delta}-F\left(x_{j}\right)\right\| \quad \text { and } \quad\left\|y-F\left(x_{j}\right)+T e_{j}\right\|, \quad 0 \leq j \leq l-1
$$

For each term we will give two types of estimates, one is true for all $0 \leq j \leq l-1$ and the other is true for $0 \leq j<l-1$.

By using (3.2) in Lemma 3.1, Assumption 2.2 (a), Lemma 3.3, and $\tau \delta \leq$ $\left\|y^{\delta}-F\left(x_{j}\right)\right\|$ for $0 \leq j<n_{\delta}$ we have

$$
\begin{aligned}
\left\|y^{\delta}-F\left(x_{j}\right)+T e_{j}\right\| & \leq \delta+\left\|y-F\left(x_{j}\right)+T e_{j}\right\| \leq \delta+C K_{0}\left\|e_{j}\right\|_{s}^{\beta}\left\|e_{j}\right\|_{-a} \\
& \leq \frac{1}{\tau}\left\|y^{\delta}-F\left(x_{j}\right)\right\|+C K_{0}\left\|e_{0}\right\|_{s}^{\beta}\left\|T e_{j}\right\| .
\end{aligned}
$$

This shows for $0 \leq j<n_{\delta}$ that

$$
\begin{align*}
& \left\|y^{\delta}-F\left(x_{j}\right)\right\| \leq \frac{\tau}{\tau-1}\left(1+C K_{0}\left\|e_{0}\right\|_{s}^{\beta}\right)\left\|T e_{j}\right\|  \tag{4.12}\\
& \left\|y^{\delta}-F\left(x_{j}\right)\right\| \geq \frac{\tau}{1+\tau}\left(1-C K_{0}\left\|e_{0}\right\|_{s}^{\beta}\right)\left\|T e_{j}\right\| \tag{4.13}
\end{align*}
$$

The inequalities (4.12), (4.13) and Lemma 4.1 imply that if $K_{0}\left\|e_{0}\right\|_{s}^{\beta}$ is sufficiently small then

$$
\begin{equation*}
\left\|T e_{j}\right\| \lesssim\left\|T e_{j+1}\right\|, \quad 0 \leq j<n_{\delta}-1 \tag{4.14}
\end{equation*}
$$

Consequently, we have from (4.12) and (4.14) that

$$
\begin{equation*}
\left\|y^{\delta}-F\left(x_{j}\right)\right\| \lesssim\left\|T e_{j+1}\right\|, \quad 0 \leq j<n_{\delta}-1 \tag{4.15}
\end{equation*}
$$

This together with (4.11) gives

$$
\begin{equation*}
\left\|y^{\delta}-F\left(x_{j}\right)\right\| \lesssim\left\|e_{0}\right\|_{\mu} s_{j+1}^{-\frac{a+\mu}{2(a+s)}}, \quad 0 \leq j<l-1 \tag{4.16}
\end{equation*}
$$

Next we estimate $\left\|y-F\left(x_{j}\right)+T e_{j}\right\|$. We have from (3.2) in Lemma 3.1, Assumption 2.2 (a), and (4.11) that

$$
\left\|y-F\left(x_{j}\right)-T e_{j}\right\| \lesssim K_{0}\left\|e_{j}\right\|_{\mu}^{\beta}\left\|e_{j}\right\|_{-a} \lesssim K_{0}\left\|e_{0}\right\|_{\mu}^{\beta}\left\|T e_{j}\right\| .
$$

Therefore, it follows from (4.14) that

$$
\begin{equation*}
\left\|y-F\left(x_{j}\right)-T e_{j}\right\| \lesssim K_{0}\left\|e_{0}\right\|_{\mu}^{\beta}\left\|T e_{j+1}\right\|, \quad 0 \leq j \leq l-1 \tag{4.17}
\end{equation*}
$$

On the other hand, by using (3.1) in Lemma 3.1 and Assumption 2.2 (a), we have

$$
\begin{aligned}
\left\|y-F\left(x_{j}\right)+T e_{j}\right\| & \leq K_{0}\left\|e_{j}\right\|^{\frac{a(1+\beta)-b}{a+\mu}}\left\|e_{j}\right\|_{-a^{\frac{\mu(1+\beta)+b}{a \mu}}} \\
& \lesssim K_{0}\left\|e_{j}\right\|^{\frac{a(1+\beta)-b}{a+\mu}}\left\|T e_{j}\right\|^{\frac{\mu(1+\beta)+b}{a+\mu}}
\end{aligned}
$$

Therefore, it follows from (4.14) and (4.11) that

$$
\begin{equation*}
\left\|y-F\left(x_{j}\right)-T e_{j}\right\| \lesssim K_{0}\left\|e_{0}\right\|_{\mu}^{1+\beta} s_{j+1}^{-\frac{\mu(1+\beta)+b}{2(a+s)}}, \quad 0 \leq j<l-1 \tag{4.18}
\end{equation*}
$$

For the term $\left\|e_{j}\right\|$, we first have from the interpolation inequality (2.6), Lemma 3.3, and Assumption 2.2 (a) that

$$
\left\|e_{j}\right\| \leq\left\|e_{j}\right\|_{s}^{\frac{a}{a+s}}\left\|e_{j}\right\|_{-a}^{\frac{s}{a+s}} \lesssim\left\|e_{0}\right\|_{s}^{\frac{a}{a+s}}\left\|T e_{j}\right\|^{\frac{s}{a+s}} .
$$

With the help of (4.13) we then obtain

$$
\begin{equation*}
\left\|e_{j}\right\| \lesssim\left\|e_{0}\right\|_{s}^{\frac{a}{a+s}}\left\|y^{\delta}-F\left(x_{j}\right)\right\|^{\frac{s}{a+s}}, \quad 0 \leq j \leq l-1 \tag{4.19}
\end{equation*}
$$

On the other hand, by using the interpolation inequality (2.6) and Assumption 2.2 (a) we also obtain for $0 \leq j \leq l-1$ that

$$
\left\|e_{j}\right\| \leq\left\|e_{j}\right\|_{\mu}^{\frac{a}{a+\mu}}\left\|e_{j}\right\|_{-a}^{\frac{\mu}{a+\mu}} \lesssim\left\|e_{j}\right\|_{\mu}^{\frac{a}{a+\mu}}\left\|T e_{j}\right\|^{\frac{\mu}{a+\mu}}
$$

This together with (4.14) and (4.11) gives

$$
\begin{equation*}
\left\|e_{j}\right\| \lesssim\left\|e_{0}\right\|_{\mu} s_{j+1}^{-\frac{\mu}{2(a+s)}}, \quad 0 \leq j<l-1 \tag{4.20}
\end{equation*}
$$

Now we use (4.15), (4.17) and (4.19) with $j=l-1$ and use (4.16), (4.18) and (4.20) for $0 \leq j<l-1$, we then obtain from (4.10) that

$$
\begin{aligned}
\left\|T e_{l}\right\| \leq & c_{3}\left\|e_{0}\right\|_{\mu} s_{l}^{-\frac{a+\mu}{2(a+s)}}+\delta+C K_{0}\left\|e_{0}\right\|_{\mu}^{1+\beta} \sum_{j=0}^{l-2} t_{j}\left(s_{l}-s_{j}\right)^{-1} s_{j+1}^{-\frac{\mu(1+\beta)+b}{2(a+s)}} \\
& +C K_{0}\left\|e_{0}\right\|_{\mu}^{\beta}\left\|T e_{l}\right\|+C K_{0}\left\|e_{0}\right\|_{s}^{\frac{a \beta}{a+s}} t_{l-1}^{\frac{a-b}{2(a+s)}}\left\|y^{\delta}-F\left(x_{l-1}\right)\right\|^{\frac{s \beta}{a+s}}\left\|T e_{l}\right\| \\
& +C K_{0}\left\|e_{0}\right\|_{\mu}^{1+\beta} \sum_{j=0}^{l-2} t_{j}\left(s_{l}-s_{j}\right)^{-\frac{b+a+2 s}{2(a+s)}} s_{j+1}^{-\frac{\mu(1+\beta)+a}{2(a+s)}}
\end{aligned}
$$

Since $\mu>s \geq(a-b) / \beta$, we can use Lemma 4.3 to derive that

$$
\begin{aligned}
\left\|T e_{l}\right\| \leq & \left(c_{3}+C K_{0}\left\|e_{0}\right\|_{\mu}^{\beta}\right)\left\|e_{0}\right\|_{\mu} s_{l}^{-\frac{a+\mu}{2(a+s)}}+\delta+C K_{0}\left\|e_{0}\right\|_{\mu}^{\beta}\left\|T e_{l}\right\| \\
& +C K_{0}\left\|e_{0}\right\|_{s}^{\frac{a \beta}{a+s}} t_{l-1}^{\frac{a-b}{2(a+s)}}\left\|y^{\delta}-F\left(x_{l-1}\right)\right\|^{\frac{s \beta}{a+s}}\left\|T e_{l}\right\| .
\end{aligned}
$$

Recall that (3.8) in Lemma 3.3 implies $t_{l-1}\left\|y^{\delta}-F\left(x_{l-1}\right)\right\|^{2} \lesssim\left\|e_{0}\right\|_{s}^{2}$. Since $s \geq(a-b) / \beta$ and $t_{l-1} \geq c_{0}>0$, we have

$$
t_{l-1}^{\frac{a-b}{2(a+s)}}\left\|y^{\delta}-F\left(x_{l-1}\right)\right\|^{\frac{s \beta}{a+s}} \leq\left(t_{l-1}\left\|y^{\delta}-F\left(x_{l-1}\right)\right\|^{2}\right)^{\frac{s \beta}{2(a+s)}} t_{l-1}^{\frac{a-b-s \beta}{2(a+s)}} \lesssim\left\|e_{0}\right\|_{s}^{\frac{s \beta}{a+s}} .
$$

Therefore, noting $\left\|e_{0}\right\|_{s} \lesssim\left\|e_{0}\right\|_{\mu}$, we obtain

$$
\begin{equation*}
\left\|T e_{l}\right\| \leq\left(c_{3}+C K_{0}\left\|e_{0}\right\|_{\mu}^{\beta}\right)\left\|e_{0}\right\|_{\mu} s_{l}^{-\frac{a+\mu}{2(a+s)}}+\delta+C K_{0}\left\|e_{0}\right\|_{\mu}^{\beta}\left\|T e_{l}\right\| . \tag{4.21}
\end{equation*}
$$

Since $l<n_{\delta}$, we have from the definition of $n_{\delta}$ and (4.12) that

$$
\begin{equation*}
\delta \leq \frac{1}{\tau}\left\|y^{\delta}-F\left(x_{l}\right)\right\| \leq \frac{1}{\tau-1}\left(1+C K_{0}\left\|e_{0}\right\|_{\mu}^{\beta}\right)\left\|T e_{l}\right\| . \tag{4.22}
\end{equation*}
$$

Combining this with (4.21) gives

$$
\left\|T e_{l}\right\| \leq\left(c_{3}+C K_{0}\left\|e_{0}\right\|_{\mu}^{\beta}\right)\left\|e_{0}\right\|_{\mu} s_{l}^{-\frac{a+\mu}{2(a+s)}}+\left(\frac{1}{\tau-1}+C K_{0}\left\|e_{0}\right\|_{\mu}^{\beta}\right)\left\|T e_{l}\right\| .
$$

Recall that $\tau>2$. Therefore, if $K_{0}\left\|e_{0}\right\|_{\mu}^{\beta}$ is sufficiently small, then we have

$$
\left\|T e_{l}\right\| \leq \frac{2 c_{3}(\tau-1)}{\tau-2}\left\|e_{0}\right\|_{\mu} s_{l}^{-\frac{a+\mu}{2(a+s)}}
$$

Since $l \geq 1$ and $s_{l} \geq t_{l-1} \geq c_{0}$, we have $1+s_{l} \leq\left(1+1 / c_{0}\right) s_{l}$. Therefore $\left\|T e_{l}\right\| \leq C_{*}\left\|e_{0}\right\|_{\mu}\left(1+s_{l}\right)^{-\frac{a+\mu}{2(a+s)}}$ if we choose $C_{*} \geq 2 c_{3}\left(1+1 / c_{0}\right)(\tau-1) /(\tau-2)$.

Finally we will use (4.9) to show the desired estimate for $\left\|\left(A^{*} A\right)^{\frac{-3-\mu}{2(a+s)}} L^{s} e_{l}\right\|$. Since we have verified the estimates for $\left\|T e_{l}\right\|$, the estimates (4.16), (4.18) and (4.20) therefore can be improved to include $j=l-1$; this is clear from the above argument. Consequently we have from (4.9) that

$$
\begin{aligned}
& \left\|\left(A^{*} A\right)^{\frac{s-\mu}{2(a+s)}} L^{s} e_{l}\right\| \\
& \quad \leq c_{3}\left\|e_{0}\right\|_{\mu}+s_{l}^{\frac{a+\mu}{2(a+s)}} \delta+C K_{0}\left\|e_{0}\right\|_{\mu}^{1+\beta} \sum_{j=0}^{l-1} t_{j}\left(s_{l}-s_{j}\right)^{-\frac{a+2 s-\mu}{2(a+s)}} s_{j+1}^{-\frac{\mu(1+\beta)+b}{2(a+s)}} \\
& \quad+C K_{0}\left\|e_{0}\right\|_{\mu}^{1+\beta} \sum_{j=0}^{l-1} t_{j}\left(s_{l}-s_{j}\right)^{-\frac{b+2 s-\mu}{2(a+s)}} s_{j+1}^{-\frac{\mu(1+\beta)+a}{2(a+s)}}
\end{aligned}
$$

It then follows from Lemma 4.3 that

$$
\left\|\left(A^{*} A\right)^{\frac{s-\mu}{2(a+s)}} L^{s} e_{l}\right\| \leq\left(c_{2}+C K_{0}\left\|e_{0}\right\|_{\mu}^{\beta}\right)\left\|e_{0}\right\|_{\mu}+s_{l}^{\frac{a+\mu}{2(a+s)}} \delta .
$$

With the help of (4.22) and the estimate on $\left\|T e_{l}\right\|$, we obtain

$$
\left\|\left(A^{*} A\right)^{\frac{s-\mu}{2(a+s)}} L^{s} e_{l}\right\| \leq\left(c_{3}+C K_{0}\left\|e_{0}\right\|_{\mu}^{\beta}\right)\left\|e_{0}\right\|_{\mu}+\frac{C_{*}}{\tau-1}\left(1+C K_{0}\left\|e_{0}\right\|_{\mu}^{\beta}\right)\left\|e_{0}\right\|_{\mu} .
$$

Since $\tau>2$, we thus obtain $\left\|\left(A^{*} A\right)^{\frac{s-\mu}{2(a+s)}} L^{s} e_{l}\right\| \leq C_{*}\left\|e_{0}\right\|_{\mu}$ for any $C_{*} \geq$ $4 c_{3}(\tau-1) /(\tau-2)$ if $K_{0}\left\|e_{0}\right\|_{\mu}^{\beta}$ is sufficiently small. The proof is therefore complete.

Now we are ready to complete the proof of Theorem [2.2, the main result in this paper.

Proof of Theorem 2.2. Considering Lemma 3.3 and Remark 3.1 it remains only to derive the order optimal convergence rates. When $n_{\delta}=0$, the proof is standard. So we may assume $n_{\delta}>0$. From Lemma 4.4 it follows that $\left\|e_{n_{\delta}-1}\right\|_{\mu} \lesssim\left\|e_{0}\right\|_{\mu}$. By using Lemma 4.1 and the definition of $n_{\delta}$ we have $\left\|y^{\delta}-F\left(x_{n_{\delta}-1}\right)\right\| \lesssim \delta$, which together with (4.13) implies that $\left\|e_{n_{\delta}-1}\right\|_{-a} \lesssim$
$\left\|T e_{n_{\delta}-1}\right\| \lesssim \delta$. Therefore, from the interpolation inequality (2.6) it follows that

$$
\left\|e_{n_{\delta}-1}\right\|_{s} \leq\left\|e_{n_{\delta}-1}\right\|_{\mu}^{\frac{a+s}{a+\mu}}\left\|e_{n_{\delta}-1}\right\|_{-a}^{\frac{\mu-s}{a+\mu}} \lesssim\left\|e_{0}\right\|_{\mu}^{\frac{a+s}{a+\mu}} \delta^{\frac{\mu-s}{a+\mu}}
$$

In view of (3.9) in Lemma 3.3, we consequently obtain $\left\|e_{n_{\delta}}\right\|_{s} \lesssim\left\|e_{0}\right\|_{\mu}^{\frac{a+s}{a+\mu}} \delta^{\frac{\mu-s}{a+\mu}}$. By using the definition of $n_{\delta}$ and (1.2) we have $\left\|y-F\left(x_{n_{\delta}}\right)\right\| \leq(1+\tau) \delta$. Observing that (3.2) in Lemma 3.1 and (3.9) in Lemma 3.3 imply

$$
\begin{aligned}
\left\|T e_{n_{\delta}}\right\| & \leq\left\|y-F\left(x_{n_{\delta}}\right)\right\|+\left\|y-F\left(x_{n_{\delta}}\right)+T e_{n_{\delta}}\right\| \\
& \leq\left\|y-F\left(x_{n_{\delta}}\right)\right\|+C K_{0}\left\|e_{n_{\delta}}\right\|_{l}^{\beta}\left\|T e_{n_{\delta}}\right\| \\
& \leq\left\|y-F\left(x_{n_{\delta}}\right)\right\|+C K_{0}\left\|e_{0}\right\|_{s}^{\beta}\left\|T e_{n_{\delta}}\right\| .
\end{aligned}
$$

Thus, if $K_{0}\left\|e_{0}\right\|_{s} \lesssim K_{0}\left\|e_{0}\right\|_{\mu}$ is sufficiently small, then $\left\|T e_{n_{\delta}}\right\| \lesssim\left\|y-F\left(x_{n_{\delta}}\right)\right\|$. Consequently $\left\|e_{n_{\delta}}\right\|_{-a} \lesssim\left\|T e_{n_{\delta}}\right\| \lesssim \delta$. Now we can use again the interpolation inequality (2.6) to derive for all $r \in[-a, s]$ that

$$
\left\|e_{n_{\delta}}\right\|_{r} \leq\left\|e_{n_{\delta}}\right\|_{s}^{\frac{a+r}{a+s}}\left\|e_{n_{\delta}}\right\|_{-a}^{\frac{s-r}{a+s}} \lesssim\left\|e_{0}\right\|_{\mu}^{\frac{a+r}{a+\mu}} \delta^{\frac{\mu-r}{a+\mu}} .
$$

The proof is therefore complete.

## 5 Conclusions

Inexact Newton regularization methods have been suggested by Hanke and Rieder in [4] and [12], respectively, for solving nonlinear ill-posed inverse problems. The convergence rates of these methods have been considered in 12,13, the results however turned out to be inferior to the so-called order optimal rates. For a long time it has been an open problem whether these inexact Newton methods are order optimal, although the numerical illustrations in [12,13] present strong indication.

Important progress has been made recently in [5] where the regularizing Levenberg-Marquardt scheme is shown to be order optimal affirmatively. In this paper we considered a general class of inexact Newton methods in which the inner schemes are defined by Landweber iteration, the implicit iteration, the asymptotic regularization and Tikhonov regularization. By establishing the monotonicity of iteration errors and deriving a series of subtle estimates, we succeeded in proving the order optimality of these methods. We also extended these order optimality results to a more general situation where the inner schemes are defined by linear regularization methods in Hilbert scales. Our theoretical findings confirm the numerical results in [12,13.

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