# AN IMPROVED A PRIORI ERROR ANALYSIS OF NITSCHE'S METHOD FOR ROBIN BOUNDARY CONDITIONS 

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#### Abstract

In a previous paper [6] we have extended Nitsche's method [8] for the Poisson equation with general Robin boundary conditions. The analysis required that the solution is in $H^{s}$, with $s>3 / 2$. Here we give an improved error analysis using a technique proposed by Gudi [5].


## 1. The method and its consistency

In the article [6] a Nitsche-type method is introduced and analyzed for the following model Poisson problem with general Robin boundary conditions: Find $u \in H^{1}(\Omega)$ such that

$$
\begin{align*}
-\Delta u & =f \quad \text { in } \Omega  \tag{1.1}\\
\frac{\partial u}{\partial n} & =\frac{1}{\epsilon}\left(u_{0}-u\right)+g \quad \text { on } \Gamma \tag{1.2}
\end{align*}
$$

where $\Omega \subset \mathbb{R}^{N}, N=2,3$, is a bounded domain with polygonal or polyhedral boundary $\Gamma, f \in L^{2}(\Omega), u_{0} \in H^{1 / 2}(\Gamma), g \in L^{2}(\Gamma)$, and $\epsilon \in \mathbb{R}, 0 \leq \epsilon \leq \infty$. The limiting values of the parameter $\epsilon$ give the Dirichlet and Neumann problems, respectively.

The error analysis presented was not entirely satisfactory. It assumed that the solution is in $H^{s}(\Omega)$ with $s>3 / 2$, which is the same condition that traditionally has been needed for discontinuous Galerkin methods [4]. For discontinuous Galerkin methods Gudi introduced a technique using a posteriori error analysis by which this assumption could be avoided [5].

The purpose of this paper is to use these arguments to improve the analysis of the Nitsche method for the above Robin problem. Below we start by recalling the method of [6]. We first recall the derivation of the method in a way that emphasizes the use of the residual, which will be crucial for the error analysis. The same notation as in [6] will be used. The finite element partitioning into simplexes is denoted by $\mathcal{T}_{h}$. This induces a mesh, denoted by $\mathcal{G}_{h}$, on the boundary $\Gamma$. By $K \in \mathcal{T}_{h}$ we denote an element of the mesh and by $E$ we denote an edge or a face in $\mathcal{G}_{h}$. By $h_{K}$ we denote the diameter of the element $K \in \mathcal{T}_{h}$, and by $\rho_{K}$ the radius of the biggest ball contained in $K$. The mesh is assumed to be regular, i.e. it holds

$$
\begin{equation*}
\sup _{K \in \mathcal{T}_{h}} \frac{h_{K}}{\rho_{K}}=\kappa<\infty \tag{1.3}
\end{equation*}
$$

By $h_{E}$ we denote the diameter of $E \in \mathcal{G}_{h}$. The finite element subspace is denoted by

$$
V_{h}:=\left\{v \in H^{1}(\Omega):\left.v\right|_{K} \in \mathcal{P}_{p}(K) \forall K \in \mathcal{T}_{h}\right\}
$$

where $\mathcal{P}_{p}(K)$ is the space of polynomials of degree $p$.

The Nitsche method is obtained as follows. Multiplying the differential equation (1.1) with a testfunction $w \in V_{h}$ and integrating by parts we have

$$
\begin{equation*}
(\nabla u, \nabla w)_{\Omega}-\left\langle\frac{\partial u}{\partial n}, w\right\rangle_{\Gamma}-(f, w)_{\Omega}=0 \tag{1.4}
\end{equation*}
$$

Defining the residual

$$
\begin{equation*}
R_{\Gamma}(v)=\epsilon\left(\frac{\partial v}{\partial n}-g\right)+v-u_{0} \tag{1.5}
\end{equation*}
$$

the boundary condition is

$$
\begin{equation*}
R_{\Gamma}(u)=0 \tag{1.6}
\end{equation*}
$$

Hence it holds

$$
\begin{equation*}
\sum_{E \in \mathcal{G}_{h}} \frac{1}{\epsilon+\gamma h_{E}}\left\langle R_{\Gamma}(u), w\right\rangle_{E}=0 \tag{1.7}
\end{equation*}
$$

and

$$
\begin{equation*}
-\sum_{E \in \mathcal{G}_{h}} \frac{\gamma h_{E}}{\epsilon+\gamma h_{E}}\left\langle R_{\Gamma}(u), \frac{\partial w}{\partial n}\right\rangle_{E}=0 \tag{1.8}
\end{equation*}
$$

Adding (1.4), (1.7) and (1.8) shows that the exact solution satisfies

$$
\begin{aligned}
(\nabla u, \nabla w)_{\Omega}-\left\langle\frac{\partial u}{\partial n}, w\right\rangle_{\Gamma}-(f, w)_{\Omega} & +\sum_{E \in \mathcal{G}_{h}} \frac{1}{\epsilon+\gamma h_{E}}\left\langle R_{\Gamma}(u), w\right\rangle_{E} \\
& -\sum_{E \in \mathcal{G}_{h}} \frac{\gamma h_{E}}{\epsilon+\gamma h_{E}}\left\langle R_{\Gamma}(u), \frac{\partial w}{\partial n}\right\rangle_{E}=0
\end{aligned}
$$

Substituting the expression (1.5) for the boundary condition and rearranging the terms, we see that the exact solution satisfies

$$
\begin{equation*}
\mathcal{B}_{h}(u, w)-\mathcal{F}_{h}(w)=0 \quad \forall w \in V_{h} \tag{1.10}
\end{equation*}
$$

where

$$
\begin{align*}
\mathcal{B}_{h}(v, w)=(\nabla v, \nabla w)_{\Omega}+ & \sum_{E \in \mathcal{G}_{h}}\left\{-\frac{\gamma h_{E}}{\epsilon+\gamma h_{E}}\left[\left\langle\frac{\partial v}{\partial n}, w\right\rangle_{E}+\left\langle v, \frac{\partial w}{\partial n}\right\rangle_{E}\right]\right.  \tag{1.11}\\
& \left.+\frac{1}{\epsilon+\gamma h_{E}}\langle v, w\rangle_{E}-\frac{\epsilon \gamma h_{E}}{\epsilon+\gamma h_{E}}\left\langle\frac{\partial v}{\partial n}, \frac{\partial w}{\partial n}\right\rangle_{E}\right\}
\end{align*}
$$

and

$$
\begin{align*}
\mathcal{F}_{h}(w)=(f, w)_{\Omega}+\sum_{E \in \mathcal{G}_{h}}\{ & \frac{1}{\epsilon+\gamma h_{E}}\left\langle u_{0}, w\right\rangle_{E}-\frac{\gamma h_{E}}{\epsilon+\gamma h_{E}}\left\langle u_{0}, \frac{\partial w}{\partial n}\right\rangle_{E}  \tag{1.12}\\
& \left.+\frac{\epsilon}{\epsilon+\gamma h_{E}}\langle g, w\rangle_{E}-\frac{\epsilon \gamma h_{E}}{\epsilon+\gamma h_{E}}\left\langle g, \frac{\partial w}{\partial n}\right\rangle_{E}\right\}
\end{align*}
$$

The above derivation shows the consistency of the
Nitsche Method [6]. Find $u_{h} \in V_{h}$ such that

$$
\begin{equation*}
\mathcal{B}_{h}\left(u_{h}, w\right)=\mathcal{F}_{h}(w) \quad \forall w \in V_{h} \tag{1.13}
\end{equation*}
$$

## 2. The new a priori error estimate

The estimate will be given in the mesh and problem dependent norm

$$
\begin{equation*}
\|v\|_{h}^{2}:=\|\nabla v\|_{0, \Omega}^{2}+\sum_{E \in \mathcal{G}_{h}} \frac{1}{\epsilon+h_{E}}\|v\|_{0, E}^{2} \tag{2.1}
\end{equation*}
$$

We recall the following discrete trace inequality which is easily proved by scaling arguments.

Lemma 2.1. There is a positive constant $C_{I}$ such that

$$
\begin{equation*}
\sum_{E \in \mathcal{G}_{h}} h_{E}\left\|\frac{\partial v}{\partial n}\right\|_{0, E}^{2} \leq C_{I}\|\nabla v\|_{0, \Omega}^{2} \quad \forall v \in V_{h} \tag{2.2}
\end{equation*}
$$

For the formulation we have the following stability result, cf. [6]. Here and in what follows $C$ denotes a generic positive constant independent of both the mesh parameter $h$ and the parameter $\epsilon$.

Lemma 2.2. Suppose that $0<\gamma<1 / C_{I}$. Then there exists a positive constant $C$ such that

$$
\begin{equation*}
\mathcal{B}_{h}(v, v) \geq C\|v\|_{h}^{2} \quad \forall v \in V_{h} \tag{2.3}
\end{equation*}
$$

By $f_{h} \in V_{h}$ and $g_{h},\left.u_{0, h} \in V_{h}\right|_{\Gamma}$ we denote the interpolants to the data. For $E \in \mathcal{G}_{h}$ we denote by $K(E) \in \mathcal{T}_{h}$ the element with $E$ as edge/face. In [6] we proved the following bound.

Lemma 2.3. For an arbitrary $v \in V_{h}$ and $E \in \mathcal{G}_{h}$ it holds

$$
\begin{align*}
& \frac{h_{E}^{1 / 2}}{\epsilon+h_{E}}\left\|R_{\Gamma}(v)\right\|_{0, E} \leq C\left(\|\nabla(u-v)\|_{0, K(E)}+h_{K}\left\|f-f_{h}\right\|_{0, K(E)}\right.  \tag{2.4}\\
& \left.\quad+\frac{1}{\left(\epsilon+h_{E}\right)^{1 / 2}}\|u-v\|_{0, E}+\frac{h_{E}^{1 / 2}}{\epsilon+h_{E}}\left\|\epsilon\left(g-g_{h}\right)+u_{0}-u_{0, h}\right\|_{0, E}\right)
\end{align*}
$$

We introduce the oscillation terms

$$
\begin{align*}
\operatorname{osc}(f) & =\left(\sum_{K \in \mathcal{T}_{h}} h_{K}^{2}\left\|f-f_{h}\right\|_{0, K}^{2}\right)^{1 / 2}  \tag{2.5}\\
\operatorname{osc}\left(\epsilon, u_{0}, g\right) & =\left(\sum_{E \in \mathcal{G}_{h}} \frac{h_{E}}{\left(\epsilon+h_{E}\right)^{2}}\left\|\epsilon\left(g-g_{h}\right)+u_{0}-u_{0, h}\right\|_{0, E}^{2}\right)^{1 / 2} \tag{2.6}
\end{align*}
$$

Lemma 2.3 then gives
Lemma 2.4. For $v \in V_{h}$ it holds

$$
\begin{equation*}
\left(\sum_{E \in \mathcal{G}_{h}} \frac{h_{E}}{\left(\epsilon+h_{E}\right)^{2}}\left\|R_{\Gamma}(v)\right\|_{0, E}^{2}\right)^{1 / 2} \leq C\left\{\|u-v\|_{h}+\operatorname{osc}(f)+\operatorname{osc}\left(\epsilon, u_{0}, g\right)\right\} \tag{2.7}
\end{equation*}
$$

We can now prove our new error estimate.
Theorem 2.1. Suppose that $0<\gamma<1 / C_{I}$. Then there exist a positive constant $C$ such that

$$
\begin{equation*}
\left\|u-u_{h}\right\|_{h} \leq C\left\{\inf _{v \in V_{h}}\|u-v\|_{h}+\operatorname{osc}(f)+\operatorname{osc}\left(\epsilon, u_{0}, g\right)\right\} \tag{2.8}
\end{equation*}
$$

Proof. We will divide the proof in 6 steps.

1. Treating the consistency by Gudi's method.

Let $v \in V_{h}$ be arbitrary. From the stability we have

$$
\begin{equation*}
C\left\|v-u_{h}\right\|_{h}^{2} \leq \mathcal{B}_{h}\left(v-u_{h}, v-u_{h}\right) \tag{2.9}
\end{equation*}
$$

Next, we denote $w=v-u_{h}$ and use (1.13)

$$
\begin{align*}
\mathcal{B}_{h}\left(v-u_{h}, v-u_{h}\right) & =\mathcal{B}_{h}\left(v-u_{h}, w\right)=\mathcal{B}_{h}(v, w)-\mathcal{B}_{h}\left(u_{h}, w\right)  \tag{2.10}\\
& =\mathcal{B}_{h}(v, w)-\mathcal{F}_{h}(w)
\end{align*}
$$

Reversing the arguments leading from (1.9) to (1.10) we see that

$$
\begin{align*}
\mathcal{B}_{h}(v, w)-\mathcal{F}_{h}(w) & =(\nabla v, \nabla w)_{\Omega}-\left\langle\frac{\partial v}{\partial n}, w\right\rangle_{\Gamma}-(f, w)_{\Omega} \\
& +\sum_{E \in \mathcal{G}_{h}} \frac{1}{\epsilon+\gamma h_{E}}\left\langle R_{\Gamma}(v), w\right\rangle_{E}  \tag{2.11}\\
& -\sum_{E \in \mathcal{G}_{h}} \frac{\gamma h_{E}}{\epsilon+\gamma h_{E}}\left\langle R_{\Gamma}(v), \frac{\partial w}{\partial n}\right\rangle_{E}
\end{align*}
$$

Substituting the boundary condition (1.2) into (1.4) we get

$$
\begin{equation*}
(\nabla u, \nabla w)_{\Omega}-\left\langle\frac{1}{\epsilon}\left(u_{0}-u\right)+g, w\right\rangle_{\Gamma}-(f, w)_{\Omega}=0 \tag{2.12}
\end{equation*}
$$

Subtracting this from the right hand side of (2.11) yields

$$
\begin{align*}
\mathcal{B}_{h}(v, w)-\mathcal{F}_{h}(w) & =(\nabla(v-u), \nabla w)_{\Omega}-\left\langle\frac{\partial v}{\partial n}-\frac{1}{\epsilon}\left(u_{0}-u\right)-g, w\right\rangle_{\Gamma} \\
& +\sum_{E \in \mathcal{G}_{h}} \frac{1}{\epsilon+\gamma h_{E}}\left\langle R_{\Gamma}(v), w\right\rangle_{E} \\
& -\sum_{E \in \mathcal{G}_{h}} \frac{\gamma h_{E}}{\epsilon+\gamma h_{E}}\left\langle R_{\Gamma}(v), \frac{\partial w}{\partial n}\right\rangle_{E}  \tag{2.13}\\
& =R_{1}+R_{2}+R_{3}+R_{4}
\end{align*}
$$

Next we estimate the terms in the right hand side above.
2. Estimates for the terms $R_{1}$ and $R_{4}$.

The first and the last term are readily estimated. By Schwarz inequality and the definition (2.1) of the norm, we have

$$
\begin{equation*}
R_{1}=(\nabla(v-u), \nabla w)_{\Omega} \leq\|u-v\|_{h}\|w\|_{h} \tag{2.14}
\end{equation*}
$$

Schwarz inequality, the discrete trace inequality (2.2), and Lemma 2.4 give

$$
\begin{align*}
R_{4} & \leq\left|\sum_{E \in \mathcal{G}_{h}} \frac{\gamma h_{E}}{\epsilon+\gamma h_{E}}\left\langle R_{\Gamma}(v), \frac{\partial w}{\partial n}\right\rangle_{E}\right| \\
& \leq\left(\sum_{E \in \mathcal{G}_{h}} \frac{\gamma^{2} h_{E}}{\left(\epsilon+\gamma h_{E}\right)^{2}}\left\|R_{\Gamma}(v)\right\|_{0, E}^{2}\right)^{1 / 2}\left(\sum_{E \in \mathcal{G}_{h}} h_{E}\left\|\frac{\partial w}{\partial n}\right\|_{0, E}^{2}\right)^{1 / 2}  \tag{2.15}\\
& \leq C\left(\|u-v\|_{h}+\operatorname{osc}(f)+\operatorname{osc}\left(\epsilon, u_{0}, g\right)\right)\|w\|_{h}
\end{align*}
$$

3. Splitting the boundary.

To treat the two remaining terms $R_{2}$ and $R_{3}$, we have to separate the cases when the edge size $h_{E}$ is smaller or greater than $\epsilon$. To this end we denote the collection of edges of size greater than $\epsilon$ by

$$
\begin{equation*}
\mathcal{G}_{h}^{\epsilon}=\left\{E \in \mathcal{G}_{h} \mid \epsilon<h_{E}\right\} \tag{2.16}
\end{equation*}
$$

and the corresponding part of the boundary by

$$
\begin{equation*}
\Gamma_{\epsilon}=\bigcup_{E \in \mathcal{G}_{h}^{\epsilon}} E \tag{2.17}
\end{equation*}
$$

We then write

$$
\begin{align*}
R_{2}+R_{3}= & -\left\langle\frac{\partial v}{\partial n}-\frac{1}{\epsilon}\left(u_{0}-u\right)-g, w\right\rangle_{\Gamma}+\sum_{E \in \mathcal{G}_{h}} \frac{1}{\epsilon+\gamma h_{E}}\left\langle R_{\Gamma}(v), w\right\rangle_{E} \\
= & \sum_{E \in \mathcal{G}_{h}}\left\{-\left\langle\frac{\partial v}{\partial n}-\frac{1}{\epsilon}\left(u_{0}-u\right)-g, w\right\rangle_{E}+\frac{1}{\epsilon+\gamma h_{E}}\left\langle R_{\Gamma}(v), w\right\rangle_{E}\right\}  \tag{2.18}\\
= & \sum_{E \subset \Gamma_{\epsilon}}\left\{-\left\langle\frac{\partial v}{\partial n}-\frac{1}{\epsilon}\left(u_{0}-u\right)-g, w\right\rangle_{E}+\frac{1}{\epsilon+\gamma h_{E}}\left\langle R_{\Gamma}(v), w\right\rangle_{E}\right\} \\
& +\sum_{E \subset \Gamma \backslash \Gamma_{\epsilon}}\left\{-\left\langle\frac{\partial v}{\partial n}-\frac{1}{\epsilon}\left(u_{0}-u\right)-g, w\right\rangle_{E}+\frac{1}{\epsilon+\gamma h_{E}}\left\langle R_{\Gamma}(v), w\right\rangle_{E}\right\} .
\end{align*}
$$

4. Estimation of the contribution to $R_{2}+R_{3}$ from the part $\Gamma_{\epsilon}$.

On $E \subset \Gamma_{\epsilon}$ it holds $\epsilon<h_{E}$ and we estimate as follows, using Lemma 2.4,

$$
\begin{aligned}
\sum_{E \subset \Gamma_{\epsilon}} & \frac{1}{\epsilon+\gamma h_{E}}\left\langle R_{\Gamma}(v), w\right\rangle_{E} \\
& \leq \sum_{E \subset \Gamma_{\epsilon}} \frac{\left(\epsilon+h_{E}\right)^{1 / 2}}{\epsilon+\gamma h_{E}}\left\|R_{\Gamma}(v)\right\|_{0, E} \cdot\left(\epsilon+h_{E}\right)^{-1 / 2}\|w\|_{0, E} \\
& \leq \sum_{E \subset \Gamma_{\epsilon}} \frac{\sqrt{2} h_{E}^{1 / 2}}{\epsilon+\gamma h_{E}}\left\|R_{\Gamma}(v)\right\|_{0, E} \cdot\left(\epsilon+h_{E}\right)^{-1 / 2}\|w\|_{0, E} \\
& \leq\left(\sum_{E \subset \Gamma_{\epsilon}} \frac{2 h_{E}}{\left(\epsilon+\gamma h_{E}\right)^{2}}\left\|R_{\Gamma}(v)\right\|_{0, E}^{2}\right)^{1 / 2}\left(\sum_{E \subset \Gamma_{\epsilon}}\left(\epsilon+h_{E}\right)^{-1}\|w\|_{0, E}^{2}\right)^{1 / 2} \\
& \leq C\left(\|u-v\|_{h}+\operatorname{osc}(f)+\operatorname{osc}\left(\epsilon, u_{0}, g\right)\right)\|w\|_{h}
\end{aligned}
$$

Next, we have to estimate

$$
\begin{equation*}
-\left\langle\frac{\partial v}{\partial n}-\frac{1}{\epsilon}\left(u_{0}-u\right)-g, w\right\rangle_{\Gamma_{\epsilon}} \tag{2.20}
\end{equation*}
$$

We substitute

$$
\begin{equation*}
\frac{1}{\epsilon}\left(u_{0}-u\right)+g=\frac{\partial u}{\partial n} \tag{2.21}
\end{equation*}
$$

which gives

$$
\begin{equation*}
-\left\langle\frac{\partial v}{\partial n}-\frac{1}{\epsilon}\left(u_{0}-u\right)-g, w\right\rangle_{\Gamma_{\epsilon}}=\left\langle\frac{\partial u}{\partial n}-\frac{\partial v}{\partial n}, w\right\rangle_{\Gamma_{\epsilon}} \tag{2.22}
\end{equation*}
$$



Figure 1. The boundary parts $\Gamma_{\epsilon}$ and $\Gamma_{\epsilon}^{+}$, and the strip $\Omega_{\epsilon}$.

Now we define the strip

$$
\begin{equation*}
\Omega_{\epsilon}=\bigcup_{\substack{K \in \mathcal{T}_{h} \\ K \cap \Gamma_{\epsilon} \neq \emptyset}} K \tag{2.23}
\end{equation*}
$$

Following $[1,2,7]$ we construct a linear finite element extension $\mathcal{E}_{h} w \in V_{h}$ of $\left.w\right|_{\Gamma_{\epsilon}}$ such that

$$
\begin{equation*}
\left.\mathcal{E}_{h} w\right|_{\Gamma_{\epsilon}}=\left.w\right|_{\Gamma_{\epsilon}} \tag{2.24}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{E}_{h} w=0 \text { in } \Omega \backslash \Omega_{\epsilon} . \tag{2.25}
\end{equation*}
$$

In $[1,2,7]$ the following estimate is derived

$$
\begin{equation*}
\left\|\nabla \mathcal{E}_{h} w\right\|_{0, \Omega_{\epsilon}} \leq C\left(\sum_{E \subset \Gamma_{\epsilon}} h_{E}^{-1}\|w\|_{0, E}^{2}\right)^{1 / 2} \tag{2.26}
\end{equation*}
$$

We denote

$$
\begin{equation*}
\Gamma_{\epsilon}^{+}=\Omega_{\epsilon} \cap \Gamma \tag{2.27}
\end{equation*}
$$

and split the boundary of $\Omega_{\epsilon}$ in three parts (cf. Figure 2)

$$
\begin{equation*}
\partial \Omega_{\epsilon}=\Gamma_{\epsilon} \cup\left\{\Gamma_{\epsilon}^{+} \backslash \Gamma_{\epsilon}\right\} \cup\left\{\partial \Omega_{\epsilon} \backslash \Gamma_{\epsilon}^{+}\right\} \tag{2.28}
\end{equation*}
$$

Note that $\mathcal{E}_{h} w \neq w$ on $\Gamma_{\epsilon}^{+} \backslash \Gamma_{\epsilon}$. Since $\left.\mathcal{E}_{h} w\right|_{\partial \Omega_{\epsilon} \backslash \Gamma_{\epsilon}^{+}}=0$, scaling and the estimate (2.26) show that

$$
\begin{align*}
& \left(\sum_{K \subset \Omega_{\epsilon}} h_{K}^{-2}\left\|\mathcal{E}_{h} w\right\|_{0, K}^{2}\right)^{1 / 2}+\left(\sum_{E \subset \Omega_{\epsilon} \backslash \Gamma_{\epsilon}^{+}} h_{E}^{-1}\left\|\mathcal{E}_{h} w\right\|_{0, E}^{2}\right)^{1 / 2}  \tag{2.29}\\
& \leq C\left\|\nabla \mathcal{E}_{h} w\right\|_{0, \Omega_{\epsilon}} \leq C\left(\sum_{E \subset \Gamma_{\epsilon}} h_{E}^{-1}\|w\|_{0, E}^{2}\right)^{1 / 2}
\end{align*}
$$

and also

$$
\begin{equation*}
\left(\sum_{E \subset \Gamma_{\epsilon}^{+} \backslash \Gamma_{\epsilon}} h_{E}^{-1}\left\|\mathcal{E}_{h} w\right\|_{0, E}^{2}\right)^{1 / 2} \leq C\left\|\nabla \mathcal{E}_{h} w\right\|_{0, \Omega_{\epsilon}} \leq C\left(\sum_{E \subset \Gamma_{\epsilon}} h_{E}^{-1}\|w\|_{0, E}^{2}\right)^{1 / 2} \tag{2.30}
\end{equation*}
$$

Further, since $\epsilon<h_{E}$, it holds

$$
\begin{equation*}
\left(\sum_{E \subset \Gamma_{\epsilon}} h_{E}^{-1}\|w\|_{0, E}^{2}\right)^{1 / 2} \leq \sqrt{2}\left(\sum_{E \subset \Gamma_{\epsilon}} \frac{1}{h_{E}+\epsilon}\|w\|_{0, E}^{2}\right)^{1 / 2} \leq \sqrt{2}\|w\|_{h} \tag{2.31}
\end{equation*}
$$

Next, integrating by parts and using (2.29)-(2.31) we estimate as follows

$$
\begin{align*}
\left\langle\frac{\partial u}{\partial n}-\right. & \left.\frac{\partial v}{\partial n}, \mathcal{E}_{h} w\right\rangle_{\Gamma_{\epsilon}^{+}} \\
= & \sum_{K \subset \Omega_{\epsilon}}\left[-\left(f+\Delta v, \mathcal{E}_{h} w\right)_{K}+\left(\nabla(u-v), \nabla \mathcal{E}_{h} w\right)_{K}\right]  \tag{2.32}\\
& +\sum_{E \subset \Omega_{\epsilon} \backslash \Gamma_{\epsilon}^{+}}\left\langle\llbracket \frac{\partial v}{\partial n} \rrbracket, \mathcal{E}_{h} w\right\rangle_{E} \\
\leq & C\left(\sum_{K \subset \Omega_{\epsilon}} h_{K}^{2}\|f+\Delta v\|_{0, K}^{2}\right)^{1 / 2}\left(\sum_{K \subset \Omega_{\epsilon}} h_{K}^{-2}\left\|\mathcal{E}_{h} w\right\|_{0, K}^{2}\right)^{1 / 2} \\
& \left.+\|\nabla(u-v)\|_{0, \Omega_{\epsilon}}\left\|\nabla \mathcal{E}_{h} w\right\|_{0, \Omega_{\epsilon}} h_{E}\left\|\llbracket \frac{\partial v}{\partial n} \rrbracket\right\|_{0, E}^{2}\right)^{1 / 2}\left(\sum_{E \subset \Omega_{\epsilon} \backslash \Gamma_{\epsilon}^{+}} h_{E}^{-1}\left\|\mathcal{E}_{h} w\right\|_{0, E}^{2}\right)^{1 / 2} \\
& +\left(\sum_{E \subset \Omega_{\epsilon} \backslash \Gamma_{\epsilon}^{+}} h^{\prime} h_{K}\left\|\left(\sum_{K \subset \Omega_{\epsilon}} h_{K}^{2}\|f+\Delta v\|_{0, K}^{2}\right)^{1 / 2}+\right\| \nabla(u-v) \|_{0, \Omega_{\epsilon}}\right. \\
\leq & C\left\{\left(\sum_{E \subset \Omega_{\epsilon} \backslash \Gamma_{\epsilon}^{+}} h_{E}\left\|\llbracket \frac{\partial v}{\partial n} \rrbracket\right\|_{0, E}^{2}\right)^{1 / 2}\right\}\|w\|_{h} .
\end{align*}
$$

From a posteriori error analysis [3, 9] we know that

$$
\begin{align*}
& \left(\sum_{E \subset \Omega_{\epsilon} \backslash \Gamma_{\epsilon}^{+}} h_{E}\left\|\llbracket \frac{\partial v}{\partial n} \rrbracket\right\|_{0, E}^{2}\right)^{1 / 2}  \tag{2.33}\\
& \leq C\left(\sum_{K \subset \Omega_{\epsilon}} h_{K}^{2}\|f+\Delta v\|_{0, K}^{2}\right)^{1 / 2}+\|\nabla(u-v)\|_{0, \Omega_{\epsilon}}
\end{align*}
$$

and

$$
\begin{equation*}
\left(\sum_{K \subset \Omega_{\epsilon}} h_{K}^{2}\|f+\Delta v\|_{0, K}^{2}\right)^{1 / 2} \leq C\left(\|\nabla(u-v)\|_{0, \Omega}+\operatorname{osc}(f)\right) \tag{2.34}
\end{equation*}
$$

Hence we have

$$
\begin{equation*}
\left\langle\frac{\partial u}{\partial n}-\frac{\partial v}{\partial n}, E_{h} w\right\rangle_{\Gamma_{\epsilon}^{+}} \leq C\left(\|u-v\|_{h}+\operatorname{osc}(f)\right)\|w\|_{h} . \tag{2.35}
\end{equation*}
$$

Since $\mathcal{E}_{h} w=w$ on $\Gamma_{\epsilon}$, we get

$$
\begin{equation*}
\left\langle\frac{\partial u}{\partial n}-\frac{\partial v}{\partial n}, w\right\rangle_{\Gamma_{\epsilon}} \leq C\left(\|u-v\|_{h}+\operatorname{osc}(f)\right)\|w\|_{h}-\left\langle\frac{\partial u}{\partial n}-\frac{\partial v}{\partial n}, \mathcal{E}_{h} w\right\rangle_{\Gamma_{\epsilon}^{+} \backslash \Gamma_{\epsilon}} \tag{2.36}
\end{equation*}
$$

For $E \subset \Gamma_{\epsilon}^{+} \backslash \Gamma_{\epsilon}$ it holds that $h_{E} \leq \epsilon \leq C h_{E}$ with a constant that only depends on the regularity constant $\kappa$. Thus we can estimate

$$
\begin{align*}
- & \left\langle\frac{\partial u}{\partial n}-\frac{\partial v}{\partial n}, \mathcal{E}_{h} w\right\rangle_{\Gamma_{\epsilon}^{+} \backslash \Gamma_{\epsilon}} \\
& =\sum_{E \subset \Gamma_{\epsilon}^{+} \backslash \Gamma_{\epsilon}}-\frac{1}{\epsilon}\left\langle R_{\Gamma}(v), \mathcal{E}_{h} w\right\rangle_{E}-\frac{1}{\epsilon}\left\langle u-v, \mathcal{E}_{h} w\right\rangle_{E} \\
7) \leq & C\left(\sum_{E \subset \Gamma_{\epsilon}^{+} \backslash \Gamma_{\epsilon}} \frac{1}{\epsilon+h_{E}}\left\|R_{\Gamma}(v)\right\|_{0, E}\left\|\mathcal{E}_{h} w\right\|_{0, E}+\frac{1}{\epsilon+h_{E}}\|u-v\|_{0, E}\left\|\mathcal{E}_{h} w\right\|_{0, E}\right)  \tag{2.37}\\
& \leq C\left(\sum_{E \subset \Gamma_{\epsilon}^{+} \backslash \Gamma_{\epsilon}} \frac{h_{E}}{\left(h_{E}+\epsilon\right)^{2}}\left\|R_{\Gamma}(v)\right\|_{0, E}^{2}+\frac{1}{h_{E}+\epsilon}\|u-v\|_{0, E}^{2}\right)^{1 / 2} \\
& \times\left(\sum_{E \subset \Gamma_{\epsilon}^{+} \backslash \Gamma_{\epsilon}} \frac{1}{h_{E}+\epsilon}\left\|\mathcal{E}_{h} w\right\|_{0, E}^{2}\right)^{1 / 2} .
\end{align*}
$$

Thus we have

$$
\begin{equation*}
\left\langle\frac{\partial u}{\partial n}-\frac{\partial v}{\partial n}, w\right\rangle_{\Gamma_{\epsilon}} \leq C\left(\|u-v\|_{h}+\operatorname{osc}(f)+\operatorname{osc}\left(\epsilon, u_{0}, g\right)\right)\|w\|_{h} \tag{2.38}
\end{equation*}
$$

which together with (2.19) gives

$$
\begin{gather*}
\sum_{E \subset \Gamma_{\epsilon}}\left\{-\left\langle\frac{\partial v}{\partial n}-\frac{1}{\epsilon}\left(u_{0}-u\right)-g, w\right\rangle_{E}+\frac{1}{\epsilon+\gamma h_{E}}\left\langle R_{\Gamma}(v), w\right\rangle_{E}\right\} \\
\leq C\left(\|u-v\|_{h}+\operatorname{osc}(f)+\operatorname{osc}\left(\epsilon, u_{0}, g\right)\right)\|w\|_{h} \tag{2.39}
\end{gather*}
$$

5. Estimation of the contribution to $R_{2}+R_{3}$ from the part $\Gamma \backslash \Gamma_{\epsilon}$.

It now holds $\epsilon \geq h_{E}$. First write

$$
\begin{align*}
\frac{\partial v}{\partial n}-\frac{1}{\epsilon}\left(u_{0}-u\right)-g & =\frac{\partial v}{\partial n}-\frac{1}{\epsilon}\left(u_{0}-v\right)-g+\frac{1}{\epsilon}(u-v)  \tag{2.40}\\
& =\frac{1}{\epsilon} R_{\Gamma}(v)+\frac{1}{\epsilon}(u-v)
\end{align*}
$$

Hence, on $E$ it holds

$$
\begin{align*}
-\langle & \left\langle\frac{\partial v}{\partial n}-\frac{1}{\epsilon}\left(u_{0}-u\right)-g, w\right\rangle_{E}+\frac{1}{\epsilon+\gamma h_{E}}\left\langle R_{\Gamma}(v), w\right\rangle_{E} \\
& =\left(\frac{1}{\epsilon+\gamma h_{E}}-\frac{1}{\epsilon}\right)\left\langle R_{\Gamma}(v), w\right\rangle_{E}-\frac{1}{\epsilon}\langle u-v, w\rangle_{E} \\
& =-\frac{\gamma h_{E}}{\left(\epsilon+\gamma h_{E}\right) \epsilon}\left\langle R_{\Gamma}(v), w\right\rangle_{E}-\frac{1}{\epsilon}\langle u-v, w\rangle_{E}  \tag{2.41}\\
& \leq \frac{\gamma h_{E}}{\left(\epsilon+\gamma h_{E}\right) \epsilon}\left\|R_{\Gamma}(v)\right\|_{0, E}\|w\|_{0, E}+\frac{1}{\epsilon}\|u-v\|_{0, E}\|w\|_{0, E}
\end{align*}
$$

Since $\epsilon+h_{E} \leq 2 \epsilon$, it holds

$$
\begin{equation*}
\frac{1}{\epsilon}\|u-v\|_{0, E}\|w\|_{0, E} \leq \frac{2}{\epsilon+h_{E}}\|u-v\|_{0, E}\|w\|_{0, E} \tag{2.42}
\end{equation*}
$$

Since $h_{E} / \epsilon \leq 1$ we estimate as follows

$$
\begin{align*}
& \frac{\gamma h_{E}}{\left(\epsilon+\gamma h_{E}\right) \epsilon}\left\|R_{\Gamma}(v)\right\|_{0, E}\|w\|_{0, E} \\
& =\frac{\gamma h_{E}\left(\epsilon+h_{E}\right)^{1 / 2}}{\left(\epsilon+\gamma h_{E}\right) \epsilon}\left\|R_{\Gamma}(v)\right\|_{0, E} \cdot\left(\epsilon+h_{E}\right)^{-1 / 2}\|w\|_{0, E} \\
& =\gamma \frac{h_{E}^{1 / 2}}{\epsilon^{1 / 2}} \frac{h_{E}^{1 / 2}}{\left(\epsilon+\gamma h_{E}\right)} \frac{\left(\epsilon+h_{E}\right)^{1 / 2}}{\epsilon^{1 / 2}}\left\|R_{\Gamma}(v)\right\|_{0, E} \cdot\left(\epsilon+h_{E}\right)^{-1 / 2}\|w\|_{0, E}  \tag{2.43}\\
& =\gamma \frac{h_{E}^{1 / 2}}{\epsilon^{1 / 2}} \frac{h_{E}^{1 / 2}}{\left(\epsilon+\gamma h_{E}\right)}\left(1+\frac{h_{E}}{\epsilon}\right)^{1 / 2}\left\|R_{\Gamma}(v)\right\|_{0, E} \cdot\left(\epsilon+h_{E}\right)^{-1 / 2}\|w\|_{0, E} \\
& \leq \sqrt{2} \gamma \frac{h_{E}^{1 / 2}}{\left(\epsilon+\gamma h_{E}\right)}\left\|R_{\Gamma}(v)\right\|_{0, E} \cdot\left(\epsilon+h_{E}\right)^{-1 / 2}\|w\|_{0, E} .
\end{align*}
$$

Combining (2.41)-(2.43) yields

$$
\begin{align*}
& \sum_{E \subset \Gamma \backslash \Gamma_{\epsilon}}\left(-\left\langle\frac{\partial v}{\partial n}-\frac{1}{\epsilon}\left(u_{0}-u\right)-g, w\right\rangle_{E}+\frac{1}{\epsilon+\gamma h_{E}}\left\langle R_{\Gamma}(v), w\right\rangle_{E}\right)  \tag{2.44}\\
& \quad \leq C\left(\|u-v\|_{h}+\operatorname{osc}(f)+\operatorname{osc}\left(\epsilon, u_{0}, g\right)\right)\|w\|_{h}
\end{align*}
$$

6. Collecting the estimates.

Adding (2.39) and (2.44) gives

$$
\begin{equation*}
R_{2}+R_{3} \leq C\left(\|u-v\|_{h}+\operatorname{osc}(f)+\operatorname{osc}\left(\epsilon, u_{0}, g\right)\right)\|w\|_{h} \tag{2.45}
\end{equation*}
$$

The assertion then follows from this and (2.13), (2.14), and (2.15).

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