Discrete maximal regularity and the finite element method for parabolic equations

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Abstract Maximal regularity is a fundamental concept in the theory of partial differential equations. In this paper, we establish a fully discrete version of maximal regularity for a parabolic equation. We derive various stability results in $L^p(0,T;L^q(\Omega))$ norm, $p,q \in (1,\infty)$ for the finite element approximation with the mass-lumping to the linear heat equation. Our method of analysis is an operator theoretical one using pure imaginary powers of operators and might be a discrete version of G. Dore and A. Venni (On the closedness of the sum of two closed operators. Math. Z., 196(2):189-201, 1987). As an application, optimal order error estimates in that norm are proved. Furthermore, we study the finite element approximation for semilinear heat equations with locally Lipschitz continuous nonlinearity and offer a new method for deriving optimal order error estimates. Some interesting auxiliary results including discrete Gagliardo-Nirenberg and Sobolev inequalities are also presented.

Keywords maximal regularity · parabolic equation · finite element method

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1 Introduction

Let Ω be a bounded domain in \mathbb{R}^d , d = 2, 3, with the boundary $\partial \Omega$. Let $J_T = (0,T)$ be a time interval with $T \in (0,\infty]$. We consider the finite element

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approximation of linear heat equation for the function u = u(x,t) of $(x,t) \in \overline{\Omega} \times [0,T)$:

$$\begin{cases} \partial_t u = \Delta u + g & \text{in } \Omega \times J_T, \\ u = 0 & \text{on } \partial\Omega \times J_T, \\ u|_{t=0} = u_0 & \text{on } \Omega, \end{cases}$$
(1)

where $\partial_t u = \partial u / \partial t$, $\Delta u = \sum_{j=1}^d \partial^2 u / \partial x_j^2$, g = g(x, t), and $u_0 = u_0(x)$; g and u_0 are prescribed functions. All functions and function spaces considered in this paper are complex-valued.

The purpose of this paper is to derive various stability estimates in the $L^p(J_T; L^q(\Omega))$ norm

$$\|v\|_{L^{p}(J_{T};L^{q}(\Omega))} = \left[\int_{0}^{T} \left(\int_{\Omega} |v(x,t)|^{q} dx\right)^{1/q} dt\right]^{1/p}$$

and discrete $L^p(J_T; L^q(\Omega))$ norm defined as (8) with $X = L^q(\Omega)$, where $p, q \in (1, \infty)$. As applications of those estimates, we also derive optimal order error estimates in those norms for the finite element approximations of (1) and semilinear heat equation

$$\begin{cases} \partial_t u = \Delta u + f(u) & \text{in } \Omega \times J_T \\ u = 0 & \text{on } \partial\Omega \times J_T, \\ u|_{t=0} = u_0 & \text{in } \Omega, \end{cases}$$
(2)

where $f : \mathbb{C} \to \mathbb{C}$ is a prescribed function. Particularly, we assume only a locally Lipschitz continuity and offer a new method of error analysis for (2).

In other words, we intend to develop a discrete version of theory of maximal regularity for evolution equations of parabolic type. To recall maximal regularity in a general context, let us consider an abstract Cauchy problem on a Banach space X as

$$\begin{cases} u'(t) = Au(t) + g(t), & t \in J_T, \\ u(0) = 0, \end{cases}$$
(3)

where A is a densely defined closed operator on X with the domain $D(A) \subset X$, $g: J_T \to X$ is a given function, $u: J_T \to X$ is an unknown function and u'(t) = du(t)/dt.

Definition 1 (Maximal regularity, MR, CMR) Let $p \in (1, \infty)$. The operator A has maximal L^p -regularity $(L^p - MR)$ on J_T , if and only if, for every $g \in L^p(J_T; X)$, there exists a unique solution $u \in W^{1,p}(J_T; X) \cap L^p(J_T; D(A))$ of (3) satisfying

$$\|u\|_{L^{p}(J_{T};X)} + \|u'\|_{L^{p}(J_{T};X)} + \|Au\|_{L^{p}(J_{T};X)} \le C_{\mathrm{MR}} \|g\|_{L^{p}(J_{T};X)}, \qquad (4)$$

where $C_{\rm MR} > 0$ denotes a constant that is independent of g. We say that A has maximal regularity (MR) if A has maximal L^p -regularity for some $p \in (1, \infty)$

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(see Lemma 1). To distinguish L^p -MR and MR from the discrete versions introduced later, we say that A has continuous maximal L^p -regularity (L^p -CMR) and continuous maximal regularity (CMR).

It is proved that the $L^q(\Omega)$ realization A of Δ with $D(A) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ has L^p -CMR for any $p, q \in (1, \infty)$ (see [14,32]). The problem (1) admits a unique solution $u \in W^{1,p}(J_T; L^q(\Omega)) \cap L^p(J_T; D(A))$ satisfying (4) with $u_0 = 0$. This result implies that $\partial_t u$ and Δu are well defined and have the same regularity as the right-hand side function g. Moreover, $\partial_t u$ and Δu cannot be in a better function space than g, since $g = \partial_t u - \Delta u$. This is not a trivial fact. For comparison, we recall the solution obtained using the analytical semigroup theory, which is a powerful method to establish the well-posedness of (1) and (2). For example, assume $g \in C^{\sigma}(\overline{J_T}; L^q(\Omega))$ for some $\sigma \in (0, 1)$, that is, assume

$$\sup_{t,s\in\overline{J_T},\ t\neq s}\frac{\|g(t)-g(s)\|_{L^q(\Omega)}}{|t-s|^{\sigma}}<\infty.$$

Then, by application of the analytical semigroup theory, we can prove that the problem (1) with $u_0 = 0$ admits a unique solution $u \in C(\overline{J_T}; X) \cap$ $C(J_T; D(A)) \cap C^1(J_T; L^q(\Omega));$ see [35, Theorems 4.3.2, 7.3.5]. However, we are able to obtain slightly less regularity $\partial_t u - \Delta u \in C(J_T; L^q(\Omega))$ than g. To obtain the same regularity $\partial_t u - \Delta u \in C^{\sigma}(\overline{J_T}; L^q(\Omega))$, we must further assume g(x,0) = 0 for all $x \in \Omega$; see [35, Theorem 4.3.5]. Therefore, $W^{1,p}(J_T; L^q(\Omega)) \cap L^p(J_T; D(A))$ is an appropriate function space to study parabolic equations such as (1). Moreover, CMR is a "stronger" property than the generation of analytical semigroup in the sense that, if A has CMR, then A generates the analytical (bounded) semigroup (cf. [15]). Although CMR is a concept for linear equations, it actually has many important applications to nonlinear equations, as reported in the literature [3, 32, 40]. Moreover, the analytic semigroup theory and its discrete counterparts play important roles in construction and study of numerical schemes for parabolic equations (see e.g. [18,20,21,37,38,46]). Therefore, it is natural to wonder whether a discrete version of CMR is available.

This study has another motivation. Considering the problem (2) with $f(u) = u|u|^{\alpha}$ for $\alpha > 0$, then without loss of the generality, we assume $0 \in \Omega$. Let $\lambda > 0$. Then the function

$$u_{\lambda}(x,t) = \lambda^{\frac{2}{\alpha}} u(\lambda x, \lambda^2 t)$$

also solves (2) where Ω and J_T are replaced, respectively, by $\Omega_{\lambda} = \{\lambda^{-1}x \mid x \in \Omega\}$ and J_{T/λ^2} . Moreover, if $p, q \in (1, \infty)$ satisfy

$$\frac{2}{\alpha} = \frac{d}{p} + \frac{2}{q},\tag{5}$$

we have

$$\|u_{\lambda}\|_{L^{p}(J_{T/\lambda^{2}};L^{q}(\Omega_{\lambda}))} = \|u\|_{L^{p}(J_{T};L^{q}(\Omega))}$$

for any $\lambda > 0$. Those p, q are called the scale invariant exponents. The function space $L^p(J_T; L^q(\Omega))$ with p, q satisfying (5) plays a crucially important role in the study of time-local and time-global well-posedness of (2). Furthermore, such a scaling argument is applied to deduce a novel numerical method for solving (2) (see [5]). Therefore, it would be interesting to derive stability and error estimates in those norms from the dual perspectives of numerical and theoretical analysis.

Based on those motivations, we studied a time discrete version of maximal regularity for (3) in an earlier study [29]. Let

$$N_T = \begin{cases} \lfloor T/\tau \rfloor & (T < \infty) \\ N_\infty = \infty & (T = \infty). \end{cases}$$
(6)

We consider the implicit θ scheme for (3) given as

$$\begin{cases} \frac{u^{n+1} - u^n}{\tau} = Au^{n+\theta} + g^{n+\theta}, & n = 0, 1, \dots, N_T - 1, \\ u^0 = 0, \end{cases}$$
(7)

where $\tau > 0$ is the time increment, $\theta \in [0, 1]$, $g = (g^n)_{n=0}^{N_T}$ is a given X^{N_T+1} -valued function, and $u = (u^n)_{n=0}^{N_T}$ is an unknown X^{N_T+1} -valued function. Set

$$v^{n+\theta} = (1-\theta)v^n + \theta v^{n+1}$$

for a sequence $v = (v^n)_n$. We moreover assume that A is bounded when $\theta \neq 1$. The function u^n might be an approximation of $u(n\tau)$ for $n = 1, \ldots, N_T$.

We introduce the space $l^p(N; X)$ by setting

$$l^{p}(N;X) = \begin{cases} X^{N+1}, & N \in \mathbb{N}, \\ l^{p}(\mathbb{N};X), & N = \infty \end{cases}$$

and let

$$\|v\|_{l^{p}_{\tau}(N;X)} = \left(\sum_{n=0}^{N-1} \|v^{n}\|_{X}^{p} \tau\right)^{1/p},$$
(8)
$$D_{\tau}v = \left(\frac{v^{n+1} - v^{n}}{\tau}\right)_{n=0}^{N-1}, \quad Av = (Av^{n})_{n=0}^{N}, v_{\theta} = (v^{n+\theta})_{n=0}^{N-1},$$

for $v = (v^n) \in l^p(N; X)$.

Discrete maximal regularity is then introduced as follows (see [29]).

Definition 2 (Discrete maximal regularity, DMR) Let $p \in (1, \infty)$. The operator A has maximal l^p -regularity $(l^p - DMR)$ on J_T if and only if, for every $g \in l^p(N_T; X)$, there exists a unique solution $u \in X^{N_T}$ of (7) satisfying

$$\|u_{\theta}\|_{l^{p}_{\tau}(N_{T};X)} + \|D_{\tau}u\|_{l^{p}_{\tau}(N_{T};X)} + \|Au_{\theta}\|_{l^{p}_{\tau}(N_{T};X)} \le C_{\text{DMR}}\|g_{\theta}\|_{l^{p}_{\tau}(N_{T};X)}, \quad (9)$$

uniformly with respect to τ , where $C_{\text{DMR}} > 0$ is independent of g. We say that A has discrete maximal regularity (DMR) if A has l^p -DMR for some $p \in (1, \infty)$.

In [6], Blunck considered the forward Euler method ($\theta = 0$) and characterized DMR by developing a discrete version of the operator-valued Fourier multiplier theorem. However, the dependence of τ on DMR inequalities is not clear since only the case $\tau = 1$ is studied. The backward Euler method ($\theta = 1$) with an arbitrary time increment τ is discussed in [4]. Ashyralyev and Sobolevskii provided no reasonable sufficient conditions for DMR. Consequently, those results cannot be applied straightforwardly to numerical analysis. In contrast to those works, we gave sufficient conditions on τ , θ , A for DMR to hold in [29]. We recall the statement below (see Lemma 6).

Spatial discretization must be addressed next. We introduce the finite element approximation L_h of Δ in $H_0^1(\Omega)$ and prove that L_h has CMR. Herein, h denotes the size parameter of a triangulation \mathcal{T}_h . As a matter of fact, Geissert studied CMR for the finite element approximation of the second order parabolic equations in the divergence form in [22,23]. He considered a smooth convex domain Ω and triangulations defined on a polyhedral approximation Ω_h of Ω . (For the Neumann boundary condition case, he considered the exactly fitted triangulation.) Therefore, combining those results with our Lemma 6, we are able to obtain DMR for the smooth domain case. In those works, the method of [39] and [42] for studying stability and analyticity in L^{∞} norm is applied. He first derived some estimates for the discrete Green function associated with the finite element operator in parabolic annuli. Then he obtained some estimates in the whole Ω by a dyadic decomposition technique. Consequently, the proofs are quite intricate. Moreover, he applied several kernel estimates for the Green function associated with a parabolic equation. Therefore, the domain and coefficients should be suitably smooth.

In the present paper, we take a completely different approach. We directly establish a discrete version of the method using pure imaginary powers of operators developed by [16]. To this end, we consider polyhedral domains and study the discrete Laplacian with mass-lumping A_h instead of the standard discrete Laplacian since the positivity-preserving property of the semigroup generated by A_h (see Lemma 9) plays an important role in our analysis. Actually, the standard discrete Laplacian has no such property (see [43]). It must be borne in mind that the L^q theory for the discrete Laplacian with mass-lumping is of great use in study of nonlinear problems, such as the finite element and finite volume approximation of the Keller-Segel system modelling chemotaxis (see [37, 38, 46]).

After having established CMR and DMR for A_h (see Theorems I, II, III and IV), we derive optimal order error estimates for the finite element approximations combined with the implicit θ method to (1) (see Theorem V). We address not only unconditionally stable cases ($\theta \in [1/2, 1]$), but also conditionally stable cases ($\theta \in [0, 1/2)$). For the latter case, we give a useful sufficient condition for the scheme to be stable. As a further application, we study the finite element approximation for (2) and prove optimal order error estimates (see Theorem VI). Since nonlinearity f is assumed to be only locally Lipschitz continuous, the solution u might blow up in some sense. Our error estimate is valid as long as u exists in contrast to [23]. To achieve such an objective, we apply the fractional powers of $-A_h$ and derive a sub-optimal error estimate in the $L^{\infty}(\Omega \times (0,T))$ norm as an intermediate result (see Theorem VII). Our proposed method is apparently new in the literature. Some auxiliary results including discrete Gagliardo-Nirenberg and Sobolev inequalities are also presented (see Lemmas 24 and 28).

We learned about [33,31] after completion of the present study. The paper [33] specifically examined the time-discrete version of L^p-L^q -maximal regularity for arbitrary $p, q \in [1, \infty]$, by discontinuous Galerkin time stepping (cf. [41]) for parabolic problems. This result is valid for $p, q = 1, \infty$. However, they did not consider the R-boundedness of sets of operators, which plays an important role in the theory of maximal regularity developed by Weis [45]. The main tools in [33] were the smoothing properties of the continuous and discrete Laplace operators. Consequently, their estimate invariably contained the logarithmic term, so that the optimal error estimate is never obtained. It was established by a related work [31] that arbitrary A-stable time-discretization preserves the time-discrete version of maximal L^p -regularity for abstract Cauchy problems and for $p \in (1,\infty)$. These results were obtained via the theory of Rboundedness. It is therefore partially the same result of our previous work [29]. An optimal error estimate was established only for semi-discrete backward Euler scheme for a semilinear parabolic problem. In contrast to these works, we deal only with the finite difference scheme in time. However, our error estimate is optimal for fully discretized problems.

The plan of this paper is as follows. In Sec. 2, we introduce the notion of finite element approximation and state main results (Theorems I–VII). We summarize some preliminary results used in the proofs of Theorems in Sec. 3. Some auxiliary lemmas related to MR, DMR and A_h are described there. A useful sufficient condition for DMR to hold is also described there (Lemma 6). In Sec. 4, we prove Theorems I–IV by a discrete version of the method of [16] using pure imaginary powers of operators. Auxiliary results, Lemmas 15, 18 and 19, themselves are of interest. The proor of error estimate (Theorem V) for the linear equation (1) is described in Sec. 5. The semilinear equation (2) is studied in Sec. 6. Therein, we also prove auxiliary results including discrete Gagliardo-Nirenberg, Sobolev inequalities and provide useful results related to the fractional powers of A_h . Combining those results, we prove the final error estimate, Theorems VI and VII.

2 Main results

Throughout this paper, Ω is assumed to be a bounded polygonal or polyhedral domain in \mathbb{R}^d , d = 2, 3, with the boundary $\partial \Omega$. We follow the notation of [1]. As an abbreviation, we write $L^q = L^q(\Omega)$, $W^{s,q} = W^{s,q}(\Omega)$ and $H^s = W^{s,2}$ for

 $q \in [1, \infty]$ and s > 0. We use $W_0^{1,q} = \{v \in W^{1,q} \mid v|_{\partial \Omega} = 0\}$ and $H_0^1 = W_0^{1,2}$. Generic positive constants which are independent of discretization parameters, h and τ , are denoted as C. Their values might be different in each appearance.

Since the boundary $\partial \Omega$ is not smooth, we make the following shape assumption on Ω .

Assumption 1 (Shape assumption on Ω) There exists $\mu > d$ satisfying

$$\|v\|_{W^{2,q}} \le C \|\Delta v\|_{L^q}, \quad \forall v \in W^{2,q} \cap W^{1,q}_0, \tag{10}$$

for $q \in (1, \mu)$, where C > 0 depends only on Ω and q.

For example, if Ω is a convex polygonal domain in \mathbb{R}^2 , then one can find $\mu > 2$ satisfying Assumption 1 (see [24]).

Let \mathcal{T}_h be a triangulation of Ω with the granularity parameter h defined below. Hereinafter, a family \mathcal{T} of triangles or tetrahedra is a triangulation of Ω if and only if

1. each element of \mathcal{T} is an open triangle or tetrahedron in Ω and

$$\Omega = \operatorname{Int}\left(\bigcup_{K\in\mathcal{T}}\overline{K}\right),\,$$

where $Int(\cdot)$ is the interior part of a set,

2. any two elements of \mathcal{T} meet only in entire common faces (when d = 3), sides or vertices.

We use the following notations:

- h = max_{K∈T_h} h_K; h_K = the diameter of a triangle or tetrahedron K;
 M
 _h = the number of nodes of T_h; N_h = the number of interior nodes;
 {P_j}_{j=1}^{N_h} = the nodes of T_h; {P_j}_{j=1}^{N_h} = the interior nodes.

We assume the following.

Assumption 2 (Regularity of $\{\mathcal{T}_h\}_h$) There exists $\nu > 0$ such that

$$h_K \leq \nu \rho_K, \quad \forall K \in \mathcal{T}_h, \quad \forall h > 0,$$

where ρ_K denotes the radius of the inscribed circle or sphere of K.

Here we consider the P_1 finite element. Let V_h be the space of continuous functions on Ω which are affine in each element $K \in \mathcal{T}_h$. For every node P_i $(j = 1, ..., \overline{N}_h), \phi_j$ is the corresponding basis of V_h , which satisfies $\phi_j(P_i) =$ $\delta_{ij},$ where δ_{ij} is Kronecker's delta. Namely, V_h is the linear space spanned by $\{\phi_j\}_{j=1}^{\overline{N}_h}$. We also set

$$S_h = \{v_h \in V_h \mid v_h|_{\partial\Omega} = 0\} = \operatorname{span}\{\phi_j\}_{j=1}^{N_h}$$

Moreover, we presume that $\{\mathcal{T}_h\}_h$ satisfies the following conditions if necessary.

(H1) (Inverse assumption) There exists $\gamma > 0$ such that

$$h \leq \gamma h_K, \quad \forall K \in \mathcal{T}_h, \quad \forall h > 0$$

(H2) (Acuteness) For each h > 0 and for each $i, j \in \{1, 2, \dots, \overline{N}_h\}$ with $i \neq j$,

$$\int_{\Omega} \nabla \phi_i \cdot \nabla \phi_j \, dx \le 0. \tag{11}$$

Remark 1 In the two-dimensional case, let $\sigma \subset \Omega$ be an edge of the triangulation \mathcal{T}_h and K and L be the triangles of \mathcal{T}_h which meet in σ . Assume that the nodes P_i and P_j be both endpoints of σ . We denote the interior angle of K opposite to the edge σ by $\alpha_{i,j}^K$. Then, the condition (11) is equivalent to the equation of $\alpha_{i,j}^K + \alpha_{i,j}^L \leq \pi$. See [30, Corollary 3.48] for the detail.

Remark 2 (Discrete maximum principle) The condition (H2) is equivalent to the discrete maximum principle, i.e., the following conditions are equivalent.

- 1. The triangulation \mathcal{T}_h fulfills the acuteness condition.
- 2. Let $u_h \in V_h$ be the solution of the following problem for $f \in L^2$ and $g_h \in V_h$:

$$\begin{cases} (\nabla u_h, \nabla v_h)_{L^2} = (f, v_h)_{L^2}, & \forall v_h \in S_h, \\ u_h|_{\partial\Omega} = g_h. \end{cases}$$

Then, $u_h \ge 0$ in Ω provided that $f \ge 0$ in Ω and $g_h \ge 0$ on $\partial \Omega$.

See [30, Theorem 3.49] for details.

Remark 3 When q = 2, A_h is a self-adjoint operator in $X_{h,2}$. Therefore, (H2) is not required in the following discussion. However, the condition (H1) is required for the inverse inequality, which implies H^1 -stability of the L^2 projection (the equation (29)) and the discrete Gagliardo-Nirenberg type inequality (Lemma 24). Therefore, this condition is imposed for the consequences of (29) and Lemma 24, for example, Theorems V–VII, even if q = 2.

We describe the method of mass-lumping. For a node P_j , we designate the corresponding barycentric domain as Λ_j ; see Figure 1 for illustration and see [20,19] for the definition. We denote the characteristic function of Λ_j by χ_j for $j = 1, \ldots, N_h$. Then, we set

$$\overline{S}_h = \operatorname{span}\{\chi_j\}_{j=1}^{N_h}$$

and define the lumping operator $M_h \colon S_h \to \overline{S}_h$ as

$$M_h v_h = \sum_{j=1}^{N_h} v_h(P_j) \chi_j.$$

Moreover, we define $K_h = M_h^* M_h$, where M_h^* is the adjoint operator of M_h with respect to the L^2 -inner product. As one might expect, M_h is invertible



Fig. 1: Barycentric domains. P_{j_1} : interior node, P_{j_2} : boundary node. •: node, o: midpoint of an edge, \otimes : barycenter of a triangle.

and therefore K_h is as well. We define the mesh-dependent norms and inner product as

 $\|v_h\|_{h,q} = \|M_h v_h\|_{L^q}, \quad (u_h, v_h)_h = (M_h u_h, M_h v_h), \qquad u_h, v_h \in S_h$

for $q \in [1, \infty]$. In fact, $\|\cdot\|_{h,q}$ is an equivalent norm to $\|\cdot\|_{L^q}$ in S_h for each $q \in [1, \infty]$ (see Lemma 12).

At this stage, we introduce a discrete Laplacian as follows. Define the operator ${\cal A}_h$ on ${\cal S}_h$ as

$$(A_h u_h, v_h)_h = -(\nabla u_h, \nabla v_h), \quad \forall v_h \in S_h,$$

for $u_h \in S_h$. We designate A_h the discrete Laplacian with mass-lumping. From the Poincaré inequality, A_h is injective so that it is invertible due to $\dim S_h < \infty$.

We are now in a position to state the main results of this study. In the theorems below, we always presume that Assumptions 1 and 2 are satisfied, unless otherwise stated explicitly. The first one is about CMR for A_h .

Theorem I (CMR for A_h) Let $T \in (0, \infty]$, $p \in (1, \infty)$ and $q \in (1, \mu)$. Assume that (H1) and (H2) are satisfied when $q \neq 2$. Then, A_h has L^p -CMR on J_T in $X_{h,q}$ uniformly for h > 0. That is, there exists C > 0 independent of h > 0 satisfying

$$\|u_h\|_{L^p(J_T;X_{h,q})} + \|u_h'\|_{L^p(J_T;X_{h,q})} + \|A_h u_h\|_{L^p(J_T;X_{h,q})} \le C \|g_h\|_{L^p(J_T;X_{h,q})}$$

where $g_h \in L^p(J_T; X_{h,q})$ and u_h is the solution of

$$\begin{cases} u'_h(t) = A_h u_h(t) + g_h(t), & t \in J_T, \\ u_h(0) = 0. \end{cases}$$
(12)

Remark 4 Since (12) is a system of (inhomegeneous) linear ordinary differential equations, the unique existence of a solution follows immediately.

Next, we state results about DMR for A_h . To state them, we set

$$\theta_q = \arccos|1 - q/2|, \tag{13}$$

$$\kappa_h = \min_{K \in \mathcal{T}_h} \kappa_K,$$

where κ_K denotes the minimum length of perpendiculars of K.

Theorem II (DMR for A_h in J_{∞}) Let $p \in (1, \infty)$, $q \in (1, \mu)$ and $\theta \in [0, 1]$. Assume that (H1) and (H2) are satisfied when $q \neq 2$. We choose ε and τ sufficiently small to satisfy

$$\frac{\tau}{\kappa_h^2} \le \frac{2\sin\theta_q - \varepsilon}{(1 - 2\theta)(d+1)^2},\tag{14}$$

when $\theta \in [0, 1/2)$. Then, A_h has l^p -DMR on J_{∞} in $X_{h,q}$ uniformly for h > 0. That is, there exists C > 0 independent of h and τ satisfying

 $\|u_{h,\theta}\|_{l^p_{\tau}(\mathbb{N};X_{h,q})} + \|D_{\tau}u_h\|_{l^p_{\tau}(\mathbb{N};X_{h,q})} + \|A_hu_{h,\theta}\|_{l^p_{\tau}(\mathbb{N};X_{h,q})} \le C\|g_{h,\theta}\|_{l^p_{\tau}(\mathbb{N};X_{h,q})},$

where $g_h = (g_h^n)_n \in l^p(\mathbb{N}; X_{h,q})$ and $u_h = (u_h^n)_n$ is the solution of

$$\begin{cases} (D_{\tau}u_h)^n = A_h u_h^{n+\theta} + g_h^{n+\theta}, & n \in \mathbb{N}, \\ u_h^0 = 0. \end{cases}$$

Theorem III (DMR for A_h **in** J_T) Let $p \in (1, \infty)$, $q \in (1, \mu)$ and $\theta \in [0,1]$. Assume that (H1) and (H2) are satisfied when $q \neq 2$. Choose ε and τ sufficiently small to satisfy (14), when $\theta \in [0,1/2)$. Then, for every T > 0 and for every $g_h \in l^p(N_T - 1; X_{h,q})$, there exists a unique solution $u_h \in l^p(N_T; X_{h,q})$ of

$$\begin{cases} (D_{\tau}u_h)^n = A_h u_h^{n+\theta} + g_h^n, & n = 0, 1, \dots, N_T - 1, \\ u_h^0 = 0, \end{cases}$$

and it satisfies

 $\|u_{h,\theta}\|_{l^p_{\tau}(N_T;X_{h,q})} + \|D_{\tau}u_h\|_{l^p_{\tau}(N_T;X_{h,q})} + \|Au_{h,\theta}\|_{l^p_{\tau}(N_T;X_{h,q})} \le C\|g_h\|_{l^p_{\tau}(N_T;X_{h,q})},$

where C > 0 is independent of g, T, h, and τ .

Theorem IV (DMR for non-zero initial value) Let $p \in (1, \infty)$ and $q \in (1, \mu)$. Assume that (H1) and (H2) are satisfied when $q \neq 2$. Then, for every T > 0, $g_h \in l^p(N_T; X_{h,q})$, and $u_{0,h} \in (X_{h,q}, D(A_h))_{1-1/p,p}$, there exists a unique solution $u_h \in l^p(N_T; X_{h,q})$ of

$$\begin{cases} (D_{\tau}u_h)^n = A_h u_h^{n+1} + g_h^{n+1}, & n = 0, 1, \dots, N_T - 1, \\ u_h^0 = u_{0,h}, \end{cases}$$

which satisfies

$$\begin{aligned} \|u_{h,1}\|_{l^{p}_{\tau}(N_{T};X_{h,q})} + \|D_{\tau}u_{h}\|_{l^{p}_{\tau}(N_{T};X_{h,q})} + \|Au_{h,1}\|_{l^{p}_{\tau}(N_{T};X_{h,q})} \\ &\leq C\left[\|g_{h,1}\|_{l^{p}_{\tau}(N_{T};X_{h,q})} + \|u_{0,h}\|_{1-1/p,p}\right], \end{aligned}$$

where C > 0 is independent of g, $u_{0,h}$, T, h, and τ .

Therein, $(X_{h,q}, D(A_h))_{1-1/p,p}$ and $\|\cdot\|_{1-1/p,p}$ respectively denote the real interpolation space and its norm. (see Subsection 6.1 for related details.)

Those theorems are applicable for error analysis of the fully discretized finite element approximation for heat equations. First, we consider a linear heat equation (1) for $T \in (0, \infty)$, $g \in L^p(J_T; L^q)$ and $u_0 \in L^q$. We further assume $g \in C^0(\overline{J}_T; L^q)$). We consider the following approximate problem to find $u_h = (u_h^n)_n \in l^p(N_T; S_h)$ satisfying

$$\begin{cases} ((D_{\tau}u_{h})^{n}, v_{h})_{h} + (\nabla u_{h}^{n+\theta}, \nabla v_{h})_{L^{2}} = (G^{n+\theta}, v_{h})_{L^{2}}, & \forall v_{h} \in S_{h}, \\ n = 0, 1 \dots, N_{T} - 1, \\ (u_{h}^{0}, v_{h})_{L^{2}} = (u_{0}, v_{h})_{L^{2}}, & \forall v_{h} \in S_{h} \end{cases}$$

$$(15)$$

where $\tau \in (0, 1)$, $\theta \in [0, 1]$, $t_n = n\tau$, and $G^n = g(\cdot, t_n)$. An alternative scheme is obtained with replacement $(G^{n+\theta}, v_h)_{L^2}$ by $(G^{n+\theta}, v_h)_h$. However, the resulting scheme has a shortcoming reported in Appendix B.

Let P_h be the L^2 -projection onto S_h defined as

$$(P_h v, v_h)_{L^2} = (v, v_h)_{L^2}, \quad \forall v_h \in S_h$$
 (16)

for $v \in L^1$.

Then, (15) is equivalently written as

$$\begin{cases} (D_{\tau}u_h)^n = A_h u_h^{n+\theta} + K_h^{-1} P_h G^{n+\theta}, & n = 0, 1, \dots, N_T - 1, \\ u_h^0 = P_h u_0. \end{cases}$$
(17)

Since A_h is invertible, there exists a unique solution of (17). We introduce

$$j_{\theta} = \begin{cases} 2, & \theta = 1/2, \\ 1, & \text{otherwise} \end{cases}$$
(18)

and $\mu_d = \max\{\mu', d/2\}$. Since $\mu' < d' \le 2 \le d < \mu$, it might be apparent that

$$\mu_d = \begin{cases} \mu', & d = 2, \\ d/2 = 3/2, & d = 3. \end{cases}$$

Theorem V (Error estimate for linear equation) Let $p \in (1, \infty)$ and $q \in (\mu_d, \mu)$. Let $u_h = (u_h^n)_n \in l^p(N_T; S_h)$ be the solution of (17) and u be that of (1). Assume $u \in W^{1,p}(J_T; W^{2,q}) \cap W^{2,p}(J_T; W^{1,q}) \cap W^{1+j_\theta,p}(J_T; L^q)$ and set $U^n = u(\cdot, t_n)$. Assume that (H1) and (H2) are satisfied. Moreover, we

choose ε and τ sufficiently small to satisfy (14), when $\theta \in [0, 1/2)$. Then, there exists a positive constant C such that

$$\left(\sum_{n=0}^{N_T-1} \|u_h^{n+\theta} - U^{n+\theta}\|_{L^q}^p \tau\right)^{1/p} \le C(h^2 + \tau^{j_\theta}).$$
(19)

The constant C is taken as

$$C = C' \left(\|u\|_{W^{1,p}(J_T;W^{2,q})} + \|\partial_t u\|_{W^{1,p}(J_T;W^{1,q})} + \|u\|_{W^{1+j_{\theta},p}(J_T;L^q)} \right),$$

where C' depends only on Ω , p, q, and θ , but is independent of h and τ .

For $q \in (1, \infty)$, let A_q be the realization of the Dirichlet Laplacian:

$$D(A_q) = W^{2,q} \cap W_0^{1,q}, \quad A_q u = \Delta u.$$
 (20)

We are assuming Assumption 1. We consider a semilinear heat equation (2) under the following basic assumptions:

$$u_0 \in (L^q, D(A_q))_{1-1/p,p},$$
(21)

$$f: \mathbb{C} \to \mathbb{C}$$
 is locally Lipschitz continuous with $f(0) = 0.$ (22)

Herein, $(L^q, D(A_q))_{1-1/p,p}$ denotes the real interpolation space [2,34,44]. Restriction f(0) = 0 is set for simplicity. It is noteworthy that the solution u of (2) might blow-up: let $T_{\infty} \in (0, \infty]$ be the life span of u (the maximal existence time of u).

To avoid unnecessary difficulties, we restrict our consideration to a semiimplicit scheme for (2) given as

$$\begin{cases} (D_{\tau}u_h)^n = A_h u_h^{n+1} + K_h^{-1} P_h f(u_h^n), & n = 0, 1, \dots, N_T - 1, \\ u_h^0 = P_h u_0, \end{cases}$$
(23)

or, equivalently,

$$\begin{cases} ((D_{\tau}u_{h})^{n}, v_{h})_{h} + (\nabla u_{h}^{n+1}, \nabla v_{h})_{L^{2}} = (f(u_{h}^{n}), v_{h})_{L^{2}}, & \forall v_{h} \in S_{h}, \\ n = 0, 1, \dots, N_{T} - 1, \\ (u_{h}^{0}, v_{h})_{L^{2}} = (u_{0}, v_{h})_{L^{2}}, & \forall v_{h} \in S_{h}. \end{cases}$$

Since A_h is invertible, there exists a unique solution of (23). Our final theorem is the following error estimate for semilinear equation. Our error estimate remains valid as long as the solution of (2) exists and requires no size condition on u_0 .

Theorem VI (Error estimate for semilinear equation) Let $p \in (1, \infty)$, $q \in (\mu_d, \mu)$ and p > 2q/(2q - d). Assume that (H1) and (H2) are satisfied. Presuming that (2) admits a sufficiently smooth solution u under the conditions (21) and (22), then, for every $T \in (0, T_{\infty})$ and the solution $u_h = (u_h^n)_{n=0}^{N_T}$ of (23), we have

$$\left(\sum_{n=1}^{N_T} \|u_h^n - U^n\|_{L^q}^p \tau\right)^{1/p} \leq C(h^2 + \tau),$$
 where $U^n = u(\cdot, n\tau).$

In the proof of Theorem VI (Sec. 6), the following sub-optimal error estimate, which is worth stating separately, will be used.

Theorem VII (L^{∞} error estimate for semilinear equation) Under the same assumptions of Theorem VI, for every $\alpha \in (0, \alpha_{p,q,d})$ and $T \in (0, T_{\infty})$, the following error estimate holds:

$$\max_{0 \le n \le N_T} \|u_h^n - U^n\|_{L^{\infty}} \le C(h^{2\alpha} + \tau),$$

where $\alpha_{p,q,d} = 1 - 1/p - d/(2q)$ and $U^n = u(\cdot, n\tau)$.

3 Preliminaries

As explained in this section, we collect some preliminary results used for this study.

3.1 Continuous maximal regularity

The definition of CMR in Definition 1 is the classical one. The weaker one is introduced in [45, Definition 4.1], which requires the inequality

$$\|u'\|_{L^p(J_T;X)} + \|Au\|_{L^p(J_T;X)} \le C \|f\|_{L^p(J_T;X)}$$
(24)

instead of (4). Also, CMR in this sense is characterized by operator-theoretical properties ([45, Theorem 4.2]). However, two inequalities (4) and (24) are equivalent if $0 \in \rho(A)$, where $\rho(A)$ denotes the resolvent set of A. Since the condition $0 \in \rho(A)$ is satisfied in our application, we ignore the differences between these definitions.

Conditions necessary for CMR to hold have been studied by many researchers (see e.g. [15,45]). Among them, we review some sufficient conditions for CMR, which will be used for this study. For the detail, see [15] and references therein.

Lemma 1 Let $T \in (0, \infty]$, X be a Banach space and A be a densely defined and closed operator on X. Assume that A has L^{p_0} -CMR on J_T for some $p_0 \in (1, \infty)$. Then, A has L^p -CMR on J_T , for any $p \in (1, \infty)$.

Lemma 2 Let $p \in (1, \infty)$, X be a Banach space and let A be a densely defined and closed operator on X. Assume that A has L^p -CMR on J_∞ . Then, A has L^p -CMR on J_T , for any T > 0.

The next lemma is the celebrated result of Dore and Venni [16, Theorem 3.2] (see also [3, Section III.4]).

Lemma 3 Let $p \in (1, \infty)$, X be a UMD space, and let A be a densely defined and closed operator on X. Assume that $-A \in \mathcal{P}(X; K) \cap \mathcal{BIP}(X; M, \theta)$ for some K > 0, $M \ge 1$, and $\theta \in [0, \pi/2)$. Then, A has L^p -CMR on J_T , for any T > 0 and $T = \infty$. Moreover, the constant $C_{\rm MR} > 0$ depends only on X, K, M, θ , and T. Herein, the sets $\mathcal{P}(X; K)$ and $\mathcal{BIP}(X; M, \theta)$ are defined as

$$\mathcal{P}(X;K) = \{A \in \mathcal{C}(X) \mid \rho(A) \subset (-\infty,0] \text{ and} \\ \|(1+\lambda)R(\lambda;A)\|_{\mathcal{L}(X)} \leq K, \ \forall \lambda \geq 0\}, \\ \mathcal{BIP}(X;M,\theta) = \{A \in \mathcal{P}(X) \mid A^{it} \in \mathcal{L}(X) \text{ and } \|A^{it}\|_{\mathcal{L}(X)} \leq Me^{\theta|t|}, \ \forall t \in \mathbb{R}\},$$

for K > 0, $M \ge 1$, and $\theta \ge 0$, where $\mathcal{C}(X)$ is the set of all closed linear operators on X with dense domains, $\mathcal{P}(X) = \bigcup_{K>0} \mathcal{P}(X;K)$. The imaginary power A^{it} is defined by H^{∞} -functional calculus (see Appendix A).

The dependence of the constant $C_{\rm MR}$ on the Banach space X derives from the boundedness of imaginary powers of the time-differential operator on $L^p(J_T; X)$. See [3, Lemma III.4.10.5] for $T < \infty$ and [25, Corollary 8.5.3] for $T = \infty$. Chasing the constants appearing in the proofs, we can obtain the following property (see [29]).

Lemma 4 Let $p \in (1, \infty)$, X be a UMD space, $X_0 \subset X$ be a closed subspace, and A be a densely defined and closed operator on X_0 . Assume that $-A \in \mathcal{P}(X_0; K) \cap \mathcal{BIP}(X_0; M, \theta)$ for some K > 0, $M \ge 1$, and $\theta \in [0, \pi/2)$. Then A has L^p -CMR on J_T , for any T > 0 and $T = \infty$. Moreover, the constant $C_{MR} > 0$ depends only on X, K, M, θ , and T, but is independent of X_0 .

In the definition of CMR (3), we consider only the zero initial value. However, in general cases, particularly in the nonlinear cases, the choice of initial values is extremely important. Therefore, we now consider the following Cauchy problem:

$$\begin{cases} u'(t) = Au(t) + g(t), & t \in J_T, \\ u(0) = u_0, \end{cases}$$
(25)

for $u_0 \in X$.

Lemma 5 Let $p \in (1, \infty)$, $T \in (0, \infty]$, X be a Banach space and A be a densely defined and closed operator. Assume that A has L^p -CMR on J_T . Then, for each $g \in L^p(J_T; X)$ and for each $u_0 \in (X, D(A))_{1-1/p,p}$, there exists a unique solution $u \in W^{1,p}(J_T; X) \cap L^p(J_T; D(A))$ of (25) satisfying

$$\begin{aligned} \|u\|_{L^{p}(J_{T};X)} + \|u'\|_{L^{p}(J_{T};X)} + \|Au\|_{L^{p}(J_{T};X)} \\ &\leq C_{\mathrm{MR}}\left(\|g\|_{L^{p}(J_{T};X)} + \|u_{0}\|_{1-1/p,p}\right), \end{aligned}$$

where $C_{\rm MR} > 0$ is independent of g and u_0 .

Herein, the norm $\|\cdot\|_{1-1/p,p}$ is the norm of the real interpolation space $(X, D(A))_{1-1/p,p}$.

3.2 Discrete maximal regularity

As in the CMR case, the weaker definition can be considered, which does not require that $0 \in \rho(A)$. Indeed, the weaker one is used in [6,29]. However, for the same reason as that presented in the previous subsection, we do not distinguish these two definitions.

We investigated a sufficient condition for DMR on J_{∞} , in the UMD case in [29]. More precisely, we proved the following result.

Lemma 6 Let $p \in (1, \infty)$, $\theta \in [0, 1]$, X be a UMD space, $X_0 \subset X$ be a closed subspace, and A be a bounded operator on X_0 . Assume that A has L^p -CMR on J_{∞} with the constant C_{MR} . Furthermore, we suppose that the following conditions (condition $(NR)_{\delta,\varepsilon}$) are satisfied when $\theta \in [0, 1/2)$:

(NR1) There exists $\delta \in (0, \pi/2)$ such that $S(A) \subset \mathbb{C} \setminus \Sigma_{\delta + \pi/2}$. (NR2) There exists $\varepsilon > 0$ such that $(1 - 2\theta)\tau r(A) + \varepsilon \leq 2 \sin \delta$.

Then, A has l^p -DMR on J_{∞} . Moreover, the constant C_{DMR} depends only on $p, \theta, \delta, \varepsilon, X$, and C_{MR} , but is independent of X_0 .

Herein, for $\omega \in (0, \pi)$, the set Σ_{ω} denotes the sector

$$\Sigma_{\omega} = \{ z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \omega \}.$$
(26)

The set $S(A) \subset \mathbb{C}$ is the numerical range of A defined as

$$S(A) = \left\{ \langle x^*, Ax \rangle \; \middle| \; \substack{x \in D(X), \ \|x\| = 1, \\ x^* \in X^*, \ \|x^*\| = 1, \ \langle x^*, x \rangle = 1. } \right\},\$$

where $\langle \cdot, \cdot \rangle$ is the duality paring ([20,35]). We set

$$r(A) = \max_{z \in S(A)} |z|.$$

Actually, DMR on finite intervals is obtainable from the infinite-interval case. The following lemma corresponds to Lemma 2. Although the inequality (28) below is slightly different from (9), it does not affect error analysis.

Lemma 7 Let $p \in (1, \infty)$, $\theta \in [0, 1]$, X be a Banach space, and A be a bounded operator on X. Assume that A has l^p -DMR on J_∞ with $C_{\text{DMR}} = C_0$. Then, for every T > 0 and for every $g \in l^p(N_T - 1; X)$, there exists a unique solution $u \in l^p(N_T; X)$ of the equation

$$\begin{cases} \frac{u^{n+1} - u^n}{\tau} = Au^{n+\theta} + g^n, \quad n = 0, 1, \dots, N_T - 1, \\ u^0 = 0, \end{cases}$$
(27)

and it satisfies

$$\|u_{\theta}\|_{l^{p}_{\tau}(N_{T};X)} + \|D_{\tau}u\|_{l^{p}_{\tau}(N_{T};X)} + \|Au_{\theta}\|_{l^{p}_{\tau}(N_{T};X)} \le C_{0}\|g\|_{l^{p}_{\tau}(N_{T};X)}.$$
 (28)

Proof Fix T > 0, $\tau > 0$, and $g \in l^p(N_T - 1; X)$ arbitrarily. Define $\tilde{g} \in l^p(\mathbb{N}; X)$ as

$$\tilde{g}^n = \begin{cases} g^n, & n = 0, 1, \dots, N_T - 1, \\ 0, & n \ge N_T, \end{cases}$$

and consider the Cauchy problem

$$\begin{cases} \frac{\tilde{u}^{n+1} - \tilde{u}^n}{\tau} = A\tilde{u}^{n+\theta} + \tilde{g}^n, \quad n = 0, 1, \dots, \\ \tilde{u}^0 = 0. \end{cases}$$

Since A has l^p -DMR on J_{∞} , we can find the corresponding solution $\tilde{u} = (\tilde{u}^n)_n \in X^{\mathbb{N}}$ satisfying

$$\|\tilde{u}_{\theta}\|_{l^{p}_{\tau}(\mathbb{N};X)} + \|D_{\tau}\tilde{u}\|_{l^{p}_{\tau}(\mathbb{N};X)} + \|A\tilde{u}_{\theta}\|_{l^{p}_{\tau}(\mathbb{N};X)} \le C_{\mathrm{DMR}}\|\tilde{g}\|_{l^{p}_{\tau}(\mathbb{N};X)}.$$

Then, $u := (\tilde{u}^n)_{n=0}^{N_T} \in l^p(N_T; X)$ is a solution of (27), and fulfills

$$\|u_{\theta}\|_{l^{p}_{\tau}(N_{T};X)} + \|D_{\tau}u\|_{l^{p}_{\tau}(N_{T};X)} + \|Au_{\theta}\|_{l^{p}_{\tau}(N_{T};X)} \le C_{\text{DMR}}\|\tilde{g}\|_{l^{p}_{\tau}(\mathbb{N};X)},$$

which implies (28). The uniqueness of the solution might be readily apparent. \Box

An a priori estimate with non-zero initial value is obtained only in the case where $\theta = 1$. See [4] for $T < \infty$ and [29] for $T = \infty$.

Lemma 8 Let $p \in (1,\infty)$, $\theta \in [0,1]$, $T \in (0,\infty]$, X be a UMD space, $X_0 \subset X$ be a closed subspace, and A be a bounded operator on X_0 . Assume that A has l^p -DMR on J_T . Then, for each $g \in l^p(N_T; X_0)$ and for each $u_0 \in (X_0, D(A))_{1-1/p,p}$, there exists a unique solution $u \in l^p(N_T; X_0)$ of the equation

$$\begin{cases} \frac{u^{n+1}-u^n}{\tau} = Au^{n+1} + g^{n+1}, & n = 0, 1, \dots, N_T - 1, \\ u^0 = u_0, \end{cases}$$

which satisfies

$$\begin{aligned} \|u_1\|_{l^p_{\tau}(N_T;X_0)} + \|D_{\tau}u\|_{l^p_{\tau}(N_T;X_0)} + \|Au_1\|_{l^p_{\tau}(N_T;X_0)} \\ &\leq C_{\text{DMR}}\left(\|g_1\|_{l^p_{\tau}(N_T;X_0)} + \|u_0\|_{1-1/p,p}\right), \end{aligned}$$

where $C_{\text{DMR}} > 0$ is independent of g, u_0 , and X_0 .

3.3 Operator-theoretical properties of A_h

A semigroup T(t) on a Lebesgue space $X = L^q(\Omega, \mu)$ $(q \in [1, \infty])$ is said to be positivity-preserving if

$$u \geq 0 \mu$$
-a.e. in $\Omega \implies T(t)u \geq 0 \mu$ -a.e. in Ω

for each t > 0 and $u \in X$. In the proofs of the following two lemmas, the discrete maximum principle (Remark 2) plays a crucially important role.

Lemma 9 ([41, Theorem 15.5]) Let $q \in [1, \infty]$. Assume that the family of triangulations $\{\mathcal{T}_h\}$ satisfies the acuteness condition (H2). Then, the semigroup e^{tA_h} generated by A_h is positivity-preserving in $X_{h,q}$.

Lemma 10 ([13, Theorem 4.1]) Let $q \in [1, \infty]$. Assume that the family of triangulations $\{\mathcal{T}_h\}$ satisfies the acuteness condition (H2). Then, A_h generates an analytic and contraction semigroup on $X_{h,q}$. Moreover, if $q \in (1, \infty)$, then A_h satisfies the condition (NR1) with the angle θ_q defined as (13).

We introduce several mesh-depending operators on S_h . The L^2 projection P_h is defined as (16). Let R_h be the Ritz projection of $W^{1,1} \to S_h$ defined as

$$(\nabla R_h u, \nabla v_h)_{L^2} = (\nabla u, \nabla v_h)_{L^2}, \quad \forall v_h \in S_h$$

for $u \in W^{1,1}$. These operators have the following well-known properties. See [28,12,7] for the proofs.

Lemma 11 Assume that $\{\mathcal{T}_h\}_h$ satisfies (H1). Then, there exists C > 0 depending only on Ω and q such that

$$\begin{aligned} \|P_{h}v\|_{L^{q}} &\leq C\|v\|_{L^{q}}, \quad \forall v \in L^{q}, \quad \forall q \in [1,\infty], \\ \|P_{h}v\|_{W^{1,q}} &\leq C\|v\|_{W^{1,q}}, \quad \forall v \in W^{1,q}, \quad \forall q \in [1,\infty], \\ \|R_{h}v\|_{W^{1,q}} &\leq C\|v\|_{W^{1,q}}, \quad \forall v \in W^{1,q}, \quad \forall q \in (1,\infty], \\ |v - P_{h}v\|_{L^{q}} &\leq Ch^{2}\|v\|_{W^{2,q}}, \quad \forall v \in W^{2,q}, \quad \forall q \in (d/2,\infty], \\ |v - R_{h}v\|_{L^{q}} &\leq Ch^{2}\|v\|_{W^{2,q}}, \quad \forall v \in D(A_{q}), \quad \forall q \in (\mu',\infty), \end{aligned}$$

$$(29)$$

where μ' is the Hölder conjugate of μ . When $q \neq 2$, (H1) is not required for all inequalities above except for (29).

Mass-lumping operator M_h and K_h have the following properties. For the proof, see [20].

Lemma 12 Let $q \in [1, \infty]$. Then, there exists C > 0 depending only on q and Ω such that

 $C^{-1} \|v_h\|_{L^q} \le \|M_h v_h\|_{L^q} \le C \|v_h\|_{L^q}, \quad v_h \in S_h, \quad q \in [1, \infty].$

Moreover, if $\{\mathcal{T}_h\}_h$ satisfies (H1) when $q \neq 2$, then there exists C > 0 depending only on q and Ω such that

 $C^{-1} \|v_h\|_{L^q} \le \|K_h v_h\|_{L^q} \le C \|v_h\|_{L^q}, \quad v_h \in S_h, \quad q \in [1, \infty].$

We use the standard discrete Laplacian L_h defined as

$$(L_h u_h, v_h) = -(\nabla u_h, \nabla v_h), \quad \forall v_h \in S_h,$$

for $u_h \in S_h$. We designate L_h the discrete Laplacian without mass-lumping. From the Poincaré inequality, L_h is injective. Consequently, it is invertible due to dim $S_h < \infty$. Then, by the definitions given above, it is apparent that

$$L_h = K_h A_h, \quad R_h = L_h^{-1} P_h A. \tag{30}$$

From these relations, the following estimate is obtained.

Lemma 13 Assume that $\{\mathcal{T}_h\}_h$ satisfies (H1) when $q \neq 2$. Then, for $q \in (1, \mu)$, there exists C > 0 satisfying

$$||v_h||_{h,q} \le C ||A_h v_h||_{h,q}, \quad \forall v_h \in S_h,$$

where C depends only on Ω and q.

Proof By (30) and Lemma 12, it suffices to show that

$$\|v_h\|_{L^q} \le C \|L_h v_h\|_{L^q}$$

for all $v_h \in S_h$. Fix $v_h \in S_h$ arbitrarily and set $f_h = L_h v_h$ and $v = A^{-1} f_h \in D(A)$. Then, noting that $P_h f_h = f_h$ and from (30), one obtains

$$v_h = L_h^{-1} P_h f_h = L_h^{-1} P_h A v = R_h v.$$

Therefore, we have

$$\|v_h\|_{L^q} \le \|R_h v\|_{W^{1,q}} \le C \|v\|_{W^{1,q}} \le C \|v\|_{W^{2,q}} \le C \|Av\|_{L^q} = C \|L_h v_h\|_{L^q}$$

by Lemma 11 and (10). \Box

Furthermore, the following estimate holds. See [37, Lemma 4.6] for the proof.

Lemma 14 Assume that $\{\mathcal{T}_h\}_h$ satisfies (H1) when $q \neq 2$. Let $q \in (\mu', \mu)$. Then, there exists C > 0 depending only on q and Ω such that

$$\|A_h^{-1}(I - K_h^{-1})v_h\|_{h,q} \le Ch^2 \|\nabla v_h\|_{L^q}, \quad v_h \in S_h.$$

4 Proofs of Theorems I, II, III and IV

The aim of this section is to establish CMR and DMR for A_h . We first consider the continuous case via the method of imaginary powers of operators. Then, we obtain DMR for A_h by our previous result (Lemma 6). We also present a useful criterion to check the condition $(NR)_{\delta,\varepsilon}$.

In view of Lemma 3, it suffices to show that

$$-A_h \in \mathcal{P}(X_{h,q};K) \cap \mathcal{BIP}(X_{h,q};M,\theta)$$

for some K > 0, $M \ge 1$, and $\theta \in [0, \pi/2)$, uniformly with respect to h. We first show that $-A_h \in \mathcal{P}(X_{h,q}; K)$.

Lemma 15 Let $q \in (1, \mu)$. Assume that the family $\{\mathcal{T}_h\}_h$ satisfies (H1) and (H2) when $q \neq 2$. Then, there exists $K_q > 0$ satisfying

$$-A_h \in \mathcal{P}(X_{h,q}; K_q),$$

where K_q is independent of h > 0.

Proof Let $T_h(t)$ be the semigroup e^{tA_h} generated by A_h in $X_{h,q}$. Then, by Lemma 10, $T_h(t)$ is an analytic and contraction semigroup. Since $T_h(t)$ is contraction semigroup, we have

$$||R(\lambda; A_h)||_{\mathcal{L}(X_{h,q})} \le \frac{1}{\lambda}, \quad \forall \lambda > 0$$

for each h > 0. In addition, by virtue of Lemma 13 and analyticity of $T_h(t)$, we have

$$||R(\lambda; A_h)f_h||_{h,q} = ||A_h^{-1}[\lambda R(\lambda; A_h) - I]f_h||_{h,q} \le C||f_h||_{h,q}, \quad \forall f_h \in S_h$$

for all $\lambda > 0$ and h > 0, where C > 0 is independent of h. Therefore, we obtain $-A_h \in \mathcal{P}(X_{h,q}; K_q)$ with $K_q = C + 1$ since $R(\cdot; A_{h,q}) \in C([0,\infty); \mathcal{L}(X_{h,q}))$. \Box

To show $-A_h \in \mathcal{BIP}(X_{h,q}; M, \theta)$, we use Duong's result, which is based on H^{∞} -functional calculus. The imaginary power is understood as the special case of the function of operators. Let X be a Banach space, $D \subset \mathbb{C}$ be a domain and $\mathcal{O}(D)$ be the space of holomorphic functions on \mathbb{C} . We set

$$H^{\infty}(D) = \mathcal{O}(D) \cap L^{\infty}(D; \mathbb{C}).$$
(31)

Then, for $A \in \mathcal{P}(X)$ and for $m \in H^{\infty}(\Sigma_{\theta})$ with suitable θ , m(A) can be defined as a linear operator on X. When we take $m(z) = z^{it}$, the imaginary power A^{it} is defined in this sense. The definition and details of the properties of m(A) have been presented in the literature [11] and in the Appendix A. We refer to [17] for the proof of the following lemma (see also [10]).

Lemma 16 ([17, Theorem 2]) Let (Ω, μ) be a σ -finite measure space and let A be a linear operator on $X = L^q(\Omega, \mu)$ for $q \in (1, \infty)$. Assume that $A \in \mathcal{P}(X)$ and that -A generates a contraction semigroup T(t) on X. Moreover, we suppose that T(t) is positivity-preserving on X. Then, for each $\theta \in (\pi/2, \pi)$, there exists M > 0 satisfying

$$\|m(A)\|_{\mathcal{L}(X)} \le M \|m\|_{L^{\infty}(\Sigma_{\theta})}$$

for all $m \in H^{\infty}(\Sigma_{\theta})$. Furthermore, M depends only on q and θ , but is independent of A and measure space (Ω, μ) .

Lemma 17 Let X and A be as in Lemma 16. Then, for each $\theta \in (\pi/2, \pi)$, there exists M > 0 such that $A \in \mathcal{BIP}(X; M, \theta)$.

Proof Let $m(z) = z^{it}$ for $z \in \Sigma_{\theta}$ and $t \in \mathbb{R}$. Here, z^{it} is defined as

 $z^{it} = e^{it(\log|z|+i\arg z)}, \quad \arg z \in (-\pi,\pi)$

for $z \in \Sigma_{\pi}$. Then, setting $z = |z|e^{i\vartheta}$ $(\vartheta \in (-\theta, \theta))$, one can readily obtain $|z^{it}| = e^{-t\vartheta}$. Therefore, we have

$$||m||_{L^{\infty}(\Sigma_{\theta})} \leq e^{|t|\theta}$$

which yields $m \in H^{\infty}(\Sigma_{\theta})$ and $A \in \mathcal{BIP}(M, \theta)$ for some M > 0 by Lemma 16. \Box

Now, we are ready to show the following lemma.

Lemma 18 (Imaginary powers of discrete Laplacian) Let $q \in (1, \mu)$. Assume that (H1) and (H2) are satisfied when $q \neq 2$. Then there exist $M_q > 0$ and $\theta_q \in (0, \pi/2)$ satisfying

$$-A_h \in \mathcal{BIP}(X_{h,q}; M_q, \theta_q)$$

where M_q and θ_q are independent of h > 0.

Proof We begin by proving that $-A_h \in \mathcal{BIP}(X_{h,q}; M, \theta)$ for each $\theta \in (\pi/2, \pi)$ and for suitable M > 0 independent of h. Let $T_h(t)$ be the semigroup e^{tA_h} generated by A_h in $X_{h,q}$. Then, by Lemma 9 and 15, we can apply Lemma 17. Therefore, for each $\theta \in (\pi/2, \pi)$, there exists M > 0 satisfying

$$-A_h \in \mathcal{BIP}(X_{h,q}; M, \theta). \tag{32}$$

Now, we show our assertion. We first assume that q = 2. In this case, $X_{h,2}$ is a Hilbert space and $-A_h$ is self-adjoint and positive definite without conditions on the triangulation by Poincáre inequality. Consequently, by Theorem 32, we have

$$\|(-A_h)^{it}\|_{\mathcal{L}(X_{2,h})} \le \int_0^\infty dE_{-A_h}(\lambda) = 1$$

for all $t \in \mathbb{R}$, which implies $-A_h \in \mathcal{BIP}(X_{2,h}; 1, 0)$. Here, E_{-A_h} is the spectral decomposition of $-A_h$. Then we presume that $q \neq 2$. Set

$$\theta_{q,r} = \frac{q^{-1} - 2^{-1}}{r^{-1} - 2^{-1}}$$

for $r \neq 2$. Since $q \neq 2$, we can choose $r \in (1, \infty)$ satisfying $\theta_{q,r} \in (0, 1)$. Then, by the Riesz-Thorin theorem, we obtain

$$\|(-A_h)^{it}\|_{X_{h,q}} \le \|(-A_h)^{it}\|_{X_{h,2}}^{1-\theta_{q,r}}\|(-A_h)^{it}\|_{X_{h,r}}^{\theta_{q,r}} \le M^{\theta_{q,r}}e^{\theta_{\theta_{q,r}}}t^{1-\theta_{q,r}}$$

for any $t \in \mathbb{R}$ and $\theta \in (\pi/2, \pi)$, where M > 0 is as in (32). Since $\theta_{q,r} \in (0, 1)$, we can take θ as

$$\frac{\pi}{2} < \theta < \frac{\pi}{2\theta_{q,r}},$$

which implies

$$-A_h \in \mathcal{BIP}(X_{h,q}; M^{\theta_{q,r}}, \theta\theta_{q,r})$$

with $\theta \theta_{q,r} < \pi/2$. This is the desired assertion. \Box

Owing to Lemma 6 and Theorem I, we are able to obtain DMR for A_h . To apply Lemma 6, it is necessary to verify that the condition $(NR)_{\delta,\varepsilon}$ is satisfied. From Lemma 10, the condition (NR1) is always satisfied. Therefore, what is left is to check the condition (NR2). We begin with the following lemma, which is a generalization of [19, Lemma 2]. No condition on the triangulation is required.

Lemma 19 Let $r \in [1, \infty)$. Then, we have

$$\|\nabla v_h\|_{L^r} \le \frac{d+1}{\kappa_h} \|v_h\|_{h,r}, \quad \forall v_h \in S_h.$$

Proof Fix $K \in \mathcal{T}_h$ arbitrarily. Then it suffices to show that

$$\|\nabla v_h\|_{L^r(K)} \le \frac{d+1}{\kappa_h} \|v_h\|_{h,r,K}, \quad \forall v_h \in S_h,$$

where $||v_h||_{h,r,K} = ||M_h v_h||_{L^r(K)}$. Let Q_j (j = 0, ..., d) be the vertex of K, λ_j be the corresponding barycentric coordinate in K, and κ_j be the length of the perpendicular from P_j in K. Then it is well-known that $|\nabla \lambda_j| = 1/\kappa_j$. Take $v_h \in S_h$ arbitrarily and set $v_j = v_h(Q_j)$. Since $v_h|_K = \sum_{j=0}^d v_j \lambda_j$, we have

$$\begin{aligned} \|\nabla v_{h}\|_{L^{r}(K)} &\leq \sum_{j=0}^{d} |v_{j}| \|\nabla \lambda_{j}\|_{L^{r}(K)} = \sum_{j=0}^{d} \frac{|u_{j}|}{\kappa_{j}} |K|^{1/r} \\ &\leq \left(\sum_{j=0}^{d} \frac{1}{\kappa_{j}^{r'}}\right)^{1/r'} \left(\sum_{j=0}^{d} |u_{j}|^{r}\right)^{1/r} |K|^{1/r} \\ &\leq \frac{(d+1)^{1/r'}}{\kappa_{h}} \left(|K|\sum_{j=0}^{d} |u_{j}|^{r}\right)^{1/r}, \end{aligned}$$
(33)

where r' is the Hölder conjugate of r. Moreover, it is readily apparent that

$$\|v_h\|_{h,r,K} = \left(\frac{1}{d+1}|K|\sum_{j=0}^d |v_j|^r\right)^{1/r}$$

This, together with (33), implies that

$$\|\nabla v_h\|_{L^r(K)} \le \frac{(d+1)^{1/r'+1/r}}{\kappa_h} \|v_h\|_{h,r,K} = \frac{d+1}{\kappa_h} \|v_h\|_{h,r,K}.$$

Thereby we complete the proof. \Box

Now, we describe a sufficient condition for (NR2) to hold.

Lemma 20 (A sufficient condition for $(NR)_{\delta,\varepsilon}$) Assume $\theta \in [0, 1/2)$ and $q \in (1, \infty)$, and Let $\theta_q = \arccos |1-2/q|$. If we choose ε and τ sufficiently small so that A_h satisfies (14), for every h, then the condition $(NR)_{\theta_q,\varepsilon}$ is fulfilled.

Proof The numerical range of A_h is expressed as

$$S(A_h) = \{ (A_h v_h, v_h^*)_h \mid v_h \in S_h, \ \|v_h\|_{h,q} = 1 \}$$

where $v_h^* \in S_h$ is defined as

$$v_h^*(P) = |v_h(P)|^{q-2} v_h(P)$$
 for every node P of \mathcal{T}_h

for $v_h \in S_h$. Therefore, by Lemma 19, we have

$$|(A_h v_h, v_h^*)_h| \le \|\nabla v_h\|_{L^q} \|\nabla v_h^*\|_{h,q'} \le \frac{(d+1)^2}{\kappa_h^2} \|v_h\|_{h,q}^q$$

for all $v_h \in S_h$. Hence we can deduce (NR2) form the assumption (14). \Box

At this stage, we can state the following proofs.

Proof (Proof of Theorem I) It is a consequence of Lemmas 2, 4, and 18. \Box

Proof (Proof of Theorem II) It is a consequence of Theorem I and Lemmas 6 and 20. \Box

Proof (Proof of Theorem III) It is a consequence of Theorem II and Lemma 7. \Box

Proof (Proof of Theorem IV) It is a consequence of Theorem II and Lemma 8. \Box

5 Proof of Theorem V

This section is devoted to error analysis of the solution $u_h = (u_h^n) \in l^p(N_T; S_h)$ of (17). We begin by presenting some lemmas.

Lemma 21 Let X be a Banach space, T > 0, $p \in (1, \infty)$ and $\tau \in (0, 1)$. Set $t_n = n\tau$ for $n = 0, 1, \ldots, N_T$. Then, there exists $C_S > 0$ satisfying

$$\left(\sum_{n=0}^{N_T-1} \|v(t_n)\|_X^p \tau\right)^{1/p} + \left(\sum_{n=1}^{N_T} \|v(t_n)\|_X^p \tau\right)^{1/p} \le C_{\mathrm{S}} \|v\|_{W^{1,p}(J_T;X)}$$
(34)

for all $v \in W^{1,p}(J_T; X)$, where C_S depends only on p, but is independent of T, τ , and X.

Proof By the Sobolev embedding $W^{1,p}(0,1;X) \hookrightarrow L^{\infty}(0,1;X)$, there exists $C_1 > 0$ such that

$$\|v\|_{L^{\infty}(0,1;X)} \le C_1 \|v\|_{W^{1,p}(0,1;X)}$$

for $v \in W^{1,p}(0,1;X)$. One can check that C_1 is independent of X. See the proof of [8, Theorem 8.8]. Then, setting $J_n = (t_n, t_{n+1})$ and considering the change of variables, we have

$$\|v(t_n)\|_X \le \|v\|_{L^{\infty}(J_n;X)} \le C_1(1+\tau)\tau^{-1/p}\|v\|_{W^{1,p}(J_n;X)}$$

for each $n \in \mathbb{N}$. Therefore, we have (34) with $C_{\rm S} = 2C_1$. \Box

The next lemma is shown readily by Taylor's theorem. Therefore, we skip the proof.

Lemma 22 Let X be a Banach space, T > 0, $p \in (1,\infty)$, $\theta \in [0,1]$ and $\tau \in (0,1)$. Set $t_n = n\tau$ for $n = 0, 1, \ldots, N_T$ and

$$r^{n} = \frac{v(t_{n+1}) - v(t_{n})}{\tau} - \left[(1 - \theta) \frac{dv}{dt}(t_{n}) + \theta \frac{dv}{dt}(t_{n+1}) \right]$$

for $v \in W^{j_{\theta}+1,p}(J_T; X)$, where j_{θ} is defined as (18). Then, there exists C > 0 such that

$$\left(\sum_{n=0}^{N_T-1} \|r^n\|_X^p \tau\right)^{1/p} \le C\tau^{j_\theta} \|v\|_{W^{j_\theta+1,p}(J_T;X)},$$

where C is independent of τ and X.

Now we can state the following proof.

Proof (Proof of Theorem V) We set $e_h^n = u_h^n - P_h U^n$ so that

$$u_h^n - U^n = e_h^n + (P_h - I)U^n.$$

Then, by Lemmas 11 and 21, we have

$$\sum_{n=0}^{N_T-1} \|(P_h - I)U^{n+\theta}\|_{L^q}^p \tau$$

$$\leq Ch^{2p} \left[(1-\theta)^p \sum_{n=0}^{N_T-1} \|U^n\|_{W^{2,q}}^p \tau + \theta^p \sum_{n=1}^{N_T} \|U^n\|_{W^{2,q}}^p \tau \right]$$

$$\leq Ch^{2p} \|u\|_{W^{1,p}(J_T;W^{2,q})}^p.$$
(35)

It remains to derive an estimation for e_h^n . Set $V^n = \partial_t u(\cdot, t_n)$ and

$$r_h^{n,\theta} = (K_h^{-1}P_hA - A_hP_h)U^{n+\theta} + P_h\left(\frac{u(t_{n+1}) - u(t_n)}{\tau}\right) - K_h^{-1}P_hV^{n+\theta}.$$

Then, by a simple computation, we have

$$\begin{cases} (D_{\tau}e_h)^n = A_h e_h^{n+\theta} + r_h^{n,\theta}, & n = 0, 1, \dots, N_T - 1, \\ e_h^0 = 0. \end{cases}$$

Therefore,

$$\begin{cases} (D_{\tau}(A_h^{-1}e_h))^n = A_h(A_h^{-1}e_h^{n+\theta}) + A_h^{-1}r_h^{n,\theta}, & n = 0, 1, \dots, N_T - 1, \\ A_h^{-1}e_h^0 = 0. \end{cases}$$

Consequently, according to Theorem III, we obtain

$$\sum_{n=0}^{N_T-1} \|e_h^{n+\theta}\|_{L^q}^p \tau = \sum_{n=0}^{N_T-1} \|A_h(A_h^{-1}e_h^{n+\theta})\|_{L^q}^p \tau \le C \sum_{n=0}^{N_T-1} \|A_h^{-1}r_h^{n,\theta}\|_{L^q}^p \tau.$$
 (36)

We divide $r_h^{n,\theta}$ into two parts as

$$r_h^{n,\theta} = r_{1,h}^{n,\theta} + r_{2,h}^{n,\theta},$$

where

$$r_{1,h}^{n,\theta} = (K_h^{-1} P_h A - A_h P_h) U^{n+\theta}, \ r_{2,h}^{n,\theta} = P_h \left(\frac{u(t_{n+1}) - u(t_n)}{\tau}\right) - K_h^{-1} P_h V^{n+\theta}.$$

We first estimate $r_{1,h}^{n,\theta}$. Noting the relation (30), we have

$$A_h^{-1}r_{1,h}^{n,\theta} = (R_h - P_h)U^{n+\theta},$$

so that

$$\left(\sum_{n=0}^{N_{T}-1} \|A_{h}^{-1}r_{1,h}^{n,\theta}\|_{L^{q}}^{p}\tau\right)^{1/p} \leq Ch^{2} \left(\sum_{n=0}^{N_{T}-1} \|U^{n+\theta}\|_{W^{2,q}}^{p}\tau\right)^{1/p} \leq Ch^{2} \|u\|_{W^{1,p}(J_{T};W^{2,q})} \tag{37}$$

by Lemma 11 and Lemma 21. Also, $A_h^{-1}r_{h,2}^{n,\theta}$ is expressed as

$$A_h^{-1}r_{2,h}^{n,\theta} = A_h^{-1}P_h\left[\frac{u(t_{n+1}) - u(t_n)}{\tau} - V^{n+\theta}\right] + A_h^{-1}(I - K_h^{-1})P_hV^{n+\theta}$$

According to Lemmas 11, 13, 14, 21, and 22, we have

$$\left(\sum_{n=0}^{N_{T}-1} \|A_{h}^{-1}r_{2,h}^{n,\theta}\|_{L^{q}}^{p}\tau\right)^{1/p} \leq C\tau^{j\theta}\|u\|_{W^{j\theta}+1,p}(J_{T};L^{q}) + Ch^{2}\left(\sum_{n=0}^{N_{T}-1} \|\nabla P_{h}V^{n+\theta}\|_{L^{q}}^{p}\tau\right)^{1/p} \leq C\tau^{j\theta}\|u\|_{W^{j\theta}+1,p}(J_{T};L^{q}) + Ch^{2}\left(\sum_{n=0}^{N_{T}-1} \|V^{n}\|_{W^{1,q}}^{p}\tau\right)^{1/p} \leq C\tau^{j\theta}\|u\|_{W^{j\theta}+1,p}(J_{T};L^{q}) + Ch^{2}\|\partial_{t}u\|_{W^{1,p}(J_{T};W^{1,q})}.$$
(38)

Combining (35), (36), (37), and (38), we obtain the error estimate (19). \Box

6 Proofs of Theorems VI and VII

This section is devoted to analysis of semilinear problems (2) and (23). We first prove several auxiliary lemmas.

6.1 Embedding and trace theorems

For $q \in (1, \infty)$, we recall that A_q denotes the realization of the Dirichlet Laplacian defined as (20). Let $D(A_q)$ be a Banach space equipped with the norm $||A_q \cdot ||_{L^q}$. This is a norm if $q \in (1, \mu)$ by the regularity assumption (10). We also set $D(A_{h,q}) = (S_h, ||A_h \cdot ||_{h,q})$, which is a Banach space for $q \in (1, \mu)$ by Lemma 13.

For $N \in \mathbb{N} \cup \{\infty\}$ and $v_h \in S_h^{N+1}$, we set

$$\|v_h\|_{Y^{p,q}_{h,\tau,N}} = \|v_{h,1}\|_{l^p_{\tau}(N;X_{h,q})} + \|A_h v_{h,1}\|_{l^p_{\tau}(N;X_{h,q})} + \|D_{\tau} v_h\|_{l^p_{\tau}(N;X_{h,q})}$$
(39)

and $Y_{h,\tau,N}^{p,q} = \left(S_h^{N+1}, \|\cdot\|_{Y_{h,\tau,N}^{p,q}}\right)$. For abbreviation, we write $Y_{h,\tau}^{p,q} = Y_{h,\tau,\infty}^{p,q}$ and

$$\|v_h\|_{Y_T} = \|v_h\|_{Y_{h,T,N_T}^{p,q}} \tag{40}$$

for T > 0, where N_T is defined as (6).

Then, we have the following embedding result.

Lemma 23 Let $q \in (\mu_d, \mu)$ and p > 2q/(2q - d). Assume that the family $\{\mathcal{T}_h\}_h$ satisfies (H1) and (H2) when $q \neq 2$. Then, the embedding

$$(X_{h,q}, D(A_{h,q}))_{1-1/p,p} \hookrightarrow L^{\infty}$$

holds uniformly for h > 0.

To show Lemma 23, we prove the discrete Gagliardo-Nirenberg type inequality. The following result is the generalization of [26, Lemma 3.3], and that the proof is almost identical. However, for the reader's convenience, we provide the proof.

Lemma 24 (Discrete Gagliardo–Nirenberg type inequality) Let $q \in (\mu_d, \mu)$. Assume that the family $\{\mathcal{T}_h\}_h$ satisfies (H1) and (H2). Then, we have

$$||v_h||_{L^{\infty}} \le C ||A_h v_h||_{h,q}^{\frac{d}{2q}} ||v_h||_{h,q}^{1-\frac{d}{2q}}, \quad \forall v_h \in S_h.$$

Proof It suffices to show that

$$\|L_h^{-1}f_h\|_{L^{\infty}} \le C \|f_h\|_{L^q}^{\frac{d}{2q}} \|L_h^{-1}f_h\|_{L^q}^{1-\frac{d}{2q}},$$

for every $f_h \in S_h$. We decompose the left-hand side as

$$\|L_h^{-1}f_h\|_{L^{\infty}} \le \|(L_h^{-1} - P_h A_q^{-1} f_h)f_h\|_{L^{\infty}} + \|P_h A_q^{-1} f_h f_h\|_{L^{\infty}} =: a + b$$

From the usual Gagliardo-Nirenberg inequality [1, Theorem 5.9] and the regularity assumption (10), we have

$$b \leq C \|A_q^{-1} f_h\|_{L^{\infty}} \leq C \|f_h\|_{L^q}^{\frac{d}{2q}} \|A_q^{-1} f_h\|_{L^q}^{1-\frac{d}{2q}} \\ \leq C \|f_h\|_{L^q}^{\frac{d}{2q}} \left(\|L_h^{-1} f_h\|_{L^q}^{1-\frac{d}{2q}} + \|(A_q^{-1} - L_h^{-1} P_h) f_h\|_{L^q}^{1-\frac{d}{2q}} \right)$$

Setting $u = A_q^{-1} f_h \in D(A_q)$, we have

$$\|(A_q^{-1} - L_h^{-1}P_h)f_h\|_{L^q} = \|u - R_h u\|_{L^q} \le Ch^2 \|u\|_{W^{2,q}} \le Ch^2 \|f_h\|_{L^q}, \quad (42)$$

by Lemma 11 and (10). Since Lemma 19 and the inverse assumption imply

$$||L_h v_h||_{L^q} \le Ch^{-2} ||v_h||_{L^q}, \quad \forall v_h \in S_h,$$

we obtain

$$\|(A_q^{-1} - L_h^{-1} P_h)f_h\|_{L^q} \le C \|L_h^{-1} f_h\|_{L^q}.$$
(43)

From (41) and (43), we have

$$b \le C \|f_h\|_{L^q}^{\frac{d}{2q}} \|L_h^{-1} f_h\|_{L^q}^{1-\frac{d}{2q}}.$$

We estimate a. The inverse assumption (H1) is well known to imply (see [9, theorem 3.2.6]) the inverse inequality

$$\|v_h\|_{L^{\infty}} \le Ch^{-d/r} \|v_h\|_{L^q}, \quad \forall v_h \in S_h,$$

where C > 0 is independent of h. This, together with (42) and (43), implies

$$a = \|P_h(L_h^{-1}P_h - A_q^{-1})f_h\|_{L^{\infty}} \le Ch^{-d/q} \|(L_h^{-1}P_h - A_q^{-1})f_h\|_{L^q}$$
$$\le C\|f_h\|_{L^q}^{\frac{d}{2q}} \|L_h^{-1}f_h\|_{L^q}^{1-\frac{d}{2q}}.$$

Therefore, we can complete the proof. \Box

Proof (Proof of Lemma 23) From the general theory of interpolation spaces, it is readily apparent that the embedding

$$(X_{h,q}, D(A_{h,q}))_{1-1/p,p} \hookrightarrow (X_{h,q}, D(A_{h,q}))_{1-1/p-\varepsilon, 1}$$

for $\varepsilon \in (0, 1 - 1/p)$, uniformly with respect to h. Take $\varepsilon = 1 - 1/p - d/(2q)$ so that $1 - 1/p - \varepsilon = d/(2q)$. Then, the assumptions q > d/2 and p > 2q/(2q - d) imply $\varepsilon \in (0, 1 - 1/p)$. Therefore, we can obtain from Lemma 24 that the embedding

$$(X_{h,q}, D(A_{h,q}))_{d/(2q),1} \hookrightarrow L^{\infty}$$

holds uniformly with respect to h, by the same argument of the embedding theorem for the Besov spaces (see [1, Theorem 7.34]). \Box

We next show the trace theorem for $Y_{h,\tau}^{p,q}$. The following result is the discrete version of the characterization of the real interpolation space via the analytic semigroup [34, Lemma 6.2].

Lemma 25 Let $q \in (1, \mu)$ and $p \in (1, \infty)$. Assume that the family $\{\mathcal{T}_h\}_h$ satisfies (H1) and (H2) when $q \neq 2$. Then there exists C > 0 depending only on p such that

$$\sup_{n \ge 1} \|v_h^n\|_{1-1/p,p} \le C \|v_h\|_{Y_{h,\tau}^{p,q}}.$$

for every $v_h \in Y_{h,\tau}^{p,q}$.

Proof Fix $v_h \in Y_{h,\tau}^{p,q}$ arbitrarily. It suffices to show that

$$\|v_h^1\|_{1-1/p,p} \le C \|v_h\|_{Y_{h,\tau}^{p,q}} \tag{44}$$

by translation. Since

$$v_h^1 = -\sum_{j=1}^n (v_h^{j+1} - v_h^j) + v_h^{n+1}$$

for $n \ge 1$, we have

$$K(t, v_h^1) \le \sum_{j=1}^n \|v_h^{j+1} - v_h^j\|_{h,q} + t \|A_h v_h^{n+1}\|_{h,q}$$
(45)

for t > 0. Here, the function

$$K(t, w_h) = \inf\{\|a_h\|_{h,q} + t\|A_h b_h\|_{h,q} \mid w_h = a_h + b_h, \ a_h, b_h \in X_{h,q}.\},\$$

$$t > 0, \ w_h \in X_{h,q}$$

is the K-function with respect to the interpolation pair $(X_{h,q}, D(A_{h,q}))$ (see [34] and [44]). Then, (45) implies that

$$\|v_{h}^{1}\|_{1-1/p,p}^{p} = \int_{0}^{\infty} \left| t^{-1+1/p} K(t, v_{h}^{1}) \right|^{p} \frac{dt}{t}$$

$$\leq \int_{0}^{\tau} |t^{-1} K(t, v_{h}^{1})|^{p} dt$$

$$+ 2^{p} \sum_{n=1}^{\infty} \left[\int_{n\tau}^{(n+1)\tau} \left(\frac{1}{t} \sum_{j=1}^{n} \|v_{h}^{j+1} - v_{h}^{j}\|_{h,q} \right)^{p} dt + \tau \|A_{h} v_{h}^{n+1}\|_{h,q}^{p} \right]$$

$$\leq 2^{p} \|A_{h} v_{h}\|_{l_{\tau}^{p}(\mathbb{N}; X_{h,q})}^{p} + 2^{p} \sum_{n=1}^{\infty} I_{n}.$$
(46)

In the last step, we used the property $K(t, v_h^1) \leq t \|A_h v_h^1\|_{h,q}$ and we defined I_n as

$$I_n = \int_{t_n}^{t_{n+1}} \left(\frac{1}{t} \sum_{j=1}^n \|v_h^{j+1} - v_h^j\|_{h,q} \right)^p dt$$

for $n \geq 1$. The term I_n is bounded as

$$I_n \le \int_{t_n}^{t_{n+1}} \left(\frac{1}{n\tau} \sum_{j=1}^n \|v_h^{j+1} - v_h^j\|_{h,q} \right)^p dt = \tau \left(\frac{1}{n} \sum_{j=1}^n \|(D_\tau v_h)^j\|_{h,q} \right)^p.$$

Therefore, we can obtain

$$\sum_{n=1}^{\infty} I_n \le \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} \|(D_{\tau} v_h)^j\|_{h,q}^p \tau$$
(47)

by the Hardy inequality [27], and inequalities (46) and (47) imply (44), with a constant C depending only on p. \Box

For $Y_{h,\tau,N}^{p,q}$, we have the following trace theorem.

Lemma 26 Let $N \in \mathbb{N}$, $q \in (1, \mu)$ and $p \in (1, \infty)$. Assume that the family $\{\mathcal{T}_h\}_h$ satisfies (H1) and (H2) when $q \neq 2$. Then, there exists C > 0 independent of N, h, and τ such that

$$\max_{0 \le n \le N} \|v_h^n\|_{1-1/p,p} \le C \left(\|v_h\|_{Y_{h,\tau,N}^{p,q}} + \|v_h^0\|_{1-1/p,p} \right)$$

for every $v_h \in Y_{h,\tau,N}^{p,q}$.

To prove this result, we need to extend each element of $Y_{h,\tau,N}^{p,q}$ to that of $Y_{h,\tau,\infty}^{p,q}$. First, we obtain the following extension lemma, which corresponds to [3, Lemma 7.2].

Lemma 27 Let X be a Banach space and A be a linear operator which has discrete maximal regularity and which satisfies $0 \in \rho(A)$. Let $N \in \mathbb{N} \cup \{\infty\}$ and set

$$\|v\|_{p,N} = \|v_1\|_{l^p_{\tau}(N;X)} + \|Av_1\|_{l^p_{\tau}(N;X)} + \|D_{\tau}v\|_{l^p_{\tau}(N;X)}$$

for $v \in X^{N+1}$ and $Y_N^p = \{v \in X^{N+1} \mid v^0 \in (X, D(A))_{1-1/p,p}, \|v\|_{p,N} < \infty\}$. Then, for $M \in \mathbb{N}$ with M < N, there exists a map $\operatorname{ext}_M \colon Y_M^p \to Y_N^p$ satisfying

$$(\operatorname{ext}_M v)^n = v^n, \quad n = 0, \dots M,$$

and

$$\| \exp_M v \|_{p,N} \le C \left(\|v\|_{p,M} + \|v^0\|_{1-1/p,p} \right)$$

where C is independent of τ and M.

Proof For $v \in Y_M^p$, we define $g \in Y_N^p$ as

$$g^{n} = \begin{cases} (D_{\tau}v)^{n} - Av^{n+1}, & n = 0, \dots, N-1, \\ 0, & \text{otherwise.} \end{cases}$$

Let V be the solution of

$$\begin{cases} (D_{\tau}V)^n = V^{n+1} + g^n, & n \in \mathbb{N}, \\ V^0 = v^0, \end{cases}$$

which is uniquely solvable by discrete maximal regularity of A. Then, if we set $ext_M v = V$, it satisfies the desired properties. Indeed, since $w^n = v^n - V^n$ satisfies

$$\begin{cases} (D_{\tau}w)^n = Aw^n, & n = 0, \dots, M-1 \\ w^0 = 0, \end{cases}$$

we can obtain $w^n = (I - \tau A)^{-n} w^0 = 0$ for n = 0, ..., M. Moreover, by discrete maximal regularity, we have

$$\|V\|_{p,N} \le C \left(\|g\|_{l^{p}_{\tau}(N;X)} + \|V^{0}\|_{1-1/p,p} \right) \le C \left(\|v\|_{p,M} + \|v^{0}\|_{1-1/p,p} \right).$$

Proof (Proof of Lemma 26.) Let $v_h \in Y_{h,\tau,N}^{p,q}$. Then, by Lemmas 25 and 27, we have

$$\max_{0 \le n \le N} \|v_h^n\|_{1-1/p,p} \le \sup_{n \in \mathbb{N}} \|(\operatorname{ext}_N v_h)^n\|_{1-1/p,p}$$
$$\le C \left(\|\operatorname{ext}_N v_h\|_{Y_{h,\tau}^{p,q}} + \|v_h^0\|_{1-1/p,p}\right)$$
$$\le C \left(\|v_h\|_{Y_{h,\tau,N}^{p,q}} + \|v_h^0\|_{1-1/p,p}\right)$$

6.2 Fractional powers

We will use the fractional power $(-A_h)^z$ for $z \in (0,1)$ and $z \in (-1,0)$; see [35]. The negative powers are defined as

$$(-A_h)^{-z}v_h = \frac{\sin(\pi z)}{\pi} \int_0^\infty t^{-z} R(t; A_h) v_h dt$$
(48)

for $z \in (0, 1)$. Since $-A_h$ is an operator of positive type, it is well-defined. One can check that $(-A_h)^{-z}$ is invertible. Consequently, the positive power $(-A_h)^z$ defined by the inverse operator of $(-A_h)^{-z}$ for $z \in (0, 1)$. Fractional powers satisfy the following interpolation properties:

$$\|(-A_h)^z v_h\|_{h,q} \le C \|v_h\|_{h,q}^{1-z} \|A_h v_h\|_{h,q}^z,$$
(49)

$$\|(-A_h)^{-z}v_h\|_{h,q} \le C \|v_h\|_{h,q}^{1-z} \|A_h^{-1}v_h\|_{h,q}^z,$$
(50)

for each $z \in (0, 1)$ and $v_h \in S_h$, uniformly for h. Consequently, we have

$$\|(-A_h)^{-z}v_h\|_{h,q} \le C \|v_h\|_{h,q}, \quad \forall v_h \in S_h$$
(51)

uniformly for h, because of Lemma 13. Below we set $(-A_h)^0 = I$ and $(-A_h)^1 = -A_h$.

Lemma 28 (Discrete Sobolev inequality) Assume that the family $\{\mathcal{T}_h\}_h$ satisfies (H1) and (H2). For every q > d/2 and $\alpha \in (d/(2q), 1)$, there exists C > 0 independent of h, which fulfills the inequality

$$||v_h||_{L^{\infty}} \le C ||(-A_h)^{\alpha} v_h||_{L^q},$$

for all $v_h \in S_h$.

Proof It suffices to show that

$$\|(-A_h)^{-\alpha}f_h\|_{L^{\infty}} \le C \|f_h\|_{h,q}, \quad \forall f_h \in S_h.$$

$$\tag{52}$$

By the definition (48), it is necessary to estimate $||R(t; A_h)f_h||_{L^{\infty}}$. Lemmas 24 and 15 imply

$$\|R(t;A_h)f_h\|_{L^{\infty}} \le C(1+t)^{-1+\frac{d}{2q}} \|f_h\|_{h,q}$$

Consequently,

$$\|(-A_{h})^{-\alpha}f_{h}\|_{L^{\infty}} \leq \frac{\sin(\pi\alpha)}{\pi} \int_{0}^{\infty} t^{-\alpha} \|R(t;A_{h})f_{h}\|_{L^{\infty}} dt$$
$$\leq C \int_{0}^{\infty} t^{-\alpha} (1+t)^{-1+\frac{d}{2q}} dt \|f_{h}\|_{h,q}.$$
(53)

Since $\alpha \in (d/(2q), 1)$, the integral in the right-hand-side of (53) is finite. Therefore, we can obtain the estimate (52). \Box **Lemma 29** Assume that the family $\{\mathcal{T}_h\}_h$ satisfies (H1) and (H2) when $q \neq 2$. For every $\beta \in (0, 1 - 1/p)$, there exists C > 0 independent of h, which satisfies

$$\|(-A_h)^{\beta} v_h\|_{h,q} \le C \|v_h\|_{1-\frac{1}{n},p}$$

for all $v_h \in S_h$. Here, the norm $\|\cdot\|_{1-\frac{1}{p},p}$ is that of $(X_{h,q}, D(A_h))_{1-\frac{1}{p},p}$.

Proof By the general embedding theorem for positive operators [34, Proposition 4.7], we have

$$(X_{h,q}, D(A_h))_{\beta,1} \hookrightarrow D((-A_h)^{\beta}).$$

Moreover, $\beta < 1 - 1/p$ implies

$$(X_{h,q}, D(A_h))_{1-\frac{1}{n},p} \hookrightarrow (X_{h,q}, D(A_h))_{\beta,1}.$$

Chasing the constants in these proofs, one can show that both embedding properties are uniform for h. Therefore, we can establish the desired estimate. \Box

Lemma 30 Assume that the family $\{\mathcal{T}_h\}_h$ satisfies (H1) and (H2). For every $\alpha \in (0, \alpha_{p,q,d})$, there exists C > 0 independent of h, which satisfies

$$\max_{0 \le n \le N} \|v_h\|_{L^{\infty}} \le C \left(\|(-A_h)^{-\alpha} v_h\|_{Y^{p,q}_{h,\tau,N}} + \|(-A_h)^{-\alpha} v_h^0\|_{1-\frac{1}{p},p} \right)$$

for all $N \in \mathbb{N}$ and $v_h \in S_h^{N+1}$.

Proof Since $\alpha + d/(2q) < 1 - 1/p$, we can find $\beta \in (0, 1)$ that satisfies

$$\frac{d}{2q} + \alpha < \beta < 1 + \alpha \quad \text{and} \quad 0 < \beta < 1 - \frac{1}{p}.$$

 $\beta - \alpha \in (d/(2q), 1)$. Then, owing to Lemmas 28, 29, and 26, we have

$$\begin{aligned} \|v_h^n\|_{L^{\infty}} &\leq C \|(-A_h)^{\beta-\alpha} v_h^n\|_{h,q} \leq C \|(-A_h)^{-\alpha} v_h^n\|_{1-\frac{1}{p},p} \\ &\leq C \left(\|(-A_h)^{-\alpha} v_h\|_{Y_{h,\tau,N}^{p,q}} + \|(-A_h)^{-\alpha} v_h^0\|_{1-\frac{1}{p},p} \right). \end{aligned}$$

for $v_h = (v_h^n)_n \in S_h^{N+1}$ and $n \in \mathbb{N}$. \Box

6.3 Completion of the proofs of Theorems VI and VII

Let u and $u_h = (u_h^n)_{n=0}^{N_T}$ be solutions of (2) and (23), respectively. Set $U^n = u(n\tau)$. We consider the error $e_h = (e_h^n)_{n=0}^{N_T} \in S_h^{N_T+1}$ defined as

$$e_h^n = u_h^n - P_h U^n \quad (n = 0, 1, \dots, N_T).$$

We first state the sub-optimal error estimate for a globally Lipschitz nonlinearity f. If f is a globally Lipschitz continuous function, then (2) admits a unique time-global solution and the solution of (23) is bounded from above uniformly in h and τ (see Remark 5). Recall that $\|\cdot\|_{Y_T}$ is defined as (39) and (40). **Lemma 31** In addition to hypotheses of Theorem VI, we assume that f is a globally Lipschitz continuous function. Then, for every $\alpha \in [0,1]$ and $T \in (0,\infty)$,

$$\|(-A_h)^{-\alpha}e_h\|_{Y_T} \le C(h^{2\alpha} + \tau).$$
(54)

Proof The proof is divided into two steps.

Step 1. We prove that there exists $T_1 = T_1(u_0, T) \in (0, T)$ satisfying

$$\|(-A_h)^{-\alpha}e_h\|_{Y_{T_1}} \le C(h^{2\alpha} + \tau).$$
(55)

The error e_h satisfies

$$\begin{cases} (D_{\tau}e_h)^n = A_h e_h^{n+1} + r_h^n, & n = 0, 1, \dots, \\ e_h^0 = 0, \end{cases}$$

where $r_h^n = F_h(u_h^n) - P_h(D_\tau U)^n + A_h P_h U^{n+1}$ and $F_h = K_h^{-1} \circ P_h \circ f$. We decompose r_h^n into two parts:

$$r_h^n = r_{1,h}^n + r_{2,h}^n, \quad r_{1,h}^n = F(u_h^n) - F(P_h U^n), \quad r_{2,h}^n = r_h^n - r_{1,h}^n$$

We perform an estimation for $r_{2,h}^n$. Let $V^n = \partial_t u(\cdot, n\tau)$. Noting that $V^{n+1} = AU^{n+1} + f(U^{n+1})$, the residual term $r_{2,h}^n$ is can be decomposed as

$$\begin{split} r_{2,h}^n &= R_{1,h}^n + R_{2,h}^n + R_{3,h}^n, \\ R_{1,h}^n &= A_h(P_h - R_h)U^{n+1}, \\ R_{2,h}^n &= (K_h^{-1} - I)P_hV^{n+1} + P_h(V^{n+1} - (D_\tau U)^n), \\ R_{3,h}^n &= [F_h(P_hU^n) - F_h(U^n)] + [F_h(U^n) - F_h(U^{n+1})]. \end{split}$$

From the interpolation property (49) and the inverse inequality, we have

$$\|(-A_h)^{\gamma} v_h\|_{h,q} \le Ch^{-2\gamma} \|v_h\|_{h,q},$$

for $\gamma \in (0,1).$ Therefore, the first term $R^n_{1,h}$ is estimated as

$$\begin{aligned} \|(-A_h)^{-\alpha} R_{1,h}^n\|_{h,q} &= \|(-A_h)^{1-\alpha} (P_h - R_h) U^{n+1}\|_{h,q} \\ &\leq C h^{-2(1-\alpha)} \cdot h^2 \|U^{n+1}\|_{W^{2,q}} \\ &= C h^{2\alpha} \|U^{n+1}\|_{W^{2,q}}. \end{aligned}$$
(56)

Similarly, from (50) and Lemmas 11 and 14, we have

$$\|(-A_h)^{-\alpha}(K_h^{-1}-I)P_hV^{n+1}\|_{h,q} \le Ch^{2\alpha}\|V^{n+1}\|_{W^{1,q}}.$$

Combining this inequality with Lemma 22, we have

$$\left(\sum_{n=0}^{N-1} \|(-A_h)^{-\alpha} R_{2,h}^n\|_{h,q}^p \tau\right)^{1/p} \le C(h^{2\alpha} + \tau).$$
(57)

Since f is globally Lipschitz continuous, we have by (51)

$$\left(\sum_{n=0}^{N-1} \|(-A_{h})^{-\alpha} R_{3,h}^{n}\|_{h,q}^{p} \tau\right)^{1/p} \leq CL \left[\left(\sum_{n=0}^{N-1} \|(P_{h}-I)U^{n}\|_{h,q}^{p} \tau\right)^{1/p} + \left(\sum_{n=0}^{N-1} \|U^{n+1} - U^{n}\|_{h,q}^{p} \tau\right)^{1/p} \right] \leq CL(h^{2} + \tau), \quad (58)$$

where L is the Lipschitz constant of f. The equations (56), (57), and (58) yield

$$\left(\sum_{n=0}^{N-1} \|(-A_h)^{-\alpha} r_{2,h}^n\|_{h,q}^p \tau\right)^{1/p} \le C(h^{2\alpha} + \tau).$$
(59)

Now, we are ready to show (55). We designate some constants appearing in this proof. Since A_h has discrete maximal regularity on J_{∞} in $X_{h,q}$ uniformly for h, there exists $C_{\text{DMR}} > 0$ depending only on p, q, Ω satisfying

$$\|v_h\|_{Y_S} \le C_{\text{DMR}} \left(\|g_{h,1}\|_{l^p_{\tau}(N_S;X_{h,q})} + \|x_h\|_{1-1/p,p} \right), \tag{60}$$

for every $g_h = (g_h^n)_n \in l^p(N_S; X_{h,q})$ and $x_h \in S_h$, where $v_h = (v_h^n)_n$ is the solution of

$$\begin{cases} (D_{\tau}v_h)^n = A_h v_h^{n+1} + g_h^{n+1}, & n = 0, \dots, N_S - 1, \\ v_h^0 = x_h. \end{cases}$$

In view of (51) and the Lipschitz continuity of f, we have

$$C_{\text{Lip}} = \sup\left\{\frac{\|(-A_h)^{-\alpha}(F_h(v_h) - F_h(w_h))\|_{h,q}}{\|v_h - w_h\|_{h,q}} \mid \begin{array}{c} h > 0, \ v_h, w_h \in S_h, \\ v_h \neq w_h \end{array}\right\} < \infty,$$

which is the Lipschitz constant of $(-A_h)^{-\alpha} \circ F_h$. Finally, we set

$$C_0 = C_{\rm DMR} C_{\rm Lip} |\Omega|^{1/q}$$

where $|\Omega|$ denotes the *d*-dimensional Lebesgue measure.

Let $e_{j,h}$ (j = 1, 2) be the solution of

$$\begin{cases} (D_{\tau}e_{j,h})^n = A_h e_{j,h}^{n+1} + r_{j,h}^n, & n = 0, \dots, N_T - 1, \\ e_{j,h}^0 = 0. \end{cases}$$
(61)

It is apparent that $e_h = e_{1,h} + e_{2,h}$. Moreover, for every $S < T_{\infty}$, one can obtain

$$\|(-A_h)^{-\alpha} e_{2,h}\|_{Y_S} \le C(h^{2\alpha} + \tau)$$
(62)

by (51) and (59).

Next, it is necessary to derive an estimation for $e_{1,h}$. Take S < T arbitrarily. Since $e_{1,h}$ is the solution of (61), discrete maximal regularity (60) and Lemma 30 yield

$$\begin{split} &\|(-A_{h})^{-\alpha}e_{1,h}\|_{Y_{S}} \\ &\leq C_{\text{DMR}}\left(\sum_{n=0}^{N_{S}-1}\|F_{h}(u_{h}^{n})-F_{h}(P_{h}U^{n})\|_{h,q}^{p}\tau\right)^{1/p} \\ &\leq C_{\text{DMR}}\left[C_{\text{Lip}}\left(\sum_{n=0}^{N_{S}-1}\|e_{1,h}^{n}\|_{h,q}^{p}\tau\right)^{1/p}+C_{\text{Lip}}\left(\sum_{n=0}^{N_{S}-1}\|e_{2,h}^{n}\|_{h,q}^{p}\tau\right)^{1/p}\right] \\ &\leq C_{\text{DMR}}C_{\text{Lip}}|\Omega|^{1/q}S^{1/p}\max_{0\leq n\leq N_{S}-1}\|e_{1,h}^{n}\|_{L^{\infty}}+C(h^{2\alpha}+\tau) \\ &\leq C_{0}S^{1/p}\|(-A_{h})^{-\alpha}e_{1,h}\|_{Y_{S}}+C(h^{2\alpha}+\tau). \end{split}$$

Consequently, taking $S = (2C_0)^{-p}$, we obtain

$$\|(-A_h)^{-\alpha}e_{h,1}\|_{Y_{T_1}} \le C(h^{2\alpha} + \tau)$$
(63)

with $T_1 = (2C_0)^{-p}$. This, together with (62), implies (55). **Step 2.** We prove (54) for any $T \in (0, \infty)$. We denote the constants appearing in Lemma 26 by $C_{\rm tr}$, and set

$$C_1 = C_0 C_{\rm tr}, \quad C_2 = C_{\rm DMR} C_{\rm tr}.$$

Then we show that

$$\|(-A_h)^{-\alpha} e_{1,h}^{+N_S}\|_{Y_{\sigma}} \le C \left(\|(-A_h)^{-\alpha} e_{1,h}\|_{Y_S} + h^{2\alpha} + \tau\right)$$
(64)

for all S < T and $\sigma \le \min\{T_1, T - S\}$. Take S < T and $\sigma \le \min\{T_1, T - S\}$ arbitrarily, and set $w_{j,h}^n = e_{j,h}^{n+N_S}$ (j = 1, 2). Then, $w_{1,h}$ satisfies

$$\begin{cases} (D_{\tau}w_{1,h})^n = A_h w_{1,h}^{n+1} + F_h(u_h^{n+N_S}) - F_h(P_h U^{n+N_S}), & n = 0, \dots, N_{T-S}, \\ w_{1,h}^0 = e_{1,h}^{N_S}. \end{cases}$$

Therefore, discrete maximal regularity (60), Lemmas 26, 30, and (63) yield

$$\begin{split} \|(-A_{h})^{-\alpha}w_{1,h}\|_{Y_{\sigma}} \\ &\leq C_{\text{DMR}} \Bigg[\left(\sum_{n=0}^{N_{\sigma}-1} \left\| (-A_{h})^{-\alpha} \left[F_{h}(u_{h}^{n+N_{S}}) - F_{h}(P_{h}U^{n+N_{S}}) \right] \right\|_{h,q}^{p} \tau \right)^{1/p} \\ &\quad + \left\| (-A_{h})^{-\alpha} e_{1,h}^{N_{S}} \right\|_{1-1/p,p} \Bigg] \\ &\leq C_{\text{DMR}} \Bigg[C_{\text{Lip}} \left(\sum_{n=0}^{N_{\sigma}-1} \|w_{1,h}^{n}\|_{h,q}^{p} \tau \right)^{1/p} + C_{\text{Lip}} \left(\sum_{n=0}^{N_{\sigma}-1} \|w_{2,h}^{n}\|_{h,q}^{p} \tau \right)^{1/p} \\ &\quad + C_{\text{tr}} \| (-A_{h})^{-\alpha} e_{1,h} \|_{Y_{S}} \Bigg] \\ &\leq C_{0} \sigma^{1/p} \left(\| (-A_{h})^{-\alpha} w_{1,h} \|_{Y_{\sigma}} + \| (-A_{h})^{-\alpha} e_{1,h}^{N_{S}} \|_{1-1/p,p} \right) + C(h^{2\alpha} + \tau) \\ &\quad + C_{\text{DMR}} C_{\text{tr}} \| (-A_{h})^{-\alpha} e_{1,h} \|_{Y_{S}} \\ &\leq \frac{1}{2} \| (-A_{h})^{-\alpha} w_{1,h} \|_{Y_{\sigma}} + \left(\frac{C_{\text{tr}}}{2} + C_{2} \right) \| (-A_{h})^{-\alpha} e_{1,h} \|_{Y_{S}} + C(h^{2\alpha} + \tau), \end{split}$$

since $\sigma \leq T_1 = (2C_0)^{-p}$. Therefore, we obtain (64). Noting that $N_{S+\sigma} \leq N_S + N_{\sigma}$, one obtains

$$||v_h||_{Y_{S+\sigma}} \le ||v_h||_{Y_S} + ||v_h^{\cdot+N_S}||_{Y_{\sigma}}$$

for $v_h \in l^p(N_S + N_{\sigma}; S_h)$ and $S, \sigma > 0$. Therefore, we can inductively establish (54) from (63) and (64). Now we can complete the proof owing to Lemma 30. \Box

Finally, we state the following proof.

Proof (Proof of Theorems VI and VII) Observe that

$$||u_h^n - U^n||_{L^q} \le ||e_h^n||_{L^q} + ||P_h U^n - U^n||_{L^q} \le ||e_h^n||_{L^q} + Ch^2 ||U^n||_{W^{2,q}}$$

by Lemma 11. Therefore, it suffices to prove

$$\left(\sum_{n=1}^{N_T} \|e_h^n\|_{L^q}^p \tau\right)^{1/p} \le C(h^2 + \tau) \quad \text{and} \quad \max_{0 \le n \le N_T} \|e_h^n\|_{L^{\infty}} \le C(h^{2\alpha} + \tau)$$

for $\alpha \in (0, \alpha_{p,q,d})$. To this end, let

$$M = \|u\|_{L^{\infty}(\Omega \times (0,T))} + \sup_{h>0} \|P_{h}u\|_{L^{\infty}(\Omega \times (0,T))}$$

for the solution u of (2) and $T \in (0, T_{\infty})$. It is apparent that M is finite since the L^2 -projection P_h is stable in the L^{∞} -norm (Lemma 11). We introduce

$$\tilde{f}(z) = \tilde{f}_M(z) = \begin{cases} f(z), & |z| \le M, \\ f\left(M\frac{z}{|z|}\right), & |z| > M. \end{cases}$$

Then, \tilde{f} is a globally Lipschitz continuous function. We consider the problems (2) and (23) with replacement of f by \tilde{f} , and denote the corresponding solutions by \tilde{u} and \tilde{u}_h , respectively. Moreover, we consider the error $\tilde{e}_h = (\tilde{e}_h^n)_{n=0}^{N_T} \in$ $S_h^{N_T+1}$, where $\tilde{e}_h^n = \tilde{u}_h^n - P_h \tilde{u}(n\tau)$. In view of Lemma 31, the following error estimate holds:

$$\|(-A_h)^{-\alpha}\tilde{e}_h\|_{Y_T} \le C(h^{2\alpha} + \tau)$$

for any $\alpha \in [0, 1]$. By setting $\alpha = 1$, we obtain

$$\left(\sum_{n=1}^{N_T} \|\tilde{e}_h^n\|_{L^q}^p \tau\right)^{1/p} \le C(h^2 + \tau).$$
(65)

Applying Lemma 30, we can deduce

$$\max_{0 \le n \le N_T} \|\tilde{e}_h^n\|_{L^\infty} \le C(h^{2\alpha} + \tau), \tag{66}$$

for $\alpha \in (0, \alpha_{p,q,d})$.

At this stage, we have $\tilde{u} = u$ by the unique solvability of (2). Indeed, $\|u\|_{L^{\infty}(\Omega \times (0,T))} \leq M$ implies $\tilde{f}(u(x,t)) = f(u(x,t))$ for every $(x,t) \in \Omega \times (0,T)$ (0,T). Moreover, according to (66), we estimate as

$$\max_{0 \le n \le N_T} \|\tilde{u}_h^n\|_{L^{\infty}} \le \max_{0 \le n \le N_T} \|\tilde{e}_h^n\|_{L^{\infty}} + \max_{0 \le n \le N_T} \|P_h U^n\|_{L^{\infty}} \le C(h^{2\alpha} + \tau) + \sup_{h \ge 0} \|P_h u\|_{L^{\infty}(\Omega \times (0,T))}$$

for $\alpha \in (0, \alpha_{p,q,d})$. Therefore, there exist $h_0 > 0$ and $\tau_0 > 0$ such that

$$\max_{0 \le n \le N_T} \|\tilde{u}_h^n\|_{L^{\infty}} \le M, \qquad \forall h \le h_0, \quad \forall \tau \le \tau_0,$$

which implies that $\tilde{f}(\tilde{u}_h^n) = f(\tilde{u}_h)$. Again, the unique solvability of (23) yields $\tilde{u}_h = u_h$ for $h \leq h_0$ and $\tau \leq \tau_0$. Hence we can replace \tilde{e}_h^n by e_h^n in (65) and (66), which completes the proof of Theorems VI and VII. \Box

Remark 5 Based on the same assumptions of Lemma 31, the solution $u_h =$ $(u_h^n)_n$ of (23) admits

$$||u_h^n||_{L^{\infty}} \le C ||u_0||_{L^{\infty}} e^{TL}.$$

We briefly show this inequality. Let $T_{h,\tau} = (I - \tau A_h)^{-1}$ and $F_h = K_h^{-1} \circ P_h \circ f$. Then the first equation of (23) is equivalent to

$$u_h^n = T_{h,\tau}^n P_h u_0 + \tau \sum_{n=0}^{n-1} T_{h,\tau}^{n-j} F_h(u_h^j)$$

for $n \in \mathbb{N}$. It follows from Lemma 10 and the Hille-Yosida theorem that

$$\|\lambda^n R(\lambda; A_h)^n\|_{\mathcal{L}(X_{h,\infty})} \le 1$$

for all $n \in \mathbb{N}$ and $\lambda > 0$. Particularly, we have

$$\|T_{h,\tau}^n\|_{\mathcal{L}(X_{h,\infty})} \le 1$$

for all $n \in \mathbb{N}$. Moreover, one can find L > 0, independent of h, such that

$$|F_h(v_h) - F_h(w_h)||_{L^{\infty}} \le L ||v_h - w_h||_{L^{\infty}}, \quad \forall h > 0$$

for $v_h, w_h \in S_h$ by the globally Lipschitz continuity of f and Lemmas 11 and 12. Then, we obtain

$$||u_h^n||_{L^{\infty}} \le C ||u_0||_{L^{\infty}} + \tau \sum_{j=0}^{n-1} L ||u_h^j||_{L^{\infty}}.$$

Therefore, the well-known discrete Gronwall lemma [36, Lemma 2.3] implies

$$||u_h^n||_{L^{\infty}} \le C ||u_0||_{L^{\infty}} e^{n\tau L} \le C ||u_0||_{L^{\infty}} e^{TL}$$

for $n \in \mathbb{N}$.

A H^{∞} -functional calculus

In this appendix, we review the notion of H^{∞} -functional calculus. We present only the definition and the theorem used for this study. For relevant details, one can refer to [11] and references therein. Throughout this section, X denotes a Banach space and Σ_{ω} is the sector defined as (26).

Definition 3 For $\omega \in (0, \pi)$, a linear operator A is of type ω if and only if

- 1. A is closed and densely defined,
- 2. $\sigma(A) \subset \overline{\Sigma}_{\omega}$,

3. for each $\theta \in (\omega, \pi]$, there exists $C_{\theta} > 0$ satisfying $||R(z; A)||_{\mathcal{L}(X)} \leq C_{\theta}/|z|$ for all $z \in \mathbb{C} \setminus \Sigma_{\theta}$ with $z \neq 0$.

Every positive type operator is of type ω for some $\omega \in (0, \pi/2)$. Now, we define the functions of operators of type ω . For $\theta \in (0, \pi)$, we set

$$\Psi(\Sigma_{\theta}) = \bigcup_{C \ge 0, \ s > 0} \left\{ f \in H^{\infty}(\Sigma_{\theta}) \ \bigg| \ |f(z)| \le C \frac{|z|^s}{1 + |z|^{2s}}, \quad \forall z \in \Sigma_{\theta} \right\},$$

where $H^{\infty}(\Sigma_{\theta})$ is defined as (31). Let $\Gamma_{\vartheta} = \{-te^{-i\vartheta} \mid -\infty < t < 0\} \cup \{te^{i\vartheta} \mid 0 \le t < \infty\}$ be a contour for $\vartheta \in (0, \pi)$, which is oriented so that the imaginary parts increase along Γ_{ϑ} .

Definition 4 Let $A \in \mathcal{P}(X; K)$ for some $K \geq 1$. Assume that A is of type ω and let $\omega < \vartheta < \theta$. Then, we define the function of operator A as

$$\psi(A) = \frac{1}{2\pi i} \int_{\Gamma_{\vartheta}} (A - zI)^{-1} \psi(z) dz$$

for $\psi \in \Psi(\Sigma_{\theta})$. We also define m(A) for $m \in H^{\infty}(\Sigma_{\theta})$ as

$$m(A) = \psi_0(A)^{-1}(\psi_0 m)(A)$$

where $\psi_0(z) = z/(1+z)^2$.

In the case in which X is a Hilbert space and the operator A is positive type and selfadjoint, we can define m(A) for $m \in L^{\infty}(\mathbb{R}^+)$ by the spectral decomposition. It is natural to wonder whether these two definitions coincide. The answer is as follows. See for example [2, Theorem 4.6.7 in Chapter III] for the proof.

Lemma 32 Let X be a Hilbert space and $A \in \mathcal{P}(X)$. Assume that A is self-adjoint and let $E_A(\lambda)$ be its spectral decomposition. Then, we have

$$m(A) = \int_0^\infty m(\lambda) dE_A(\lambda)$$

for $m \in H^{\infty}(\Sigma_{\theta})$.

B Remark on the scheme (17)

An alternate of the scheme (17) is given as

$$(D_{\tau}u_h)^n = A_h u_h^{n+\theta} + P_h G^{n+\theta}, \tag{67}$$

or, equivalently,

$$((D_{\tau}u_h)^n, v_h)_h = -(\nabla u_h^{n+\theta}, \nabla v_h)_{L^2} + (P_h G^{n+\theta}, v_h)_h, \quad \forall v_h \in S_h.$$

If taking (67) instead of the first equation of (17), we can only obtain the following error estimate:

$$\left(\sum_{n=0}^{N_T-1} \|u_h^{n+\theta} - U^{n+\theta}\|_{L^q}^p \tau\right)^{1/p} \le C(h+\tau^{j_\theta}),\tag{68}$$

since Lemma 14 is not available. This shortcoming is confirmed by numerical examples as follows.

Let us consider the following two-dimensional heat equation in $\varOmega=(0,1)^2\colon$

$$\begin{cases} \frac{\partial u}{\partial t}(x, y, t) = \Delta u(x, y, t) + g(x, y, t), & (x, y) \in \Omega, \ 0 < t \le T, \\ u(x, y, t) = 0, & (x, y) \in \partial\Omega, \ 0 < t \le T, \\ u(x, y, 0) = x^{5/2}(1-x)^{5/2}y(1-y) & (x, y) \in \Omega, \end{cases}$$
(69)

where T > 0 and

$$g(x,y,t) = x^{1/2}(1-x)^{1/2}e^t \left[x^2(1-x)^2 y(1-y) - \frac{5}{4}(3-4x)(1-4x)y(1-y) + 2 \right].$$

The exact solution is $u(x, y, t) = x^{5/2}(1-x)^{5/2}y(1-y)e^t$. We approximate the equation (69) by the schemes (17) and (67) with meshes such as Figure 2, which satisfies the conditions (H1) and (H2).

We consider the case for $\theta = 0$, 1/2 and 1. When $\theta = 1/2$ and $\theta = 1$, we take τ as $\tau = h$ or $\tau = h^2$. In the case for $\theta = 0$, τ should be chosen to satisfy the condition (14). We take $\varepsilon = \sin \theta_q$ and

$$\tau = \frac{\sin \theta_q}{(1 - 2\theta)(d+1)^2} \kappa_h^2,$$

so that τ satisfies $\tau = O(h^2)$ by the inverse assumption. We set the parameters as follows:

•
$$(p,q) = (4,2),$$

• $T = 0.1 \ (\theta = 0)$ or $T = 0.5 \ (\theta = 1/2, 1)$.





1

 $\substack{0.1\\h}$

0.01

Fig. 3: Behavior of L^4 - L^2 -errors.

0.01

0.1

h

Behavior of the errors is shown in Figure 3. In these figures, cases 1–5 mean the following cases:

$$\begin{array}{ll} {\rm case} \ {\bf 1} : \ \theta = 0 \ (\tau = O(h^2)), \\ {\rm case} \ {\bf 2} : \ \theta = 1/2, \ \tau = h, \\ {\rm case} \ {\bf 3} : \ \theta = 1/2, \ \tau = h^2, \\ {\rm case} \ {\bf 4} : \ \theta = 1, \ \tau = h, \\ {\rm case} \ {\bf 5} : \ \theta = 1, \ \tau = h^2. \end{array}$$

Let us consider the order of the error. In case 4 with the scheme (17), for example, from Theorem V and $\tau = h$, we have

(The error)
$$\leq C(h^2 + \tau) \leq Ch$$

if h is sufficiently small. We summarize these theoretical orders and results in Table 1. When we use the scheme (17), the orders correspond to the theoretical bounds. In the case for the scheme (67), all orders are expected to be O(h). However, except for case 4, the orders are apparently $O(h^2)$. It is of course no problem since the error estimate (68) is just an upper bound. In case 4, it also seems that the order is $O(h^2)$. However, when we compute (67) in case 4 for smaller h, the error decreases more slowly. It seems to approach $O(h^{\alpha})$ for some $\alpha \in [1, 2)$: Figure 4. We leave more rigorous error estimates for the scheme (67) as a subject for future work.

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1

| 1. | | | | | - | | | | |
|-----------------|----------------|--------------------|----------|----------|---|-----------------|--------------------|--------|----------|
| | conditions | | bounds | results | | condi | tions | bounds | results |
| | $\theta = 0$ | $\tau \propto h^2$ | $O(h^2)$ | $O(h^2)$ | | $\theta = 0$ | $\tau \propto h^2$ | O(h) | $O(h^2)$ |
| | $\theta = 1/2$ | $\tau = h$ | $O(h^2)$ | $O(h^2)$ |] | $\theta = 1/2$ | au = h | O(h) | $O(h^2)$ |
| | | $\tau = h^2$ | $O(h^2)$ | $O(h^2)$ |] | | $\tau = h^2$ | O(h) | $O(h^2)$ |
| | $\theta = 1$ | $\tau = h$ | O(h) | O(h) |] | $\theta = 1$ | $\tau = h$ | O(h) | ? |
| | | $\tau = h^2$ | $O(h^2)$ | $O(h^2)$ | 1 | 0 - 1 | $\tau = h^2$ | O(h) | $O(h^2)$ |
| (a) Scheme (17) | | | | | | (b) Scheme (67) | | | |

Table 1: The convergence rates: theoretical bounds and results.



Fig. 4: Behavior of L^4 - L^2 -errors in case 4 with scheme (17) for smaller h.

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