Analysis of Lavrentiev-Finite Element Methods for Data Completion Problems

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May 13, 2016

Abstract

The variational finite element solution of Cauchy's problem, expressed in the Steklov-Poincaré framework and regularized by the Lavrentiev method, has been introduced and computationally assessed in [Inverse Problems in Science and Engineering, **18**, 1063–1086 (2011)]. The present work concentrates on the numerical analysis of the semi-discrete problem. We perform the mathematical study of the error to rigorously establish the convergence of the global bias-variance error.

KEYWORDS: CAUCHY PROBLEM, ILL POSED PROBLEM, LAVRENTIEV REGULARIZATION, FINITE ELEMENTS, BIAS-VARIANCE DECOMPOSITION, ERROR ANALYSIS

1 Introduction

Solving data completion problems consists in implementing numerical procedures for reconstructing missing data on a portion of the boundary of the domain which is inaccessible to measurements. The success of the (computational) reconstruction depends on the data that users are able to collect along the accessible boundary; these are Cauchy's data. As a matter of fact, 'Cauchy's problems' is an alternative terminology for these models. The amount of necessary boundary data is tightly related to the (partial) differential equation considered. When the differential equation is elliptic such as the Laplace equation, the data completion problem turns out to be severely ill-posed (see [33, 9]). In particular, numerical schemes cannot be accurate and stable at the same time. These two properties should be carefully balanced to ensure satisfactory results. A wide literature exists on the subject, in both theoretical and numerical analysis, and also in computations and applications. A non-exhaustive recent bibliography is [2, 28, 19, 43, 32, 37, 38, 25, 42, 27, 14, 41]. A list of earlier works is found in [4]. Various computational methods have been designed and different

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frameworks, variational or not, have been introduced. Many have been experimented and evaluated with varying success. The quasi-reversibility [15, 16, 23], the complex method [19], the boundary integral method [14], the method of the fundamental solution [44, 48], the least-squares or optimal control formulation [47] are among popular approaches.

The variational framework elaborated in [8], letting aside its computational relevance, brings about great facilities to the analysis and the proof of important properties related to the severe ill-posedness of the problem. The variational problem introduced is set on the incomplete boundary and is the result of the Kohn-Vogelius duplicating procedure associated to the Steklov-Poincaré condensation concept. We refer to [4, 10, 26, 7] for a comprehensive exposition, especially when Lavrentiev regularization is used. A finite-element discretization has also been presented in [3]. This reference discusses related computational issues such as existence and uniqueness criteria for the resulting linear system, and presents a wide range of numerical experiments that show the reliability and relevance of the numerical method. The ultimate goal, namely the convergence analysis of the full approximation, turns out to be really hard, and thus was missing so far. Methodologically, we choose to explore the problem gradually([1]) and the present work is a first step in this direction. We consider the (semi-discrete) problem, where the use of finite elements is confined to the incomplete boundary. This problem inherits and still suffers from the main instability features of the continuous completion process. We focus on the complications specifically arising from the ill-posedness and attempt to handle them in a mathematical way.

The outline of the paper is as follows. Section 2 recalls the variational formulation of Cauchy's problem, where the unknown is the missing Dirichlet boundary condition, and lists useful properties of the variational solution. Section 3 mainly describes the Lavrentiev regularization and its Galerkin finite element approximation. In Section 4, we conduct the numerical analysis of the discrete solution and state a convergence result in polygonally shaped domains. This study uses sharp tools from the theory of elliptic regularity. We address afterward the effect of noisy data and discuss how they affect the selection of the regularization parameter and the mesh-size, to guarantee satisfactory computations. Finally, we investigate the issue of local convergence, away from the incomplete boundary. By means of a Carleman inequality, Hölderian convergence rates with respect to the regularizing parameter ϱ and mesh-size h are therefore derived.

Notation.— Let Ω be a bounded Lipschitzian domain in $\mathbb{R}^d$, $d = 2, 3$. The symbol $\boldsymbol{x}$ denotes the generic point of Ω. As usual, $L^2(\Omega)$ is the Lebesgue space of square integrable functions, with inner product $(\cdot,\cdot)_{L^2(\Omega)}$ and associated norm $\|\cdot\|_{L^2(\Omega)}$. The Sobolev space $H^1(\Omega)$ is the space of the functions that are in $L^2(\Omega)$ as well as their first order derivatives. Let $\Upsilon \subset \partial\Omega$ be a closed subset

[1] Actually, have we the choice?

(surface or curve) of the boundary. The space $H^{1/2}(\Upsilon)$ is the set of the traces over Υ of all the functions of $H^1(\Omega)$ and we consider the notation $H^{-1/2}(\Upsilon)$ for the dual space of $H^{1/2}(\Upsilon)$. These are the basic functional tools we use repeatedly here. We refer to [1] for further constructions of fractional Sobolev spaces.

2 Setting of the problem

Let Ω be a given domain, with boundary $\Gamma = \partial\Omega$. To avoid unnecessary technicalities we assume that Γ is divided into two disjoint components Γ_C and Γ_I (where C stands for complete and I for incomplete) as indicated in Fig. 1. We have then $\overline{\Gamma}_C \cap \overline{\Gamma}_I = \emptyset$. *Unless explicitly stated, both portions of the boundary* Γ_I *and* Γ_C *are taken smooth.* We focus on the case when all measurements are recorded on the complete boundary Γ_C whereas data are missing on Γ_I, the incomplete boundary. The data completion model we study is therefore designed to recover the missing data on Γ_I, by using the abundant Cauchy data on Γ_C.

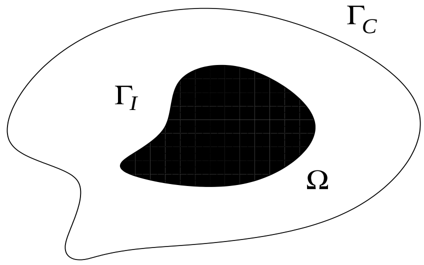

Figure 1: The boundary Γ_C is accessible to measures and Γ_I is out of reach

Let the conductivity function a be given in $L^\infty(\Omega)$; it is supposed smooth enough, say $a \in \mathscr{C}^1(\overline{\Omega})$, with $a(\boldsymbol{x}) \geq a_* > 0$ for all $\boldsymbol{x} \in \Omega$. The following norm, equivalent to the $H^1(\Omega)$ norm for functions that vanish on Γ_I (or on Γ_C), is well adapted to our problem:

$$|v|_{a,H^1(\Omega)} = \|\sqrt{a}\nabla v\|_{\boldsymbol{L}^2(\Omega)}, \tag{1}$$

and shall be used in this work.

Now, let a pair of Cauchy's boundary conditions $(g, \varphi) \in H^{1/2}(\Gamma_C) \times H^{-1/2}(\Gamma_C)$ be prescribed. The Data Completion problem we propose to solve consists in the following: *find* $u \in H^1(\Omega)$ *solution of*

$$\begin{aligned} -\operatorname{div}(a\nabla u) = 0 \qquad & \text{in } \Omega, \\ u = g, \quad a\partial_{\boldsymbol{n}} u = \varphi \qquad & \text{on } \Gamma_C. \\ u = ? \qquad & \text{on } \Gamma_I. \end{aligned} \tag{2}$$

This Cauchy problem is ill-posed in the sense that it may not have a solution. If it has one then by Holmgren's theorem it is unique (see [34]). Thus there is always uniqueness but not always existence. Existence depends on the data (g, φ), but the condition that guarantees existence can hardly be checked on the data. Furthermore, the set of data (g, φ) for which (2) has a solution is dense in $H^{1/2}(\Gamma_C) \times H^{-1/2}(\Gamma_C)$, but the complementary set of pairs (g, φ) that does not yield existence is also dense in $H^{1/2}(\Gamma_C) \times H^{-1/2}(\Gamma_C)$.

In reference [8], problem (2) is set in a variational formulation derived by associating the duplication argument of Kohn-Vogelius [40] with the condensation Steklov-Poincaré approach (see [46]). This variational formulation uses a pair of functions: the solution $u_D(\mu, g) \in H^1(\Omega)$ of a Poisson problem with pure Dirichlet boundary conditions,

$$\begin{aligned} -\operatorname{div}(a\nabla u_D(\mu,g)) &= 0 && \text{in } \Omega, \\ u_D(\mu,g) &= g && \text{on } \Gamma_C, \\ u_D(\mu,g) &= \mu && \text{on } \Gamma_I, \end{aligned} \tag{3}$$

and the solution $u_N(\mu, \varphi) \in H^1(\Omega)$ of a Poisson problem with mixed Dirichlet–Neumann boundary conditions,

$$\begin{aligned} -\operatorname{div}(a\nabla u_N(\mu,\varphi)) &= 0 && \text{in } \Omega, \\ a\partial_{\boldsymbol{n}} u_N(\mu,\varphi) &= \varphi && \text{on } \Gamma_C, \\ u_N(\mu,\varphi) &= \mu && \text{on } \Gamma_I. \end{aligned} \tag{4}$$

Both functions $u_D(\mu, g)$ and $u_N(\mu, \varphi)$ are well defined owing to the coerciveness of these problems (3) and (4). The variational formulation we propose is then based on the following result:

Proposition 2.1 *Problem* (2) *has a solution if and only if there exists $\lambda \in H^{1/2}(\Gamma_I)$ for which the following fluxes identity holds:*

$$a\partial_{\boldsymbol{n}} u_D(\lambda, g) = a\partial_{\boldsymbol{n}} u_N(\lambda, \varphi), \qquad \text{on } \Gamma_I. \tag{5}$$

If (5) *is satisfied then we have $u = u_D(\lambda, g) = u_N(\lambda, \varphi)$.*

Proof: Assume that (5) holds, then the difference $w = u_D(\lambda, g) - u_N(\lambda, \varphi)$ satisfies the homogeneous data completion problem

$$\begin{aligned} -\operatorname{div}(a\nabla w) &= 0 && \text{in } \Omega, \\ w = 0, \quad a\partial_{\boldsymbol{n}} w &= 0 && \text{on } \Gamma_I. \end{aligned}$$

Holmgren's theorem implies that necessarily $w = 0$. The converse is obvious. Details can be found in [8]. ∎

The variational translation of the fluxes equality (5) prescribed on the incomplete boundary Γ_I reads then as: *find $\lambda \in H^{1/2}(\Gamma_I)$ such that*

$$s(\lambda, \mu) = \ell(\mu), \qquad \forall \mu \in H^{1/2}(\Gamma_I), \tag{6}$$

where the bilinear and linear forms $s(\cdot, \cdot)$ and $\ell(\cdot)$ are respectively defined by: $\forall \chi, \mu \in H^{1/2}(\Gamma_I)$,

$$\begin{aligned} s(\chi, \mu) &= \int_\Omega a\nabla u_D(\chi) \cdot \nabla u_D(\mu)\, d\boldsymbol{x} - \int_\Omega a\nabla u_N(\chi) \cdot \nabla u_N(\mu)\, d\boldsymbol{x}, \\ \ell(\mu) &= -\int_\Omega a\nabla \breve{u}_D(g) \cdot \nabla u_D(\mu)\, d\boldsymbol{x} - \langle \varphi, u_N(\mu) \rangle_{1/2,\Gamma_C}. \end{aligned}$$

Here $u_N(\mu)$ (respectively, $u_D(\mu)$) stands for $u_N(\mu,0)$ (respectively, $u_D(\mu,0)$) and $\breve{u}_N(\varphi)$ (respectively, $\breve{u}_D(g)$) replaces $u_N(0,\varphi)$ (respectively, $u_D(0,g)$). The forms $s(\cdot,\cdot)$ and $\ell(\cdot)$ are made of two contributions each, $(s_D(\cdot,\cdot), s_N(\cdot,\cdot))$ and $(\ell_D(\cdot), \ell_N(\cdot))$ with no ambiguity about their definitions. The equation (6) is derived by suitable applications of Green's formula and a substitution of (5). Since $s_D(\mu,\mu) = |u_D(\mu)|^2_{a,H^1(\Omega)}$ for all $\mu \in H^{1/2}(\Gamma_I)$ (with the norm defined in (1)), the mapping $\mu \mapsto \sqrt{s_D(\mu,\mu)}$ is a norm equivalent to the natural norm $\|\cdot\|_{H^{1/2}(\Gamma_I)}$ (see [46]). For convenience, it is denoted by $\|\cdot\|_{s_D}$ and it will be used in the sequel as norm on $H^{1/2}(\Gamma_I)$,

$$\|\mu\|_{s_D} = \sqrt{s_D(\mu,\mu)}, \qquad \forall \mu \in H^{1/2}(\Gamma_I), \tag{7}$$

with associated scalar product $s_D(\cdot,\cdot)$.
Regarding the symmetric bilinear form $s(\cdot,\cdot)$, there holds that

$$s(\lambda,\mu) \le |u_D(\lambda)|_{a,H^1(\Omega)}|u_D(\mu)|_{a,H^1(\Omega)} + |u_N(\lambda)|_{a,H^1(\Omega)}|u_N(\mu)|_{a,H^1(\Omega)}.$$

Considering the bound,

$$|u_N(\mu)|_{a,H^1(\Omega)} \le |u_D(\mu)|_{a,H^1(\Omega)} = \|\mu\|_{s_D},$$

we finally obtain that

$$s(\lambda,\mu) \le 2|u_D(\lambda)|_{a,H^1(\Omega)}|u_D(\mu)|_{a,H^1(\Omega)} = 2\,\|\lambda\|_{s_D}\|\mu\|_{s_D}. \tag{8}$$

Thus $s(\cdot,\cdot)$ is continuous with respect to $\|\cdot\|_{s_D}$. Next, owing to the obvious identity

$$\int_\Omega a\nabla u_N(\mu)\cdot\nabla(u_D(\mu) - u_N(\mu))\, d\boldsymbol{x} = 0,$$

it can be stated that

$$s(\mu,\mu) = \int_\Omega a\nabla\big(u_D(\mu) - u_N(\mu)\big)\cdot\nabla\big(u_D(\mu) - u_N(\mu)\big)\, d\boldsymbol{x}.$$

Hence, we deduce that

$$s(\mu,\mu) = |u_N(\mu) - u_D(\mu)|^2_{a,H^1(\Omega)}. \tag{9}$$

It follows that the symmetric form $s(\cdot,\cdot)$ is non-negative. Moreover, the condition $s(\mu,\mu) = 0$ implies that $u_N(\mu) = u_D(\mu) (= w)$. As a result, w satisfies the data completion problem with homogeneous Cauchy data on Γ_C. This problem has $w = 0$ as the only solution. This yields that $\mu = 0$. Therefore $s(\cdot,\cdot)$ determines an inner product with associated norm denoted by $\|\cdot\|_s$,

$$\|\mu\|_s = \sqrt{s(\mu,\mu)}, \qquad \forall \mu \in H^{1/2}(\Gamma_I), \tag{10}$$

but we must keep in mind that it is not a Hilbertian norm on $H^{1/2}(\Gamma_I)$. The, with this very weak norm, the following sharp bound holds

$$\sup_{\mu\in H^{1/2}(\Gamma_I)} \frac{\ell(\mu)}{\|\mu\|_s} = |\breve{u}_D(g) - \breve{u}_N(\varphi)|_{a,H^1(\Omega)}. \tag{11}$$

This is useful when studying noisy data. The proof, which is not straightforward, can be found in [10, Lemma 2.1].

3 Lavrentiev-Finite Element Method

The inexact Cauchy conditions (g, φ) have inevitably dramatic effects on the quality of approximation of the problem. Indeed, even if (g, φ) belongs to the set where existence of a solution of problem (6) is ensured, this existence may typically fail when (g, φ) is slightly perturbed, since the set of non existence is dense in $H^{1/2}(\Gamma_C) \times H^{-1/2}(\Gamma_C)$. Crude computations are therefore risky and should be avoided. Regularization is mandatory. To dampen the pollution that may damage the solution λ, because Cauchy's data suffer from noise, Lavrentiev method turns out to be well suited to the symmetric positive-definite problem (6), as illustrated in [10].

3.1 Lavrentiev Regularization

Let ϱ be a small positive real parameter and consider the regularized problem: *find* $\lambda_\varrho \in H^{1/2}(\Gamma_I)$ *such that,*

$$\varrho s_D(\lambda_\varrho, \mu) + s(\lambda_\varrho, \mu) = \ell(\mu), \qquad \forall \mu \in H^{1/2}(\Gamma_I). \tag{12}$$

The term $\varrho s_D(\cdot, \cdot)$ ensures the ellipticity of the problem, λ_ϱ exists in $H^{1/2}(\Gamma_I)$ whatever the data $\ell(\cdot)$. For compatible data (g, φ), the convergence proof of the Lavrentiev solution λ_ϱ towards λ may be found in [4]. We refer also to a more extensive study in [10] for the regularization strategy; the Lavrentiev method strengthened by the Morozov Discrepancy Principle, and for the selection of the parameter ϱ.

Lemma 3.1 *Assume that problem* (6) *has a solution* $\lambda \in H^{1/2}(\Gamma_I)$. *There holds that*

$$\lim_{\varrho \to 0} \frac{\|\lambda - \lambda_\varrho\|_{s_D}}{\|\lambda\|_{s_D}} = 0. \tag{13}$$

Moreover, we have

$$\lim_{\varrho \to 0} \frac{1}{\sqrt{\varrho}} \frac{\|\lambda - \lambda_\varrho\|_{s}}{\|\lambda\|_{s_D}} = 0. \tag{14}$$

Proof: The limit (13) is proved in [4, Proposition 3.2]. Let us prove (14). By subtracting (6) from (12), we obtain

$$\varrho s_D(\lambda_\varrho - \lambda, \mu) + s(\lambda_\varrho - \lambda, \mu) = -\varrho s_D(\lambda, \mu).$$

The choice $\mu = \lambda_\varrho - \lambda$ gives

$$\varrho \|\lambda_\varrho - \lambda\|_{s_D}^2 + \|\lambda_\varrho - \lambda\|_s^2 = \frac{\varrho}{2} \left[s_D(\lambda, \lambda) + s_D(\lambda_\varrho - \lambda, \lambda_\varrho - \lambda) - s_D(\lambda_\varrho, \lambda_\varrho) \right].$$

This yields

$$\frac{\varrho}{2} \|\lambda_\varrho - \lambda\|_{s_D}^2 + \|\lambda_\varrho - \lambda\|_s^2 = \frac{\varrho}{2} (\|\lambda\|_{s_D}^2 - \|\lambda_\varrho\|_{s_D}^2). \tag{15}$$

Then the limit (14) follows by using (13). ∎

Remark 3.1 Note that (15) implies the particular bound

$$\forall \varrho > 0, \quad \|\lambda_\varrho\|_{s_D} \le \|\lambda\|_{s_D}. \tag{16}$$

Remark 3.2 Apart from the convergence of λ_ϱ towards λ in the strong norm $\|\cdot\|_{s_D}$, no Hölderian estimate can be expected. Without further regularity assumption on λ, the convergence result (13) is the best we can aim at. In contrast, the gap $(\lambda_\varrho - \lambda)$ decays faster than $\sqrt{\varrho}$, with respect to the norm $\|\cdot\|_s$. The extreme weakness of the norm $\|\cdot\|_s$ explains this result.

3.2 Finite Element semi-discretization

In practice, running computations for the numerical simulation of (12) prompts practitioners to select a fitting approximation method. Here, we propose a finite element method of degree one. However, since the numerical analysis of the full discretization of u_D and u_N is highly technical, in this work we study a semi-discretization where only λ_ϱ is discretized while u_D and u_N are computed by solving exactly (3) and (4) with these discrete boundary functions. Moreover, to avoid inessential complications and to focus on difficulties inherent to Cauchy's problem, especially the approximation effects on the solution λ_ϱ, we assume that the boundary Γ_I is polygonal when $d = 2$ or polyhedral when $d = 3$. Thus there is no error in approximating the boundary, but the price to pay is that a corner domain generally reduces the regularity of the solution of such problems as (3) and (4). And as convergence results for finite element methods are tightly connected to the regularity of the underlying functions (here u_D and u_N), corners of the polygonal or polyhedral domain will be responsible for lowering these convergence rates (see [49]), thus making the ultimate results possibly suboptimal. We henceforth call the approximation a *semi-discretization procedure.*

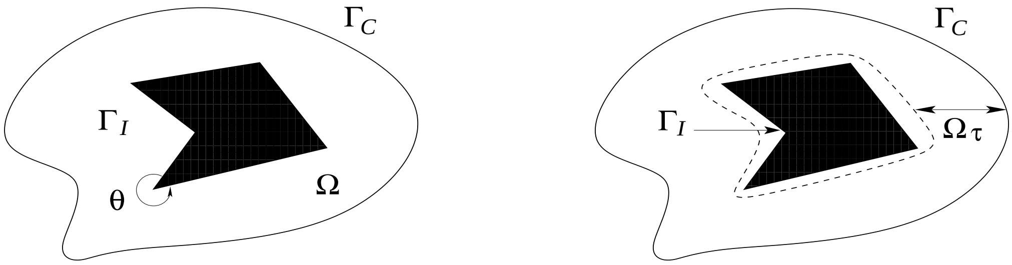

Figure 2: Polygonal incomplete boundary Γ_I (left). The sub-domain Ω_τ (right).

Let $h > 0$ be the discretization parameter and let $\mathcal{T}_h$ be a subdivision of Γ_I made of simplicial elements with maximum mesh size (i.e., diameter h),

$$\Gamma_I = \bigcup_{\kappa \in \mathcal{T}_h} \kappa.$$

We assume that $\mathcal{T}_h$ is compatible with the polyhedral shape of Γ_I in the sense that each κ is located in exactly one face of Γ_I. Each face is the exact union of elements in a subset of $\mathcal{T}_h$. In addition, we suppose that $\mathcal{T}_h$ is regular in the sense of Ciarlet [20]; there exists a constant $\sigma > 0$, independent of h such that

$$\frac{h_\kappa}{\rho_\kappa} := \sigma_\kappa \leq \sigma, \qquad \forall \kappa \in \mathcal{T}_h,$$

where h_κ denotes the diameter of κ and ρ_κ the diameter of the ball inscribed in κ.
Let $\mathcal{P}_1$ be the space of polynomials with degree ≤ 1. We introduce the following finite elements subspace on Γ_I

$$H_h = \Big\{ \psi_h \in \mathscr{C}(\Gamma_I) \ : \ \forall \kappa \in \mathcal{T}_h, \quad (\psi_h)_{|\kappa} \in \mathcal{P}_1 \Big\}.$$

The discretization transforms the problem (12) into a linear system, where ill-posedness is circumvented by the Lavrentiev regularization method. The (semi) discrete problem to explore reads as follows: *find* $\lambda_{\varrho,h} \in H_h$ *such that*

$$\varrho s_D(\lambda_{\varrho,h}, \mu_h) + s(\lambda_{\varrho,h}, \mu_h) = \ell(\mu_h), \qquad \forall \mu_h \in H_h. \tag{17}$$

This is nothing but the Ritz-Galerkin discretization with exact integration

$$\begin{aligned}(1+\varrho)\int_\Omega a\nabla u_D(\lambda_{\varrho,h}) \cdot \nabla u_D(\mu_h)\, d\boldsymbol{x} - \int_\Omega a\nabla u_N(\lambda_{\varrho,h}) \cdot \nabla u_N(\mu_h)\, d\boldsymbol{x} \\ = -\int_\Omega a\nabla \breve{u}_D(g) \cdot \nabla u_D(\mu_h)\, d\boldsymbol{x} - (\varphi, u_N(\mu_h))_{1/2,\Gamma_C}, \qquad \forall \mu_h \in H_h.\end{aligned}$$

In the remainder of this work, we investigate the behavior of $\lambda_{\varrho,h}$ as the parameters ϱ and h decay both to zero. Our aim is to exhibit a sufficient condition on these parameters that ensures the convergence of the Lavrentiev-finite element solution $\lambda_{\varrho,h}$ towards λ, the exact solution of (6), in the case when this solution exists.

Remark 3.3 Note that Problem (17) still has a solution when $\varrho = 0$, whatever the Cauchy boundary conditions. The reason is that $\|\cdot\|_s$ is a norm on the finite dimensional space H_h and therefore the matrix of the linear system (17) is invertible. However, this invertibility is not uniform with respect to h, things may go wrong for small h, and the computed solution may blow up and regularization (by discretization) needs to be strengthened.

4 Numerical Analysis

The purpose of this section is to bound the error $(\lambda_\varrho - \lambda_{\varrho,h})$, called the bias in the ill-posed problems community. The variance, i.e., the error caused by noisy data is addressed later on. We thus assume for a while that the data are exact and problem (6) has a unique solution λ. The starting point of the analysis is Céa's abstract result (see [20]).

Lemma 4.1 *The following inequality holds for all $\mu_h \in H_h$:*

$$\|\lambda_\varrho - \lambda_{\varrho,h}\|_{s_D}^2 + \frac{1}{\varrho}\|\lambda_\varrho - \lambda_{\varrho,h}\|_s^2 \leq \|\lambda_\varrho - \mu_h\|_{s_D}^2 + \frac{1}{\varrho}\|\lambda_\varrho - \mu_h\|_s^2.$$

Remark 4.1 The right-hand side of Céa's estimate shows the critical effect of the division by the parameter ϱ. Indeed, in the computations ϱ is usually small, and the numerical analysis must take into account its asymptotic decay to zero. Therefore the order of $\|\lambda_\varrho - \mu_h\|_s^2$ as h tends to zero must be strong enough to compensate the factor $1/\varrho$. As in Remark 3.2, our hope is to obtain the best convergence rate of this term with respect to h.

In view of Lemma 4.1, a bound is needed for the approximation error of λ_ϱ by a suitably chosen μ_h. The L^2-orthogonal projection on H_h will be selected, i.e. $(\mu_h = \pi_h\lambda_\varrho)$. The point is hence to derive optimal approximation errors, for functions with low regularity. Let q be real-number in $[0,1]$, we have

$$\forall \mu \in H^q(\Gamma_I), \quad \|\mu - \pi_h\mu\|_{L^2(\Gamma_I)} \leq Ch^q\|\mu\|_{H^q(\Gamma_I)}. \tag{18}$$

This requires the use of some special 'interpolation' operators, rather than the standard Lagrange interpolant. Indeed, the fact that point-wise values of μ may not be accessible because μ is not necessarily smooth suggests using instead some sort of regularized approximation operator such as the one introduced by Scott & Zhang type [50] and extended by Girault & Lions in [29] to L^1 functions. The proof of the approximation result (18) in fractional Sobolev spaces follows [21]. Adapting those proofs to our non planar polyhedral geometry of Γ_I is checked out in [11]. Moreover, optimal estimates hold in some dual Sobolev spaces. The proof is short and may be found in [13], but for the sake of completeness, we provide it here.

Lemma 4.2 *For any real numbers $p, q \in [0,1]$, the following estimate holds for all $\mu \in H^q(\Gamma_I)$*

$$\|\mu - \pi_h\mu\|_{H^{-p}(\Gamma_I)} \leq Ch^{q+p}\|\mu\|_{H^q(\Gamma_I)}. \tag{19}$$

Proof: We use the duality Aubin-Nitsche argument. Let $\chi \in H^p(\Gamma_I)$, we have

$$\int_{\Gamma_I} (\mu - \pi_h\mu)\chi \, d\gamma = \int_{\Gamma_I} (\mu - \pi_h\mu)(\chi - \pi_h\chi) \, d\gamma \leq \|\mu - \pi_h\mu\|_{L^2(\Gamma_I)}\|\chi - \pi_h\chi\|_{L^2(\Gamma_I)}.$$

By estimate (18), we derive the bound

$$\int_{\Gamma_I} (\mu - \pi_h \mu)\chi \, d\gamma \leq Ch^q \|\mu\|_{H^q(\Gamma_I)} h^p \|\chi\|_{H^p(\Gamma_I)} = Ch^{q+p} \|\mu\|_{H^q(\Gamma_I)} \|\chi\|_{H^p(\Gamma_I)}.$$

Observing that

$$\|\mu - \pi_h \mu\|_{H^{-p}(\Gamma_I)} = \sup_{\chi \in H^p(\Gamma_I)} \frac{1}{\|\chi\|_{H^p(\Gamma_I)}} \int_{\Gamma_I} (\mu - \pi_h \mu)\chi \, d\gamma,$$

completes the proof. ∎

The convergence of $(\mu - \pi_h \mu)$ in norms stronger than the L^2 norm, such as the $H^{1/2}$ norm, follows from the stability of the operator π_h established in the early works of Crouzeix & Thomée in [22, 1987] or more recent publications by Bramble & Pasciak and Bank & Yserentant in [17, 6]. Thus, at the cost of a mild assumption on the meshes, by using these results and the equivalence of the norms $\|\cdot\|_{s_D}$ and $\|\cdot\|_{H^{1/2}(\Gamma_I)}$, we have the stronger convergence,

$$\lim_{h \to 0} \frac{\|\mu - \pi_h \mu\|_{s_D}}{\|\mu\|_{s_D}} = 0. \tag{20}$$

We do not describe accurately the criterion on the meshes, suffices it to know that the result is available for a large class of meshes such as graded and refined meshes used in adaptativity.

Let us choose $\mu_h = \pi_h \lambda_\varrho$ in Lemma 4.1. The convergence (20) can be used for the term $\|\lambda_\varrho - \mu_h\|_{s_D}$. But in contrast, convergence of $\|\lambda_\varrho - \mu_h\|_s$, that follows from the continuity of the bilinear form $s(\cdot,\cdot)$, is not sufficient because of the division by ϱ. Recalling that both ϱ and h tend to 0, a far better estimate of $\|\lambda_\varrho - \mu_h\|_s$ is required to compensate for this division. *The quality of the approximation depends in general upon the regularity of the function to approximate, λ_ϱ in our case. However, due to the particular expression of the norm $\|\cdot\|_s$, the effective regularity is the smoothness of either $u_D(\lambda_\varrho)$ or $u_N(\lambda_\varrho)$ <u>away from Γ_I</u>. Each of these functions enjoys more regularity than expected at the vicinity of Γ_C.*
In the sequel, we need some elliptic regularity results for solutions to Poisson problems in polyhedral domains. Even for boundary data that are very smooth, the regularity of the Poisson solutions may be limited by the geometry. According to [35, 36], and assuming that the boundary data are smooth enough on Γ_C and Γ_I, the solution of the Poisson problem belongs to the Sobolev spaces $H^{3/2+p}(\Omega)$, for any $p < p^*$, where the number $p^* \in]0, 1/2[$ depends on the geometry. Such is the case for $u_D(\mu, g)$ and $u_N(\mu, \varphi)$, even if g, φ and μ are very smooth. No better results can be expected. Then, p^* is the Sobolev exponent that drives and limits the regularity of the solution of Poisson problems, because of the geometry of Γ_I. This exponent will be used in several places.

Remark 4.2 *Things are well understood and documented in two dimensions, where p^* depends on the angles of the closed polygonal boundary Γ_I* (see [24, 30]). *Indeed, let θ_i be the measure of the* *(internal) angle at the corners of Γ_I (recall that Γ_C is assumed to be smooth). There is of course a finite number i^* of corners and we denote by θ the largest angle, that is $\theta = \max_{1\le i\le i^*} \theta_i$* (see Fig. 2)*. This angle lies in the interval $]\pi, 2\pi[$. The angle 2π corresponds to a cusp which is not allowed, since Γ_I is a Lipschitz boundary. Then, the real number p^*, is given by*

$$p^* = \frac{\pi}{\theta} - \frac{1}{2} \in \ ]0, \frac{1}{2}[.$$

4.1 Technical lemmas

Let μ be given in $H^{1/2}(\Gamma_I)$, our aim is to extract more regularity than expected on the function $u_D(\mu) - u_N(\mu)$, in the vicinity of Γ_I. This is related to the elliptic regularity in non smooth geometries of the following problem: *for ψ given in $H^{1/2}(\Gamma_C)$, find $v \in H^1(\Omega)$ solution of,*

$$\begin{aligned} -\operatorname{div}(a\nabla v) = 0 \qquad & \text{in } \Omega,\\ v = \psi \qquad & \text{on } \Gamma_C,\\ v = 0 \qquad & \text{on } \Gamma_I. \end{aligned}$$

This problem has a unique solution and this solution satisfies

$$\|v\|_{H^1(\Omega)} \le C\|\psi\|_{H^{1/2}(\Gamma_C)}. \tag{21}$$

Considering that ψ has no further smoothness than being in $H^{1/2}(\Gamma_C)$, the function v cannot be more globally regular than being in $H^1(\Omega)$. Nevertheless, far away from Γ_C, there is no reason why v will not be smoother. This is the object of the following lemma:

Lemma 4.3 *Let $p < p^*$ be any positive real number. Then, there exists a constant C, depending only on Ω, such that*

$$\|a\partial_{\boldsymbol{n}} v\|_{H^p(\Gamma_I)} \le C\|\psi\|_{H^{1/2}(\Gamma_C)}. \tag{22}$$

Proof: Let ξ be a smooth cut-off function defined in $\overline{\Omega}$, that takes the value one in a neighborhood V_I of Γ_I, the value zero in a neighborhood V_C of Γ_C and intermediate values elsewhere in Ω, i.e.,

$$\xi(\boldsymbol{x}) = 0, \quad \forall \boldsymbol{x} \in V_C, \qquad \xi(\boldsymbol{x}) = 1, \quad \forall \boldsymbol{x} \in V_I, \qquad 0 \le \xi(\boldsymbol{x}) \le 1, \quad \forall \boldsymbol{x} \in \overline{\Omega}.$$

Then, define $v_\xi = \xi v$. Given that $\operatorname{div}(a\nabla v) = 0$, it can be checked out that

$$\begin{aligned} -\operatorname{div}(a\nabla v_\xi) = -v\operatorname{div}(a\nabla\xi) - 2a\nabla v\nabla\xi \qquad & \text{in } \Omega,\\ v_\xi = 0 \qquad & \text{on } \Gamma_C,\\ v_\xi = 0 \qquad & \text{on } \Gamma_I. \end{aligned}$$

The right-hand side of the first equation lies in $L^2(\Omega)$. Therefore, the elliptic regularity results of [24] imply that v_ξ belongs to $H^{3/2+p}(\Omega)$ with the following stability, consequence of (21):

$$\|v_\xi\|_{H^{3/2+p}(\Omega)} \le C\|v\|_{H^1(\Omega)} \le C\|\psi\|_{H^{1/2}(\Gamma_C)}.$$

Owing to the fact that v_ξ coincides with v in V_I, the estimate (22) is obtained by a trace theorem. ∎

Recall that $u_N(\mu)$ is a lifting to the whole domain Ω of a given trace μ on Γ_I. The next result provides a bound of the trace of $u_N(\mu)$ on Γ_C by the weak norm $\|\mu\|_s$.

Lemma 4.4 *The following holds: for all $\mu \in H^{1/2}(\Gamma_I)$,*

$$\|u_N(\mu)\|_{H^{1/2}(\Gamma_C)} \leq C\|\mu\|_s. \tag{23}$$

Proof: The trace theorem applied to the function $(u_D(\mu) - u_N(\mu))$ yields that

$$\|u_N(\mu)\|_{H^{1/2}(\Gamma_C)} \leq C|u_D(\mu) - u_N(\mu)|_{a,H^1(\Omega)}.$$

Then, (23) follows from (9) and (10). ∎

From these two lemmas, we deduce the following estimate which will have a preponderant impact on the subsequent analysis.

Lemma 4.5 *Let $p < p^*$ be any positive real number. Then, there exists a constant C such that: forall $\mu \in H^{1/2}(\Gamma_I)$,*

$$\|a\partial_{\boldsymbol{n}}(u_D(\mu) - u_N(\mu))\|_{H^p(\Gamma_I)} \leq C\|\mu\|_s.$$

The constant C depends on p.

Proof: Let μ be given in $H^{1/2}(\Gamma_I)$ and set $v = u_D(\mu) - u_N(\mu)$. We have

$$\begin{aligned} -\operatorname{div}(a\nabla v) &= 0 \qquad \text{in } \Omega, \\ v &= -u_N(\mu) \qquad \text{on } \Gamma_C, \\ v &= 0 \qquad \text{on } \Gamma_I. \end{aligned}$$

We know from the Lemma 4.3 that $(a\partial_{\boldsymbol{n}} v)$ lies in $H^p(\Gamma_I)$ and

$$\|a\partial_{\boldsymbol{n}} v\|_{H^p(\Gamma_I)} \leq C\|u_N(\mu)\|_{H^{1/2}(\Gamma_C)}.$$

The proof is completed by substituting the estimate of Lemma 4.4. ∎

Remark 4.3 The estimate of Lemma 4.5 is strong because the norm $\|\cdot\|_s$ is very weak. Indeed, the severe ill posedness of Cauchy's problem suggests that this norm is weaker than any Sobolev dual norm $\|\cdot\|_{H^\sigma(\Gamma_I)}$ with $\sigma < 0$. No particular difficulty arises in checking out this claim when Γ_I is smooth.

4.2 Convergence Results

We have at hand the tools for deriving bounds of the approximation error, $\|\lambda_\varrho - \pi_h\lambda_\varrho\|_s$, without any particular smoothness requirement on λ_ϱ. This is the object of the next lemma.

Lemma 4.6 *Let $p < p^*$ be any positive real number. There holds that*

$$\|\lambda_\varrho - \pi_h \lambda_\varrho\|_s \le C h^{1/2+p} \|\lambda_\varrho\|_{s_D}.$$

Proof: Let η the residual function, $\eta = \lambda_\varrho - \pi_h \lambda_\varrho$. We start then from

$$\|\eta\|_s^2 = \int_\Omega a \nabla u_D(\eta) \cdot \nabla u_D(\eta) \, d\boldsymbol{x} - \int_\Omega a \nabla u_N(\eta) \cdot \nabla u_N(\eta) \, d\boldsymbol{x}.$$

Applying Green's formula, and after setting once more $v = u_D - u_N$, we obtain

$$\|\eta\|_s^2 = \int_{\Gamma_I} (a \partial_{\boldsymbol{n}} v(\eta)) \eta \, d\gamma = \int_{\Gamma_I} (a \partial_{\boldsymbol{n}} v(\eta))(\lambda_\varrho - \pi_h \lambda_\varrho) \, d\gamma.$$

But according to Lemma 4.3, $a \partial_{\boldsymbol{n}} v(\eta)$ belongs to $H^p(\Gamma_I)$, therefore we derive by duality that

$$\|\eta\|_s^2 \le \|a \partial_{\boldsymbol{n}} v(\eta)\|_{H^p(\Gamma_I)} \|\lambda_\varrho - \pi_h \lambda_\varrho\|_{H^{-p}(\Gamma_I)}.$$

Owing to (19) with $q = 1/2$, we get the following estimate:

$$\|\eta\|_s^2 \le C \|a \partial_{\boldsymbol{n}} v(\eta)\|_{H^p(\Gamma_I)} h^{1/2+p} \|\lambda_\varrho\|_{H^{1/2}(\Gamma_I)} = C h^{1/2+p} \|a \partial_{\boldsymbol{n}} v(\eta)\|_{H^p(\Gamma_I)} \|\lambda_\varrho\|_{s_D}.$$

Finally, the bound of Lemma 4.5 implies

$$\|\eta\|_s^2 \le C h^{1/2+p} \|\eta\|_s \|\lambda_\varrho\|_{s_D}.$$

The proof is complete. ∎

Remark 4.4 At least in two dimensions, a sharp study of the geometrical singularities responsible for the limitation of the regularity allows to obtain a quasi-optimal estimate for the critical Sobolev regularity exponent p^*. In fact, following [5], it should be possible to state that

$$\|\lambda_\varrho - \pi_h \lambda_\varrho\|_s \le C \, h^{1/2+p^*} |\log h|^b \|\lambda_\varrho\|_{s_D},$$

for some $b > 0$, but that is beyond the scope of this work.

By combining Lemma 4.6 and the convergence (20), we come up with an estimate of the error between the Lavrentiev and Lavrentiev-finite elements solutions.

Proposition 4.7 *Let $p < p^*$. There exist two functions ϵ_ϱ and ϵ_h that tend to zero for small ϱ and h such that,*

$$\|\lambda_\varrho - \lambda_{\varrho,h}\|_{s_D} \le C \left(\epsilon_\varrho + \epsilon_h + \sqrt{\frac{h^{1+2p}}{\varrho}} \right) \|\lambda\|_{s_D}.$$

Proof: First, by combining (16) (where the Lavrentiev solution is dominated by the exact one) and Lemma 4.6, we infer,

$$\frac{1}{\sqrt{\varrho}} \|\lambda_\varrho - \pi_h \lambda_\varrho\|_s \le C \sqrt{\frac{h^{1+2p}}{\varrho}} \|\lambda\|_{s_D},$$

with p is a real number such that $p < p^*$. Next, we bound $\|\lambda_\varrho - \pi_h\lambda_\varrho\|_{s_D}$ as follows:

$$\|\lambda_\varrho - \pi_h\lambda_\varrho\|_{s_D} \leq \|(\lambda_\varrho - \lambda) - \pi_h(\lambda_\varrho - \lambda)\|_{s_D} + \|\lambda - \pi_h\lambda\|_{s_D}.$$

Owing to the uniform stability of π_h given by (20), we deduce that

$$\|\lambda_\varrho - \pi_h\lambda_\varrho\|_{s_D} \leq C\|\lambda_\varrho - \lambda\|_{s_D} + \|\lambda - \pi_h\lambda\|_{s_D} = (\epsilon_\varrho + \epsilon_h)\|\lambda\|_{s_D}.$$

The proof is complete after using (13), another application of (20), and Lemma 4.1. ∎

Remark 4.5 A sufficient condition to ensure the convergence of the finite element solution $\lambda_{\varrho,h}$ towards the Lavrentiev solution λ_ϱ is to fix the mesh size $h = h(\varrho)$ so that

$$\lim_{\varrho \to 0} \frac{h^{1+2p}}{\varrho} = 0.$$

Now, when evaluated with respect to the weaker norm $\|\cdot\|_s$, the error is bounded as follows:

$$\|\lambda_\varrho - \lambda_{\varrho,h}\|_s \leq C\left((\epsilon_\varrho + \epsilon_h)\sqrt{\varrho} + h^{1/2+p}\right)\|\lambda\|_{s_D}. \tag{24}$$

This error decays towards zero provided that each of ϱ and h tends to zero independently. Notice that the overall estimates derived here do not require any additional smoothness on the exact solution λ.

Remark 4.6 We have already pointed out that a polygonal or polyhedral boundary Γ_I slows down substantially the convergence rate of the finite element method. Indeed, assume that Γ_I is smooth and the problem is discretized by curved finite elements of degree one, as defined in [12, section 2] and [20, Section 4.3]). On one hand, since these finite elements fit the exact shape of Γ_I, they induce no approximation error of the boundary. On the other hand, the smoothness of the exact solution is not limited by the boundary. As a result, the analysis elaborated above permits to take a full advantage of the finite element approximation estimates for smooth functions. The final estimate is hence changed to

$$\|\lambda_\varrho - \lambda_{\varrho,h}\|_{s_D} \leq C\left(\epsilon_\varrho + \epsilon_h + \sqrt{\frac{h^5}{\varrho}}\right)\|\lambda\|_{s_D}.$$

We refer to [26, Chapter 5] where this particular case is discussed.

The final error bound evaluates the difference between λ, the solution of equation (6), and $\lambda_{\varrho,h}$, the regularized-discrete solution of problem (17).

Theorem 4.8 *Let $p < p^*$ be any positive real number. There exists two functions ϵ_ϱ and ϵ_h that decay both towards zero, for small ϱ and h respectively, such that*

$$\|\lambda - \lambda_{\varrho,h}\|_{s_D} \leq C\left(\epsilon_\varrho + \epsilon_h + \sqrt{\frac{h^{1+2p}}{\varrho}}\right)\|\lambda\|_{s_D}.$$

Proof: The estimate is ensued after assembling results of Lemma 3.1 and Proposition 4.7. ∎

4.3 Noisy Data

Crude computations with problem (6), or rather with its discrete version, are certainly affected by perturbations that fatally alter the data (g, φ), with high risk of irrelevant results. Thus, regularization is necessary for a safe solution. We focus on this issue in the sequel and try to control the effects of disturbances on the Lavrentiev-finite element approximation due to inexact Cauchy data.

Assume we are given the polluted data $(g_\epsilon, \varphi_\epsilon) = (g, \varphi) + (\delta g, \delta\varphi)$, with a known noise level $\epsilon > 0$. This means that

$$\|\breve{u}_D(\delta g) - \breve{u}_N(\delta\varphi)\|_{H^1(\Omega)} \leq \epsilon. \tag{25}$$

It is true that the noise level should be evaluated in the natural norms of $(\delta g, \delta\varphi)$ which means that

$$\|\delta g\|_{H^{1/2}(\Gamma_C)} + \|\delta\varphi\|_{H^{-1/2}(\Gamma_C)} \leq C\epsilon.$$

This bound implies necessarily (25). Conversely, it may be ensued from (25) if a statistical independence assumption is added on the errors δg and $\delta\varphi$. However, from a practical point of view, the (statistical) evaluation in (25) is easier to obtain. Furthermore, as will be seen later on it is well fitted to the Kohn-Vogelius approach we follow here. We refer to [10, Remark 5.1] for more clues on this issue.

We consider now the Lavrentiev regularization in the finite element context applied to the perturbed problem as follows: *find* $\lambda^\epsilon_{\varrho,h} \in H_h$ *such that*

$$\varrho s_D(\lambda^\epsilon_{\varrho,h}, \mu_h) + s(\lambda^\epsilon_{\varrho,h}, \mu_h) = \ell_\epsilon(\mu_h), \qquad \forall \mu_h \in H_h. \tag{26}$$

The linear form ℓ_ϵ is given by the same formula as ℓ with $(g_\epsilon, \varphi_\epsilon)$ instead of (g, φ). Our aim here is to estimate the difference $\lambda_{\varrho,h} - \lambda^\epsilon_{\varrho,h}$, called the variance.

Lemma 4.9 *There holds that*

$$\|\lambda_{\varrho,h} - \lambda^\epsilon_{\varrho,h}\|_{s_D} \leq \frac{\epsilon}{2\sqrt{\varrho}}.$$

Proof: For the sake of simplicity, we set $\eta_\epsilon = \lambda^\epsilon_{\varrho,h} - \lambda_{\varrho,h}$. By subtracting (17) from (26), we obtain

$$\varrho s_D(\eta_\epsilon, \mu_h) + s(\eta_\epsilon, \mu_h) = \ell_\epsilon(\mu_h) - \ell(\mu_h), \qquad \forall \mu_h \in H_h.$$

Let us set $\delta\ell = \ell_\epsilon - \ell$. The choice $\mu_h = \eta_\epsilon$ gives

$$\varrho\|\eta_\epsilon\|^2_{s_D} + \|\eta_\epsilon\|^2_s = (\delta\ell)(\eta_\epsilon). \tag{27}$$

Using the stability (11) and accounting for (25) implies that

$$|(\delta\ell)(\eta_\epsilon)| \leq \|\breve{u}_D(\delta g) - \breve{u}_N(\delta\varphi)\|_{H^1(\Omega)}\|\eta_\epsilon\|_s \leq \epsilon\|\eta_\epsilon\|_s.$$

Then Cauchy-Schwarz inequality yields that

$$|(\delta\ell)(\eta_\epsilon)| \leq \frac{\epsilon^2}{4} + \|\eta_\epsilon\|_s^2.$$

By substituting this inequality into identity (27), we obtain

$$\varrho\|\eta_\epsilon\|_{s_D}^2 \leq \frac{\epsilon^2}{4},$$

hence the desired result. The proof is complete. ∎

Remark 4.7 A by product of the proof is a bound on the weak norm of η_ϵ,

$$\|\eta_\epsilon\|_s = \|\lambda_{\varrho,h} - \lambda_{\varrho,h}^\epsilon\|_s \leq \epsilon. \tag{28}$$

We are well equipped to give an estimate of the global error and state a sufficient condition to ensure a convergence result when all the parameters involved (ϱ, h, ϵ) decay towards zero. We need the bias-variance decomposition

$$\|\lambda - \lambda_{\varrho,h}^\epsilon\|_{s_D} \leq \|\lambda - \lambda_{\varrho,h}\|_{s_D} + \|\lambda_{\varrho,h} - \lambda_{\varrho,h}^\epsilon\|_{s_D}.$$

We have the following final result, a straightforward consequence of the results of Theorem 4.8 and Lemma 4.9.

Theorem 4.10 *Let $p < p^*$ be any positive real number. There exist two functions ϵ_ϱ and ϵ_h, both tending to zero, for small ϱ and h respectively, such that*

$$\|\lambda - \lambda_{\varrho,h}^\epsilon\|_{s_D} \leq C\left(\epsilon_\varrho + \epsilon_h + \sqrt{\frac{h^{1+2p}}{\varrho}}\right)\|\lambda\|_{s_D} + \frac{\epsilon}{2\sqrt{\varrho}}.$$

Corollary 4.11 *Let $p < p^*$ be any positive real number. There holds*

$$\|u - u_D(\lambda_{\varrho,h}^\epsilon, g)\|_{H^1(\Omega)} + \|u - u_N(\lambda_{\varrho,h}^\epsilon, \varphi)\|_{H^1(\Omega)} \leq C\left[\left(\epsilon_\varrho + \epsilon_h + \sqrt{\frac{h^{1+2p}}{\varrho}}\right)\|\lambda\|_{s_D} + \frac{\epsilon}{2\sqrt{\varrho}}\right].$$

Remark 4.8 In general, the noise level ϵ drives the convergence. The above theorem brings about a (sufficient) criterion for choosing the regularization parameters (ϱ, h) that ensure the convergence of the Lavrentiev-finite-element approximation towards the exact solution. A sufficient condition is to choose $\varrho = \varrho(\epsilon)$ and $h = h(\epsilon)$ which satisfy

$$\lim_{\epsilon\to 0} \varrho = 0, \quad \lim_{\epsilon\to 0} \frac{\epsilon}{\sqrt{\varrho}} = 0, \quad \lim_{\epsilon\to 0} \frac{h^{1+2p}}{\varrho} = 0.$$

This suggests that ϱ cannot be chosen arbitrarily small, but depends on the noise magnitude ϵ. In our opinion, the condition on (ϱ, ϵ) is more stringent and will prevail over the condition linking (h, ϱ). This means that h is in general small enough for fulfilling the last limit.

5 Local Super-Convergence

The study conducted above concludes to the convergence of the regularization-discretization method. No convergence rates, of any form Hölderian or logarithmic, can be supplied in the global domain without additional assumptions on the solution of (6). However, super-convergence holds far away from the incomplete boundary, as observed in [18, 7]. We propose to establish super-convergence results for $\lambda_{\varrho,h}$ and their incidence on the semi-discrete solution. The ideas we follow may be found in [18] or in [39]. They have been successfully extended for the Lavrentiev solution of problem (12) in [7].

The subsequent analysis needs slightly more regularity on the Neumann data φ; it must be in $L^2(\Gamma_C)$ instead of $H^{-1/2}(\Gamma_C)$. This is consistent with the actual measurements on φ, since noise affecting Neumann data is also estimated in the Lebesgue L^2-norm. Consequently, we assume that

$$\|\delta g\|_{H^{1/2}(\Gamma_C)} + \|\delta\varphi\|_{L^2(\Gamma_C)} \leq C\epsilon. \tag{29}$$

The pillar tool here is a fitting Carleman inequality. Let then $\psi \in \mathscr{C}^2(\overline{\Omega})$ be a smooth function defined in $\overline{\Omega}$ that satisfies the following properties:

$$|\nabla\psi(\boldsymbol{x})| > 0,\ \ \forall \boldsymbol{x} \in \overline{\Omega}, \qquad \psi(\boldsymbol{x}) > 1,\ \ \forall \boldsymbol{x} \in \overline{\Omega} \setminus \Gamma_I \qquad \psi(\boldsymbol{x}) = 1,\ \ \forall \boldsymbol{x} \in \Gamma_I. \tag{30}$$

Thus the maximum of ψ is attained on Γ_C. We shall use the following Carleman estimate, valid in a bounded Lipschitz domain: for large $\zeta > 0$,

$$\begin{aligned}\int_\Omega \big[a(\nabla v)^2 + \zeta^2 v^2\big]\, e^{2\zeta\psi}\, d\boldsymbol{x} \leq C\Big(\frac{1}{\zeta}\int_\Omega \big[-\operatorname{div}(a\nabla v) + v\big]^2 e^{2\zeta\psi}\, d\boldsymbol{x} \\ + \int_{\partial\Omega} \big[(a\partial_{\boldsymbol{n}} v)^2 + \zeta^2 v^2\big]\, e^{2\zeta\psi}\, d\gamma\Big), \qquad \forall v \in H^2(\Omega).\end{aligned} \tag{31}$$

The constant C is independent of ζ. We refer to [51] for the proof. Next, for a given small parameter $\tau > 0$, we define

$$\Omega_\tau = \big\{\boldsymbol{x} \in \Omega\ :\ \psi(\boldsymbol{x}) \geq 1 + \tau\big\}.$$

Owing to the last condition in (30), the closure $\overline{\Omega}_\tau$ does not intersect Γ_I. Moreover, τ may be chosen small enough so that $\Omega \setminus \Omega_\tau$ determines a thin tubular neighborhood of Γ_I. The right plot of Fig. 2 illustrates the geometry of Ω_τ. Now, for a given pair (τ, ν) with $\tau > \nu > 0$, we shall use the smooth cut-off function $\xi_{\tau,\nu}$ defined by the following:

$$\xi_{\tau,\nu}(\boldsymbol{x}) = 1, \quad \forall \boldsymbol{x} \in \overline{\Omega}_\tau, \qquad \xi_{\tau,\nu}(\boldsymbol{x}) = 0, \quad \forall \boldsymbol{x} \in \overline{\Omega} \setminus \overline{\Omega}_\nu, \qquad 0 \leq \xi_{\tau,\nu} \leq 1, \quad \forall \boldsymbol{x} \in \overline{\Omega}. \tag{32}$$

Notice that the support of $\xi_{\tau,\nu}$ is contained in $\overline{\Omega}_\nu$.

5.1 The Bias

The local super-convergence analysis begins with the bias. We mainly resume and adapt the arguments exposed and sharpened in [7]. The following estimate holds:

Lemma 5.1 *Let $\beta > 0$ be a small parameter. There exists $q = q(\beta) \in [0, 1/2[$ and a constant $C = C(\beta)$ such that*

$$|u_N(\lambda_{\varrho,h}, \varphi) - u|_{a,H^1(\Omega_\beta)} \le C\varrho^q \sqrt{1 + \frac{h^{1+2p}}{\varrho}} \|\lambda\|_{s_D}, \tag{33}$$

for all pairs (ϱ, h) of small enough numbers.

Proof: In addition to the notation $\eta_\varrho = \lambda_{\varrho,h} - \lambda_\varrho$, we denote

$$w_{N,\varrho} = u_N(\lambda_{\varrho,h}, \varphi) - u_N(\lambda_\varrho, \varphi) = u_N(\lambda_{\varrho,h} - \lambda_\varrho) = u_N(\eta_\varrho).$$

Then, we choose (τ, ν) with $\beta > \tau > \nu > 0$, and consider $\xi_{\tau,\nu}$, the cut-off function defined in (32). To alleviate the presentation, we drop the indices ${}_N$ and ${}_{\tau,\nu}$ and we use ξw_ϱ instead of $\xi_{\tau,\nu} w_{N,\varrho}$. As stated in Section 4, the function w_ϱ cannot be expected to belong to $H^2(\Omega)$ for at least one of two reasons: the trace η_ϱ of w_ϱ on Γ_I lies in $H^{1/2}(\Gamma_I)$ and Γ_I is polygonal or polyhedral. However these sources of singularities are both made ineffective by multiplication with the cut-off function ξ and the product ξw_ϱ is indeed in $H^2(\Omega)$. We can therefore apply Carleman's estimate (31) to $\xi w_\varrho \in H^2(\Omega)$. It yields

$$\begin{aligned}\int_{\Omega_\tau} \left[a(\nabla(\xi w_\varrho))^2 + \zeta^2 (\xi w_\varrho)^2\right] e^{2\zeta\psi}\, d\boldsymbol{x} \le C\Big(\frac{1}{\zeta}\int_\Omega \left[-\operatorname{div}(a\nabla(\xi w_\varrho)) + \xi w_\varrho\right]^2 e^{2\zeta\psi}\, d\boldsymbol{x}\\ + \int_{\Gamma_C} \left[(a\partial_{\boldsymbol{n}}(\xi w_\varrho))^2 + \zeta^2(\xi w_\varrho)^2\right] e^{2\zeta\psi}\, d\gamma\Big).\end{aligned}$$

By carrying out the calculations, observing that $(a\partial_{\boldsymbol{n}} w_\varrho)_{|\Gamma_I} = (a\partial_{\boldsymbol{n}} w_{N,\varrho})_{|\Gamma_I} = 0$ together with the fact that $\xi \equiv 1$ in Ω_τ, we derive,

$$\begin{aligned}\int_{\Omega_\tau} \left[a(\nabla(w_\varrho))^2 + \zeta^2 (w_\varrho)^2\right] e^{2\zeta\psi}\, d\boldsymbol{x} \le C\Big(\frac{1}{\zeta}\int_{\Omega_\nu\setminus\Omega_\tau} \left[a(\nabla w_\varrho)^2 + (w_\varrho)^2\right] e^{2\zeta\psi}\, d\boldsymbol{x}\\ + \frac{1}{\zeta}\int_{\Omega_\tau} (w_\varrho)^2 e^{2\zeta\psi}\, d\boldsymbol{x} + \zeta^2 \int_{\Gamma_C} (w_\varrho)^2 e^{2\zeta\psi}\, d\gamma\Big).\end{aligned}$$

By choosing a large enough ζ, the integral over Ω_τ in the right-hand side can be absorbed by the same in the left-hand side, and we have

$$\begin{aligned}\int_{\Omega_\tau} \left[a(\nabla(w_\varrho))^2 + \zeta^2 (w_\varrho)^2\right] e^{2\zeta\psi}\, d\boldsymbol{x} \le C\Big(\frac{1}{\zeta}\int_{\Omega_\nu\setminus\Omega_\tau} \left[a(\nabla w_\varrho)^2 + (w_\varrho)^2\right] e^{2\zeta\psi}\, d\boldsymbol{x}\\ + \zeta^2 \int_{\Gamma_C} (w_\varrho)^2 e^{2\zeta\psi}\, d\gamma\Big).\end{aligned}$$

The integral in the left-hand side can be restricted to Ω_β; thus denoting $\sigma = \max_{\boldsymbol{x}\in\Gamma_C} \psi(\boldsymbol{x}) - 1 > 0$ and accounting for the specification of $\Omega_\beta, \Omega_\tau$ and Ω_ν, we obtain

$$e^{2\zeta(1+\beta)} \int_{\Omega_\beta} \left[a(\nabla w_\varrho)^2 + \zeta^2 (w_\varrho)^2\right] \, d\boldsymbol{x} \le C\Big(\frac{1}{\zeta} e^{2\zeta(1+\tau)} \int_{\Omega_\nu \setminus \Omega_\tau} [a(\nabla w_\varrho)^2 + (w_\varrho)^2] \, d\boldsymbol{x} \\ + \zeta^2 e^{2\zeta(1+\sigma)} \int_{\Gamma_C} (w_\varrho)^2 \, d\gamma\Big).$$

Then, obvious simplifications lead to

$$\int_{\Omega_\beta} \left[a(\nabla w_\varrho)^2 + \zeta^2 (w_\varrho)^2\right] \, d\boldsymbol{x} \le C\Big(\frac{1}{\zeta} e^{2\zeta(\tau-\beta)} \int_{\Omega_\nu \setminus \Omega_\tau} [a(\nabla w_\varrho)^2 + (w_\varrho)^2] \, d\boldsymbol{x} \\ + \zeta^2 e^{2\zeta(\sigma-\beta)} \int_{\Gamma_C} (w_\varrho)^2 \, d\gamma\Big).$$

On one hand the stability of $w_\varrho (= w_{N,\varrho})$ with respect to η_ϱ, and Lemma 4.4 on the other hand, give

$$\int_{\Omega_\beta} \left[a(\nabla w_\varrho)^2 + \zeta^2 (w_\varrho)^2\right] \, d\boldsymbol{x} \le C\Big(\frac{1}{\zeta} e^{-2\zeta(\beta-\tau)} \|\eta_\varrho\|^2_{s_D} + \zeta^2 e^{2\zeta(\sigma-\beta)} \|\eta_\varrho\|^2_s\Big).$$

In view of (24), we have the following bound

$$\int_{\Omega_\beta} \left[a(\nabla w_\varrho)^2 + \zeta^2 (w_\varrho)^2\right] \, d\boldsymbol{x} \le C\Big(\frac{1}{\zeta} e^{-2\zeta(\beta-\tau)} \|\eta_\varrho\|^2_{s_D} + \zeta^2 e^{2\zeta(\sigma-\beta)} (\varrho + h^{1+2p}) \|\lambda\|^2_{s_D}\Big).$$

Now, let $t = \frac{1}{\zeta} e^{-2\zeta(\beta-\tau)}$. Since $\beta - \tau$ is positive, this quantity decays towards zero for large ζ and the above estimate can be transformed into

$$|w_\varrho|_{a,H^1(\Omega_\beta)} \le C\left(t\|\eta_\varrho\|^2_{s_D} + \frac{(\varrho + h^{1+2p})}{t^s} \|\lambda\|^2_{s_D}\right)^{1/2}, \tag{34}$$

where[2] $s = [(\sigma-\beta)/(\beta-\tau)]_+$ and $C = C(\beta,\tau)$ is a positive constant.
Let us select t so as to minimize the right-hand side of (34). With this minimum value (34) becomes,

$$|w_\varrho|_{a,H^1(\Omega_\beta)} \le C(\varrho + h^{1+2p})^{\frac{1}{2(1+s)}} \|\eta_\varrho\|_{s_D}^{\frac{s}{1+s}} \|\lambda\|_{s_D}^{\frac{1}{1+s}}.$$

Hence, according to Proposition 4.7, we deduce

$$|w_\varrho|_{a,H^1(\Omega_\beta)} \le C(\varrho + h^{1+2p})^{\frac{1}{2(1+s)}} \left(1 + \frac{h^{1+2p}}{\varrho}\right)^{\frac{s}{2(1+s)}} \|\lambda\|_{s_D}.$$

To close the proof of (33), we need the estimate established in [7, Theorem 3.2],

$$|u_N(\lambda_\varrho, \varphi) - u|_{a,H^1(\Omega_\beta)} \le C\varrho^q \|\lambda\|_{s_D},$$

with the same exponent $q = \frac{1}{2(1+s)}$. The proof is complete by the triangle inequality. ∎

[2]The symbol α_+ stand for any real number strictly larger than α.

Remark 5.1 The following bound holds true:

$$|u_D(\lambda_{\varrho,h}, g) - u|_{a,H^1(\Omega_\beta)} \leq C\varrho^q \sqrt{1 + \frac{h^{1+2p}}{\varrho}} \|\lambda\|_{s_D}.$$

As a result, a sufficient condition to ensure the convergence in the sub-region Ω_β is as follows:

$$\lim_{\varrho \to 0} \frac{h^{1+2p}}{\varrho^{1-2q}} = 0.$$

This is a weaker condition on the mesh-size h than the one reported earlier for the whole domain Ω.

5.2 The variance

In the same spirit, we derive a sharp convergence rate for the variance error, away from the incomplete boundary. We retain the notation of Section 4.3 and set $\eta_\epsilon = \lambda_{\varrho,h} - \lambda^\epsilon_{\varrho,h}$. Then, we define

$$w_{N,\epsilon} = u_N(\lambda_{\varrho,h}, \varphi) - u_N(\lambda^\epsilon_{\varrho,h}, \varphi_\epsilon) = u_N(\eta_\epsilon, \delta\varphi) = u_N(\eta_\epsilon) + \breve{u}_N(\delta\varphi).$$

We need the following preliminary result:

Lemma 5.2 *The function $w_{N,\epsilon}$ satisfies,*

$$\|w_{N,\epsilon}\|_{H^{1/2}(\Gamma_C)} \leq C\epsilon.$$

Proof: The inequality (29) gives the straightforward bound,

$$\|\breve{u}_N(\delta\varphi)\|_{H^{1/2}(\Gamma_C)} \leq C\|\delta\varphi\|_{H^{-1/2}(\Gamma_C)} \leq C\epsilon.$$

On the other hand, owing to the stability of the trace and the fact that $u_D(\eta_\epsilon)$ vanishes on Γ_C, $u_N(\eta_\epsilon)$ satisfies,

$$\|u_N(\eta_\epsilon)\|_{H^{1/2}(\Gamma_C)} \leq C|u_D(\eta_\epsilon) - u_N(\eta_\epsilon)|_{a,H^1(\Omega)} = \|\eta_\epsilon\|_s \leq C\epsilon.$$

The last bound is given in (28). The proof is obtained by the triangle inequality. ∎

Lemma 5.3 *Let $\beta > 0$ be a small parameter. There exists a constant $C = C(\beta)$ such that*

$$|u_N(\lambda^\epsilon_{\varrho,h}, \varphi_\epsilon) - u_N(\lambda_{\varrho,h}, \varphi)|_{a,H^1(\Omega_\beta)} \leq C\epsilon\varrho^{-\frac{1}{2}+q},$$

where $q = q(\beta)$ is defined as in Lemma 5.1.

Proof: We retain the notation of Lemma 5.1. The argument of the proof in inspired by [7, Theorem 3.5]. We apply Carleman's estimate to $\xi_{\tau,\nu} w_{N,\epsilon}$. Again, we drop the indices $_N$ and $_{\tau,\nu}$. By taking

$v = \xi w_\epsilon$ (that belongs to $H^2(\Omega)$) in (31), and proceeding as in the proof of Lemma 5.1, we arrive at

$$\int_{\Omega_\beta} \big[a(\nabla w_\epsilon)^2 + \zeta^2 (w_\epsilon)^2\big] \ d\boldsymbol{x} \leq C\Big(\frac{1}{\zeta} e^{2\zeta(\tau-\beta)} \int_{\Omega_\nu \setminus \Omega_\tau} [a(\nabla w_\epsilon)^2 + (w_\epsilon)^2] \, d\boldsymbol{x} \\ + e^{2\zeta(\sigma-\beta)} \int_{\Gamma_C} [(a\partial_{\boldsymbol{n}} w_\epsilon)^2 + \zeta^2 (w_\epsilon)^2] \, d\gamma\Big).$$

In the right-hand side, the first integral is bounded by using Lemma 4.9. Owing to (29), in the last integral the normal derivative $(a\partial_{\boldsymbol{n}} w_\epsilon) = \delta\varphi$ is bounded by $C\epsilon$ and the bound on the trace w_ϵ comes from Lemma 5.2. Assembling all the bounds leads to

$$\int_{\Omega_\beta} \big[a(\nabla w_\epsilon)^2 + (w_\epsilon)^2\big] \, d\boldsymbol{x} \leq C\Big(\frac{1}{\zeta} e^{-2\zeta(\beta-\tau)} \frac{\epsilon^2}{\varrho} + (1+\zeta^2) e^{2\zeta(\sigma-\beta)} \epsilon^2\Big).$$

By introducing again $t = \frac{1}{\zeta} e^{-2\zeta(\beta-\tau)}$, we obtain

$$|w_\epsilon|^2_{a,H^1(\Omega_\beta)} \leq C\epsilon^2 (\frac{t}{\varrho} + \frac{1}{t^s}).$$

Then the choice of t that minimizes the above right-hand side yields

$$|w_\epsilon|_{a,H^1(\Omega_\beta)} \leq C\epsilon\varrho^{-\frac{s}{2(s+1)}}.$$

Given that $q = \frac{1}{2(1+s)}$, the final bound is expressed by $C\epsilon\varrho^{-\frac{1}{2}+q}$. The proof is complete. ∎

5.3 Super-Convergence

According to the bias-variance decomposition principle, and after assembling the results of Lemmas 5.1 and 5.3, we are able to derive local convergence rates for the Lavrentiev-finite elements solution.

Theorem 5.4 *Let $\beta > 0$ be a small parameter. There exists $q = q(\beta) \in [0, 1/2[$ and a constant $C = C(\beta)$ such that the following bound holds*

$$|u_N(\lambda^\epsilon_{\varrho,h}, \varphi_\epsilon) - u|_{a,H^1(\Omega_\beta)} \leq C\varrho^q \left(\sqrt{1 + \frac{h^{1+2p}}{\varrho}} \|\lambda\|_{s_D} + \epsilon\varrho^{-\frac{1}{2}}\right).$$

6 Conclusion

The variational formulation, proposed in [8] for the Cauchy problem, provides a suitable framework where Lavrentiev's regularization has been successfully applied and studied (see [10]). The present contribution, which yields the expected error estimates, establishes that this same framework is well adapted to the convergence analysis of a semi-discrete Galerkin Finite Elements method. This is a first step in the theoretical confirmation of the work in [3] where the computational efficiency of the fully discrete scheme has been assessed. We emphasize the fact that the overall results obtained are valid in two and three dimensions. The next step will consist in extending the present analysis to the fully discrete finite element approximation where all the intermediate Poisson problems are also discretized. This work in progress, which will be the subject of a forthcoming article, is based on non-standard, sharp local convergence results such as those proved by A. Schatz and his co-workers, see for instance [45, 52].

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