A direct linear inversion for discontinuous elastic parameters recovery from internal displacement information only

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Abstract

The aim of this paper is to present and analyze a new direct method for solving the linear elasticity inverse problem. Given measurements of some displacement fields inside a medium, we show that a stable reconstruction of elastic parameters is possible, even for discontinuous parameters and without boundary information. We provide a general approach based on the weak definition of the stiffness-to-force operator which conduces to see the problem as a linear system. We prove that in the case of shear modulus reconstruction, we have an L^2-stability with only one measurement under minimal smoothness assumptions. This stability result is obtained though the proof that the linear operator to invert has closed range. We then describe a direct discretization which provides stable reconstructions of both isotropic and anisotropic stiffness tensors.

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Contents

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1 Introduction

Elastography is an imaging modality that aims at reconstructing the mechanical properties of tissues. The local values of the elastic parameters can be used as a discriminatory criterion to differentiate healthy tissues from diseased tissues [30]. Elasticity imaging emerged in the late 80's and early 90's as a way to improve the diagnostics on ultrasound images [22]. A variety of techniques have been developed since then to assess the elastic parameters of tissues in vivo. For a comprehensive list of the different seminal works on the subject, we refer the reader to the reviews [16, 27, 15, 34].

    Most of the elastography methods are based on the following four steps:

(i) Perturb a medium with a mechanical stimulation (static, harmonic, or transient);

(ii) Image the deformation of the medium (usually via ultrasound imaging, magnetic resonance imaging, or optical coherence tomography);

(iii) Reconstruct the displacement field or some of its components in the medium;

(iv) Reconstruct the mechanical properties of the medium by solving an inverse problem.

In most cases, the scale of the imaging resolution and the amplitude of the displacement field justify the use of linear elasticity model:

$$-\nabla \cdot (\mathbf{C} : \nabla^s \mathbf{u}) = \mathbf{f},$$

where $\mathbf{C} = \{\mathbf{C}_{ijkl}\}_{1 \leq i,j,k,l \leq d}$ is the order four unknown elasticity tensor in dimension $d \in \{2, 3\}$, $\nabla^s \mathbf{u} := (\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2$ is the strain tensor associated to the displacement field $\mathbf{u}$. The internal force density $\mathbf{f}$ depends on the type of source excitation: In the elastostatic regime, $\mathbf{f}$ is zero, in elastodynamics, $\mathbf{f} = \partial_{tt} \mathbf{u}$ or $\mathbf{f} = -\omega^2 \mathbf{u}$ in the time harmonic regime.

1.1 Scientific context

We consider the problem of reconstructing the elasticity tensor $\mathbf{C}$ from the knowledge of a finite number n of displacement fields $\{\mathbf{u}^\ell\}_{\ell=1}^n$ solutions of the system of linear elasticity

$$-\nabla \cdot (\mathbf{C} : \nabla^s \mathbf{u}^\ell) = \mathbf{f}^\ell,$$

where the force densities $\mathbf{f}^\ell$ are assumed to be known. In the isotropic elastic cases, the tensor $\mathbf{C}$ can be written as

$$\mathbf{C} = 2\mu \mathbf{I} + \lambda I \otimes I,$$

where μ and λ are the Lamé coefficients, $\mathbf{I}$ is the identity tensor $\mathbf{I}_{ijkl} = \delta_{ik}\delta_{jl}$ and I is the identity matrix $I_{ij} = \delta_{ij}$. Note that some results about the stability of this inverse problem can be found in [6, 35, 9].

Before reviewing the different inversion methods already developed for the fourth step, it is important to have in mind the methods available for the reconstruction of the displacement field (third step). Displacement field reconstructions methods fall into two categories:

(i) Methods that, given images of the unperturbed and the perturbed medium, use a mathematical treatment to recover the geometrical transformation between the images. Such methods can be based, for exemple, on speckle correlation technique [32], optimal control [3] or optimal transport [20].

(ii) Direct reconstruction of the displacement field (or one of its components) during the imaging procedure. Since ultrasound and OCT are imaging modalities that rely on the computation of a *travel time* in a single scattering regime, axial displacements that are one or two orders of magnitude below the resolution of the imaging modality can be directly reconstructed by measuring a phase shift of the backscattered echo, with a very high frame rate (~ 10 KHz for ultrasound [29], ~ 700 Hz for OCT [33, 26]). Although generally only the axial displacement is recovered by this method, a smart illumination sequence allows for a recovery of the axial and the lateral displacements [31, 8]. Step 2 and 3 are therefore performed simultaneously.

A variety of methods are already available to perform the recovery problem (fourth step), depending on the type of mechanical stimulation, the data available (full internal displacement field or partial displacement field, single or multiple measurements), or the used model (linear compressible elasticity or incompressible Navier equation). Most inversion algorithms roughly fall into one of these categories:

(i) Resolution of a first-order transport equation [21, 23, 24, 7];

(ii) Algebraic inversions [10, 29, 12, 9];

(iii) Iterative inversions [2, 4, 5].

First order methods and algebraic inversions are stable under some regularity assumptions on the elastic parameters of the medium and the reconstructed displacement field, but their performances decrease when the Lamé parameters are not locally differentiable, which is often the case in biological media, or when the reconstruction of the displacement field is noisy. Moreover, they assume boundary knowledge which is usually not available in biomedical applications.

Iterative inversions assume less regularity for the elastic parameters, but are computationally more costly, since a forward problem needs to be simulated at each step. In practice, it is difficult to use this approach because some boundary information is required.

In clinical applications, the current state of the art for ultrasound and OCT based shear modulus imaging [28, 11, 14, 26] is the algebraic inversion method developed in [12, 25]. It relies on the assumption that the medium is *locally homogeneous* (but not necessarily isotropic [17]) and is based on the computation of the group speed of a shear wave. In this *locally homogeneous* case, the different polarizations of the elastic waves are decoupled, and only one component of the displacement field is required to compute the shear modulus. More precisely, the shear wave equation is

$$-\nabla \cdot (\mu \nabla^s \mathbf{u}) = -\partial_{tt}\mathbf{u}$$

and if μ is constant almost everywhere, one can assume that

$$-\mu \nabla \cdot (\nabla^s \mathbf{u}) \approx -\partial_{tt}\mathbf{u} \quad \text{almost everywhere.} \tag{1}$$

This approximation is in general false but it allows to simply approach μ as the square of the group speed of the shear wave. One just has to observe shear waves displacement using a fast enough imaging method. Another possible technique is to directly use the approximation

$$\mu \approx \frac{|\partial_{tt}\mathbf{u}|}{|\nabla \cdot (\nabla^s \mathbf{u})|} \tag{2}$$

at positions and times such that $\nabla \cdot (\nabla^s \mathbf{u})$ does not vanish. These methods have the advantages of being able to reconstruct a good image of the shear modulus from small sub-wavelength displacement fields, at a very low computational cost (no matrix inversion needed). Nevertheless, the method fails to quantitatively reconstruct the shear modulus where the medium exhibits discontinuities or strong variations. It is also not applicable to elastostatic experiments as the term $\partial_{tt}\mathbf{u}$ must not vanish.

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The method that we propose in this article is directly inspired by the previous formulae. If one defines the linear operator $A_{\mathbf{u}} : \mu \mapsto -\nabla \cdot (\mu \nabla^s \mathbf{u})$, the approximation made in (1) is in fact a diagonal approximation of $A_{\mathbf{u}}$ defining the diagonal operator $D_{\mathbf{u}} = \mu \mapsto -\mu \nabla \cdot (\nabla^s \mathbf{u})$ and the inverse formula (2) is equivalent to $D_{\mathbf{u}}(\mu) \approx -\partial_{tt}\mathbf{u}$.

As we can imagine, approaching $A_{\mathbf{u}}$ by $D_{\mathbf{u}}$ can be very optimistic in strongly heterogeneous media. In this article, we directly study the operator $A_{\mathbf{u}}$ in order to stably invert it when it is possible.

1.2 Outline of the article and the main results

In this paper, we study a new direct inversion method for reconstructing coefficients of the elasticity tensor from internal fields measurements. The outline of the paper is the following:

(i) we introduce a general weak formulation for the inverse problem (Section 2);

(ii) we theoretically study the operator to invert (null space, closed range property, stability of the inversion) in the isotropic shear modulus imaging case (Section 3);

(iii) we study the numerical performance of the method in the previous case as well as in some more general frameworks, in particular, for reconstructing both Lamé coefficients and anisotropic media (Section 4).

The strength of this direct inversion method is the fact that it combines the low computational cost of an algebraic inversion method (compared to the costly iterative methods) without requiring the high regularity assumptions on the coefficients to be reconstructed.

The determination of the null space of the operator to invert (Theorem 3) and the main stability result (Theorem 6) for the inversion are obtained under a weak regularity assumption on the coefficients of the elasticity tensor, allowing the reconstruction of discontinuous coefficients. To the best of our knowledge, it is the first time that a non-iterative inversion method is theoretically studied for discontinuous elastic coefficients. The main consequence of this theoretical study is that the stable reconstruction of a discontinuous shear modulus is possible from one single measurement.

The numerical experiments shown in Section 4 are also new. We introduce a $\mathcal{P}^1 - \mathcal{P}^0$ finite elements basis for the resolution of the inverse problem. We show that the sharp reconstruction of discontinuous coefficients from a minimal number of measurements is possible, and therefore that the theoretical results of Section 3 numerically hold in a more general setting.

2 The inverse problem

2.1 The direct weak formulation

Consider a smooth elastic medium $\widetilde{\Omega} \subset \mathbb{R}^d$, $d = 2$ or 3 with linear elastic properties described by the unknown elasticity tensor $\mathbf{C}(x) \in T^4_{\text{sym}}$. The space T^4_{sym} as well as the different tensor products are all defined in Definition A.1. We assume that the

unknown tensor $\mathbf{C}$ belongs to $L^\infty(\widetilde{\Omega}, T^4_{\text{sym}})$. Consider now that one has measured internal displacement field $\mathbf{u}$, which corresponds to the internal force density $\mathbf{f}$ in some smooth subdomain of interest $\Omega \subset \widetilde{\Omega}$. The field $\mathbf{u} \in H^1(\widetilde{\Omega}, \mathbb{R}^d)$ satisfies the linear elasticity equation

$$-\nabla \cdot (\mathbf{C} : \nabla^s \mathbf{u}) = \mathbf{f} \quad \text{in } \Omega,$$

in the sense of distributions, i.e., in $\mathcal{D}'(\Omega, \mathbb{R}^d)$. In the case of multiple measurements, we assume knowledge of a finite number n of force densities $\mathbf{f}^\ell$ and the corresponding displacement fields $\mathbf{u}^\ell$ satisfying linear elasticity equation

$$-\nabla \cdot (\mathbf{C} : \nabla^s \mathbf{u}^\ell) = \mathbf{f}^\ell \quad \text{in } \Omega,$$

in the sense of distributions. As $\mathbf{C} \in L^\infty(\Omega, T^4_{\text{sym}})$ and $\nabla^s \mathbf{u} \in L^2(\Omega, \mathbb{R}^{d \times d}_{\text{sym}})$, the previous equation makes sense in $H^{-1}(\Omega, \mathbb{R}^d)$ writing

$$\int_\Omega (\mathbf{C} : \nabla^s \mathbf{u}^\ell) : \nabla^s \mathbf{v} = \left\langle \mathbf{f}^\ell, \mathbf{v} \right\rangle_{H^{-1}, H^1_0}, \quad \forall \mathbf{v} \in H^1_0(\Omega, \mathbb{R}^d). \tag{3}$$

Here, $\langle \, , \, \rangle_{H^{-1}, H^1_0}$ denotes the duality pairing between H^{-1} and H^1_0. Note that, by considering this problem in H^{-1} (taking test functions in H^1_0), we naturally forget what happens on the boundary. This classical weak formulation naturally introduces a bilinear form $a_{\mathbf{C}}$ such that the forward problem reads

$$a_{\mathbf{C}}(\mathbf{u}, \mathbf{v}) = l(\mathbf{v}), \quad \forall \mathbf{v} \in H^1_0(\Omega, \mathbb{R}^d).$$

The method that we present is based on the simple idea of changing the point of view and, given a vector field $\mathbf{u}$, writing (3) as a bilinear form acting on $(\mathbf{C}, \mathbf{v})$ instead of $(\mathbf{u}, \mathbf{v})$:

$$a_{\mathbf{u}}(\mathbf{C}, \mathbf{v}) = l(\mathbf{v}), \quad \forall \mathbf{v} \in H^1_0(\Omega, \mathbb{R}^d). \tag{4}$$

In order to stay in a Hilbert space framework, we make the non-restrictive assumption that the strain tensor $\nabla^s \mathbf{u}$ is bounded. We will stand under this hypothesis in the whole paper.

Definition 2.1 (Stiffness-to-force operator)**.** *If $\nabla^s \mathbf{u} \in L^\infty(\Omega, \mathbb{R}^{d \times d}_{\text{sym}})$, then (4) canonically defines the bounded operator:*

$$A_{\mathbf{u}} : L^2(\Omega, T^4_{sym}) \longrightarrow H^{-1}(\Omega, \mathbb{R}^d)$$
$$\mathbf{C} \longmapsto -\nabla \cdot (\mathbf{C} : \nabla^s \mathbf{u}),$$

which is called the stiffness-to-force operator.

Hence, the general inverse problem that we want to solve simply reads $A_{\mathbf{u}}\mathbf{C} = \mathbf{0}$ in the elastostatic case and $A_{\mathbf{u}}\mathbf{C} = \mathbf{f}$ in the elastodynamic case.

In most of the cases, we do not look for a general tensor $\mathbf{C}(x) \in T^4_{\text{sym}}$ and we know, a priori, that it can be decomposed as a sum of known directions:

$$\mathbf{C}(x) = \sum_{k=1}^N \mu^{(k)}(x) \mathbf{C}^k, \quad \forall x \in \Omega,$$

where $\mu^{(k)}$ are **unknown** functions of $L^2(\Omega)$ and $\mathbf{C}^k$ are **known** constant tensors. For instance, in isotropic cases $\mathbf{C}(x) = 2\mu(x)\mathbf{I}$ or $\mathbf{C}(x) = 2\mu(x)\mathbf{I} + \lambda(x)I \otimes I$, where μ and λ are the two Lamé parameters. Hence, the reconstruction of $\mathbf{C}(x)$ can be obtained from the reconstruction of the $\mu^{(k)}(x)$ solutions of the variational problem:

$$\sum_{k=1}^{N} \int_{\Omega} \mu^{(k)}(x)\big(\mathbf{C}^k : \nabla^s \mathbf{u}^\ell(x)\big) : \nabla^s \mathbf{v}(x) \mathrm{d}x = \big\langle \mathbf{f}^\ell, \mathbf{v} \big\rangle_{H^{-1}, H_0^1} \quad \forall \mathbf{v} \in H_0^1(\Omega, \mathbb{R}^d),$$

or equivalently

$$\sum_{k=1}^{N} \left\langle A_{\mathbf{u}^\ell}^{\mathbf{C}^k}(\mu^{(k)}), \mathbf{v} \right\rangle_{H^{-1}, H_0^1} = \langle \mathbf{f}^\ell, \mathbf{v} \rangle_{H^{-1}, H_0^1}.$$

Here, for all $\mathbf{u} \in W^{1,\infty}(\Omega, \mathbb{R}^d)$ and $\mathbf{C} \in T_{\mathrm{sym}}^4$, the bounded linear operator $A_{\mathbf{u}}^{\mathbf{C}} : L^2(\Omega) \to H^{-1}(\Omega, \mathbb{R}^d)$ is defined by

$$A_{\mathbf{u}}^{\mathbf{C}}(\mu) = -\nabla \cdot (\mu \mathbf{C} : \nabla^s \mathbf{u}).$$

The general recovery problem with multiple measurements reads as the following system:

$$\begin{pmatrix} A_{\mathbf{u}^1}^{\mathbf{C}^1} & \cdots & A_{\mathbf{u}^1}^{\mathbf{C}^N} \\ \vdots & & \vdots \\ A_{\mathbf{u}^n}^{\mathbf{C}^1} & \cdots & A_{\mathbf{u}^n}^{\mathbf{C}^N} \end{pmatrix} \begin{pmatrix} \mu^{(1)} \\ \vdots \\ \mu^{(N)} \end{pmatrix} = \begin{pmatrix} \mathbf{f}^1 \\ \vdots \\ \mathbf{f}^n \end{pmatrix}.$$

Remark 2.1. *As we will see in Section 4, this formulation is naturally adapted to a finite element discretization when looking for the coefficients $\mu^{(k)}(x)$ in $\mathcal{P}^0$ using test functions in $\mathcal{P}_0^1$.*

2.2 Existing stability results

Although the question of the injectivity is very hard without extra regularity assumptions on $\mathbf{C}$, there exists some stability results for the reconstruction of the tensor $\mathbf{C}$. The most important one can be found in [9]. We include here, for the sake of completeness, the following stability result for data with $W^{2,\infty}$ regularity for $\mathbf{u}$.

Theorem 1 (see [9]). *Let $(\mathbf{u}^1, \ldots, \mathbf{u}^n)$ and $(\tilde{\mathbf{u}}^1, \ldots, \tilde{\mathbf{u}}^n)$ be two families of displacement fields of size $n = d(d+1)/2 + N/d$ and $A := (A_{\mathbf{u}^1}, \ldots, A_{\mathbf{u}^n})$, $\tilde{A} := (A_{\tilde{\mathbf{u}}^1}, \ldots, A_{\tilde{\mathbf{u}}^n})$ be the corresponding multiple data stiffness-to-force operators. If the tensors $\mathbf{C}$ and $\widetilde{\mathbf{C}}$ satisfy*

$$A\mathbf{C} = \mathbf{0} \quad and \quad \tilde{A}\widetilde{\mathbf{C}} = \mathbf{0},$$

under some extra assumptions on the linear independence of these families of displacement fields, then $\mathbf{C}$ and $\widetilde{\mathbf{C}}$ can each be uniquely reconstructed over Ω up to a multiplicative constant. Moreover, if we assume that $\|\mathbf{C}\|_{L^\infty(\Omega)} = \|\widetilde{\mathbf{C}}\|_{L^\infty(\Omega)}$, then

$$\|\mathbf{C} - \widetilde{\mathbf{C}}\|_{L^\infty(\Omega)} + \|\nabla \cdot \mathbf{C} - \nabla \cdot \widetilde{\mathbf{C}}\|_{L^\infty(\Omega)} \le k \sum_{\ell=1}^{n} \|\nabla^s \mathbf{u}^\ell - \nabla^s \tilde{\mathbf{u}}^{(\ell)}\|_{W^{1,\infty}(\Omega)},$$

where k does not depend on $\mathbf{C}$ and $\widetilde{\mathbf{C}}$.

We refer the reader to [9] for more details.

Remark 2.2. *If we assume that the elasticity tensor $\mathbf{C}$ is of the form $\mathbf{C} = 2\mu\mathbf{I}$ or $\mathbf{C} = 2\mu\mathbf{I} + \lambda I \otimes I$, in dimension 2, then Theorem 1 implies that one needs at least 4 sets of measurements in order to reconstruct $\mathbf{C}$ up to a multiplicative constant. Moreover, one needs $\nabla^s \mathbf{u}^\ell$ to be Lipschitz.*

2.3 Classical elastic media inversion problems

2.3.1 Shear modulus inversion

In the ideal case where λ is equal to zero or it is assumed to be known in the medium, the elasticity equation reads as

$$-\nabla \cdot (\mu\nabla^s \mathbf{u}^\ell) = \mathbf{f}^\ell \quad \text{in } H^{-1}(\Omega, \mathbb{R}^d)$$

which corresponds to the previous model with $N = 1$, $\mu = \mu^{(1)}$ and $\mathbf{C}^1 = \mathbf{I}$. Note that in the static case ($\mathbf{f}^\ell = 0$), the recovery problem is equivalent to finding μ in the null space of $(A_{\mathbf{u}}^{\mathbf{I}})$. In particular, *formally*, if $\mathbf{u}^\ell$ is smooth and if $\nabla^s \mathbf{u}^\ell$ is invertible, then

$$\nabla \cdot (\mu\nabla^s \mathbf{u}^\ell) = 0 \quad \text{and} \quad \mu > 0$$

implies that

$$\mu\nabla^s \mathbf{u}^\ell \left(\nabla \log(\mu) + (\nabla^s \mathbf{u}^\ell)^{-1} \Delta^s \mathbf{u}^\ell \right) = 0,$$

which suggests that this equation has a non-trivial solution if and only if there exists φ such as

$$(\nabla^s \mathbf{u}^\ell)^{-1} \Delta \mathbf{u}^\ell = \nabla\varphi.$$

In that case,

$$N(A_{\mathbf{u}^\ell}^{\mathbf{I}}) = \text{span} \left\{ \exp(-\varphi) \right\}.$$

The problem of showing that the null space of $A_{\mathbf{u}^\ell}^{\mathbf{I}}$ is at most of dimension one has been solved in [10] in the case of *smooth coefficients*. The aim of the next section is to generalize this approach for *discontinuous* strain tensors.

Remark 2.3. *The method developed in [10] can be numerically implemented by using the Helmholtz decomposition of $(\nabla^s \mathbf{u}^\ell)^{-1} \Delta \mathbf{u}^\ell$. Moreover, it suggests also that only one set of data is required to reconstruct μ up to a multiplicative constant.*

2.3.2 Inversion of Lamé coefficients

In the general isotropic case, i.e, $P = 2$, $\mu^{(1)} = \mu$, $\mathbf{C}^1 = \mathbf{I}$, $\mu^{(2)} = \lambda$ and $\mathbf{C}^2 = I \otimes I$, the coefficient λ is associated to the operator $A_{\mathbf{u}}^{I \otimes I}$ defined by

$$A_{\mathbf{u}}^{I \otimes I}(\lambda) = \nabla \left(\lambda\nabla \cdot \mathbf{u} \right).$$

Formally, its null space is at most of dimension one and is given by

$$N \left(A_{\mathbf{u}}^{I \otimes I} \right) = \text{Span} \left\{ \frac{1}{\nabla \cdot \mathbf{u}} \right\}.$$

In practice, this shows that the reconstruction of Lamé coefficients (μ, λ) requires at least two sets of data $\mathbf{u}^1$ and $\mathbf{u}^2$:

$$\begin{pmatrix} 2A_{\mathbf{u}^1}^{\mathbf{I}} & A_{\mathbf{u}^1}^{I \otimes I} \\ 2A_{\mathbf{u}^2}^{\mathbf{I}} & A_{\mathbf{u}^2}^{I \otimes I} \end{pmatrix} \begin{pmatrix} \mu \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f}^{(1)} \\ \mathbf{f}^{(2)} \end{pmatrix},$$

which satisfy the necessary condition $N\left(A_{\mathbf{u}^1}^{I \otimes I}\right) \neq N\left(A_{\mathbf{u}^2}^{I \otimes I}\right)$ or in other terms,

$$\forall c \in \mathbb{R}, \quad \nabla \cdot \mathbf{u}^1 \neq c \nabla \cdot \mathbf{u}^2.$$

2.3.3 Anisotropic medium inversion

The last example is an anisotropic medium such that the tensor takes 3 independent directions:

$$\mathbf{C}(x) = \mu^{(1)}(x)\mathbf{C}^1 + \mu^{(2)}(x)\mathbf{C}^2 + \mu^{(3)}(x)\mathbf{C}^3,$$

where the tensors $\mathbf{C}^1$, $\mathbf{C}^2$ and $\mathbf{C}^3$ are defined by

$$\mathbf{C}^1 : A = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{C}^2 : A = \begin{pmatrix} 0 & 0 \\ 0 & (A)_{22} \end{pmatrix}, \tag{5}$$

and

$$\mathbf{C}^3 : A = \begin{pmatrix} 0 & (A_{12} + A_{21})/2 \\ (A_{12} + A_{21})/2 & 0 \end{pmatrix}. \tag{6}$$

for any squared matrix A.

This is an ideal case, and doesn't necessarily correspond to a biomedical imaging application. It is used as a an example to show the versatility of our method. Anisotropic shear wave imaging is of great use in cardiac imaging. The anisotropic model for the myocardium and the imaging of the degree of anisotropy will be investigated in a forthcoming paper.

2.4 Regularity of the coefficients of elasticity tensor

The choice of the functional spaces for the elasticity tensor's coefficients and for $\mathbf{u}$ is a crucial question. The standard theory of elliptic systems shows that the regularity of $\mathbf{C}$'s coefficients determines the regularity of the solution $\mathbf{u}$ of the linear elasticity equation. For instance, it is well-known that, under some ellipticity conditions, if the coefficients of $\mathbf{C}$ are in L^∞, the solution $\mathbf{u}$ is in $H^1(\Omega, \mathbb{R}^2)$ and therefore no more than L^2 regularity can be expected for $\nabla^s \mathbf{u}$. The standard Hölder theory for elliptic systems tells us that if the coefficients are piecewise Hölder continuous, then the same regularity can be expected for $\nabla^s \mathbf{u}$.

As we mainly focus on imaging mechanical properties of biological tissues, we should use an appropriate model for the elastic coefficients. Typically, it is not realistic to assume that the elastic coefficients are everywhere differentiable, since biological tissues are often constituted of different types of embedded materials which exhibit discontinuities.

A good acceptable model for a biological medium is to assume that the biological parameters are piecewise smooth with smooth discontinuity surfaces. Out of these discontinuities, we suppose a Sobolev type smoothness. We call such a space of function $W_{\mathrm{pw}}^{1,p}(\Omega)$, and give its precise definition in Definition 3.1.

We will also use spaces that include discontinuous functions and that are more general than those in $W^{1,p}_{\mathrm{pw}}(\Omega)$. We introduce the subspace $SBV(\Omega) \subset BV(\Omega)$ of the functions of bounded variations whose discontinuity sets have no Cantor parts. The full precise definition is given in Definition 3.2.

The relations between the functional spaces that we use are the following:

$$W^{1,p}(\Omega) \subset W^{1,p}_{\mathrm{pw}}(\Omega) \subset SBV^p(\Omega) \subset L^p(\Omega).$$

Remark 2.4. *The condition that the coefficients of the tensor belong to $W^{1,p}_{pw}(\Omega)$ or $SBV(\Omega)$ makes an important difference between this work and the aforementioned theoretical works on elastography. Under this assumption, we cannot assume that $\nabla^s \mathbf{u} \in W^{1,p}$ and the analysis becomes more complicated. This is the reason for Section 3 to be quite lengthy and technical.*

3 Shear modulus imaging: invertibility and stability in the isotropic case

In this section, we study the so-called *shear-to-force operator*

$$A^{\mathbf{I}}_{\mathbf{u}} : L^2(\Omega) \longrightarrow H^{-1}(\Omega, \mathbb{R}^d)$$
$$\mu \longmapsto -\nabla \cdot (\mu \nabla^s \mathbf{u}).$$

The outline of this section is the following:

(i) In Subsection 3.2, we study the null space of $A^{\mathbf{I}}_{\mathbf{u}}$ and we extend the results of [10] by showing that under low regularity assumptions for $\mathbf{u}$ (typically, SBV type regularity for $\nabla^s \mathbf{u}$), the null space is of dimension zero or one.

(ii) In Subsection 3.3, we study the solvability of the inverse problem by giving sufficient conditions on $\nabla^s \mathbf{u}$ for the operator $A^{\mathbf{I}}_{\mathbf{u}}$ to be of closed range, therefore ensuring the continuity of the inverse on the orthogonal of the null space. We first prove that the operator $A^{\mathbf{I}}_{\mathbf{u}}$ as closed range under invertibility and $W^{1,p}$ smoothness assumption for $\nabla^s \mathbf{u}$, for some $p > d$. We then relax the regularity assumption to a piecewise regularity $\nabla^s \mathbf{u} \in W^{1,p}_{\mathrm{pw}}(\Omega)$, ensuring the solvability of the inverse problem when looking for piecewise smooth shear modulii.

(iii) In Subsection 3.4, we give quantitative results on the stability of the inversion of $A^{\mathbf{I}}_{\mathbf{u}}$.

3.1 Spaces of discontinuous functions

In order to prove invertibility and stability of the inverse problem under minimal smoothness assumptions on the coefficients and the data, we introduce here two spaces of discontinuous functions.

3.1.1 The space $W_{\mathbf{pw}}^{1,p}(\Omega)$

Definition 3.1. *A function f is said to be in $W_{pw}^{1,p}(\Omega)$ for $1 \leq p \leq +\infty$, if there exists a smooth covering $\Omega_1, \ldots, \Omega_k$, for $k \geq 1$, such that*

(i) Ω_i is a smooth open connected subdomain of Ω for every $i \in \{1, \ldots, k\}$;

(ii) $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$;

(iii) $\displaystyle\bigcup_{i=1}^{k} \overline{\Omega_i} = \overline{\Omega}$;

(iv) $\forall i \in \{1, \ldots, k\}, \quad f\big|_{\Omega_i} \in W^{1,p}(\Omega_i)$.

In order to have an even more general set of discontinuous functions, we introduce the following space inspired by the space $SBV(\Omega)$.

3.1.2 The space $\mathrm{SBV}^p(\Omega)$

Since the derivative of a function $f \in BV(\Omega)$ can be decomposed as:

$$Df = \nabla f \mathcal{H}^d + [f]\mathbf{n}_S \mathcal{H}_S^{d-1} + D_c f,$$

where $\mathcal{H}^d$ is the Lebesgue measure on Ω, $\mathcal{H}_S^{d-1}$ is the surface Hausdorff measure on a rectifiable surface S, $\mathbf{n}_S$ is a normal vector defined almost everywhere on S, $f \in L^1(\Omega)$ is the smooth derivative of f, $[f] \in L^1(S, \mathcal{H}_S^{d-1})$ is the jump of f across S and $D_c f$ is a vector measure supported on a set of Hausdorff dimension less than $(d-1)$, which means that its $(d-1)$-Hausdorff-measure is zero. The well-known space $SBV(\Omega)$ introduced by De Giorgi and Ambrosio [1] is the subclass of $BV(\Omega)$ of functions whose derivative Cantor parts are zero: $D_c f = \mathbf{0}$. Following this idea, we introduce a very large piecewise-$W^{1,p}$ class of functions:

Definition 3.2. *For $1 \leq p \leq +\infty$, we define*

$$\mathrm{SBV}^p(\Omega) = \left\{ f \in \mathrm{SBV}(\Omega) \cap L^p(\Omega), \ \nabla f \in L^p(\Omega, \mathbb{R}^d) \right\},$$

where ∇f is the Lebesgue part of the measure Df.

Note that we clearly have the inclusion $W_{\mathrm{pw}}^{1,p}(\Omega) \subset SBV^p(\Omega)$.

3.2 Null space of the shear-to-force operator

In this subsection, we prove Theorems 2 and 3, which give simple conditions on $S := \nabla^S \mathbf{u}$ in order to ensure that the operator $A_{\mathbf{u}}^{\mathbf{I}}$ has a null space of dimension zero or one.

Theorem 2 (Characterization in $W^{1,p}$). *Assume that $\nabla^s \mathbf{u} \in L^\infty(\Omega, \mathbb{R}_{sym}^{d \times d}) \cap W^{1,p}(\Omega, \mathbb{R}^{d \times d})$ for some $p > d$ and that $|\det \nabla^s \mathbf{u}| \geq c > 0$. Then the space*

$$K_{\mathbf{u}} := \left\{ \mu \in L^2(\Omega), \ \nabla \cdot (\mu \nabla^s \mathbf{u}) = 0 \right\},$$

is of dimension zero or one. In the second case, there exists a positive continuous function μ_0, such that $K_{\mathbf{u}} = span\{\mu_0\}$. If Ω is Lipschitz, then μ_0 belongs to $W^{1,p}(\Omega)$.

Theorem 3 (Characterization in SBV^p). *Assume that $\nabla^s \mathbf{u} \in L^\infty(\Omega, \mathbb{R}^{d \times d}_{sym}) \cap SBV^p(\Omega, \mathbb{R}^{d \times d})$ for some $p > d$ and that $|\det \nabla^s \mathbf{u}| \geq c > 0$. Then, the space*

$$K_{\mathbf{u}} := \left\{ \mu \in L^2(\Omega), \ \nabla \cdot (\mu \nabla^s \mathbf{u}) = 0 \right\}$$

is of dimension zero or one.

Proof. Denote Σ the closure of the discontinuity surface of S. The open set $\Omega \backslash \Sigma$ can be decomposed as a countable union of connected open sets:

$$\Omega \backslash \Sigma = \bigcup_{i \in I} \Omega_i.$$

One may apply Theorem 2 on each subset and say that there exists some $\nu_i \in \mathcal{C}^0(\Omega_i)$ such that any solution of the problem is written as

$$\mu = \sum_{i \in I} \alpha_i e^{\nu_i} \mathbf{1}_{\Omega_i} \quad \text{in } \Omega \backslash \Sigma,$$

where α_i's are some real numbers.

  We show now that these numbers are linked by the jump condition over Σ. Consider two subdomain Ω_i and Ω_j in contact in the sense that their common boundary

$$\Sigma_{ij} := \partial \Omega_i \cap \partial \Omega_j$$

is of positive surface measure: $\mathcal{H}^{d-1}(\partial \Omega_i \cap \partial \Omega_j) > 0$. As Σ is rectifiable, there exists $x_0 \in \Sigma_{ij}$ and $B := B(x_0, \varepsilon)$ such that $\Omega_i^B := \Omega_i \cap B$ and $\Omega_j^B := \Omega_j \cap B$ are Lipschitz domains. As μ and S are $W^{1,p}$ in Ω_i^B and Ω_j^B, so is the product μS and it admits two-sided traces $\mu_i S_i$ and $\mu_j S_j$ defined as functions of $L^p(\Sigma_{ij} \cap B)$. From the variational formulation, the jump condition at $\Sigma_{ij} \cap B$ reads as

$$\mu_i S_i \nu = \mu_j S_j \nu \quad \text{almost everywhere on } \Sigma_{ij} \cap B.$$

This jump condition gives a vectorial equation linking α_i and α_j which is

$$\alpha_i e^{\nu_i} S_i \nu = \alpha_j e^{\nu_j} S_j \nu. \tag{7}$$

As ν_i, ν_j are bounded in B and $|\det S_i|$, $|\det S_j| \geq c > 0$, there exists $c' > 0$ such that $|e^{\nu_i} S_i \nu| \geq c'$ and $|e^{\nu_j} S_j \nu| \geq c'$. A first consequence is that if one $\alpha_i = 0$ then they are all zero and $\mu = 0$.

  Now consider another solution $\mu' = \sum_{i \in I} \beta_i e^{\nu_i} \mathbf{1}_{\Omega_i}$ and assume that $\frac{\mu'}{\mu}$ is not constant. There exist Ω_i, Ω_j in contact such that $\beta_i / \alpha_i \neq \beta_j / \alpha_j$. Using (7) for both couples (α_i, α_j) and (β_i, β_j), it follows that there exists $\gamma \neq 0$ such that $\alpha_j = \gamma \alpha_i$ and $\beta_j = \gamma \beta_i$, which leads to $\beta_i / \alpha_i = \beta_j / \alpha_j$. Since this is absurd, μ'/μ is constant. $\qquad \square$

Proof of Theorem 2   As $|\det S| \geq c > 0$, there exists $S^{-1} \in L^\infty(\Omega, \mathbb{R}^{d \times d}_{sym})$ such that $SS^{-1} = I$ almost everywhere in Ω. We successively apply Propositions 3.1 and 3.3 with $\mathbf{b} = -S^{-1} \nabla \cdot S$, which ends the proof. $\qquad \square$

Proposition 3.1 (Decomposition). *Assume that $S \in L^\infty(\Omega, \mathbb{R}^{d \times d}_{sym}) \cap W^{1,p}(\Omega, \mathbb{R}^{d \times d})$ for some $p \in [2, +\infty]$ and there exists $S^{-1} \in L^\infty(\Omega, \mathbb{R}^{d \times d})$ such that $SS^{-1} = I$ almost everywhere on Ω. Then, any solution μ of $\nabla \cdot (\mu S) = 0$ is in $W^{1,1}(\Omega)$ and satisfies*

$$\nabla \mu + \mu S^{-1}(\nabla \cdot S) = \mathbf{0}.$$

Proof. As $(\nabla \cdot S) \in L^p(\Omega, \mathbb{R}^d)$, we have for every $\mathbf{v} \in L^\infty(\Omega, \mathbb{R}^d) \cap W^{1,p}(\Omega, \mathbb{R}^d)$ that $\nabla \cdot (S\mathbf{v}) \in L^p(\Omega)$ and $\nabla \cdot (S\mathbf{v}) = (\nabla \cdot S) \cdot \mathbf{v} + S : \nabla \mathbf{v}$. Then, we write

$$\int_\Omega \mu \nabla \cdot (S\mathbf{v}) = \int_\Omega \mu (\nabla \cdot S) \cdot \mathbf{v}.$$

For any $\mathbf{w} \in \mathcal{D}(\Omega, \mathbb{R}^d)$, the test function $\mathbf{v} = S^{-1}\mathbf{w}$ belongs to $L^\infty(\Omega, \mathbb{R}^d)$ and $\nabla \mathbf{v} = \nabla S^{-1} \cdot \mathbf{w} + S^{-1} \cdot \nabla \mathbf{w} \in L^p(\Omega, \mathbb{R}^{d \times d})$, and can be used in the previous equation to get,

$$\int_\Omega \mu \nabla \cdot \mathbf{w} = \int_\Omega \mu (\nabla \cdot S) \cdot S^{-1}\mathbf{w}, \quad \forall \mathbf{w} \in \mathcal{D}(\Omega, \mathbb{R}^d).$$

This means

$$\nabla \mu = -\mu S^{-1}(\nabla \cdot S).$$

$\square$

Proposition 3.2 (Regularity). *Take $p > d$ and consider $\mathbf{b} \in L^p(\Omega, \mathbb{R}^d)$. Any solution of*

$$\begin{cases} \mu \in L^2(\Omega), \\ \nabla \mu = \mu \mathbf{b}, \end{cases}$$

belongs to $\mathcal{C}^0(\Omega)$. If Ω is Lipschitz, then it belongs to $W^{1,p}(\Omega)$.

Proof. Consider a ball $B \subset \Omega$. As $p > d$, the injection $W^{1,p}(B) \hookrightarrow \mathcal{C}^0(B)$ holds. Let us prove that $\mu \in W^{1,p}(B)$. First, note that as $p \geq 2$, $\nabla \mu = \mu \mathbf{b} \in L^1(B, \mathbb{R}^d)$ and so $\mu \in W^{1,1}(B)$. Call now $q^* = \sup\{q \geq 1, \mu \in W^{1,q}(B)\}$.

Suppose that $q^* \leq d$. For any $1 \leq q < q^*$, $\mu \in W^{1,q}(B) \hookrightarrow L^r(B)$ with $\frac{1}{r} = \frac{1}{q} - \frac{1}{d}$ and $\mu \mathbf{b}$ belongs to $L^s(B, \mathbb{R}^d)$ with $\frac{1}{s} = \frac{1}{r} + \frac{1}{p} = \frac{1}{q} - \frac{1}{d} + \frac{1}{p}$. Let $\beta = \frac{1}{d} - \frac{1}{p} > 0$. We get that $\mu \in W^{1,s}(B)$ with $\frac{1}{s} = \frac{1}{q} - \beta$. One can choose q such that $s > q^*$ which contradicts the definition of q^*. Then $q^* > d$.

Considering that $\mu \in W^{1,q}(B)$ for some $q \in (d, q^*]$ and that for such q, $W^{1,q}(B) \hookrightarrow L^\infty(B)$, we get that $\mu \mathbf{b} \in L^p(B, \mathbb{R}^d)$ and so $\mu \in W^{1,p}(B)$. As a consequence, μ is continuous in Ω.

If Ω is Lipschitz, one can restart the proof replacing B by Ω to obtain that $\mu \in W^{1,p}(\Omega)$.

$\square$

Proposition 3.3 (Existence of non-zero solutions). *Take $p > d$ and consider $\mathbf{b} \in L^p(\Omega, \mathbb{R}^d)$. The problem*

$$\begin{cases} \mu \in L^2(\Omega), \\ \nabla \mu = \mu \mathbf{b} \quad in \ \mathcal{D}'(\Omega, \mathbb{R}^d), \end{cases}$$

admits a non-zero solution if and only if the vector field **b** *is conservative which means that*

$$\nabla \nu = \mathbf{b} \quad in \; \mathcal{D}'(\Omega, \mathbb{R}^d),$$

admits a continuous solution. In this case, the set of solutions is given by $\{\alpha e^\nu, \; \alpha \in \mathbb{R}\}$. Moreover, if Ω is Lipschitz, then ν is bounded and there exists a constant $m > 0$ such that $\mu_0 := e^\nu \geq m$.

Proof. If the equation $\nabla \nu = \mathbf{b}$ admits a solution $\nu \in \mathcal{C}^0(\Omega)$, then e^ν is continuous and positive in Ω. It satisfies in the weak sense $\nabla(e^\nu) = e^\nu \mathbf{b}$. Its inverse $e^{-\nu}$ has the same properties. Take $\mu \in L^2(\Omega)$ a solution of $\nabla \mu = \mu \mathbf{b}$ and define $\alpha = \mu e^{-\nu} \in L^2_{\text{loc}}(\Omega)$, then in the weak sense, $\nabla \alpha = e^{-\nu} \nabla \mu - \mu e^{-\nu} \mathbf{b} = 0$, so, as Ω is connected, α is constant in Ω. This proves the first part of the result.

Assume now that $\nabla \nu = \mathbf{b}$ has no solution in $\mathcal{C}^0(\Omega)$ and consider a solution $\mu \in L^2(\Omega)$ of the equation $\nabla \mu = \mu \mathbf{b}$. Using Proposition 3.2, it follows that $\mu \in \mathcal{C}^0(\Omega)$.

Suppose that μ does not vanish in Ω, then $\mu > 0$ in Ω (take $-\mu$ if $\mu < 0$), then $\nu := \ln \mu$ is continuous and satisfies $\nabla \nu = \mathbf{b}$, which is impossible.

As a consequence, μ does vanish somewhere in Ω. If $\mu \neq 0$, then there exists a ball $B \subset \Omega$ such that $\mu > 0$ (take $-\mu$ if $\mu < 0$) in B and μ vanishes somewhere on ∂B. Inside B, $\nu := \ln \mu$ is continuous and satisfies $\nabla \nu = \mathbf{b} \in L^p(B, \mathbb{R}^d)$ so $\nu \in W^{1,p}(B) \hookrightarrow L^\infty(B)$. Thus, $\mu = e^\nu \geq e^{-\|\nu\|_{L^\infty(B)}} > 0$ on B, which contradicts the fact that μ vanishes somewhere on ∂B. Finally, it follows that $\mu = 0$ is the only solution. $\qquad \square$

3.3   Closed range property of the shear-to-force operator

In the case of existence of a non-trivial null space for the shear-to-force operator (elastostatic case), we study the possibility of a stable reconstruction of the parameter μ in $L^2(\Omega)$. We recall that a linear operator $A : H \to E$ where H is an Hilbert space and E a Banach space has closed range if $R(A) := A(H)$ is closed in E. The following proposition gives an equivalent definition of this property:

Proposition 3.4 (Closed range operator). *Let H and E be en Hilbert and a Banach space. A linear operator $A : H \to E$ has closed range if and only if there exists a constant $c > 0$ such that*

$$\forall x \in N(A)^\perp, \quad \|x\|_H \leq c \, \|Ax\|_E .$$

In particular, this implies the existence of a bounded inverse operator from $R(A)$ to $N(A)^\perp$. We refer the reader to [13, Section 2.7] for more details.

Theorem 4 (Closed range with strain in $W^{1,p}$). *Take $\mathbf{u}$ such that $S := \nabla^s \mathbf{u} \in W^{1,p}(\Omega)$, and that $|\det \nabla^s \mathbf{u}| > m > 0$ in Ω. If $N\left(A_{\mathbf{u}}^{\mathbf{I}}\right) \neq \{0\}$, then $A_{\mathbf{u}}^{\mathbf{I}} : L^2(\Omega) \to H^{-1}(\Omega, \mathbb{R}^d)$ has closed range.*

Proof. According to Theorem 2, there exists $\mu_0 \in W^{1,p}(\Omega)$ such that $N\left(A_{\mathbf{u}}^{\mathbf{I}}\right) = \text{Span}\,\{\mu_0\}$. By construction of μ_0, there exists a constant $\tilde{m} \geq 0$ such that $\mu_0 > \tilde{m}$ in Ω. Take

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$\mathbf{f} \in R(A_{\mathbf{u}}^{\mathbf{I}})$ and $\mu \in \{\mu_0\}^{\perp}$ such that $A_{\mathbf{u}}^{\mathbf{I}}(\mu) = \mathbf{f}$. Define $\alpha = \frac{\mu}{\mu_0} \in L^2(\Omega)$. One can write

$$-\nabla \cdot (\alpha \mu_0 S) = \mathbf{f} \quad \text{in } H^{-1}(\Omega, \mathbb{R}^d),$$
$$-\mu_0 S \nabla \alpha = \mathbf{f} \quad \text{in } H^{-1}(\Omega, \mathbb{R}^d),$$

which makes sense because $\mu_0 S \in W^{1,p}(\Omega)$ (see Lemma A.1) and $\nabla \cdot (\mu_0 S) = 0$. Multiplying by $\mu_0^{-1} S^{-1} \in W^{1,p}$ yields

$$\nabla \alpha = \frac{S^{-1}}{\mu_0} \mathbf{f} \quad \text{in } H^{-1}(\Omega, \mathbb{R}^d).$$

Using Lemma A.1, it follows that

$$\|\nabla \alpha\|_{H^{-1}(\Omega)} \leq \left\| \frac{S^{-1}}{\mu_0} \right\|_{W^{1,p}(\Omega)} \|\mathbf{f}\|_{H^{-1}(\Omega)}.$$

Since $\mu = \alpha \mu_0 \in \{\mu_0\}^{\perp}$, we have $\int_{\Omega} \alpha \mu_0^2 = 0$ and by Lemma A.2, there exists $c > 0$ such that

$$\|\alpha\|_{L^2(\Omega)} \leq c \|\nabla \alpha\|_{H^{-1}(\Omega)}$$
$$\leq c \left\| \frac{S^{-1}}{\mu_0} \right\|_{W^{1,p}(\Omega)} \|\mathbf{f}\|_{H^{-1}(\Omega)}.$$

Turning now to $\mu = \alpha \mu_0$, we obtain that

$$\|\mu\|_{L^2(\Omega)} \leq \|\alpha\|_{L^2(\Omega)} \|\mu_0\|_{L^\infty(\Omega)}$$
$$\leq c \|\mu_0\|_{L^\infty(\Omega)} \left\| \frac{S^{-1}}{\mu_0} \right\|_{W^{1,p}(\Omega)} \|\mathbf{f}\|_{H^{-1}(\Omega)},$$
$$\leq c \|\mu_0\|_{L^\infty(\Omega)} \left\| \frac{S^{-1}}{\mu_0} \right\|_{W^{1,p}(\Omega)} \|\nabla \cdot (\mu S)\|_{H^{-1}(\Omega)}.$$

$\square$

Theorem 5 (Closed range with strain in $W_{\text{pw}}^{1,p}$). *Take $\mathbf{u}$ such that $S := \nabla^s \mathbf{u} \in W_{pw}^{1,p}(\Omega, \mathbb{R}^{d \times d})$, and that $|\det \nabla^s \mathbf{u}| \geq m > 0$ in Ω. If $N\left(A_{\mathbf{u}}^{\mathbf{I}}\right) \neq \{0\}$, then $A_{\mathbf{u}}^{\mathbf{I}} : L^2(\Omega) \to H^{-1}(\Omega, \mathbb{R}^d)$ has closed range.*

Proof. According to Theorem 3, there exists $\mu_0 \in L^2(\Omega)$ and $\|\mu_0\|_{L^2(\Omega)} = 1$ and such that $N(A_{\mathbf{u}}^{\mathbf{I}}) = \text{span} \{\mu_0\}$. By construction, there exists $m > 0$ such that $|\mu_0| \geq \tilde{m}$. Take $\mathbf{f} \in R(A_{\mathbf{u}}^{\mathbf{I}})$ and $\mu \in L^2(\Omega)$ such that $A_{\mathbf{u}}^{\mathbf{I}} \mu = \mathbf{f}$. Define $\alpha = \frac{\mu}{\mu_0} \in L^2(\Omega)$. As $S \in W_{\text{pw}}^{1,p}(\Omega, \mathbb{R}^{d \times d})$, there exists a domain decomposition $\Omega_1, \ldots, \Omega_k$. Note that, for α_i and $\mathbf{f}_i$ the restrictions to Ω_i, we have for any i,

$$A_{\mathbf{u}}^{\mathbf{I}}(\alpha_i \mu_0) = \mathbf{f}_i \quad \text{in } H^{-1}(\Omega_i).$$

Following the proof of the previous theorem, we can control $\nabla \alpha_i$ in H^{-1} norm. There exists $c_i > 0$ such that

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$$\|\nabla \alpha_i\|_{H^{-1}(\Omega_i)} \leq c_i \|\mathbf{f}\|_{H^{-1}(\Omega_i)}.$$

Denote $\alpha_{\Omega_i} := |\Omega_i|^{-1} \int_{\Omega_i} \alpha_i$. From Lemma A.2, it follows that

$$\|\alpha_i - \alpha_{\Omega_i}\|_{L^2(\Omega_i)} \leq c_i \|\mathbf{f}\|_{H^{-1}(\Omega_i)}.$$

Then, taking $C_1 = \max\{c_i\}$ gives

$$\|\alpha_i - \alpha_{\Omega_i}\|_{L^2(\Omega_i)} \leq C_1 \|\mathbf{f}\|_{H^{-1}(\Omega)} \quad \forall i = 1 \ldots k. \tag{8}$$

Consider now the following decomposition:

$$\|\alpha\|_{L^2(\Omega)}^2 = \sum_{i=1}^k \|\alpha_i\|_{L^2(\Omega_i)}^2 \leq \sum_{i=1}^k \|\alpha_i - \alpha_{\Omega_i}\|_{L^2(\Omega_i)}^2 + |\Omega_i| \alpha_{\Omega_i}^2. \tag{9}$$

It remains to prove that the mean values α_{Ω_i} are controlled by $\mathbf{f}$. Going back to the variational formulation,

$$\int_\Omega \alpha \mu_0 S : \nabla \mathbf{v} = \langle \mathbf{f}, \mathbf{v} \rangle_{H^{-1}, H_0^1},$$

we decompose it as follows:

$$\sum_{i=1}^k \int_{\Omega_i} \alpha_i \mu_0 S : \nabla \mathbf{v} = \langle \mathbf{f}, \mathbf{v} \rangle_{H^{-1}, H_0^1},$$

$$\sum_{i=1}^k \int_{\Omega_i} (\alpha_i - \alpha_{\Omega_i}) \mu_0 S : \nabla \mathbf{v} + \alpha_{\Omega_i} \int_{\Omega_i} \mu_0 S : \nabla \mathbf{v} = \langle \mathbf{f}, \mathbf{v} \rangle_{H^{-1}, H_0^1}.$$

Recalling that $\mu_0 S$ is divergence free, we write $\int_{\Omega_i} \mu_0 S : \nabla \mathbf{v} = -\int_{\partial\Omega_i} \mu_0 S \mathbf{n}_i \cdot \mathbf{v}$ to get that

$$\sum_{i=1}^k \alpha_{\Omega_i} \int_{\partial\Omega_i} \mu_0 S \mathbf{n}_i \cdot \mathbf{v} = \sum_{i=1}^k \int_{\Omega_i} (\alpha_i - \alpha_{\Omega_i}) \mu_0 S : \nabla \mathbf{v} - \langle \mathbf{f}, \mathbf{v} \rangle_{H^{-1}, H_0^1}.$$

Hence, we obtain that

$$\sum_{i=1}^k \alpha_{\Omega_i} \int_{\partial\Omega_i} \mu_0 S \mathbf{n}_i \cdot \mathbf{v} \leq \left(1 + \|\mu_0 S\|_{L^\infty(\Omega)} \sum_{i=1}^k c_i \right) \|\mathbf{f}\|_{H^{-1}(\Omega)} \|\mathbf{v}\|_{H_0^1(\Omega)}.$$

Consider now a boundary $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_i \neq \emptyset$. Remark that the left and right normal traces of $\mu_0 S$ are the same (divergence free jump condition). That is to say that $\mu_0 S \mathbf{n}_i = -\mu_0 S \mathbf{n}_j$ on Γ_{ij} and $\mu_0 S \mathbf{n}_i$ belongs to $H^{\frac{1}{2}}(\Gamma_{ij})$. Consider the continuous extension operator $R_{ij} : H^{\frac{1}{2}}(\Gamma_{ij}) \to H_0^1(\Omega)$ defined by $R_{ij} u|_{\Gamma_{ij}} = u$ and such that $R_{ij} u$ vanishes on all other boundaries Γ_{pq} where $(p, q) \neq (i, j)$. These operators exist because the boundaries are distant from one another and their continuity constants can be chosen without being dependent on Γ_{ij}.

Taking now the test function $\mathbf{v}_{ij} = R_{ij}(\alpha_{\Omega_i} - \alpha_{\Omega_j}) \mu_0 S \mathbf{n}_i \in H_0^1(\Omega, \mathbb{R}^d)$ and using it in the last equation gives

$$|\alpha_{\Omega_i}-\alpha_{\Omega_j}|^2 \int_{\Gamma_{ij}} |\mu_0 S\mathbf{n}_i|^2 \leq c_R \left(1 + \|\mu_0 S\|_{L^\infty(\Omega)} \sum_{i=1}^k c_i\right) \|\mathbf{f}\|_{H^{-1}(\Omega)} |\alpha_{\Omega_i}-\alpha_{\Omega_j}| \, \|\mu_0 S\mathbf{n}_i\|_{H^{\frac{1}{2}}(\Gamma_{ij})},$$

where c_R is such that $\|R_{ij}u\|_{H_0^1(\Omega)} \leq c_R \|u\|_{H^{\frac{1}{2}}(\Gamma_{ij})}$ for all Γ_{ij} and $u \in H^{\frac{1}{2}}(\Gamma_{ij})$. Note that the constant $\int_{\Gamma_{ij}} |\mu_0 S\mathbf{n}_i|^2$ cannot be zero because $|\mu_0| \geq \tilde{m} > 0$ and $|\det S| \geq m > 0$. To summarise, we have shown that there exists a constant $C_2 > 0$ depending only on μ_0, S, and the decomposition $(\Omega_i)_{i=1}^k$ such that

$$|\alpha_{\Omega_i} - \alpha_{\Omega_j}| \leq C_2 \|\mathbf{f}\|_{H^{-1}(\Omega)}, \tag{10}$$

for all i, j such that Ω_i and Ω_j share a boundary. This clearly can be extended to non-adjacent subdomains by transitivity and triangular inequality.

We now use the fact that $\mu \in \{\mu_0\}^\perp$, that is, $\int_\Omega \alpha\mu_0^2 = 0$. In other terms,

$$\sum_{i=1}^k \int_{\Omega_i} \alpha_i \mu_0^2 = 0,$$

or

$$\sum_{i=1}^k \alpha_{\Omega_i} \int_{\Omega_i} \mu_0^2 = -\sum_{i=1}^k \int_{\Omega_i} (\alpha_i - \alpha_{\Omega_i})\mu_0^2.$$

We deduce from (8) that there exists a constant $C_3 > 0$ such that

$$\left|\sum_{i=1}^k \alpha_{\Omega_i} \int_{\Omega_i} \mu_0^2\right| \leq C_3 \|\mathbf{f}\|_{H^{-1}(\Omega)}. \tag{11}$$

From (10) and (11), we can now bound all the α_{Ω_i} by just writing

$$\left|\sum_{i=1}^k \alpha_{\Omega_i} \int_{\Omega_i} \mu_0^2\right| = \left|\alpha_{\Omega_j} \int_\Omega \mu_0^2 + \sum_{i\neq j}(\alpha_{\Omega_i} - \alpha_{\Omega_j}) \int_{\Omega_i} \mu_0^2\right| \leq C_3 \|\mathbf{f}\|_{H^{-1}(\Omega)},$$

$$|\alpha_{\Omega_j}| \int_\Omega \mu_0^2 \leq C_3 \|\mathbf{f}\|_{H^{-1}(\Omega)} + \sum_{i\neq j} |\alpha_{\Omega_i} - \alpha_{\Omega_j}| \int_{\Omega_i} \mu_0^2,$$

and finally obtaining

$$|\alpha_{\Omega_j}| \leq (C_3 + kC_2) \|\mathbf{f}\|_{H^{-1}(\Omega)}, \quad \forall j.$$

Combining (8), (9), and the previous inequality, we arrive at

$$\|\alpha\|_{L^2(\Omega)} \leq C \|\mathbf{f}\|_{H^{-1}(\Omega)}.$$

Turning to $\mu = \alpha\mu_0$, we have

$$\|\mu\|_{L^2(\Omega)} \leq \|\alpha\|_{L^2(\Omega)} \|\mu_0\|_{L^\infty(\Omega)}$$
$$\leq C \|\mu_0\|_{L^\infty(\Omega)} \|\mathbf{f}\|_{H^{-1}(\Omega)}$$
$$\leq C \|\mu_0\|_{L^\infty(\Omega)} \|A_{\mathbf{u}}^{\mathbf{I}}\mu\|_{H^{-1}(\Omega)}.$$

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Hence, the proof is complete.

$\square$

3.4 Stability estimates in $L^2(\Omega)$

Theorem 6 (Stability estimate for the null space estimation)**.** *Consider a displacement field $\mathbf{u}$ such that $\nabla^s \mathbf{u} \in L^\infty(\Omega, \mathbb{R}^{d \times d}_{sym})$ and $A^{\mathbf{I}}_{\mathbf{u}}$ has closed range. Take $\mu \in L^2(\Omega)$ such that $\|\mu\|_{L^2(\Omega)} = 1$ and $A^{\mathbf{I}}_{\mathbf{u}}(\mu) = 0$. Take $\tilde{\mathbf{u}} \in H^1(\Omega, \mathbb{R}^d)$ such that $\nabla^s \tilde{\mathbf{u}} \in L^\infty(\Omega, \mathbb{R}^{d \times d}_{sym})$ and consider*

$$\tilde{\mu} := \operatorname*{argmin}_{\|\mu'\|_{L^2(\Omega)}=1, \ \int_\Omega \mu'\mu>0} \left\| A^{\mathbf{I}}_{\tilde{\mathbf{u}}}(\mu') \right\|_{H^{-1}(\Omega)}.$$

Then

$$\|\tilde{\mu} - \mu\|_{L^2(\Omega)} \le C \|\nabla^s \tilde{\mathbf{u}} - \nabla^s \mathbf{u}\|_{L^\infty(\Omega)}$$

for some constant C independent of $\tilde{\mathbf{u}}$ and $\tilde{\mu}$.

Proof. Write $\tilde{\mu} = \alpha\mu + \nu$, with $\nu \perp \mu$, $\alpha \in [0, 1]$. Pythagoras theorem gives $\alpha^2 + \|\nu\|^2_{L^2(\Omega)} = 1$ and $\|\tilde{\mu} - \mu\|^2_{L^2(\Omega)} = (\alpha - 1)^2 + \|\nu\|^2_{L^2(\Omega)} = (\alpha - 1)^2 + (1 - \alpha^2) = 2(1 - \alpha)$. Since $1 - \alpha \le 1 - \alpha^2 = \|\nu\|^2_{L^2(\Omega)}$,

$$\|\tilde{\mu} - \mu\|^2_{L^2(\Omega)} \le 2 \|\nu\|^2_{L^2(\Omega)}.$$

Since $A^{\mathbf{I}}_{\mathbf{u}}$ has the closed range property, Theorem 4 yields

$$
\begin{aligned}
\|\nu\|_{L^2(\Omega)} &\le c \left\| A^{\mathbf{I}}_{\mathbf{u}}(\nu) \right\|_{H^{-1}(\Omega)} \\
&\le c \left\| A^{\mathbf{I}}_{\mathbf{u}}(\tilde{\mu} - \alpha\mu) \right\|_{H^{-1}(\Omega)} \\
&\le c \left\| A^{\mathbf{I}}_{\mathbf{u}}(\tilde{\mu}) \right\|_{H^{-1}(\Omega)} \\
&\le c \left(\left\| A^{\mathbf{I}}_{\tilde{\mathbf{u}}}(\tilde{\mu}) \right\|_{H^{-1}(\Omega)} + \left\| \left[A^{\mathbf{I}}_{\mathbf{u}} - A^{\mathbf{I}}_{\tilde{\mathbf{u}}} \right](\tilde{\mu}) \right\|_{H^{-1}(\Omega)} \right) \\
&\le c \left(\left\| A^{\mathbf{I}}_{\tilde{\mathbf{u}}}(\mu) \right\|_{H^{-1}(\Omega)} + \left\| \left[A^{\mathbf{I}}_{\mathbf{u}} - A^{\mathbf{I}}_{\tilde{\mathbf{u}}} \right](\tilde{\mu}) \right\|_{H^{-1}(\Omega)} \right) \\
&\le c \left(\left\| \left[A^{\mathbf{I}}_{\tilde{\mathbf{u}}} - A^{\mathbf{I}}_{\mathbf{u}} \right](\mu) \right\|_{H^{-1}(\Omega)} + \left\| \left[A^{\mathbf{I}}_{\mathbf{u}} - A^{\mathbf{I}}_{\tilde{\mathbf{u}}} \right](\tilde{\mu}) \right\|_{H^{-1}(\Omega)} \right).
\end{aligned}
$$

Since

$$
\begin{aligned}
\left\| \left[A^{\mathbf{I}}_{\tilde{\mathbf{u}}} - A^{\mathbf{I}}_{\mathbf{u}} \right](\mu) \right\|_{H^{-1}(\Omega)} &= \left\| \nabla \cdot \left[(\nabla^s \tilde{\mathbf{u}} - \nabla^s \mathbf{u}) \mu \right] \right\|_{H^{-1}(\Omega)} \\
&\le \left\| (\nabla^s \tilde{\mathbf{u}} - \nabla^s \mathbf{u}) \mu \right\|_{L^2(\Omega)} \\
&\le \left\| \nabla^s \tilde{\mathbf{u}} - \nabla^s \mathbf{u} \right\|_{L^\infty(\Omega)} \|\mu\|_{L^2(\Omega)} \\
&\le \left\| \nabla^s \tilde{\mathbf{u}} - \nabla^s \mathbf{u} \right\|_{L^\infty(\Omega)}
\end{aligned}
$$

and

$$\left\| \left[A^{\mathbf{I}}_{\mathbf{u}} - A^{\mathbf{I}}_{\tilde{\mathbf{u}}} \right](\tilde{\mu}) \right\|_{H^{-1}(\Omega)} \le \left\| \nabla^s \tilde{\mathbf{u}} - \nabla^s \mathbf{u} \right\|_{L^\infty(\Omega)},$$

the following holds:

$$\|\tilde{\mu} - \mu\|_{L^2(\Omega)} \le 2\sqrt{2}c \, \|\nabla^s\tilde{\mathbf{u}} - \nabla^s\mathbf{u}\|_{L^\infty(\Omega)} .$$

$\square$

Theorem 7 (General stability estimate). *Consider two displacement fields* $\mathbf{u}, \tilde{\mathbf{u}}$ *such that* $\nabla^s\mathbf{u}$ *and such that* $\nabla^s\tilde{\mathbf{u}} \in L^\infty(\Omega, \mathbb{R}^{d\times d}_{sym})$ *and* $A^{\mathbf{I}}_{\mathbf{u}}$ *has closed range. Take a real number* $r > 0$ *and* $\mu, \tilde{\mu} \in L^2(\Omega)$ *respectively solutions of*

$$A^{\mathbf{I}}_{\mathbf{u}}(\mu) = \mathbf{f}, \quad \int_\Omega \mu = 1, \quad \|\mu\|_{L^2(\Omega)} \le r,$$

$$A^{\mathbf{I}}_{\tilde{\mathbf{u}}}(\tilde{\mu}) = \tilde{\mathbf{f}}, \quad \int_\Omega \tilde{\mu} = 1, \quad \|\tilde{\mu}\|_{L^2(\Omega)} \le r.$$

There exists a constant $C > 0$ *independent on* $\tilde{\mathbf{u}}, \tilde{\mu}, \tilde{\mathbf{f}}$ *such that,*

$$\|\tilde{\mu} - \mu\|_{L^2(\Omega)} \le C \left(\left\|\tilde{\mathbf{f}} - \mathbf{f}\right\|_{H^{-1}(\Omega)^d} + r \, \|\nabla^s\tilde{\mathbf{u}} - \nabla^s\mathbf{u}\|_{L^\infty(\Omega)} \right).$$

Proof. By difference, we write that $A^{\mathbf{I}}_{\mathbf{u}}(\tilde{\mu} - \mu) = \tilde{\mathbf{f}} - \mathbf{f} + (A^{\mathbf{I}}_{\mathbf{u}} - A^{\mathbf{I}}_{\tilde{\mathbf{u}}})\tilde{\mu}$. If $N(A^{\mathbf{I}}_{\mathbf{u}}) = \{0\}$ then $\tilde{\mu} - \mu \in N(A^{\mathbf{I}}_{\mathbf{u}})^\perp$ and applying the closed range property in the same manner than for the previous Theorem we get

$$\|\tilde{\mu} - \mu\|_{L^2(\Omega)} \le C \left(\left\|\tilde{\mathbf{f}} - \mathbf{f}\right\|_{H^{-1}(\Omega)} + r \, \|\nabla^s\tilde{\mathbf{u}} - \nabla^s\mathbf{u}\|_{L^\infty(\Omega)} \right).$$

If $N(A^{\mathbf{I}}_{\mathbf{u}}) = \text{Span}\{\mu_0\}$, where $\|\mu_0\|_{L^2(\Omega)} = 1$ and $m_0 := \int_\Omega \mu_0 \ne 0$, we decompose μ as $\mu = \alpha\mu_0 + \nu$ and $\tilde{\mu}$ as $\tilde{\mu} = \tilde{\alpha}\mu_0 + \tilde{\nu}$ where $\nu, \tilde{\nu} \in N(A^{\mathbf{I}}_{\mathbf{u}})^\perp$. Then we have

$$\|\nu - \tilde{\nu}\|_{L^2(\Omega)} \le C \left(\left\|\tilde{\mathbf{f}} - \mathbf{f}\right\|_{H^{-1}(\Omega)} + r \, \|\nabla^s\tilde{\mathbf{u}} - \nabla^s\mathbf{u}\|_{L^\infty(\Omega)} \right). \tag{12}$$

By Pythagoras' theorem, $\|\tilde{\mu} - \mu\|^2_{L^2(\Omega)} = (\tilde{\alpha} - \alpha)^2 + \|\tilde{\nu} - \nu\|^2_{L^2(\Omega)}$. Now using that

$$\int_\Omega (\tilde{\mu} - \mu) = (\tilde{\alpha} - \alpha) \int_\Omega \mu_0 + \int_\Omega (\tilde{\nu} - \nu) = 0,$$

we get that

$$(\tilde{\alpha} - \alpha)^2 m_0^2 \le |\Omega| \, \|\tilde{\nu} - \nu\|^2_{L^2(\Omega)},$$

and hence,

$$\|\tilde{\mu} - \mu\|^2_{L^2(\Omega)} \le \left(1 + \frac{|\Omega|}{m_0^2} \right) \|\tilde{\nu} - \nu\|^2_{L^2(\Omega)}. \tag{13}$$

We conclude by combining inequalities (12) and (13). $\square$
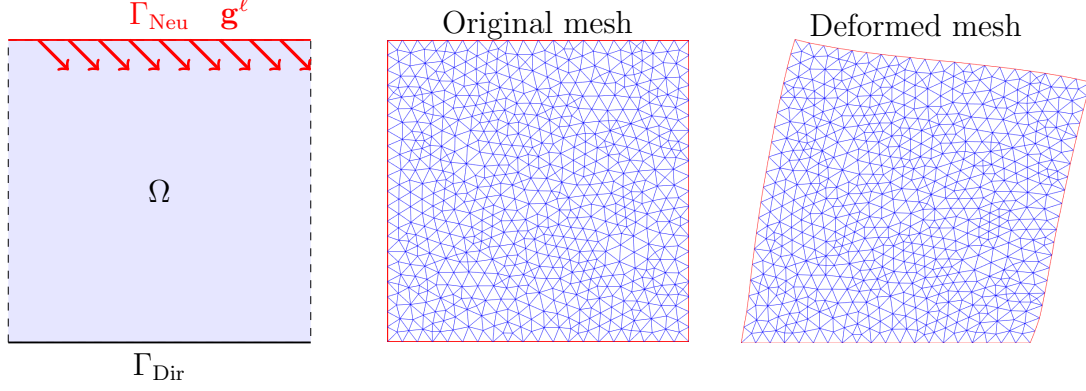
Figure 4.1: Numerical experimental setting: a non-structured meshing of the domain Ω and the computed elastic deformation.

4 Numerical experiments in the static case

The objective here is to numerically reconstruct an elasticity tensor $\mathbf{C}(x)$ in a smooth domain $\Omega \subset \widetilde{\Omega} \in \mathbb{R}^2$ from the knowledge of a set of data $\{(\mathbf{u}^\ell, \mathbf{f}^\ell)\}_{\ell=1}^n$ satisfying the linear elasticity equation

$$-\nabla \cdot \left(\mathbf{C} : \nabla^s \mathbf{u}^\ell\right) = \mathbf{0}.$$

4.1 Forward problem and data generation

In order to generate different displacement fields $\mathbf{u}^\ell \in H^1(\Omega, \mathbb{R}^2)$ of static elastic deformation, we use the classic finite elements approach to solve the boundary-value problem

$$\begin{cases} -\nabla \cdot \left(\mathbf{C} : \nabla^s \mathbf{u}^\ell\right) = \mathbf{0} & \text{in } \widetilde{\Omega}, \\ \mathbf{u}^\ell = \mathbf{0} & \text{on } \Gamma_{\text{Dir}}, \\ \left(\mathbf{C} : \nabla^s \mathbf{u}^\ell\right) \cdot \mathbf{n} = \mathbf{g}^\ell & \text{on } \Gamma_{\text{Neu}}, \\ \left(\mathbf{C} : \nabla^s \mathbf{u}^\ell\right) \cdot \mathbf{n} = \mathbf{0} & \text{elsewhere on } \partial\widetilde{\Omega}, \end{cases}$$

where $\mathbf{g}^\ell$ could be any surface force density. In the simulations, we use $\widetilde{\Omega} = (-1, 1)^2$ and Γ_{Dir} and Γ_{Neu} are as described in Figure 4.1.

The computations of direct data are made using the $\mathcal{P}^0 - \mathcal{P}^1$ finite elements method. The solution $\mathbf{u}^\ell$ is then interpolated and recorded on a structured Cartesian grid over $\widetilde{\Omega}$. The inverse problem is solved on a subdomain $\Omega \subset \widetilde{\Omega}$ endowed with a new non-structured mesh. This prevents from committing an inverse crime in inverting the problem using the same discrete operator as the one used for the direct problem.

4.2 Finite elements discretization

We assume here without loss of generality that the chosen domain $\Omega \in \mathbb{R}^2$ is polygonal that admits an exact triangular mesh $\mathcal{T}_h = \{T_i\}_{i=1}^{N_{\mathcal{T}}}$ for any small maximum edge length

$h > 0$. More precisely, $\mathcal{T}_h$ is a set of N_T open triangles such that $T_i \cap T_j = \emptyset$ if $i \neq j$ and $\overline{\Omega} = \bigcup_{i=1}^{N_T} \overline{T_i}$. Let us introduce the classic function spaces associated to $\mathcal{T}_h$:

(i) The space $\mathcal{P}^0(\mathcal{T}_h)$ is the set of functions that are constant on each triangle:

$$\mathcal{P}^0(\mathcal{T}_h) = \left\{ u \in L^2(\Omega, \mathbb{R}), \ \forall i, \ u|_{T_i} \text{ is constant} \right\}.$$

Its canonical basis $(\xi_j)_{j=1}^{N_T}$ is given by $\xi_j := \mathbf{1}_{T_i}$.

(ii) The space $\mathcal{P}^1(\mathcal{T}_h)$ is the set of continuous functions that are linear on each triangle T_i:

$$\mathcal{P}^1(\mathcal{T}_h) = \left\{ v \in H^1(\Omega), \ \forall i, \ v|_{T_i} \text{ is linear} \right\}.$$

Its canonical basis $(e_i)_{i=1}^{N_n}$ is defined by $e_i(x_j) = \delta_{ij}$ for any node x_j of the triangulation $\mathcal{T}_h$.

(iii) The space $\mathcal{P}^1(\mathcal{T}_h, \mathbb{R}^2)$ is the set of vector-valued $\mathcal{P}^1(\mathcal{T}_h)$ functions. Its canonical basis is denoted $(\mathbf{e}_i)_{i=1}^{2N_n}$.

(iv) The space $\mathcal{P}_0^1(\mathcal{T}_h)$, (resp. $\mathcal{P}_0^1(\mathcal{T}_h, \mathbb{R}^2)$) is the space of $\mathcal{P}^1(\mathcal{T}_h)$ (resp. $\mathcal{P}^1(\mathcal{T}_h, \mathbb{R}^2)$) functions that vanish on $\partial\Omega$:

$$\mathcal{P}_0^1(\mathcal{T}_h) := \mathcal{P}^1(\mathcal{T}_h) \cap H_0^1(\Omega) \quad \text{and} \quad \mathcal{P}_0^1(\mathcal{T}_h, \mathbb{R}^2) := \mathcal{P}^1(\mathcal{T}_h) \cap H_0^1(\Omega, \mathbb{R}^2).$$

(v) Its canonical basis is denoted by $(\tilde{e}_i)_{i=1}^{N_{\text{int}}} \subset (e_i)_{i=1}^{N_n}$ (resp. $(\tilde{\mathbf{e}}_i)_{i=1}^{2N_{\text{int}}} \subset (\mathbf{e}_i)_{i=1}^{2N_n}$) where N_{int} is the number of internal nodes of the mesh.

Scalar functions $\mu^{(k)}$ and displacement fields $\mathbf{u}^\ell$ are then projected respectively on the bases of $\mathcal{P}^0(\mathcal{T}_h)$ and $\mathcal{P}^1(\mathcal{T}_h)$:

$$\mu^{(k)}(x) = \sum_{j=1}^{N_T} \mu_j^{(k)} \xi_j(x) \quad \text{and} \quad \mathbf{u}^\ell(x) = \sum_{i=1}^{2N_n} u_i^\ell \mathbf{e}_i(x).$$

4.3   Discrete formulation of the inverse problem

We assume the knowledge of a model for the elasticity tensors $\mathbf{C}$ of the form

$$\mathbf{C}(x) = \sum_{k=1}^{N} \mu^{(k)}(x) \mathbf{C}^k,$$

where all the unknown scalar fields $\mu^{(k)}$ belong to $L^2(\Omega)$ and the constant tensors $\mathbf{C}^k$ are known. Recall that the reconstruction problem of each fields $\mu^{(k)}$ reads as the linear problem

$$
\begin{pmatrix} A_{\mathbf{u}^1}^{\mathbf{C}^1} & \cdots & A_{\mathbf{u}^1}^{\mathbf{C}^N} \\ \vdots & & \vdots \\ A_{\mathbf{u}^n}^{\mathbf{C}^1} & \cdots & A_{\mathbf{u}^n}^{\mathbf{C}^N} \end{pmatrix} \begin{pmatrix} \mu^1 \\ \vdots \\ \mu^N \end{pmatrix} = \begin{pmatrix} \mathbf{f}^1 \\ \vdots \\ \mathbf{f}^n \end{pmatrix}, \tag{14}
$$

where the operator $A_{\mathbf{u}}^{\mathbf{C}}$ is defined by $A_{\mathbf{u}}^{\mathbf{C}}\mu = -\nabla \cdot (\mu \mathbf{C} : \nabla^s \mathbf{u})$, or in a weak sense by

$$\langle A_{\mathbf{u}}^{\mathbf{C}}\mu, \mathbf{v} \rangle_{H^{-1},H_0^1} = \int_\Omega \mu(x)(\mathbf{C} : \nabla^s \mathbf{u}(x)) : \nabla^s \mathbf{v}(x)\mathrm{d}x, \quad \forall \mathbf{v} \in H_0^1(\Omega, \mathbb{R}^d). \qquad (15)$$

These operators admit a straightforward finite elements discretization defining the matrices $\mathbb{A}_{\mathbf{u}}^{\mathbf{C}} \in \mathbb{R}^{N_{\mathrm{int}} \times N_T}$ and $\mathbb{F}^\ell \in \mathbb{R}^{N_{\mathrm{int}}}$ as

$$\begin{aligned}
(\mathbb{A}_{\mathbf{u}}^{\mathbf{C}})_{ij} &:= \langle A_{\mathbf{u}}^{\mathbf{C}}\xi_j, \tilde{\mathbf{e}}_i \rangle_{H^{-1},H_0^1} = \int_\Omega \xi_j(x)(\mathbf{C} : \nabla^s \mathbf{u}(x)) : \nabla^s \tilde{\mathbf{e}}_i(x)\mathrm{d}x, \\
\mathbb{F}_i^\ell &:= \langle \mathbf{f}^\ell, \tilde{\mathbf{e}}_i \rangle_{H^{-1},H_0^1}.
\end{aligned} \qquad (16)$$

Introducing now the block matrices

$$\mathbb{A} := \begin{pmatrix} \mathbb{A}_{\mathbf{u}^1}^{\mathbf{C}^1} & \cdots & \mathbb{A}_{\mathbf{u}^1}^{\mathbf{C}^N} \\ \vdots & & \vdots \\ \mathbb{A}_{\mathbf{u}^n}^{\mathbf{C}^1} & \cdots & \mathbb{A}_{\mathbf{u}^n}^{\mathbf{C}^N} \end{pmatrix} \in \mathbb{R}^{nN_{\mathrm{int}} \times NN_T}, \quad \mathbb{F} := \begin{pmatrix} \mathbb{F}^1 \\ \vdots \\ \mathbb{F}^n \end{pmatrix} \in \mathbb{R}^{nN_{\mathrm{int}}}, \quad \mathbb{M} = \begin{pmatrix} \mu^1 \\ \vdots \\ \mu^N \end{pmatrix} \in \mathbb{R}^{NN_T},$$

the general inverse problem admits a simple discrete projection on the finite elements spaces and reads as

$$\mathbb{A}\mathbb{M} = \mathbb{F}. \qquad (17)$$

Note that, in the static case, $\mathbb{F} = 0$ leads to an eigenvector problem. In this case, the formulation becomes

$$\mathbb{A}\mathbb{M} = 0, \quad \|\mathbb{M}\|_2 = 1. \qquad (18)$$

As will be seen later, in practice it might be more convenient to impose a positivity constraint over $\mathbb{M} := (\mu^1, \ldots, \mu^n)^T$ of the form

$$\mu^{(k)} \geq \mu_{\mathrm{min}}^{(k)} > 0, \quad \forall k, \qquad (19)$$

since the unknown elastic parameters are indeed positive valued functions.

4.4 Least squares approach and regularization

We recall that, we have proved for the shear modulus case that the reconstruction problem of μ from $\{(\mathbf{u}^\ell, \mathbf{f}^\ell)\}_{\ell=1:n}$ is theoretically well-posed in the continuous setting. This is not clearly the case in the discretized version as the linear system $\mathbb{A}\mathbb{M} = \mathbb{F}$ has NN_T degrees of freedom with only nN_n equations. In practice, we compute the fields μ by minimizing a regularized mean squares functional of the form

$$J(\mu) = \|\mathbb{A}\mathbb{M} - \mathbb{F}\|_2^2 + R_{\mathrm{TV}}(\mathbb{M}),$$

where the regularization $R_{\mathrm{TV}}(\mathbb{M})$ penalizes the total variation of each $\mu^{(k)}$. More precisely, in the case where $\mu \in \mathcal{P}^0(\mathcal{T}_h)$ that admits the decomposition $\mu(x) = \sum_{j=1}^{N_T} \mu_j \xi_j(x)$, we can show that the TV semi-norm can be directly expressed as a linear L^1-penalization. As it is clear that $\mathcal{P}^0(\mathcal{T}_h) \subset BV(\Omega)$, we have

$$|\mu|_{TV(\Omega)} = \int_\Omega |D\mu|.$$

Call $E = \{(i,j) \in \{1,\ldots,N_n\}^2, \ i \neq j, \ \partial T_i \cap \partial T_j \neq \emptyset\}$ the set of all the oriented internal edges. As μ is constant on each triangle T_i, the Radon measure derivative $D\mu$ is given by

$$D\mu = \frac{1}{2} \sum_{(i,j)\in E} (\mu_i - \mu_j) \mathcal{H}^1_{\partial T_i \cap \partial T_j} \nu_{ij},$$

where ν_{ij} is the normal vector from triangles T_i to T_j and $\mathcal{H}^1_{\partial T_i \cap \partial T_j}$ is the restriction of the dimension one Hausdorff measure to the edge (i,j). Hence,

$$|\mu|_{TV(\Omega)} = \int_\Omega |D\mu| = \frac{1}{2} \sum_{(i,j)\in E} |\mu_i - \mu_j| \mathcal{H}^1(\partial T_i \cap \partial T_j).$$

We observe that if one defines the linear operator $L : \mathcal{P}^0(\mathcal{T}_h) \to \mathbb{R}^{\text{card}(E)}$ by

$$(L\mu)_{(i,j)\in E} = (\mu_i - \mu_j) \mathcal{H}^1(\partial T_i \cap \partial T_j),$$

then

$$|\mu|_{TV(\Omega)} = \frac{1}{2} \|L\mu\|_{\ell^1(E)}.$$

Finally, it can be shown that the TV-regularization term $R_{\text{TV}}(\mathbb{M})$ can be expressed under the form

$$R_{\text{TV}}(\mathbb{M}) = \sum_{k=1}^n \varepsilon_{\text{TV}}^{(k)} \left\| L\mu^{(k)} \right\|_{\ell^1(E)}, \tag{20}$$

where $\varepsilon_{\text{TV}}^{(k)}$ are regularizing parameters. The reconstruction of μ can then be computed by minimizing the functional

$$J(\mathbb{M}) = \|\mathbb{A}\mathbb{M} - \mathbb{F}\|_2^2 + \sum_{k=1}^n \varepsilon_{\text{TV}}^{(k)} \left\| L\mu^{(k)} \right\|_{\ell^1(E)}, \tag{21}$$

subject to $\mathbb{M} \geq \mathbb{M}_{\min}$ where $\mathbb{M}_{\min} := (\mu^{(1)}U, \ldots, \mu^{(n)}U)^T$ with $U := (1,\ldots,1) \in \mathbb{R}^{N_T}$.

Remark 4.1. *In practice the minimisation of* (21) *can be achieved with any efficient optimisation routine. Here, we used the CVX Matlab toolbox* [19, 18] *which is well-adapted to this kind of convex optimization problems under linear constraints.*

Remark 4.2. *As the given displacement fields* $\mathbf{u}^\ell$ *are in general noisy, which significantly affect the eigenvalues of the associated operator* $\mathbb{A}$, *it could be convenient to introduce a beforehand smoothing of these vector fields. More precisely, a natural way is to consider an elastic regularization* $\mathbf{u}^\ell_{\varepsilon_{elas}}$ *defined by*

$$\mathbf{u}^\ell_{\varepsilon_{elas}} = argmin_{\mathbf{u}} \left\{ \frac{1}{\varepsilon_{elas}} \|\mathbf{u} - \mathbf{u}^\ell\|_{L^2(\Omega)}^2 + \|\nabla^s \mathbf{u}\|_{L^2(\Omega)}^2 \right\}.$$

This is also equivalent to compute $\mathbf{u}^\ell_{\varepsilon_{elas}}$ *in the finite elements context as the solution of the following linear system*

$$\mathbf{u}^\ell_{\varepsilon_{elas}} = (\varepsilon_{elas} M + L)^{-1} (M\mathbf{u}^\ell),$$

where M *and* L *are respectively the mass and the vector stiffness matrix and* $\varepsilon_{elas} > 0$ *is a regularization parameter.*

4.5 Numerical experiments

The motivation is now to present some numerical experiments in the static case where the tensor $\mathbf{C}$ is assumed to be of the form:

(i) A shear modulus reconstruction only: $\mathbf{C} = \mu\mathbf{I}$;

(ii) A two Lamé parameters reconstruction: $\mathbf{C} = 2\mu\mathbf{I} + \lambda I \otimes I$;

(iii) An anisotropic stiffness reconstruction: $\mathbf{C} = \mu^{(1)}\mathbf{C}^1 + \mu^{(2)}\mathbf{C}^2 + \mu^{(3)}\mathbf{C}^3$, where tensors $\mathbf{C}^1$, $\mathbf{C}^2$ and $\mathbf{C}^3$ are defined by (5) and (6).

We only present in this paper some numerical experiments in the static case but other experiments have be done in the harmonic regime with similar results. In each case, we then use the following additional constraints on $\mu^{(k)}$:

$$\mu^{(k)} \geq 1.$$

We first consider the simplest case of shear medium $\mathbf{C} = \mu\mathbf{I}$ in order to illustrate and analyze the influence of each of the regularization parameters ε_{TV} and $\varepsilon_{\mathrm{elas}}$ on the reconstruction. In particular, we will see that the reconstruction of μ is very accurate as soon as the choice of ε_{TV} and $\varepsilon_{\mathrm{elas}}$ is appropriate.

We show that our methodology still works in the case of more complex tensor $\mathbf{C}$. In particular, we highlight that the reconstruction of (λ, μ) in the isotropic elastic case and $(\mu^{(1)}, \mu^{(2)}, \mu^{(3)})$ in the anisotropic shear case are also accurate provided that the number of the sets of data $\{\mathbf{u}^\ell\}$ is sufficiently large.

4.5.1 Shear modulus inversion

In this subsection, we first focus on the case $\mathbf{C} = \mu(x)\mathbf{I}$, where we consider three different choices for the shear modulus (μ_1, μ_2, μ_3), which are illustrated in Figure 4.2. In each case, we compute the direct elastic vector fields $\mathbf{u}^{(1)}$ associated to the same boundary conditions. Each solution is plotted in Figure 4.3 and we can observe the similarity of the different elastic fields $\mathbf{u}^{(1)}$. Notice that the mesh used to compute the elastic vector fields has been build such as the characteristic size of each triangle is about $h = 0.01$.

About the reconstruction of the shear modulus μ, we recall that we need to fix only the two regularization parameters ε_{TV} and $\varepsilon_{\mathrm{elas}}$. In each case, we also use the same mesh where the triangles have now a characteristic size of the order of $h = 0.03$.

The first experiments illustrated in Figure 4.4 have been done with $\varepsilon_{\mathrm{elas}} = 10^{-5}$. Each column corresponds to different values of ε_{TV} which are respectively equal to $\varepsilon_{TV} = 10^{-6}$, $\varepsilon_{TV} = 10^{-5}$, and $\varepsilon_{TV} = 10^{-4}$. Each line corresponds to the data associated with μ, $\tilde{\mu}$, and μ_3.

We can observe that the reconstruction is perturbed if ε_{TV} is too small and becomes very quantitative with an appropriate choice of ε_{TV}. These first experiments show the advantage of the TV regularization which preserves the discontinuities. Finally, it shows the real possibility of reconstructing a non-smooth shear modulus μ with only one set of data $\{\mathbf{u}^{(1)}\}$.

The second experiments (presented in Figure 4.5) show the influence of an elastic regularization on the data. Indeed, the estimation can be noisy and need in practice to
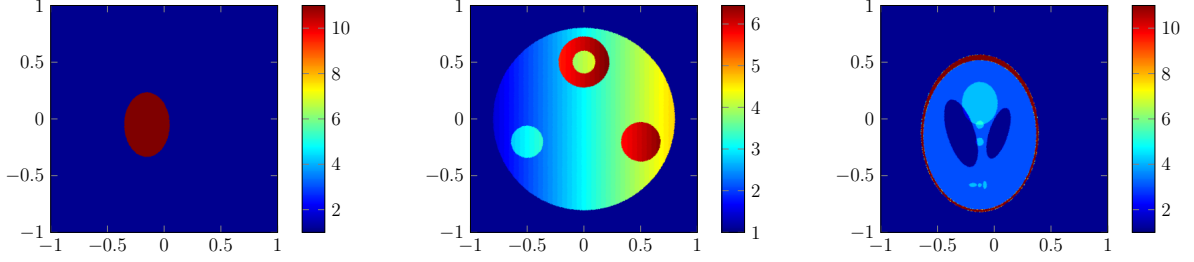
Figure 4.2: Examples of shear modulus phantoms: μ, $\tilde{\mu}$ and μ_3
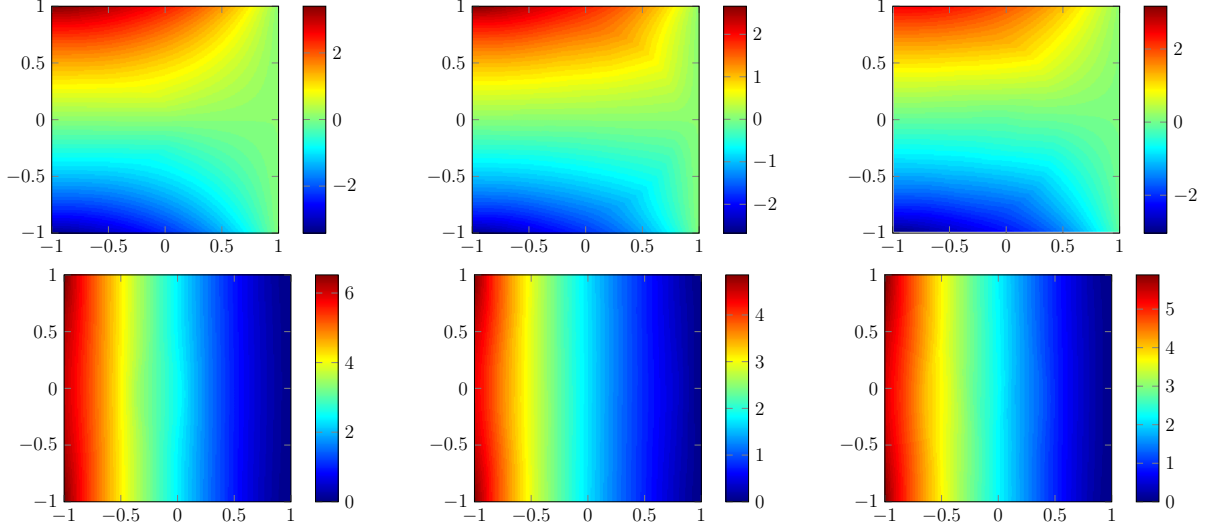

Figure 4.3: Lines: first and second components of vector fields u; Each column (from left to right) corresponds to the use of μ, $\tilde{\mu}$ and μ_3, respectively.

be regularized. We then try here to understand the influence of an elastic regularization on the quality of the reconstruction. We then fix the value of $\varepsilon_{TV} = 10^{-4}$ and compare the reconstruction of μ obtained with $\varepsilon_{\mathrm{elas}} = 10^{-5}$, $\varepsilon_{\mathrm{elas}} = 10^{-4}$ and $\varepsilon_{\mathrm{elas}} = 10^{-3}$. It then clearly appears that the effect of the elastic regularization is to smooth the reconstruction of the shear modulus μ.

Finally, as expected by our theoretical results, these experiments clearly demonstrate the ability of our methodology to reconstruct non-smooth shear modulus μ using only one set of data $\{\mathbf{u}^\ell\}$.

4.5.2 Two Lamé coefficients inversion

We now consider the case of isotropic elasticity tensor

$$\mathbf{C} = 2\mu\mathbf{I} + \lambda I \otimes I.$$

The numerical reconstructions of Lamé coefficients are presented in Figures 4.6 and 4.7, where two different choices of Lamé coefficients are used. In all experiments, we take the regularization parameters: $\varepsilon_{TV} = 10^{-4}$ and $\varepsilon_{\mathrm{elas}} = 10^{-4}$. Moreover, each column
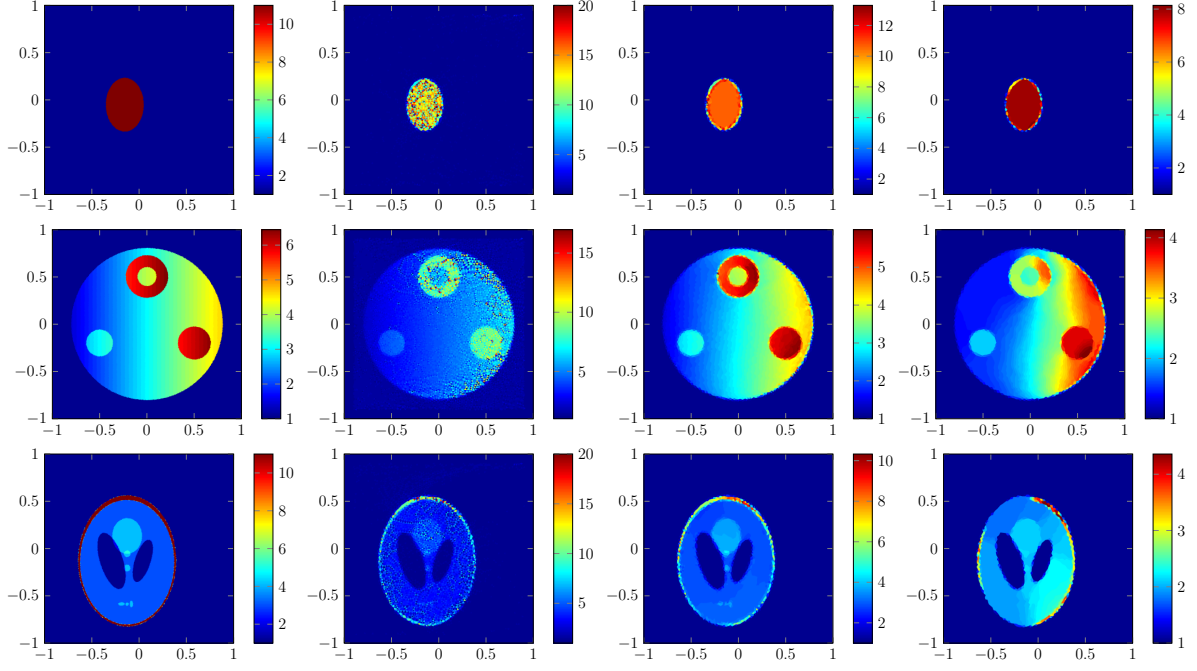
Figure 4.4: Reconstruction of μ: influence of the parameter ε_{TV} ; Lines: shear modulus μ, $\tilde{\mu}$ and μ_3. Columns: $\varepsilon_{TV} = 10^{-6}$, $\varepsilon_{TV} = 10^{-4}$ and $\varepsilon_{TV} = 10^{-3}$. In each case, we use $\varepsilon_{\text{elas}} = 10^{-5}$.
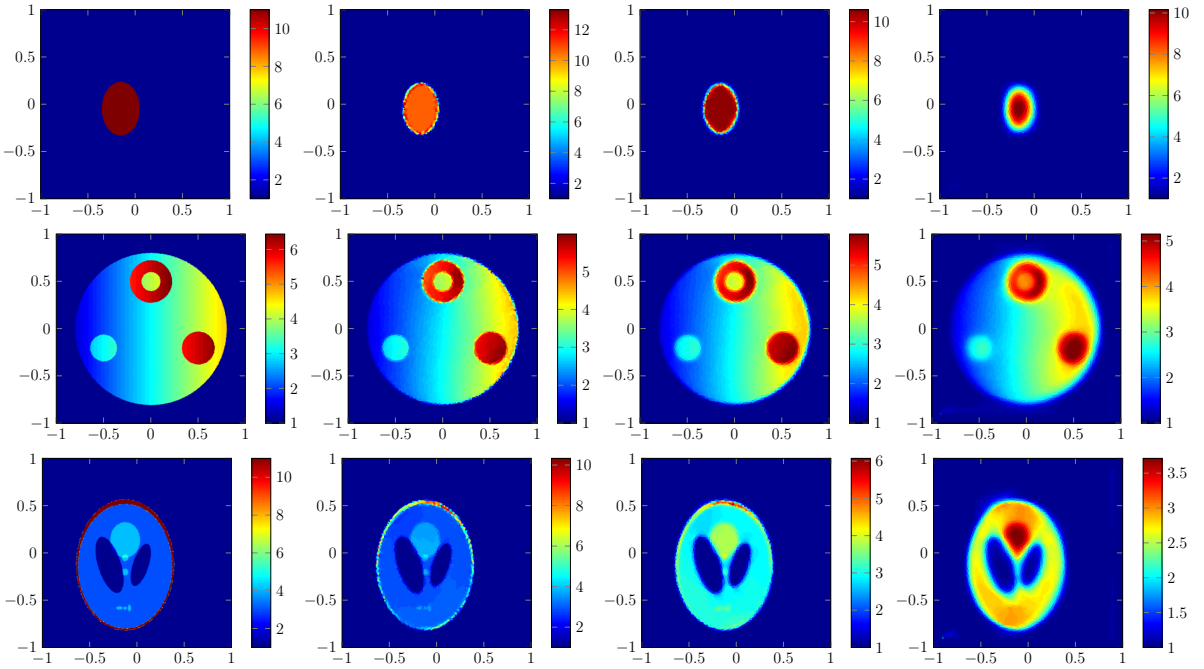

Figure 4.5: Reconstruction of μ: influence of the parameter $\varepsilon_{\text{elas}}$; Lines: shear modulus μ, $\tilde{\mu}$ and μ_3. Columns: $\varepsilon_{\text{elas}} = 10^{-5}$, $\varepsilon_{\text{elas}} = 10^{-4}$ and $\varepsilon_{\text{elas}} = 10^{-3}$. In each case, we use $\varepsilon_{TV} = 10^{-4}$.
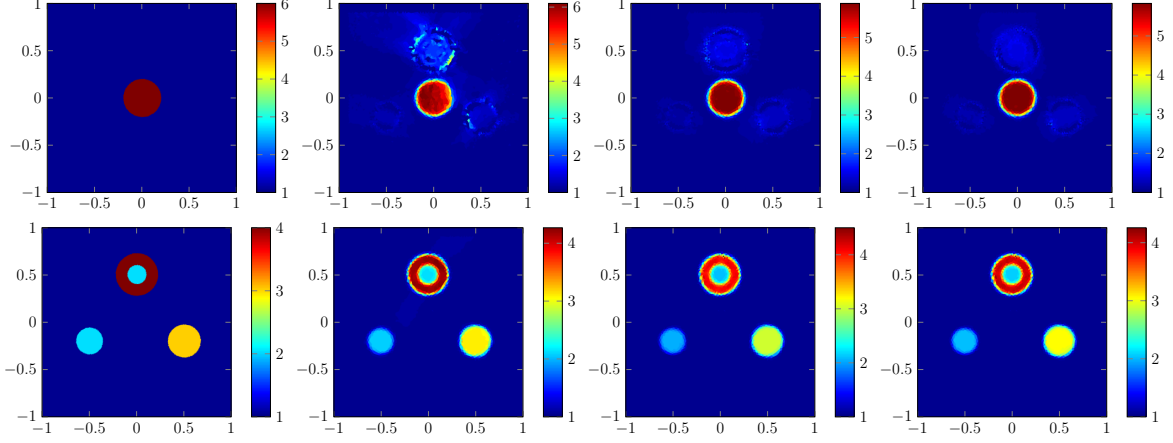
Figure 4.6: Reconstruction of the Lamé coefficients (λ_1, μ_1): influence of the number of data $\{\mathbf{u}^\ell\}$; Lines: λ and μ; From left to right: Exact Lamé coefficients and their reconstructions obtained respectively with 1, 2 and 4 sets of data $\{\mathbf{u}^\ell\}$. Here, we used $\varepsilon_{TV} = 10^{-4}$ and $\varepsilon_{\mathrm{elas}} = 10^{-4}$.

corresponds to the numerical reconstruction of (λ, μ) obtained respectively with $n = 1$, $n = 2$ and $n = 4$ sets of data $\{\mathbf{u}^\ell\}$. We also plot the exact Lamé coefficient on the first column.

Notice that in the case of one set of data, we succeeded in reconstructing a first rough approximation of (λ, μ). Finally, using $n = 2$ and $n = 4$ sets of data leads to a precise reconstruction of (λ, μ) even for complex Lamé coefficients.

4.5.3 Anisotropic tensor inversion

The last example concerns the case of an anisotropic shear tensor

$$\mathbf{C} = \mu^{(1)}\mathbf{C}^1 + \mu^{(2)}\mathbf{C}^2 + \mu^{(3)}\mathbf{C}^3.$$

The motivation is to show that our methodology can be adapted to any kind of model for the elasticity tensor C. Like previously, we use $\varepsilon_{TV} = 10^{-4}$, $\varepsilon_{\mathrm{elas}} = 10^{-4}$, and the reconstructions obtained with different number of data sets are plotted on each column of Figure 4.8.

Notice that as in the case of an isotropic elastic medium, we successfully reconstructed a quantitative approximation of the scalar fields $\mu^{(k)}$ even in the case of one set of data.

5 Concluding remarks

In this paper, we have introduced a new direct and stable method for reconstructing discontinuous elastic parameters from internal measurements of the displacement fields. We have proved an L^2-stability result with only one measurement. We have described a direct discretization of the inverse problem in both the isotropic and anisotropic cases. We have presented a variety of numerical results to illustrate the performance of our approach. In a forthcoming paper, we will apply our approach to real and clinical data
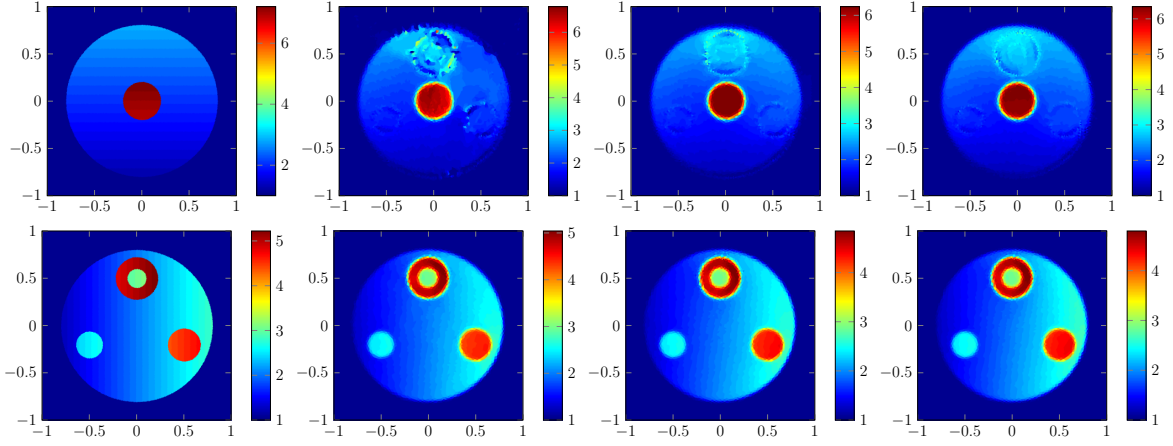
Figure 4.7: Reconstruction of the Lamé coefficients (λ_2, μ_2): influence of the number of data $\{\mathbf{u}^\ell\}$; Lines: λ and μ; From left to right: Exact Lamé coefficients and their reconstructions obtained respectively with 1, 2 and 4 sets of data $\{\mathbf{u}^\ell\}$. Here, we used $\varepsilon_{TV} = 10^{-4}$ and $\varepsilon_{\mathrm{elas}} = 10^{-4}$.
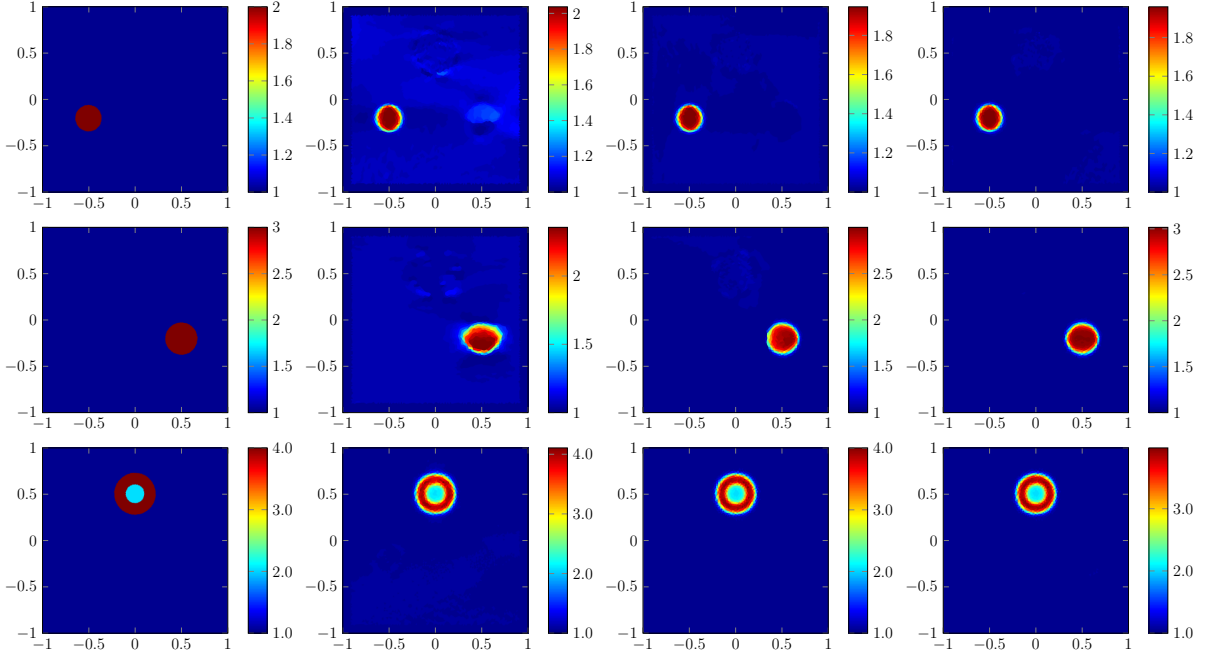

Figure 4.8: Reconstruction of the anisotropic coefficients $(\mu^{(1)}, \mu^{(2)}, \mu^{(3)})$: influence of the number of data $\{\mathbf{u}^\ell\}$; Lines: $\mu^{(1)}$, $\mu^{(1)}$ and $\mu^{(3)}$; From left to right: Exact coefficients $\mu^{(k)}$ and their reconstructions obtained respectively with 1, 2 and 4 sets of data $\{\mathbf{u}^\ell\}$. Here, we used $\varepsilon_{TV} = 10^{-4}$ and $\varepsilon_{\mathrm{elas}} = 10^{-4}$.

using only measurements of one component of the displacement field and to shear wave spectroscopy.

A   Notations and tools

A.1   Tensor notations

Definition A.1. *We denote by $\mathbb{R}^{d \times d}$ the space of real matrices and $\mathbb{R}^{d \times d}_{sym}$ the space of real symmetric matrices. Notice that $\mathbb{R}^{d \times d}_{sym} \sim \mathbb{R}^{d(d+1)/2}$. We denote by $T^4 = \mathbb{R}^{d^4}$ the space of order 4 real tensors. We recall that*

(i) $A : B = \sum_{ij} A_{ij} B_{ij} \in \mathbb{R}$ for $A, B \in \mathbb{R}^{d \times d}$;

(ii) $(A \otimes B)_{ijkl} = A_{ij} B_{kl} \in T^4$ for $A, B \in \mathbb{R}^{d \times d}$;

(iii) $(\mathbf{A} : B)_{ij} = \sum_{kl} \mathbf{A}_{ijkl} B_{kl} \in \mathbb{R}^{d \times d}$ for $\mathbf{A} \in T^4$ and $B \in \mathbb{R}^{d \times d}$;

(iv) $(B : \mathbf{A})_{ij} = \sum_{kl} B_{kl} \mathbf{A}_{klij} \in \mathbb{R}^{d \times d}$ for $\mathbf{A} \in T^4$ and $B \in \mathbb{R}^{d \times d}$;

(v) $(\mathbf{A} : \mathbf{B})_{ijkl} = \sum_{mn} \mathbf{A}_{ijmn} \mathbf{B}_{mnkl} \in T^4$ for $\mathbf{A}, \mathbf{B} \in T^4$;

(vi) $\mathbf{A} | \mathbf{B} = \sum_{ijkl} \mathbf{A}_{ijkl} \mathbf{B}_{ijkl} \in \mathbb{R}$ for $\mathbf{A}, \mathbf{B} \in T^4$.

We define T^4_{sym} to be the space of all tensors $\mathbf{T}$ such that for any symmetric matrix $S \in \mathbb{R}^{d \times d}_{sym}$, the matrix $\mathbf{T} : S$ is also symmetric and for any antisymmetric matrix A, we have $\mathbf{T} : A = 0$. Remark that in dimension two, $T^4_{sym} \sim \mathbb{R}^6$ and in dimension 3, $T^4_{sym} \sim \mathbb{R}^{21}$.

A.2   Sobolev spaces

Definition A.2. *For any Lipschitz domain $\Omega \subset \mathbb{R}^d$, we define*

$$W^{1,p}(\Omega) := \left\{ u \in L^p(\Omega), \ |\nabla u| \in L^2(\Omega) \right\}.$$

We also define the following space:

$$H^1_0(\Omega, \mathbb{R}^d) := \left\{ \mathbf{u} \in L^2(\Omega, \mathbb{R}^d), \ |\nabla \mathbf{u}| \in L^2(\Omega), \mathbf{u}|_{\partial \Omega} = \mathbf{0} \right\},$$

equipped with the norm:

$$\|\mathbf{u}\|_{H^1_0(\Omega)} := \|\nabla^s \mathbf{u}\|_{L^2(\Omega)},$$

where $\nabla^s \mathbf{u} = (\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2$.

Remark A.1. *The fact that this definition for the norm is correct is a direct consequence of Korn's inequality and Poincaré's inequality.*

Proposition A.1. *Properties of $W^{1,p}(\Omega)$: If $p > d$, the following results hold.*

(i) $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$;

(ii) If $u, v \in W^{1,p}(\Omega)$, then $uv \in W^{1,p}(\Omega)$;

(iii) If $u \in W^{1,p}(\Omega)$ and $\varphi \in H_0^1(\Omega)$, then $u\varphi \in H_0^1(\Omega)$;

(iv) $u \in W^{1,p}(\Omega)$, $f \in H^{-1}(\Omega)$ implies that $uf \in H^{-1}(\Omega)$ and

$$\|uf\|_{H^{-1}} \leq C\|u\|_{W^{1,p}}\|f\|_{H^{-1}}$$

for some constant C independent of u and f.

Lemma A.2 (∇ has a closed range in $\{\mu_0\}^\perp$). *Let Ω be a Lipschitz domain of $\mathbb{R}^d$ and $\mu_0 \in L^\infty(\Omega)$ be such that $\mu_0 \geq m \geq 0$. Then, there exists a constant $c > 0$ such that*

$$\forall \mu \in \{\mu_0\}^\perp, \quad \|\mu\|_{L^2(\Omega)} \leq c\,\|\nabla\mu\|_{H^{-1}(\Omega)}.$$

Proof. Suppose that this is false. Take a sequence (μ_n) such that $\|\mu_n\|_{L^2(\Omega)} = 1$ and $\|\nabla\mu_n\|_{H^{-1}(\Omega)} \to 0$. Up to an extraction $\mu_n \overset{L^2(\Omega)}{\rightharpoonup} \mu$ and $\int_\Omega \mu_n\mu_0 \to \int_\Omega \mu\mu_0 = 0$. Moreover, $\nabla\mu = 0$ and so μ is constant. Then $\mu = 0$. As the embedding $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ is compact, we get that $\|\mu_n\|_{H^{-1}(\Omega)} \to 0$. Saying now that

$$\|\mu_n\|_{L^2(\Omega)}^2 = \|\mu_n\|_{H^{-1}(\Omega)}^2 + \|\nabla\mu_n\|_{H^{-1}(\Omega)}^2,$$

we arrive at a contradiction. $\qquad\square$

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