A PROJECTION METHOD FOR NAVIER–STOKES EQUATIONS WITH A BOUNDARY CONDITION INCLUDING THE TOTAL PRESSURE

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ABSTRACT. We consider a projection method for time-dependent incompressible Navier–Stokes equations with a total pressure boundary condition. The projection method is one of the numerical calculation methods for incompressible viscous fluids often used in engineering. In general, the projection method needs additional boundary conditions to solve a pressure-Poisson equation, which does not appear in the original Navier–Stokes problem. On the other hand, many mechanisms generate flow by creating a pressure difference, such as water distribution systems and blood circulation. We propose a new additional boundary condition for the projection method with a Dirichlet-type pressure boundary condition and no tangent flow. We demonstrate stability for the scheme and establish error estimates for the velocity and pressure under suitable norms. A numerical experiment verifies the theoretical convergence results. Furthermore, the existence of a weak solution to the original Navier–Stokes problem is proved by using the stability.

1. Introduction

Let $T > 0$ and let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ $(d = 2, 3)$ with the boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ (see also Section 2.1 for the precise assumption). We consider the following Navier–Stokes problem: Find two functions $u : \Omega \times [0, T] \to \mathbb{R}^d$ and $p : \Omega \times [0, T] \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u + \dfrac{1}{\rho}\nabla p = f & \text{in } \Omega \times (0, T), \\ \operatorname{div} u = 0 & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \Gamma_1 \times (0, T), \\ u \times n = 0 & \text{on } \Gamma_2 \times (0, T), \\ p + \dfrac{\rho}{2}|u|^2 = p^b & \text{on } \Gamma_2 \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\nu, \rho > 0$, $f : \Omega \times (0, T) \to \mathbb{R}^d$, $p^b : \Gamma_2 \times (0, T) \to \mathbb{R}$, $u_0 : \Omega \to \mathbb{R}^d$, n is the unit outward normal vector for Γ and "$\times$" is the cross product in $\mathbb{R}^d$ [1]. The functions u and p are the velocity and the pressure of the flow governed by (1.1), respectively. For Γ_2, we assume a boundary condition including a pressure value $p + \frac{\rho}{2}|u|^2$, which is called the total pressure or stagnation pressure. Usual pressure is often called static pressure to distinguish it from the total pressure. In an experimental measurement of the total and static pressure using a Pitot tube, the boss measurement is dependent on the yaw angle of the Pitot tube. Then, the effect on the total pressure $p + \frac{\rho}{2}|u|^2$ is smaller than the effect on the usual pressure p [22, Section 7.15]. The boundary condition on Γ_2 in (1.1) is introduced in [2], and the existence of a weak velocity solution is proved in [5, 25]. We will show the existence in a different way (Corollary 2.24). The

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[1]If $d = 2$, then we define

$$v \times w := v_x w_y - v_y w_x \in \mathbb{R} \qquad \text{for all } v = (v_x, v_y), w = (w_x, w_y) \in \mathbb{R}^2.$$

stationary case has been studied in [3, 4, 7, 12, 25]. In [7, 8], the finite element discretization problems with this type of boundary condition are proposed.

Next, we introduce a projection method for (1.1). The projection method is one of the numerical schemes for Navier–Stokes equations [11, 30]. Error analysis in the case of the full Dirichlet boundary condition for the velocity is carried out in [1, 26, 27, 28, 29]. In the case of a boundary condition for the static pressure, the finite element analysis of a projection method is proposed in [17, 18]. For the nonlinear term in the first equation of (1.1), it holds that

$$(u \cdot \nabla)u = (\nabla \times u) \times u + \frac{1}{2}\nabla |u|^2$$

(cf. [14]). Hence, if we put $D(v, w) := (\nabla \times v) \times w$ and $P = p + \frac{\rho}{2}|u|^2$, then (1.1) is equivalent to the following[2]:

$$\begin{cases} \dfrac{\partial u}{\partial t} + D(u,u) - \nu \Delta u + \dfrac{1}{\rho}\nabla P = f & \text{in } \Omega \times (0,T), \\ \operatorname{div} u = 0 & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \Gamma_1 \times (0,T), \\ u \times n = 0 & \text{on } \Gamma_2 \times (0,T), \\ P = p^b & \text{on } \Gamma_2 \times (0,T), \\ u(0) = u_0 & \text{in } \Omega \end{cases} \tag{1.2}$$

In [10], a projection method for the rotation form (the first equation of (1.2)) using the total pressure is introduced to avoid checkerboard oscillation of pressure in the finite difference method.

Let $\tau (:= T/N < 1, N \in \mathbb{N})$ be a time increment and let $t_k := k\tau$ $(k = 0, 1, \dots, N)$. We set $u_0^* := u_0$ and calculate u_k^*, u_k, p_k $(k = 1, 2, \dots, N)$ by repeatedly solving the following problems (Step 1) and (Step 2).

(Step 1) Find $u_k^* : \Omega \to \mathbb{R}^d$ such that

$$\begin{cases} \dfrac{u_k^* - u_{k-1}}{\tau} + D(u_{k-1}^*, u_k^*) - \nu \Delta u_k^* = f(t_k) & \text{in } \Omega, \\ u_k^* = 0 & \text{on } \Gamma_1, \\ u_k^* \times n = 0 & \text{on } \Gamma_2, \\ \operatorname{div} u_k^* = 0 & \text{on } \Gamma_2. \end{cases} \tag{1.3}$$

(Step 2) Find $P_k :\to \mathbb{R}$ and $u_k :\to \mathbb{R}^d$ such that

$$\begin{cases} -\dfrac{\tau}{\rho}\Delta P_k = -\operatorname{div} u_k^* & \text{in } \Omega, \\ \dfrac{\partial P_k}{\partial n} = 0 & \text{on } \Gamma_1, \\ P_k = p^b(t_k) & \text{on } \Gamma_2, \end{cases} \tag{1.4}$$

$$u_k = u_k^* - \frac{\tau}{\rho}\nabla P_k \quad \text{in } \Omega. \tag{1.5}$$

For the velocity boundary condition on Γ_2, we can rewrite the third and fourth equations of (1.3) by using $\kappa := \operatorname{div} n = (d-1)\times$(mean curvature) as follows:

[2]If $d = 2$, then $\nabla \times v$ and $(\nabla \times v) \times w$ denote the scalar and vector functions, respectively, defined as follows: for all $v = (v_x, v_y), w = (w_x, w_y) \in \mathbb{R}^2$,

$$\nabla \times v := \partial_x v_y - \partial_y v_x, \quad (\nabla \times v) \times w := (w_y(\partial_y v_x - \partial_x v_y), w_x(\partial_x v_y - \partial_y v_x)).$$

Remark 1.1. *If $v \in C^1(\overline{\Omega})$ satisfies that $v \times n = 0$ on Γ_2, then we have*

$$\frac{\partial v}{\partial n} \cdot n + \kappa v \cdot n = \operatorname{div} v \qquad \text{on } \Gamma_2.$$

For the proof, see [23, Lemma 7]. *Hence, the third and fourth equations of (1.3) are equivalent to the following equations:*

$$u_k^* \times n = 0, \quad \frac{\partial u_k^*}{\partial n} \cdot n + \kappa u_k^* \cdot n = 0 \qquad \text{on } \Gamma_2.$$

In particular, if Γ_2 is flat, then it holds that

$$u_k^* \times n = 0, \quad \frac{\partial u_k^*}{\partial n} \cdot n = 0 \qquad \text{on } \Gamma_2.$$

Remark 1.2. *By replacing u_{k-1} in the first equation of (1.3) with (1.5) at the previous step, it holds that for all $k = 1, 2, \ldots, N$,*

$$\frac{u_k^* - u_{k-1}^*}{\tau} + D(u_{k-1}^*, u_k^*) - \nu \Delta u_k^* + \frac{1}{\rho} \nabla P_{k-1} = f(t_k) \qquad \text{in } \Omega.$$

It follows from (1.4) and (1.5) that $\operatorname{div} u_k = 0$ in Ω, $u_k \cdot n = 0$ on Γ_1. Hence, by (1.3), (1.4), and (1.5), it holds that for all $k = 1, 2, \ldots, N$,

$$\begin{cases} \dfrac{u_k^* - u_{k-1}^*}{\tau} + D(u_{k-1}^*, u_k^*) - \nu \Delta u_k^* + \dfrac{1}{\rho} \nabla P_{k-1} = f(t_k) & \text{in } \Omega, \\ \operatorname{div} u_k = 0 & \text{in } \Omega, \\ u_k^* = 0 & \text{on } \Gamma_1, \\ u_k^* \times n = 0 & \text{on } \Gamma_2, \\ P_k = p^b(t_k) & \text{on } \Gamma_2, \end{cases}$$

where $P_0 := 0$. Compare with (1.2).

In this paper, we demonstrate solvability (Proposition 2.9) and stability (Theorem 2.15) of the projection method and establish error estimates in suitable norms (Theorems 2.17 and 2.22). Furthermore, we prove the existence of a weak solution of (1.1) with a different approach than [5, 25] by using the stability result (Corollary 2.24).

The organization of this paper is as follows. In Section 2, we introduce the notations used in this work, the weak formulations of the Navier–Stokes equations (1.2), and the projection method (1.3), (1.4), and (1.5). We also prove the existence of the weak solution to the scheme and provide the main results. Section 3 is devoted to proving that the solution to the scheme is bounded in suitable norms and converges to the solution to (1.2) in a strong topology as $\tau \to 0$. We also establish error estimates in suitable norms between the solutions to the Navier–Stokes equations and the projection method. In Section 4, we show a numerical example of the projection method and the numerical errors between the Navier–Stokes equations and the projection method using the P2/P1 finite element method. We conclude this paper with several comments on future works in Section 5. In the Appendix, four lemmas in Section 2 and 3 and the existence result of the weak solution to (1.2) are proved.

2. Preliminaries

In this section, we introduce the notations used in this work and the weak formulations of the Navier–Stokes equations (1.2) and the projection method (1.3), (1.4), and (1.5).

2.1. **Notation.** We prepare the function spaces and the notation to be used throughout the paper. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ $(d = 2, 3)$. For the boundary $\Gamma = \partial\Omega$, we assume that there exist two relatively open subsets Γ_1, Γ_2 of Γ such that Γ_2 has a finite number of connected components that are piecewise $C^{1,1}$-class and

$$|\Gamma \setminus (\Gamma_1 \cup \Gamma_2)| = 0, \quad |\Gamma_1|, |\Gamma_2| > 0, \quad \Gamma_1 \cap \Gamma_2 = \emptyset, \quad \mathring{\overline{\Gamma_1}} = \Gamma_1, \quad \mathring{\overline{\Gamma_2}} = \Gamma_2,$$

where $\overline{A}$ is the closure of $A \subset \Gamma$ with respect to Γ, $\mathring{A}$ is the interior of A with respect to Γ, and $|A|$ is the $(d-1)$-dimensional Hausdorff measure of A. For an integer $m \geq 1$ and a real number $p \in [1, \infty]$, we use the usual Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{m,p}(\Omega)$ together with their standard norms and write $H^0(\Omega) := L^2(\Omega), H^m(\Omega) := W^{m,2}(\Omega)$. We use the same notation $(\cdot, \cdot)$ to represent the $L^2(\Omega)$ inner product for scalar-, vector-, and matrix-valued functions. For a normed space X, the dual pairing between X and the dual space X^* is denoted by $\langle \cdot, \cdot \rangle_X$, and we simply write $L^2(X)$ and $L^\infty(X)$ as $L^2(0, T; X)$ and $L^\infty(0, T; X)$, respectively. $\mathcal{D}'(\Omega)$ denotes the space of distributions on Ω.

We use the following notations:

$$\begin{aligned}
\|\cdot\|_m&\text{: the norms in } H^m(\Omega),\\
C^\infty(\overline{\Omega}) &:= \{\varphi|_\Omega \mid \varphi : \mathbb{R}^d \to \mathbb{R} \text{ is infinitely differentiable}\}\\
C_0^\infty(\Omega) &:= \{\varphi \in C^\infty(\overline{\Omega}) \mid \operatorname{supp}(\varphi) \subset \Omega\}\\
H &:= \{\varphi \in H^1(\Omega)^d \mid \varphi = 0 \text{ on } \Gamma_1, \varphi \times n = 0 \text{ on } \Gamma_2\},\\
V &:= \{\varphi \in H \mid \operatorname{div}\varphi = 0 \text{ in } L^2(\Omega)\},\\
H^1_{\Gamma_2}(\Omega) &:= \{\psi \in H^1(\Omega) \mid \psi = 0 \text{ on } \Gamma_2\}.
\end{aligned}$$

The dual spaces H^* and V^* are equipped with the dual norms

$$\|F\|_{H^*} := \sup_{0 \neq \varphi \in H} \frac{\langle F, \varphi \rangle_H}{\|\varphi\|_1}, \qquad \|G\|_{V^*} := \sup_{0 \neq \varphi \in V} \frac{\langle G, \varphi \rangle_V}{\|\varphi\|_1},$$

respectively, for all $F \in H^*$ and $G \in V^*$.

We also use the Lebesgue space $L^2(\Gamma)$ and Sobolev space $H^{1/2}(\Gamma)$ defined on Γ. The norm $\|\eta\|_{H^{1/2}(\Gamma)}$ is defined by

$$\|\eta\|_{H^{1/2}(\Gamma)} := \left(\|\eta\|^2_{L^2(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|\eta(x) - \eta(y)|^2}{|x - y|^d} ds(x) ds(y) \right)^{1/2},$$

where ds denotes the surface measure of Γ. For function spaces defined on Γ_2, we use $L^2(\Gamma_2)$ and $H^{1/2}(\Gamma_2)$.

We define a bilinear form $a : H^1(\Omega)^d \times H^1(\Omega)^d \to \mathbb{R}$, a seminorm on $H^1(\Omega)^d$, for $u, v \in H^1(\Omega)^d$,

$$a(u, v) := \int_\Omega (\operatorname{div} u)(\operatorname{div} v) dx + \int_\Omega (\nabla \times u) \cdot (\nabla \times v) dx, \qquad \|u\|_a := \sqrt{a(u, u)}.$$

Let p_d be

$$p_d := \begin{cases} 2 + \varepsilon & \text{if } d = 2, \\ 3 & \text{if } d = 3, \end{cases}$$

where $\varepsilon > 0$ is arbitrarily small. It follows from the Sobolev embeddings that $H^1(\Omega) \subset L^{p_d}(\Omega)$ and the embedding is continuous [9, Theorem III.2.34]. We define a trilinear operator $d : L^{p_d}(\Omega)^d \times H \times H \to \mathbb{R}$ for $u \in L^{p_d}(\Omega)^d$ and $v, w \in H$,

$$d(u, v, w) := \int_\Omega u \cdot (\nabla \times (v \times w)) dx.$$

We note that for all $u \in H^1(\Omega)^d$ and $v, w \in H$,

$$d(u, v, w) = -\int_\Gamma (u \times \nu) \cdot (v \times w) ds + \int_\Omega ((\nabla \times u) \times v) \cdot w dx = \int_\Omega D^1(u, v) \cdot w dx.$$

For two sequences $(x_k)_{k=0}^N$ and $(y_k)_{k=1}^N$ in a Banach space E, we define a piecewise linear interpolant $\hat{x}_\tau \in W^{1,\infty}(0,T;E)$ of $(x_k)_{k=0}^N$ and a piecewise constant interpolant $\bar{y}_\tau \in L^\infty(0,T;E)$ of $(y_k)_{k=1}^N$, respectively, by

$$\begin{aligned} \hat{x}_\tau(t) &:= x_{k-1} + \frac{t-t_{k-1}}{\tau}(x_k - x_{k-1}) && \text{for } t \in [t_{k-1}, t_k] \text{ and } k = 1,2,\dots,N, \\ \bar{y}_\tau(t) &:= y_k && \text{for } t \in (t_{k-1}, t_k] \text{ and } k = 1,2,\dots,N. \end{aligned}$$

We define a backward difference operator by

$$D_\tau x_k := \frac{x_k - x_{k-1}}{\tau}, \qquad D_\tau y_l := \frac{y_l - y_{l-1}}{\tau}$$

for $k = 1,2,\dots,N$ and $l = 2,3,\dots,N$. Then, the sequence $(D_\tau x)_k := D_\tau x_k$ satisfies $\frac{\partial \hat{x}_\tau}{\partial t} = (\overline{D_\tau x})_\tau$ on (t_{k-1}, t_k) for all $k = 1,2,\dots,N$. For a function $F \in C([0,T];E)$, we define $F_\tau \in L^\infty(0,T;E)$ as the piecewise constant interpolant of $(F(t_k))_{k=1}^N$, i.e.,

$$F_\tau(t) := F(t_k) \qquad \text{for } t \in (t_{k-1}, t_k] \text{ and } k = 1,2,\dots,N.$$

2.2. **Preliminary results.** Let $\gamma_0 \in B(H^1(\Omega), H^{1/2}(\Gamma))$ be the standard trace operator. It is known that (see e.g. [31, Theorem 1.2]) there exists a linear continuous operator $\gamma_n : M := \{\varphi \in L^2(\Omega)^d \mid \operatorname{div}\varphi \in L^2(\Omega)\} \to H^{-1/2}(\Gamma)$ such that $\gamma_n u = u \cdot n|_\Gamma$ for all $u \in C^\infty(\overline{\Omega})^d$, where $H^{-1/2}(\Gamma) := H^{1/2}(\Gamma)^*$. Then, the following generalized Gauss divergence formula holds:

$$\int_\Omega v \cdot \nabla q dx + \int_\Omega (\operatorname{div} v) q dx = \langle \gamma_n v, \gamma_0 q \rangle_{H^{1/2}(\Gamma)} \qquad \text{for all } v \in M, q \in H^1(\Omega).$$

The composition of the trace operator γ_0 and the restriction $H^{1/2}(\Gamma) \to H^{1/2}(\Gamma_2)$ is denoted by $\psi \mapsto \psi|_{\Gamma_2}$. This map is continuous from $H^1(\Omega)$ to $H^{1/2}(\Gamma_2)$. The kernel of this map is $H^1_{\Gamma_2}(\Omega)$. We simply write ψ instead of $\psi|_{\Gamma_2}$ when there is no ambiguity.

We recall the following lemmas that are necessary for the existence and the uniqueness of a solution to the Stokes problem.

Lemma 2.1. [13, Corollary 4.1] *Let $(X, \|\cdot\|_X)$ and $(Q, \|\cdot\|_Q)$ be two real Hilbert spaces. Let $a_X : X \times X \to \mathbb{R}$ and $b : X \times Q \to \mathbb{R}$ be bilinear and continuous maps and let $F \in X^*$. If there exist two constants $\alpha > 0$ and $\beta > 0$ such that for all $v \in V$ and $q \in Q$,*

$$a_X(v,v) \geq \alpha \|v\|_X^2, \qquad \sup_{0 \neq v \in X} \frac{b(v,q)}{\|v\|_X} \geq \beta \|q\|_Q$$

where $V = \{v \in X \mid b(v,q) = 0 \text{ for all } q \in Q\}$, then there exists a unique solution $(u,p) \in X \times Q$ to the following problem:

$$\begin{cases} a_X(u,v) + b(v,p) = F(v) & \text{for all } v \in X, \\ b(u,q) = 0 & \text{for all } q \in Q. \end{cases}$$

Furthermore, there exists a constant $c > 0$ independent of F such that

$$\|u\|_X + \|p\|_Q \leq c \|F\|_{X^*}.$$

Lemma 2.2. [6, proof of Theorem 2.1] *There exists a constant $c = c(\Omega, \Gamma_1, \Gamma_2) > 0$ such that for all $q \in L^2(\Omega)$,*

$$\|q\|_0 \leq c \sup_{0 \neq \varphi \in H} \frac{|(q, \operatorname{div}\varphi)|}{\|\varphi\|_1}.$$

The following embedding theorem show the continuity and the coercivity of the bilinear form $a : H \times H \to \mathbb{R}$.

Lemma 2.3. *There exists a constant $c_a = c_a(\Omega, \Gamma_1, \Gamma_2) > 0$ such that for all $\varphi_1, \varphi_2, \varphi \in H$,*

$$a(\varphi_1, \varphi_2) \leq \|\varphi_1\|_a \|\varphi_2\|_a \leq c_a \|\varphi_1\|_1 \|\varphi_2\|_1, \qquad \frac{1}{c_a}\|\varphi\|_1^2 \leq \|\varphi\|_a^2.$$

The first inequality holds from the Cauchy–Schwarz inequality. For the proof of the second inequality, see [7, Lemma 2.11] and [23, Lemma 5]. The following embedding theorem is called the Poincaré inequality.

Lemma 2.4 (Poincaré's inequality)**.** [13, Lemma 3.1] *There exists a constant $c = c(\Omega, \Gamma_1, \Gamma_2) > 0$ such that for all $\varphi \in H^1_{\Gamma_2}(\Omega)$,*

$$\|\varphi\|_0 \leq c\|\nabla\varphi\|_0.$$

We prepare the following lemma to use the Aubin–Nitsche trick.

Lemma 2.5. *We define an operator $T : L^2(\Omega)^d \ni e \mapsto (w, r) \in H \times L^2(\Omega)$ as follows:*

$$\begin{cases} a(w, \varphi) - (r, \operatorname{div}\varphi) = (e, \varphi) & \text{for all } \varphi \in H, \\ \operatorname{div} w = 0 & \text{in } L^2(\Omega). \end{cases} \tag{2.6}$$

Then, T is a linear and continuous operator and there exists a constant $c = c(\Omega, \Gamma_1, \Gamma_2) > 0$ such that for all $e \in L^2(\Omega)^d$ and $(w, r) = T(e)$,

$$\|w\|_1 + \|r\|_0 \leq c\|e\|_{H^*}, \qquad \frac{1}{c}\|e\|_{V^*} \leq \|w\|_1 \leq c\|e\|_{V^*}.$$

By Lemmas 2.1, 2.2, and 2.3, the operator T is well-posed and continuous. See the Appendix for the proof of the inequalities. Next, we show the following two lemmas for the operator d.

Lemma 2.6. *It holds that for all $u \in L^{p_d}(\Omega)^d, v, v_1, v_2 \in H^1(\Omega)^d$*

$$d(u, v, v) = 0, \qquad d(u, v_1, v_2) = -d(u, v_2, v_1).$$

By the definition of the operator d, it is easy to check Lemma 2.6.

Lemma 2.7. *There exists a constant $c_d = c_d(\Omega, \Gamma_1, \Gamma_2) > 0$ such that*

$$d(u, v, w) \leq \begin{cases} c_d\|u\|_{L^{p_d}}\|v\|_1\|w\|_1 \text{ for all } u \in L^{p_d}(\Omega)^d, v, w \in H, \\ c_d\|u\|_0\|v\|_1\|w\|_2 \text{ for all } u \in L^{p_d}(\Omega)^d, v \in H, w \in H \cap H^2(\Omega)^d, \\ c_d\|u\|_1\|v\|_1\|w\|_1 \text{ for all } u \in H^1(\Omega)^d, v, w \in H, \\ c_d\|u\|_1\|v\|_2\|w\|_0 \text{ for all } u \in H^1(\Omega)^d, v \in H \cap H^2(\Omega)^d, w \in H \\ c_d\|u\|_2\|v\|_1\|w\|_0 \text{ for all } u \in H^2(\Omega)^d, v, w \in H. \end{cases}$$

See the Appendix for the proof. Finally, we recall the discrete Gronwall inequality.

Lemma 2.8. [21, Lemma 5.1] *Let $\tau, \beta > 0$ and let nonnegative sequences $(a_k)_{k=0}^N$, $(b_k)_{k=0}^N$, $(c_k)_{k=0}^N$, $(\alpha_k)_{k=0}^N \subset \{x \in \mathbb{R} \mid x \geq 0\}$ satisfy that*

$$a_n + \tau\sum_{k=0}^m b_k \leq \tau\sum_{k=0}^m \alpha_k a_k + \tau\sum_{k=0}^m c_k + \beta \qquad \text{for all } m = 0, 1, \ldots, N.$$

If $\tau\alpha_k < 1$ for all $k = 0, 1, \ldots, N$, then we have

$$a_n + \tau\sum_{k=0}^m b_k \leq e^C\left(\tau\sum_{k=0}^m c_k + \beta\right) \qquad \text{for all } m = 0, 1, \ldots, N,$$

where $C := \tau\sum_{k=0}^N \frac{\alpha_k}{1-\tau\alpha_k}$.

2.3. **Weak formulations of (1.2), (1.3), (1.4), and (1.5).** We assume $\nu = \rho = 1$ and the following conditions for f, p^b, and u_0:

$$f \in L^2(H^*), \quad p^b \in L^2(H^1(\Omega)), \quad u_0 \in L^{p_d}(\Omega)^d. \tag{2.7}$$

To define weak formulations of the Navier–Stokes equations (1.2) and the projection method (1.3), (1.4), and (1.5), we prepare the following equation:

Proposition 2.9. *It holds that for all $u \in H^2(\Omega)$ and $\varphi \in H$,*

$$-(\Delta u, \varphi) = a(u, \varphi) - \int_{\Gamma_2} (\operatorname{div} u)\varphi \cdot n ds. \tag{2.8}$$

Proof. It holds that $-\Delta u = \nabla \times (\nabla \times u) - \nabla(\operatorname{div} u)$ for all $u \in C^2(\overline{\Omega})^d$. Hence, we have for all $u \in C^2(\overline{\Omega})^d$ and $\varphi \in C^1(\overline{\Omega})^d$,

$$(-\Delta u, \varphi) = a(u, \varphi) + \int_{\Gamma} (\nabla \times u) \cdot (\varphi \times n) ds - \int_{\Gamma} (\operatorname{div} u)\varphi \cdot n ds,$$

which also holds for all $\varphi \in H^2(\Omega)$ and $\psi \in H^1(\Omega)$ since the two spaces $C^2(\overline{\Omega})$ and $C^1(\overline{\Omega})$ are dense in $H^2(\Omega)$ and $H^1(\Omega)$, respectively. By the definition of H, equation (2.8) holds for all $u \in H^2(\Omega)$ and $\varphi \in H$. □

By Proposition 2.9 and the Gauss divergence formula, it holds that for all $u \in H^2(\Omega)^d, P \in H^1(\Omega)$, and $\varphi \in V$ with $\operatorname{div} u = 0$ in $H^1(\Omega)$,

$$(D(u, u) - \Delta u + \nabla P, \varphi) = a(u, \varphi) + d(u, u, \varphi) - (P, \operatorname{div} \varphi) + \int_{\Gamma_2} P\varphi \cdot n ds.$$

Hence, a weak formulation of (1.2) is as follows: Find $u \in L^2(H^1(\Omega)^d)$ and $P \in L^1(L^2(\Omega))$ such that $\frac{\partial u}{\partial t} \in L^1(H^*)$ and for all $\varphi \in H$,

$$\begin{cases} \left\langle \dfrac{\partial u}{\partial t}, \varphi \right\rangle_H + a(u, \varphi) + d(u, u, \varphi) - (P, \operatorname{div} \varphi) = \langle f, \varphi \rangle_H - \displaystyle\int_{\Gamma_2} p^b \varphi \cdot n ds, \\ \operatorname{div} u = 0 \qquad \text{in } L^2(\Omega), \end{cases} \tag{NS}$$

in $L^1(0, T)$, with $u(0) = u_0$ on H^*. In main convergence theorems (Theorems 2.17 and 2.22), we assume that (NS) has a unique solution and that the solution is as smooth as needed.

On the other hand, by Proposition 2.9, we have for all $u \in H^2(\Omega)^d$ and $\varphi \in H$,

$$(D(u, u) - \Delta u, \varphi) = a(u, \varphi) + d(u, u, \varphi) - \int_{\Gamma_2} (\operatorname{div} u)\varphi \cdot n ds.$$

Hence, a weak formulation of (1.3), (1.4), and (1.5) with the initial datum $u_0 (=: u_0^*)$ is as follows:

Problem 2.10. *Let $(f_k)_{k=1}^N \subset H^*$ and $(p_k^b)_{k=1}^N \subset H^1(\Omega)$. For all $k = 1, 2, \ldots, N$, find $(u_k^*, P_k, u_k) \in H \times H^1(\Omega) \times L^2(\Omega)^d$ such that $P_k - p_k^b \in H^1_{\Gamma_2}(\Omega)$ and for all $\varphi \in H$ and $\psi \in H^1_{\Gamma_2}(\Omega)$,*

$$\begin{cases} \dfrac{1}{\tau}(u_k^* - u_{k-1}, \varphi) + a(u_k^*, \varphi) + d(u_{k-1}^*, u_k^*, \varphi) = \langle f_k, \varphi \rangle_H \\ \tau(\nabla P_k, \nabla \psi) = -(\operatorname{div} u_k^*, \psi) \\ u_k = u_k^* - \tau \nabla P_k \text{ in } L^2(\Omega)^d. \end{cases} \tag{PM}$$

Remark 2.11. *For $f \in L^2(H^*)$ and $p^b \in L^2(H^1(\Omega))$, we set for all $k = 1, 2, \ldots, N$,*

$$f_k := \frac{1}{\tau} \int_{t_{k-1}}^{t_k} f(s) ds, \qquad p_k^b := \frac{1}{\tau} \int_{t_{k-1}}^{t_k} p^b(s) ds. \tag{2.9}$$

Here, it holds that $\bar{f}_\tau \in L^2(H^*)$ and $\bar{p}_\tau^b \in L^2(H^1(\Omega))$:

$$\|\bar{f}_\tau\|_{L^2(H^*)} \le \|f\|_{L^2(H^*)}, \qquad \|\bar{p}_\tau^b\|_{L^2(H^1)} \le \|p^b\|_{L^2(H^1)}.$$

In Theorems 2.17 and 2.22, we assume $f \in C([0, T]; H^*), p^b \in C([0, T]; H^1(\Omega))$ to use $f(t_k)$ and $p^b(t_k)$ for all $k = 1, 2, \ldots, N$ (Hypothesis 2.16). Then, we set for all $k = 1, 2, \ldots, N$,

$$f_k := f(t_k), \qquad p_k^b := p^b(t_k),$$

which implies that $\bar{f}_\tau = f_\tau \in L^2(H^*)$ and $\bar{p}_\tau^b = p_\tau^b \in L^2(H^1(\Omega))$.

We show the existence and uniqueness of the solution to (PM) in the following proposition.

Proposition 2.12. *For all $(f_k)_{k=1}^N \subset H^*$, $(p_k^b)_{k=1}^N \subset H^1(\Omega)^d$, and $u_0 \in L^{p_d}(\Omega)^d$, Problem 2.10 has a unique solution.*

Proof. By Lemmas 2.3, 2.6, and 2.7, if $u_{k-1}^* \in L^{p_d}(\Omega)^d$ are known, then it holds that for all $v, \varphi \in H$,

$$\frac{1}{\tau}(v,\varphi) + a(v,\varphi) + d(u_{k-1}^*, v, \varphi) \le \left(\frac{1}{\tau} + c_a + c_d\|u_{k-1}^*\|_{L^{p_d}}\right)\|v\|_1\|\varphi\|_1,$$
$$\frac{1}{\tau}(v,v) + a(v,v) + d(u_{k-1}^*, v, v) \ge \frac{1}{c_a}\|v\|_1^2,$$

which implies that the mapping $H \times H \ni (v,\varphi) \mapsto \frac{1}{\tau}(v,\varphi) + a(v,\varphi) + d(u_{k-1}^*,\, v, \varphi) \in \mathbb{R}$ is a continuous and coercive bilinear form. On the other hand, if $u_{k-1} \in L^2(\Omega)^d$, then the mapping $H \ni \varphi \mapsto \langle f(t_k), \varphi\rangle_H + \tau^{-1}(u_{k-1}, \varphi) \in \mathbb{R}$ is a functional on H. By the Lax–Milgram theorem, there exists a unique solution $u_k^* \in H \subset L^{p_d}(\Omega)^d$ to the first equation of (PM). Since $\operatorname{div} u_k^* \in L^2(\Omega)$, by the Poincaré inequality and the Lax–Milgram theorem, the second equation of (PM) also has a unique solution $P_k \in H^1(\Omega)$. Furthermore, we obtain $u_k := u_k^* - \tau\nabla P_k \in L^2(\Omega)^d$. Therefore, since $u_0 (= u_0^*) \in L^{p_d}(\Omega)^d$, (PM) has a unique solution $(u_k^*, P_k, u_k)_{k=1}^N \subset H \times H^1(\Omega) \times L^2(\Omega)^d$. □

Remark 2.13. *The function space $L^2(\Omega)^d$ has the following orthogonal decomposition:*

$$L^2(\Omega)^d = U \oplus \nabla(H^1_{\Gamma_2}(\Omega)),$$

where $U := \{\varphi \in L^2(\Omega)^d \mid \operatorname{div}\varphi = 0$ *in* $L^2(\Omega), \langle\gamma_n\varphi, \psi\rangle_{H^{1/2}(\Gamma)} = 0$ *for all* $\psi \in H^1_{\Gamma_2}(\Omega)\}$ [18, Proposition 4.1]. *By the second and third equation of (PM) and the Gauss divergence formula, it holds that for all $k = 1, 2, \ldots, N$ and $\psi \in H^1_{\Gamma_2}(\Omega)$,*

$$(u_k, \nabla\psi) = (u_k^*, \nabla\psi) - \tau(\nabla P_k, \nabla\psi) = -(\operatorname{div} u_k^*, \psi) - \tau(\nabla P_k, \nabla\psi) = 0,$$

which implies that $u_k \in U$. Since the third equation of (PM) is equivalent to

$$u_k^* - \tau\nabla p^b(t_k) = u_k + \tau\nabla(P_k - p^b(t_k)) \qquad \text{in } L^2(\Omega)^d,$$

Step 2 ((1.4) and (1.5)) is the projection of $u_k^* - \tau\nabla p^b(t_k)$ to the divergence-free space U.

Remark 2.14. *By replacing u_{k-1} in the first equation of (PM) with the third equation of (PM) at the previous step, it holds that for all $k = 1, 2, \ldots, N$, $\varphi \in H$, and $\psi \in H^1_{\Gamma_2}(\Omega)$,*

$$\begin{cases} \dfrac{1}{\tau}(u_k^* - u_{k-1}^*, \varphi) + a(u_k^*, \varphi) + d(u_{k-1}^*, u_k^*, \varphi) + (\nabla P_{k-1}, \varphi) = \langle f_k, \varphi\rangle_H \\ \tau(\nabla P_k, \nabla\psi) = -(\operatorname{div} u_k^*, \psi) \end{cases}$$

where $P_0 := 0$ (cf. [27]*). Ones can calculate $(u_k^*, P_k)_{k=1}^N$ without the velocity $(u_k)_{k=1}^M$. Since the calculation $u_k = u_k^* - \tau\nabla P_k$ is not used, this formulation is suitable for numerical calculations such as the finite element method (see Section 4).*

On the other hand, by replacing u_k^* in the first term of the first equation of (PM) with the third equation of (PM) at the same step, it holds that for all $k = 1, 2, \ldots, N$, $\varphi \in H$, and $\psi \in H^1_{\Gamma_2}(\Omega)$,

$$\begin{cases} \dfrac{1}{\tau}(u_k - u_{k-1}, \varphi) + a(u_k^*, \varphi) + (\nabla P_k, \varphi) = \langle f_k, \varphi\rangle_H - d(u_{k-1}^*, u_k^*, \varphi) \\ \tau(\nabla P_k, \nabla\psi) + (\operatorname{div} u_k^*, \psi) = 0 \\ u_k = u_k^* - \tau\nabla P_k \ \text{ in } L^2(\Omega)^d. \end{cases} \tag{2.10}$$

This formulation is helpful to prove stability and convergence results.

2.4. **Main theorems for stability and convergence.** We show the stability of the projection method (PM) and establish error estimates in suitable norms between the solutions to the Navier–Stokes equations (NS) and the projection method (PM).

Theorem 2.15. *Under the condition (2.7), we set $f_k \in H^*$ and $p_k^b \in H^1(\Omega)^d$ as (2.9) for all $k = 1, 2, \ldots, N$. Then, there exists a constant $c > 0$ independent of τ such that*

$$\begin{aligned}&\|\bar{u}_\tau\|_{L^\infty(L^2)} + \|\bar{u}_\tau^*\|_{L^\infty(L^2)} + \|\bar{u}_\tau^*\|_{L^2(H^1)} + \frac{1}{\sqrt{\tau}}\|\bar{u}_\tau - \bar{u}_\tau^*\|_{L^2(L^2)} \\ &\leq c\left(\|u_0\|_0 + \|f\|_{L^2(H^*)} + \|p^b\|_{L^2(H^1)}\right).\end{aligned}$$

For a convergence theorem, we assume:

Hypothesis 2.16. *The solution (u, P) to (NS) satisfies*

$$u \in C([0,T]; H \cap H^2(\Omega)^d) \cap H^1(L^2(\Omega)^d) \cap H^2(H^*), \qquad P \in C([0,T]; H^1(\Omega)).$$

We also assume $f \in C([0,T]; H^*)$ and $p^b \in C([0,T]; H^1(\Omega))$ and set in Problem 2.10 for all $k = 1, 2, \ldots, N$,

$$f_k := f(t_k), \qquad p_k^b := p^b(t_k).$$

Theorem 2.17. *Under Hypothesis 2.16, there exist two constants $c, \tau_0 > 0$ independent of τ such that for all $0 < \tau < \tau_0$,*

$$\begin{aligned}\|u - \bar{u}_\tau\|_{L^\infty(L^2)} + \|u - \bar{u}_\tau^*\|_{L^\infty(L^2)} + \|u - \bar{u}_\tau^*\|_{L^2(H^1)} &\leq c\sqrt{\tau}, \\ \|\bar{u}_\tau - \bar{u}_\tau^*\|_{L^2(L^2)} &\leq c\tau.\end{aligned}$$

Remark 2.18. *For regularity of the solution (u, P) to (NS), see* [5, Theorem 1.3] *and* [24, Theorems 4.2 and 4.3]. *In the case of the homogeneous Dirichlet boundary condition on the whole boundary Γ, high regularity properties of the solution to the Navier–Stokes equations are proved in* [9, Theorem V.2.10].

Remark 2.19. *If $u \in C([0,T]; H \cap H^2(\Omega)^d)$, then $|u|^2 \in C([0,T]; H^1(\Omega))$, and hence, $p \in C([0,T]; H^1(\Omega))$ is equivalent to $P = p + \frac{1}{2}|u|^2 \in C([0,T]; H^1(\Omega))$.*

Furthermore, we assume the following regularity assumptions:

Hypothesis 2.20 (Regularity of the Stokes problem)**.** *There exists a constant $c = c(\Omega, \Gamma_1, \Gamma_2) > 0$ such that*

$$\|w\|_2 + \|r\|_1 \leq c\|e\|_0.$$

for all $e \in L^2(\Omega)^d$ and $(w, r) = T(e)$.

Hypothesis 2.21. *The solution (u, P) to (NS) satisfies*

$$u \in H^1(H^1(\Omega)^d) \cap H^2(L^2(\Omega)^d) \cap H^3(H^*), \qquad P \in H^1(H^1(\Omega)).$$

Then we can improve the convergence rate:

Theorem 2.22. *Under Hypothesis 2.16 and 2.20, there exist two constants $\tau_1, c > 0$ independent of τ such that for all $0 < \tau < \tau_1$,*

$$\|u - \bar{u}_\tau\|_{L^2(L^2)} + \|u - \bar{u}_\tau^*\|_{L^2(L^2)} \leq c\tau.$$

Furthermore, if we also assume Hypothesis 2.21, then there exist two constants $\tau_2, c > 0$ independent of τ such that for all $0 < \tau < \tau_2 (\leq \tau_1)$,

$$\|P - \bar{P}_\tau\|_{L^2(L^2)} \leq c\sqrt{\tau}.$$

Remark 2.23. *Hypothesis 2.20 holds, e.g., if Ω is of class $C^{2,1}$* [4, Theorem 1.2].

2.5. **Main result for existence of a weak solution to (1.2).** Using Theorem 2.15, we prove that there exists a solution to a weak formulation of (1.2) weaker than (NS). Putting $\varphi := v \in V$ in the first equation of (NS), we obtain the following equation: for all $v \in V$,

$$\left\langle \frac{\partial u}{\partial t}, v \right\rangle_V + a(u,v) + d(u,u,v) = \langle f, v\rangle_H - \int_{\Gamma_2} p^b v \cdot n ds \tag{2.11}$$

in $L^1(0,T)$.

Corollary 2.24. *Under the condition (2.7), there exists a solution $u \in L^2(V) \cap L^\infty(L^2(\Omega)^d) \cap C([0,T];V^*)$ to (2.11) with $u(0) = u_0$ such that $\frac{\partial u}{\partial t} \in L^{4/p_d}(V^*)$.*

Remark 2.25. *For $f \in L^2(L^2(\Omega)^d)$, local existence and uniqueness of a weak solution u to (2.11) with $u_0 \in H$ are proved in* [5, Theorem 1.3]*. By* [23, Lemma 4]*:*

$$a(u,v) = \sum_{i,j=1}^d \int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} dx + \int_{\Gamma_2} \kappa u \cdot v ds \quad \text{for all } u, v \in H,$$

where $\kappa := \operatorname{div} n = (d-1)\times$(mean curvature) (cf. Remark 1.1), (2.11) is equivalent to

$$\begin{aligned} &\left\langle \frac{\partial u}{\partial t}, v \right\rangle_V + \sum_{i,j=1}^d \int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} dx + \int_{\Gamma_2} \kappa u \cdot v ds + d(u,u,v) \\ =&\langle f, v\rangle_H - \int_{\Gamma_2} p^b v \cdot n ds \quad \text{for all } v \in V \end{aligned} \tag{2.12}$$

in $L^1(0,T)$. It is known [25, Theorem 5.1] *that there exists a weak solution u to (2.12) with $u_0 \in U$, where U is defined in Remark 2.13.*

3. Proofs

In this section, we prove that the solution to (PM) is bounded in suitable norms (Theorem 2.15) and error estimates (Theorems 2.17 and 2.22) in suitable norms between the solutions to (NS) and (PM).

3.1. **Stability.** We prepare the following useful lemma for the proofs of Theorems 2.15, 2.17, and 2.22.

Lemma 3.1. *Let $v_0 \in L^2(\Omega)^d, (F_k, G_k, Q_k)_{k=1}^N \subset H^* \times H^* \times H^1(\Omega)$ and let $(v_k^*, v_k, q_k)_{k=1}^N \in H \times L^2(\Omega)^d \times H^1(\Omega)$ satisfy that for all $k = 1, 2, \ldots, N$, $\varphi \in H$, and $\psi \in H^1_{\Gamma_2}(\Omega)$,*

$$\begin{cases} \dfrac{1}{\tau}(v_k - v_{k-1}, \varphi) + a(v_k^*, \varphi) - (q_k, \operatorname{div}\varphi) = \langle F_k + G_k, \varphi\rangle_H, \\ \tau(\nabla q_k, \nabla\psi) + (\operatorname{div} v_k^*, \psi) = -\tau(\nabla Q_k, \nabla \psi), \\ v_k = v_k^* - \tau\nabla(q_k + Q_k) \text{ in } L^2(\Omega)^d. \end{cases} \tag{3.13}$$

If we assume that for all $\delta > 0$ there exist a constant $A_\delta > 0$ independent of k and τ, and a sequence $(\beta_k)_{k=1}^N \subset \mathbb{R}$ such that

$$\langle G_k, v_k^*\rangle_H \le \delta \|v_k^*\|_1^2 + A_\delta(\|v_{k-1}^*\|_0^2 + \beta_k^2) \qquad \text{for all } k = 1, 2, \ldots, N, \tag{3.14}$$

where $v_0^* := v_0$, then there exist two constants $\tau_0, c > 0$ independent of τ such that for all $0 < \tau < \tau_0$,

$$\begin{aligned} &\|\bar{v}_\tau\|^2_{L^\infty(L^2)} + \|\bar{v}^*_\tau\|^2_{L^\infty(L^2)} + \|\bar{v}^*_\tau\|^2_{L^2(H^1)} + \tau\left\|\frac{\partial \hat{v}_\tau}{\partial t}\right\|^2_{L^2(L^2)} + \frac{1}{\tau}\|\bar{v}_\tau - \bar{v}^*_\tau\|^2_{L^2(L^2)} \\ \le\, & c\left(\|v_0\|_0^2 + \|\bar{F}_\tau\|^2_{L^2(H^*)} + \tau\|\bar{Q}^b_\tau\|^2_{L^2(H^1)} + \|\bar{\beta}_\tau\|^2_{L^2(0,T)}\right). \end{aligned} \tag{3.15}$$

In particular, if $\langle G_k, v_k^*\rangle_H \le 0$ for all $k = 1, 2, \ldots, N$, then $\tau_0 = T$.

Proof. Putting $\varphi := v_k^*$ and $\psi := q_k$ and adding the two equations, we obtain for all $k = 1, 2, \ldots, N$,

$$\begin{aligned}&\frac{1}{\tau}(v_k - v_{k-1}, v_k^*) + \|v_k^*\|_a^2 + \tau\|\nabla q_k\|_0^2 + \tau(\nabla Q_k, \nabla q_k)\\ &= \langle F_k + G_k, v_k^*\rangle_H \le \frac{c_a}{2}\|F_k\|_{H^*}^2 + \frac{1}{2c_a}\|v_k^*\|_1 + \langle G_k, v_k^*\rangle_H.\end{aligned}$$

Here, by Lemma 2.3 and the third equation of (3.13), it holds that

$$\begin{aligned}&\frac{1}{\tau}(v_k - v_{k-1}, v_k^*) + \|v_k^*\|_a^2 + \tau\|\nabla q_k\|_0^2 + \tau(\nabla Q_k, \nabla q_k)\\ &= \frac{1}{\tau}(v_k - v_{k-1}, v_k) + \frac{1}{\tau}(v_k - v_{k-1}, v_k^* - v_k) + \|v_k^*\|_a^2 + \tau\|\nabla(q_k + Q_k)\|_0^2 - \tau(\nabla Q_k, \nabla(q_k + Q_k))\\ &\ge \frac{1}{2\tau}(\|v_k\|_0^2 - \|v_{k-1}\|_0^2 + \|v_k - v_{k-1}\|_0^2) - \frac{3}{8\tau}\|v_k - v_{k-1}\|_0^2 - \frac{2}{3\tau}\|v_k^* - v_k\|_0^2 + \frac{1}{c_a}\|v_k^*\|_1^2\\ &\quad + \tau\|\nabla(q_k + Q_k)\|_0^2 - 3\tau\|\nabla Q_k\|_0^2 - \frac{\tau}{12}\|\nabla(q_k + Q_k)\|_0^2\\ &= \frac{1}{2\tau}\left(\|v_k\|_0^2 - \|v_{k-1}\|_0^2 + \frac{\tau^2}{4}\|D_\tau v_k\|_0^2 + \frac{1}{2}\|v_k^* - v_k\|_0^2\right) + \frac{1}{c_a}\|v_k^*\|_1^2 - 3\tau\|\nabla Q_k\|_0^2.\end{aligned}$$

Hence, we have for all $k = 1, 2, \ldots, N$,

$$\begin{aligned}&\|v_k\|_0^2 - \|v_{k-1}\|_0^2 + \frac{\tau^2}{4}\|D_\tau v_k\|_0^2 + \frac{1}{2}\|v_k^* - v_k\|_0^2 + \frac{\tau}{c_a}\|v_k^*\|_1^2\\ &\le c_a\tau\|F_k\|_{H^*}^2 + 6\tau^2\|\nabla Q_k\|_0^2 + 2\tau\langle G_k, v_k^*\rangle_H.\end{aligned} \tag{3.16}$$

By summing up (3.16) for $k = 1, 2, \ldots, m$ with an arbitrary natural number $m \le N$, it holds that

$$\begin{aligned}&\|v_m\|_0^2 - \|v_0\|_0^2 + \tau\sum_{k=1}^m\left(\frac{\tau}{4}\|D_\tau v_k\|_0^2 + \frac{1}{2\tau}\|v_k^* - v_k\|_0^2 + \frac{1}{c_a}\|v_k^*\|_1^2\right)\\ &\le \tau\sum_{k=1}^m\left(c_a\|F_k\|_{H^*}^2 + 6\tau\|\nabla Q_k\|_0^2 + 2\langle G_k, v_k^*\rangle_H\right).\end{aligned} \tag{3.17}$$

From the assumption (3.14) with $\delta := \frac{1}{4c_a}$;

$$\langle G_k, v_k^*\rangle_H \le \frac{\|v_k^*\|_1^2}{4c_a} + A_{\frac{1}{4c_a}}(2\|v_{k-1}\|_0^2 + 2\|v_{k-1} - v_{k-1}^*\|_0^2 + \beta_k^2),$$

we obtain

$$\begin{aligned}&\|v_m\|_0^2 - \|v_0\|_0^2 + \tau\sum_{k=1}^m\left(\frac{\tau}{4}\|D_\tau v_k\|_0^2 + \frac{1 - 8\tau A_{\frac{1}{4c_a}}}{2\tau}\|v_k - v_k^*\|_0^2 + \frac{1}{2c_a}\|v_k^*\|_1^2\right)\\ \le\,&\tau\sum_{k=0}^{m-1}4A_{\frac{1}{4c_a}}\|v_k\|_0^2 + \tau\sum_{k=1}^m\left(c_a\|F_k\|_{H^*}^2 + 6\tau\|\nabla Q_k\|_0^2 + 2A_{\frac{1}{4c_a}}\beta_k^2\right),\end{aligned}$$

where we have used $v_0 - v_0^* = 0$. By the discrete Gronwall inequality, if $\tau \le \tau_0 := 1/(16A_{\frac{1}{4c_a}})$, then it holds that for all $m = 0, 1, \ldots, N$,

$$\begin{aligned}&\|v_m\|_0^2 + \tau\sum_{k=1}^m\left(\frac{\tau}{4}\|D_\tau v_k\|_0^2 + \frac{1}{4\tau}\|v_k - v_k^*\|_0^2 + \frac{1}{2c_a}\|v_k^*\|_1^2\right)\\ \le\;&\exp\left(\frac{16}{3}A_{\frac{1}{4c_a}}\right)\left\{\|v_0\|_0^2 + \tau\sum_{k=1}^m\left(c_a\|F_k\|_{H^*}^2 + 6\tau\|\nabla Q_k\|_0^2 + 2A_{\frac{1}{4c_a}}\beta_k^2\right)\right\},\end{aligned}$$

which implies that

$$\begin{aligned}&\|\bar{v}_\tau(t)\|_0^2+\int_0^t\left(\tau\left\|\frac{\partial\hat{v}_\tau}{\partial t}(s)\right\|_0^2+\frac{1}{\tau}\|\bar{v}_\tau(s)-\bar{v}_\tau^*(s)\|_0^2+\|\bar{v}_\tau^*(s)\|_1^2\right)ds\\ \le\quad & c_1\left\{\|v_0\|_0^2+\int_0^t(\|\bar{F}_\tau(s)\|_{H^*}^2+\tau\|\bar{Q}_\tau(s)\|_1^2+\bar{\beta}_\tau^2(s))ds\right\}\end{aligned}$$

for all $t\in(0,T]$, where $c_1:=\exp(16A_{\frac{1}{4c_a}}/3)\times\max\{c_a,6,2A_{\frac{1}{4c_a}}\}\times\max\{4,2c_a\}$. Hence,

$$\|\bar{v}_\tau\|_{L^\infty(L^2)}^2\le M,\qquad \tau\left\|\frac{\partial\hat{v}_\tau}{\partial t}\right\|_{L^2(L^2)}^2+\frac{1}{\tau}\|\bar{v}_\tau-\bar{v}_\tau^*\|_{L^2(L^2)}^2+\|\bar{v}_\tau^*\|_{L^2(H^1)}^2\le M, \tag{3.18}$$

where $M:=c_1(\|v_0\|_0^2+\|\bar{F}_\tau\|_{L^2(H^*)}^2+\tau\|\bar{Q}_\tau\|_{L^2(H^1)}^2+\|\bar{\beta}_\tau\|_{L^2(0,T)}^2)$. If $\langle G_k,v_k^*\rangle_H\le 0$ for all $k=1,2,\dots,N$, then we immediately obtain (3.15) for all $0<\tau<T$ from (3.17).

Since it holds that for all $m=1,2,\dots,N$,

$$\|v_m^*\|_0^2\le 2(\|v_m\|_0^2+\|v_m-v_m^*\|_0^2)\le 2\max_{k=1,\dots,N}\|v_k\|_0^2+\tau\sum_{k=1}^N\frac{2}{\tau}\|v_k-v_k^*\|_0^2,$$

we obtain for all $0<\tau<\tau_0$,

$$\|\bar{v}_\tau^*\|_{L^\infty(L^2)}^2\le 2\|\bar{v}_\tau\|_{L^\infty(L^2)}^2+\frac{2}{\tau}\|\bar{v}_\tau-\bar{v}_\tau^*\|_{L^2(L^2)}^2\le 4M.$$

□

By using Lemma 3.1, we prove Theorem 2.15.

Proof of Theorem 2.15. We set $(F_k)_{k=1}^N,(G_k)_{k=1}^N\subset H^*$ defined by

$$\langle F_k,\varphi\rangle_H:=\langle f_k,\varphi\rangle_H-(\nabla p_k^b,\varphi),\qquad\langle G_k,\varphi\rangle_H:=-d(u_{k-1}^*,u_k^*,\varphi)$$

for all $k=1,2,\dots,N$ and $\varphi\in H$. From Problem 2.10 and the condition (2.7), if we set $q_k:=P_k-p_k^b$, then $(u_k^*,u_k,q_k)_{k=1}^N\subset H\times L^2(\Omega)^d\times H^1_{\Gamma_2}(\Omega)$ satisfies that for all $k=1,2,\dots,N$,

$$\begin{cases}\dfrac{1}{\tau}(u_k-u_{k-1},\varphi)+a(u_k^*,\varphi)-(q_k,\operatorname{div}\varphi)=\langle F_k+G_k,\varphi\rangle,\\ \tau(\nabla q_k,\nabla\psi)+(\operatorname{div}u_k^*,\psi)=-(\nabla p_k^b,\nabla\psi),\\ u_k=u_k^*-\tau\nabla(q_k+p_k^b)\text{ in }L^2(\Omega)^d,\end{cases}$$

with $u_0\in L^{p_d}(\Omega)^d(\subset L^2(\Omega)^d)$. By Lemma 2.6, it holds that

$$\langle G_k,u_k^*\rangle_H=-d(u_{k-1}^*,u_k^*,u_k^*)=0\qquad\text{for all }k=1,2,\dots,N.$$

Therefore, by Lemma 3.1 and Remark 2.11, we conclude the proof. □

3.2. **Convergence.** In this section, we assume Hypothesis 2.16. We calculate the error estimates in suitable norms between the solutions to (NS) and (PM). By Hypothesis 2.16 and the first equation of (NS), it holds that $\frac{\partial u}{\partial t}\in C([0,T];H^*)$ and, for all $\varphi\in H$ and $k=1,2,\dots,N$,

$$\frac{1}{\tau}(u(t_k)-u(t_{k-1}),\varphi)+a(u(t_k),\varphi)+d(u_{k-1}^*,u_k^*,\varphi)+(\nabla P(t_k),\varphi)=\langle f(t_k)-R_k-R_k^{\text{n.l.}},\varphi\rangle_H,$$

where $R_k,R_k^{\text{n.l.}}\in H^*$ defined by

$$\begin{aligned}\langle R_k,\varphi\rangle_H&:=\left\langle\frac{\partial u}{\partial t}(t_k)-\frac{u(t_k)-u(t_{k-1})}{\tau},\varphi\right\rangle_H,\\ \langle R_k^{\text{n.l.}},\varphi\rangle_H&:=d(u(t_k),u(t_k),\varphi)-d(u_{k-1}^*,u_k^*,\varphi)\end{aligned}$$

for all $\varphi \in H$. If we put $e_0 = 0$, $e_k := u_k - u(t_k) \in L^2(\Omega)^d$, $e_k^* := u_k^* - u(t_k) \in H$, and $q_k := P_k - P(t_k) \in H^1_{\Gamma_2}(\Omega)$ for $k = 1, 2, \ldots, N$, by (2.10), then it holds that for all $k = 1, 2, \ldots, N$, $\varphi \in H$, and $\psi \in H^1_{\Gamma_2}(\Omega)$,

$$\begin{cases} \dfrac{1}{\tau}(e_k - e_{k-1}, \varphi) + a(e_k^*, \varphi) - (q_k, \operatorname{div}\varphi) = \langle R_k + R_k^{\text{n.l.}}, \varphi\rangle_H \\ \tau(\nabla q_k, \nabla\psi) + (\operatorname{div} e_k^*, \psi) = -\tau(\nabla P(t_k)\nabla\psi), \\ e_k = e_k^* - \tau\nabla(q_k + P(t_k)) \text{ in } L^2(\Omega)^d, \end{cases} \tag{3.19}$$

where we have used $(\nabla q_k, \varphi) = -(q_k, \operatorname{div}\varphi)$.

In order to prove Theorems 2.17 and 2.22, we prepare Lemmas 3.2 and 3.3. See the Appendix for the proofs.

Lemma 3.2. *(i) Under Hypothesis 2.16, we have*

$$\|\bar{R}_\tau\|^2_{L^2(H^*)} \le \frac{\tau^2}{3}\left\|\frac{\partial^2 u}{\partial t^2}\right\|^2_{L^2(H^*)}.$$

(ii) Furthermore, if Hypothesis 2.21 holds, then we have

$$\sum_{k=2}^N \tau\|D_\tau R_k\|^2_{H^*} \le \frac{2}{3}\tau^2\left\|\frac{\partial^3 u}{\partial t^3}\right\|^2_{L^2(H^*)}.$$

Lemma 3.3. *Let $(E, (\cdot,\cdot)_E)$ be a Hilbert space and let $x \in C([0,T];E)$ satisfy that $\frac{\partial x}{\partial t} \in L^2(0,T;E)$.*
(i) It holds that for all $k = 1, 2, \ldots, N$,

$$\|D_\tau x(t_k)\|_E \le \frac{1}{\sqrt{\tau}}\left\|\frac{\partial x}{\partial t}\right\|_{L^2(t_{k-1},t_k;E)}.$$

(ii) It holds that

$$\|x - x_\tau\|_{L^\infty(E)} \le \sqrt{\tau}\left\|\frac{\partial x}{\partial t}\right\|_{L^2(E)}, \qquad \|x - x_\tau\|_{L^2(E)} \le \frac{\tau}{\sqrt{2}}\left\|\frac{\partial x}{\partial t}\right\|_{L^2(E)}.$$

By using Lemmas 3.1, 3.2 and 3.3, we prove Theorem 2.17.

Proof of Theorem 2.17. For all $\delta > 0$ and $k = 1, 2, \ldots, N$, by Lemmas 2.6, 2.7 and 3.3, we have

$$\begin{aligned} \langle R_k^{\text{n.l.}}, e_k^*\rangle_H &= -d(u_{k-1}^*, e_k^*, e_k^*) - d(e_{k-1}^*, u(t_k), e_k^*) + \tau d(D_\tau u(t_k), u(t_k), e_k^*) \\ &\le c_d\|e_{k-1}^*\|_0\|u(t_k)\|_2\|e_k^*\|_1 + c_d\tau\|D_\tau u(t_k)\|_0\|u(t_k)\|_2\|e_k^*\|_1 \\ &\le \frac{\delta}{2}\|e_k^*\|_1^2 + \frac{c_d^2\|u(t_k)\|_2^2}{2\delta}\|e_{k-1}^*\|_0^2 + \frac{\delta}{2}\|e_k^*\|_1^2 + \frac{c_d^2\|u(t_k)\|_2^2}{2\delta}\tau^2\|D_\tau u(t_k)\|_0^2 \\ &\le \delta\|e_k^*\|_1^2 + \frac{c_d^2 c_{\max}^2}{2\delta}\|e_{k-1}^*\|_0^2 + \frac{c_d^2 c_{\max}^2}{2\delta}\tau\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(t_{k-1},t_k;L^2(\Omega)^d)} \end{aligned} \tag{3.20}$$

where $c_{\max} := \|u\|_{C([0,T],H^2(\Omega)^d)}$. By (3.19) and Lemmas 3.1, 3.2 and 3.3, there exist two constants $\tau_0, c_1 > 0$ such that for all $0 < \tau < \tau_0$,

$$\begin{aligned} &\|\bar{e}_\tau\|^2_{L^\infty(L^2)} + \|\bar{e}_\tau^*\|^2_{L^\infty(L^2)} + \|\bar{e}_\tau^*\|^2_{L^2(H^1)} + \frac{1}{\tau}\|\bar{e}_\tau - \bar{e}_\tau^*\|^2_{L^2(L^2)} \\ \le{}& c_1\left(\|\bar{R}_\tau\|^2_{L^2(H^*)} + \tau\|P_\tau\|^2_{L^2(H^1)} + \tau^2\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(L^2)}\right) \\ \le{}& c_1\left(\frac{\tau^2}{3}\left\|\frac{\partial^2 u}{\partial t^2}\right\|^2_{L^2(H^*)} + 2\tau\|P\|^2_{L^2(H^1)} + \tau^3\left\|\frac{\partial P}{\partial t}\right\|^2_{L^2(H^1)} + \tau^2\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(L^2)}\right), \end{aligned}$$

which implies that

$$\|\bar{u}_\tau - u_\tau\|_{L^\infty(L^2)} + \|\bar{u}^*_\tau - u_\tau\|_{L^\infty(L^2)} + \|\bar{u}^*_\tau - u_\tau\|_{L^2(H^1)} \le c_2\sqrt{\tau},$$
$$\|\bar{u}_\tau - \bar{u}^*_\tau\|_{L^2(L^2)} \le c_2\tau$$

for a constant $c_2 > 0$, where we have used $\bar{e}_\tau = \bar{u}_\tau - u_\tau$ and $\bar{e}^*_\tau = \bar{u}^*_\tau - u_\tau$. By the triangle inequality and Lemma 3.3, it holds that $\|u - \bar{u}_\tau\|_{L^\infty(L^2)} + \|u - \bar{u}^*_\tau\|_{L^\infty(L^2)} \le c_2\sqrt{\tau} + 2\sqrt{\tau}\|\frac{\partial u}{\partial t}\|_{L^2(L^2)}$. To complete the first inequality of Theorem 2.17, it is sufficient to prove that $\|u - u_\tau\|_{L^2(H^1)} \le c_3\sqrt{\tau}$ for a constant $c_3 > 0$. Since $u(t) \in H \cap H^2(\Omega)^d$ and $\operatorname{div} u(t) = 0 \in H^1(\Omega)$ for all $t \in [0,T]$, by Proposition 2.9, Lemmas 2.3 and 3.3, we find that

$$\begin{aligned}
&\|u - u_\tau\|^2_{L^2(H^1)} = \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \|u(t) - u(t_k)\|_1^2 dt \\
&\le c_a \sum_{k=1}^N \int_{t_{k-1}}^{t_k} a(u(t) - u(t_k), u(t) - u(t_k)) dt = c_a \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \left(-\Delta(u(t) - u(t_k)), u(t) - u(t_k)\right) dt \\
&\le 2\sqrt{d} c_a c_{\max} \int_0^T \|u(t) - u(t_k)\|_0 dt \le 2\sqrt{dT} c_a c_{\max} \|u - u_\tau\|_{L^2(L^2)} \le \sqrt{2dT} c_a c_{\max} \tau \left\|\frac{\partial u}{\partial t}\right\|_{L^2(L^2)}.
\end{aligned}$$

□

We improve the error estimates for the velocity and pressure in the $L^2(L^2)$-norm. In order to prove Theorem 2.22, we prepare Proposition 3.4 and Lemma 3.5.

Proposition 3.4. *Under Hypothesis 2.20, for all $e \in L^2(\Omega)$, the pair of functions $(w, r) = T(e)$ belongs to $H^2(\Omega) \times H^1_{\Gamma_2}(\Omega)$.*

Proof. By Hypothesis 2.20, $(w, r) \in H^2(\Omega)^d \times H^1(\Omega)$. Since it holds that for all $\varphi \in H$,

$$0 = a(w, \varphi) - (r, \operatorname{div}\varphi) - (e, \varphi) = \int_\Omega (\nabla \times (\nabla \times w) + \nabla r - e) \cdot \varphi dx - \int_{\Gamma_2} r\varphi \cdot n ds,$$

we obtain $r \in H^1_{\Gamma_2}(\Omega)$. □

Lemma 3.5. *Under the assumption of Lemma 3.1 and Hypothesis 2.20, if we assume the following conditions: if $(w_k, r_k) := T(v_k)$ for all $k = 0, 1, \ldots, N$, then for all $\delta > 0$ there exist a constant $A_\delta > 0$ independent of k and τ, and a sequence $(\gamma_k)_{k=1}^N \in \mathbb{R}$ such that for all $k = 1, 2, \ldots, N$,*

$$\langle G_k, w_k\rangle_H \le \delta(\|v^*_{k-1}\|_0^2 + \|v^*_k\|_0^2) + A_\delta(\|w_k\|_1^2 + \gamma_k^2), \tag{3.21}$$

then there exist two constants $\tau_0, c > 0$ independent of τ such that for all $0 < \tau < \tau_0$,

$$\|\bar{v}_\tau\|^2_{L^2(L^2)} \le c(\|v_0\|^2_{V^*} + \tau\|v^*_0\|_0^2 + \|\bar{v}^*_\tau - \bar{v}_\tau\|^2_{L^2(L^2)} + \|\bar{F}_\tau\|^2_{L^2(H^*)} + \|\bar{\gamma}_\tau\|^2_{L^2(0,T)}).$$

Proof. Let $(w_k, r_k) := T(v_k)$ for all $k = 0, 1, \ldots, N$. It follows from Proposition 3.4 that $r_k \in H^1_{\Gamma_2}(\Omega)$. The first equation of (3.13) implies that for all $k = 1, 2, \ldots, N$,

$$\frac{1}{\tau}(v_k - v_{k-1}, w_k) + a(v^*_k, w_k) = \langle F_k, w_k\rangle_H + \langle G_k, w_k\rangle_H, \tag{3.22}$$

where we have used $\operatorname{div} w_k = 0$ in $L^2(\Omega)$. By Lemma 2.5, we obtain

$$\begin{aligned}
(v_k - v_{k-1}, w_k) &= a(w_k, w_k) - (r_k, \operatorname{div} w_k) - a(w_{k-1}, w_k) + (r_{k-1}, \operatorname{div} w_k) \\
&= \frac{1}{2}\left(\|w_k\|_a^2 - \|w_{k-1}\|_a^2 + \|w_k - w_{k-1}\|_a^2\right) \ge \frac{c_1}{2}\left(\|w_k\|_1^2 - \|w_{k-1}\|_1^2\right)
\end{aligned}$$

where $c_1 := \min\{c_a, c_a^{-1}\}$. For the second term of the left hand side of (3.22), by the definition of the operator T, we have

$$a(v^*_k, w_k) = \|v_k\|_0^2 + (v_k, v^*_k - v_k) - (\nabla r_k, v^*_k - v_k),$$

where we have used the third equation of (3.13) and $(\nabla r_k, v_k) = (\nabla r_k, v_k^*) - \tau(\nabla r_k, \nabla(q_k + Q_k)) = 0$. By Hypothesis 2.20, it holds that

$$\begin{aligned}|(v_k, v_k^* - v_k) - (\nabla r_k, v_k^* - v_k)| &\le (\|v_k\|_0 + \|\nabla r_k\|_0)\|v_k^* - v_k\|_0 \le c_2\|v_k\|_0\|v_k^* - v_k\|_0 \\ &\le \frac{1}{4}\|v_k\|_0^2 + c_2^2\|v_k^* - v_k\|_0^2\end{aligned}$$

for a constant $c_2 > 0$. Hence, we have

$$a(v_k^*, w_k) \ge \frac{3}{4}\|v_k\|_0^2 - c_2^2\|v_k^* - v_k\|_0^2.$$

For the first term of the right hand side of (3.22), by Lemma 2.5, we have

$$\langle F_k, w_k\rangle_H \le \|F_k\|_{H^*}\|w_k\|_1 \le c_3\|F_k\|_{H^*}\|v_k\|_0 \le \frac{1}{4}\|v_k\|_0^2 + c_3^2\|F_k\|_{H^*}^2$$

for a constant $c_3 > 0$. Hence, we have that for all $k = 1, 2, \ldots, N$,

$$\|w_k\|_1^2 - \|w_{k-1}\|_1^2 + \frac{\tau}{c_1}\|v_k\|_0^2 \le \frac{2\tau}{c_1}(c_2^2\|v_k^* - v_k\|_0^2 + c_3^2\|F_k\|_{H^*}^2 + \langle G_k, w_k\rangle_H).$$

By summing up for $k = 1, 2, \ldots, m$ with an arbitrary natural number $m \le N$, it holds that

$$\|w_m\|_1^2 - \|w_0\|_1^2 + \frac{\tau}{c_1}\sum_{k=1}^m \|v_k\|_0^2 \le \frac{2\tau}{c_1}\sum_{k=1}^m (c_2^2\|v_k^* - v_k\|_0^2 + c_3^2\|F_k\|_{H^*}^2 + \langle G_k, w_k\rangle_H).$$

From the assumption (3.21) with $\delta := \frac{1}{16}$, we obtain for all $m = 1, 2, \ldots, N$,

$$\begin{aligned}\sum_{k=1}^m \langle G_k, w_k\rangle_H &\le \sum_{k=1}^m \left\{\frac{1}{16}(\|v_{k-1}^*\|_0^2 + \|v_k^*\|_0^2) + A_{\frac{1}{16}}(\|w_k\|_1^2 + \gamma_k^2)\right\} \\ &\le \frac{1}{16}\|v_0^*\|_0^2 + \frac{1}{4}\sum_{k=1}^m (\|v_k\|_0^2 + \|v_k^* - v_k\|_0^2) + A_{\frac{1}{16}}\sum_{k=1}^m (\|w_k\|_1^2 + \gamma_k^2)\end{aligned}$$

and hence,

$$\begin{aligned}&\|w_m\|_1^2 - \|w_0\|_1^2 + \frac{\tau}{2c_1}\sum_{k=1}^m \|v_k\|_0^2 \\ &\le \tau\sum_{k=1}^m \frac{2A_{\frac{1}{16}}}{c_1}\|w_k\|_1^2 + \frac{\tau}{8c_1}\|v_0^*\|_0^2 + \tau\sum_{k=1}^m c_4(\|v_k^* - v_k\|_0^2 + \|F_k\|_{H^*}^2 + \gamma_k^2),\end{aligned}$$

where $c_4 := 2c_1^{-1}\max\{c_2^2 + 1/4, c_3^2, A_{\frac{1}{16}}\}$. By the discrete Gronwall inequality, if $\tau \le \tau_0 := c_1/A_{\frac{1}{16}}$, then we have

$$\begin{aligned}&\|w_N\|_1^2 + \frac{\tau}{2c_1}\sum_{k=1}^N \|v_k\|^2 \\ &\le \exp\left(\frac{4A_{\frac{1}{16}}}{c_1}\right)\left\{\|w_0\|_1^2 + \frac{\tau}{8c_1}\|v_0^*\|_0^2 + \tau\sum_{k=1}^N c_4\left(\|v_k^* - v_k\|_0^2 + \|F_k\|_{H^*}^2 + \gamma_k^2\right)\right\}.\end{aligned}$$

Therefore, by Lemma 2.5, we obtain

$$\|\bar{v}_\tau\|_{L^2(L^2)}^2 \le c_5\Big(\|v_0\|_{V^*}^2 + \tau\|v_0^*\|_0^2 + \|\bar{v}_\tau^* - \bar{v}_\tau\|_{L^2(L^2)}^2 + \|F_\tau\|_{L^2(H^*)}^2 + \|\bar{\gamma}_\tau\|_{L^2}^2\Big)$$

for a constant $c_5 > 0$. □

We prove the first inequality of Theorem 2.22

Proof of the first inequality of Theorem 2.22. We apply Lemmas 3.5 for (3.19). Let $(w_k, r_k) := T(e_k)$ for all $k = 0, 1, \ldots, N$. It holds that for all $k = 1, 2, \ldots, N$,

$$\langle R_k^{\text{n.l.}}, w_k\rangle_H = -d(e_{k-1}^*, u_k^*, w_k) - d(u(t_{k-1}), e_k^*, w_k) + \tau d(D_\tau u(t_k), u(t_k), w_k).$$

Hypothesis 2.20 and Theorem 2.17 implies that there exists a constant $c_1 > 0$ such that $\|w_k\|_2 \leq c_1\sqrt{\tau}$ for all $k = 1, 2, \dots, N$. It holds that for all $\delta > 0$,

$$\begin{aligned}
-d(e^*_{k-1}, u^*_k, w_k) &= -d(e^*_{k-1}, u(t_k), w_k) - d(e^*_{k-1}, e^*_k, w_k) \\
&\leq c_d\|e^*_{k-1}\|_0\|u(t_k)\|_2\|w_k\|_1 + c_d\|e^*_{k-1}\|_0\|e^*_k\|_1\|w_k\|_2 \\
&\leq c_d c_{\max}\|e^*_{k-1}\|_0\|w_k\|_1 + c_d c_1\sqrt{\tau}\|e^*_{k-1}\|_0\|e^*_k\|_1 \\
&\leq \frac{\delta}{2}\|e^*_{k-1}\|_0^2 + \frac{c_d^2 c_{\max}^2}{2\delta}\|w_k\|_1^2 + \frac{\delta}{2}\|e^*_{k-1}\|_0^2 + \frac{c_d^2 c_1^2}{2\delta}\tau\|e^*_k\|_1^2 \\
&\leq \delta\|e^*_{k-1}\|_0^2 + \frac{c_d^2 c_{\max}^2}{2\delta}\|w_k\|_1^2 + \frac{c_d^2 c_1^2}{2\delta}\tau\|e^*_k\|_1^2, \\
-d(u(t_{k-1}), e^*_k, w_k) &= d(u(t_{k-1}), w_k, e^*_k) \leq c_d\|u(t_{k-1})\|_2\|w_k\|_1\|e^*_k\|_0 \\
&\leq \delta\|e^*_k\|_0^2 + \frac{c_d^2 c_{\max}^2}{4\delta}\|w_k\|_1^2 \\
\tau d(D_\tau u(t_k), u(t_k), w_k) &\leq c_d\tau\|D_\tau u(t_k)\|_0\|u(t_k)\|_2\|w_k\|_1 \leq c_d c_{\max}\tau\|D_\tau u(t_k)\|_0\|w_k\|_1 \\
&\leq \frac{c_d^2 c_{\max}^2}{4\delta}\|w_k\|_1^2 + \delta\tau^2\|D_\tau u(t_k)\|_0^2,
\end{aligned}$$

where $c_{\max} := \|u\|_{C([0,T];H^2(\Omega)^d)}$. Hence, by Lemma 3.3, it holds that for all $k = 1, 2, \dots, N$,

$$\begin{aligned}
\langle R^{\text{n.l.}}_k, w_k\rangle_H &\leq \delta\|e^*_{k-1}\|_0^2 + \delta\|e^*_k\|_0^2 + c_\delta(\|w_k\|_1^2 + \tau\|e^*_k\|_1^2 + \tau^2\|D_\tau u(t_k)\|_0^2) \\
&\leq \delta\|e^*_{k-1}\|_0^2 + \delta\|e^*_k\|_0^2 + c_\delta\left(\|w_k\|_1^2 + \tau\|e^*_k\|_1^2 + \tau\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(t_{k-1},t_k;L^2(\Omega)^d)}\right),
\end{aligned}$$

where $c_\delta := \max\{c_d^2 c_{\max}^2/\delta, c_d^2 c_1^2/(2\delta), \delta\}$. By Lemma 3.5, there exist two constants $c_2, \tau_0 > 0$ such that for all $0 < \tau < \tau_0$,

$$\|\bar{e}_\tau\|^2_{L^2(L^2)} \leq c_2\left(\|\bar{e}^*_\tau - \bar{e}_\tau\|^2_{L^2(L^2)} + \|\bar{R}_\tau\|^2_{L^2(H^*)} + \tau\|\bar{e}^*_\tau\|^2_{L^2(H^1)} + \tau^2\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(L^2)}\right).$$

By Lemma 3.2 and Theorem 2.17, there exists a constant $c_3 > 0$ such that for all $0 < \tau < \tau_0$,

$$\|u_\tau - \bar{u}_\tau\|_{L^2(L^2)} \leq c_3\tau.$$

By Lemma 3.3 and Theorem 2.17, we obtain the first inequality of Theorem 2.22;

$$\|u - \bar{u}_\tau\|_{L^2(L^2)} + \|u - \bar{u}^*_\tau\|_{L^2(L^2)} \leq 2\|u - u_\tau\|_{L^2(L^2)} + 2\|u_\tau - \bar{u}_\tau\|_{L^2(L^2)} + \|\bar{u}_\tau - \bar{u}^*_\tau\|_{L^2(L^2)} \leq c_4\tau$$

for a constant $c_4 > 0$. □

To prove the second inequality of Theorem 2.22, we prepare the following two lemmas:

Lemma 3.6. *Under Hypothesis 2.16, there exists a constant $c > 0$ independent of τ such that*

$$\|D_\tau e_1\|_{V^*} \leq c\sqrt{\tau}, \qquad \|D_\tau e_1\|_0 + \|D_\tau e^*_1\|_0 \leq c, \qquad \|D_\tau e^*_1\|_1 \leq \frac{c}{\sqrt{\tau}}.$$

Proof. By (3.19) and (3.16) with $k := 1$ in the proof of Lemma 3.1, we obtain

$$\|e_1\|_0^2 + \frac{1}{2}\|e_1 - e^*_1\|_0^2 + \frac{\tau}{c_a}\|e^*_1\|_1^2 \leq c_a\tau\|R_1\|^2_{H^*} + 6\tau^2\|\nabla P(t_1)\|_0^2 + 2\tau\langle R^{\text{n.l.}}_1, e^*_1\rangle_H.$$

Putting $k := 1$ and $\delta := \frac{1}{4c_a}$ in (3.20), it holds that

$$\langle R^{\text{n.l.}}_1, e^*_1\rangle_H \leq \frac{1}{4c_a}\|e^*_1\|_1^2 + 2c_a c_d^2 c_{\max}^2\tau\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(0,t_1;L^2(\Omega)^d)}$$

where $c_{\max} := \|u\|_{C([0,T];H^2(\Omega)^d)}$. Hence, by Lemma 3.2, we have

$$\begin{aligned}
&\|e_1\|_0^2 + \frac{1}{2}\|e_1 - e_1^*\|_0^2 + \frac{1}{2c_a}\tau\|e_1^*\|_1^2 \\
&\le c_a\tau\|R_1\|_{H^*}^2 + 6\tau^2\|\nabla P(t_1)\|_0^2 + 4c_a c_d^2 c_{\max}^2\tau^2 \left\|\frac{\partial u}{\partial t}\right\|_{L^2(0,t_1;L^2(\Omega)^d)}^2 \\
&\le \frac{c_a\tau^2}{3}\left\|\frac{\partial u}{\partial t}\right\|_{L^2(H^*)}^2 + 6\tau^2\|P\|_{C([0,T];H^1)}^2 + 4c_a c_d^2 c_{\max}^2\tau^2\left\|\frac{\partial u}{\partial t}\right\|_{L^2(L^2)}^2 \le c_2\tau^2
\end{aligned}$$

where $c_2 := c_a(\frac{1}{3} + 4c_d^2 c_{\max}^2)\left\|\frac{\partial u}{\partial t}\right\|_{L^2(L^2)}^2 + 6\|P\|_{C([0,T];H^1)}^2$, which implies that $\|D_\tau e_1\|_0 = \tau^{-1}\|e_1\|_0 \le \sqrt{c_2}$, $\|D_\tau e_1^*\|_1 \le \sqrt{2c_a c_2}\tau^{-1/2}$ and

$$\|D_\tau e_1^*\|_0 = \frac{1}{\tau}\|e_1^*\|_0 \le \frac{1}{\tau}(\|e_1\|_0 + \|e_1 - e_1^*\|_0) \le (1+\sqrt{2})\sqrt{c_2}.$$

On the other hand, by (3.19) and Lemmas 2.3, 3.2,

$$\begin{aligned}
\|D_\tau e_1\|_{V^*} &= \sup_{0\ne\varphi\in V}\frac{|(e_1 - e_0, \varphi)|}{\tau\|\varphi\|_1} \\
&= \sup_{0\ne\varphi\in V}\frac{|-a(e_1^*,\varphi) + (q_1, \operatorname{div}\varphi) + \langle R_1,\varphi\rangle_H - d(u_0,e_1^*,\varphi) + \tau d(D_\tau u(t_1),u(t_1),\varphi)|}{\|\varphi\|_1} \\
&\le c_a\|e_1^*\|_1 + \|R_1\|_{H^*} + c_d(\|u_0\|_1\|e_1^*\|_1 + \tau\|D_\tau u(t_1)\|_0\|u(t_1)\|_2) \\
&\le c_a\|e_1^*\|_1 + \sqrt{\frac{\tau}{3}}\left\|\frac{\partial u}{\partial t}\right\|_{L^2(0,t_1;H^*)} + c_d c_{\max}\left(\|e_1^*\|_1 + \sqrt{\tau}\left\|\frac{\partial u}{\partial t}\right\|_{L^2(0,t_1,L^2(\Omega)^d)}\right) \\
&\le \sqrt{\tau}\left\{(c_a + c_d c_{\max})\sqrt{2c_a c_2} + \left(\frac{1}{\sqrt{3}} + c_d c_{\max}\right)\left\|\frac{\partial u}{\partial t}\right\|_{L^2(L^2)}\right\},
\end{aligned}$$

where $c_{\max} := \|u\|_{C([0,T];H^2(\Omega)^d)}$. □

Lemma 3.7. *Under Hypothesis 2.16, 2.20, and 2.21, there exist two constants $c, \tau_0 > 0$ independent of τ such that for all $0 < \tau < \tau_0$,*

$$\left\|\frac{\partial \hat{e}_\tau}{\partial t}\right\|_{L^2(L^2)} \le c\sqrt{\tau}.$$

Proof. By (3.19), it holds that $(D_\tau e_k^*, D_\tau q_k, D_\tau e_k)_{k=2}^N \subset H \times H^1_{\Gamma_2}(\Omega) \times L^2(\Omega)^d$ and for all $k = 2, 3, \dots, N$, $\varphi \in H$, and $\psi \in H^1_{\Gamma_2}(\Omega)$,

$$\begin{cases}
\left(\dfrac{D_\tau e_k - D_\tau e_{k-1}}{\tau}, \varphi\right) + a(D_\tau e_k^*, \varphi) - (D_\tau q_k, \operatorname{div}\varphi) = \langle D_\tau R_k + D_\tau R_k^{\text{n.l.}}, \varphi\rangle_H, \\
\tau(\nabla D_\tau q_k, \nabla\psi) + (\operatorname{div} D_\tau e_k^*, \psi) = -(\nabla D_\tau P(t_k), \nabla\psi), \\
D_\tau e_k = D_\tau e_k^* - \tau\nabla D_\tau(q_k + P(t_k)) \text{ in } L^2(\Omega)^d
\end{cases} \tag{3.23}$$

with $D_\tau e_1 = \tau^{-1}(e_1 - e_0) = \tau^{-1}e_1$. It holds for all $k = 2, 3, \dots, N$ and $\varphi \in H$,

$$\begin{aligned}
\tau\langle D_\tau R_k^{\text{n.l.}}, \varphi\rangle_H = &-\tau d(u_{k-2}^*, D_\tau e_k^*, \varphi) - \tau d(D_\tau u_{k-1}^*, e_k^*, \varphi) + d(e_{k-2}^*, u(t_{k-1}), \varphi) \\
&- d(e_{k-1}^*, u(t_k), \varphi) - \tau d(D_\tau u(t_{k-1}), u(t_{k-1}), \varphi) + \tau d(D_\tau u(t_k), u(t_k), \varphi).
\end{aligned} \tag{3.24}$$

Here, by Lemma 2.7, the right hand side except for the first and second terms are evaluated from above for all $k = 2, 3, \ldots, N$, $\varphi \in H$ and $\delta > 0$,

$$\begin{aligned}
&d(e^*_{k-2}, u(t_{k-1}), \varphi) - d(e^*_{k-1}, u(t_k), \varphi) - \tau d(D_\tau u(t_{k-1}), u(t_{k-1}), \varphi) + \tau d(D_\tau u(t_k), u(t_k), \varphi) \\
&\le c_d c_{\max} \left(\|e^*_{k-2}\|_0 + \|e^*_{k-1}\|_0 + \tau \|D_\tau u(t_{k-1})\|_0 + \tau \|D_\tau u(t_k)\|_0 \right) \|\varphi\|_1 \\
&\le \frac{\delta}{2} \|\varphi\|_1^2 + \frac{2c_d^2 c_{\max}^2}{\delta} \sum_{i=0}^{1} \left(\|e^*_{k-i-1}\|_0^2 + \tau^2 \|D_\tau u(t_{k-i})\|_0^2 \right),
\end{aligned} \tag{3.25}$$

where $c_{\max} := \|u\|_{C([0,T];H^2(\Omega)^d)}$. By Lemma 2.6, it holds that

$$-\tau d(u^*_{k-2}, D_\tau e^*_k, D_\tau e^*_k) = 0.$$

By Theorem 2.17, there exist two constants $\tau_1, c_1 > 0$ such that $\|\bar{e}^*_\tau\|_{L^2(H^1)} \le c_1$ for all $0 < \tau < \tau_1$, and hence for all $k = 1, 2, \ldots, N$, $\|e^*_k\|_1 \le c_1$ and

$$\begin{aligned}
-\tau d(D_\tau u^*_{k-1}, e^*_k, D_\tau e^*_k) &= -\tau d(D_\tau u(t_{k-1}), e^*_k, D_\tau e^*_k) - d(D_\tau e^*_{k-1}, e^*_k, e^*_k) + d(D_\tau e^*_{k-1}, e^*_k, e^*_{k-1}) \\
&\le c_d \tau \|D_\tau u(t_{k-1})\|_1 \|e^*_k\|_1 \|D_\tau e^*_k\|_1 + c_d \|D_\tau e^*_{k-1}\|_1 \|e^*_k\|_1 \|e^*_{k-1}\|_1 \\
&\le c_d c_1 \tau \|D_\tau u(t_{k-1})\|_1 \|D_\tau e^*_k\|_1 + c_d c_1 \|D_\tau e^*_{k-1}\|_1 \|e^*_{k-1}\|_1 \\
&\le \frac{\delta}{2} \|D_\tau e^*_k\|_1^2 + \frac{c_d^2 c_1^2}{2\delta} \tau^2 \|D_\tau u(t_{k-1})\|_1^2 + \delta \|D_\tau e^*_{k-1}\|_1^2 + \frac{c_d^2 c_1^2}{4\delta} \|e^*_{k-1}\|_1^2.
\end{aligned}$$

Hence, by (3.24) with $\varphi := D_\tau e^*_k$ and Lemma 3.3, for all $0 < \tau < \tau_1$, $k = 2, 3, \ldots, N$ and $\delta > 0$,

$$\begin{aligned}
&\tau \langle D_\tau R^{\text{n.l.}}_k, D_\tau e^*_k \rangle_H \le \delta (\|D_\tau e^*_k\|_1^2 + \|D_\tau e^*_{k-1}\|_1^2) + c_\delta \sum_{i=0}^{1} \left(\|e^*_{k-i-1}\|_1^2 + \tau^2 \|D_\tau u(t_{k-i})\|_1^2 \right) \\
&\le \delta (\|D_\tau e^*_k\|_1^2 + \|D_\tau e^*_{k-1}\|_1^2) + c_\delta \sum_{i=0}^{1} \left(\|e^*_{k-i-1}\|_1^2 + \tau \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(t_{k-i-1}, t_{k-i}; H^1(\Omega)^d)} \right),
\end{aligned}$$

where $c_\delta := \delta^{-1}(2c_d^2 c_{\max}^2 + 2^{-1} c_d^2 c_1^2)$. Putting $\delta := 1/(4c_a)$, by (3.23) and (3.16) in the proof of Lemma 3.1, we have for all $0 < \tau < \tau_1$ and $k = 2, 3, \ldots, N$,

$$\begin{aligned}
&\|D_\tau e_k\|_0^2 - \|D_\tau e_{k-1}\|_0^2 + \frac{1}{2} \|D_\tau e^*_k - D_\tau e_k\|_0^2 + \frac{\tau}{c_a} \|D_\tau e^*_k\|_1^2 \\
&\le c_a \tau \|D_\tau R_k\|_{H^*}^2 + 6\tau^2 \|\nabla D_\tau P(t_k)\|_0^2 + 2\tau \langle D_\tau R^{\text{n.l.}}_k, D_\tau e^*_k \rangle_H \\
&\le c_a \tau \|D_\tau R_k\|_{H^*}^2 + 6\tau^2 \|\nabla D_\tau P(t_k)\|_0^2 + \frac{\tau}{2c_a} (\|D_\tau e^*_k\|_1^2 + \|D_\tau e^*_{k-1}\|_1^2) \\
&\quad + 2c_{\frac{1}{4c_a}} \tau \sum_{i=0}^{1} \left(\|e^*_{k-i-1}\|_1^2 + \tau \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(t_{k-i-1}, t_{k-i}; H^1(\Omega)^d)} \right).
\end{aligned}$$

Summing up for $k = 2, 3, \ldots, m$ with an arbitrary natural number $m \le N$, by Lemmas 3.2 and 3.3, it holds that

$$\begin{aligned}
&\|D_\tau e_m\|_0^2 + \frac{\tau}{2c_a} \|D_\tau e^*_m\|_1^2 + \tau \sum_{k=2}^{m} \frac{1}{2\tau} \|D_\tau e^*_k - D_\tau e_k\|_0^2 \\
&\le \|D_\tau e_1\|_0^2 + \frac{\tau}{2c_a} \|D_\tau e^*_1\|_1^2 + \tau \sum_{k=2}^{m} (c_a \|D_\tau R_k\|_{H^*}^2 + 6\tau \|D_\tau P(t_k)\|_1^2) \\
&\quad + 4c_{\frac{1}{4c_a}} \tau \sum_{k=1}^{m} \left(\|e^*_k\|_1^2 + \tau \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(t_{k-1}, t_k; H^1(\Omega)^d)} \right) \\
&\le c_2 \left\{ \|D_\tau e_1\|_0^2 + \tau \|D_\tau e^*_1\|_1^2 + \tau^2 \left\| \frac{\partial^3 u}{\partial t^3} \right\|^2_{L^2(H^*)} + \tau \left\| \frac{\partial P}{\partial t} \right\|^2_{L^2(H^1)} + \|\bar{e}^*_\tau\|^2_{L^2(H^1)} + \tau^2 \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(H^1)} \right\},
\end{aligned}$$

where $c_2 := \max\{2^{-1}c_a^{-1}, c_a, 6, 4c_{\frac{1}{4c_a}}\}$. Hence, by Lemma 3.6, there exist two constants $c_3 > 0$ such that for all $0 < \tau < \tau_1$,

$$\max_{k=1,\ldots,N} \|D_\tau e_k\|_0^2 + \tau \sum_{k=2}^{N} \frac{1}{\tau}\|D_\tau e_k - D_\tau e_k^*\|_0^2 \le c_3. \tag{3.26}$$

To use Lemma 3.5 for (3.23), we set $(w_k, r_k) = T(D_\tau e_k)$ for all $k = 1, 2, \ldots, N$. By Hypothesis 2.20 and (3.26), there exists a constant $c_4 > 0$ such that for all $0 < \tau < \tau_1$ and $k = 1, 2, \ldots, N$, $\|w_k\|_2 \le c_4$, and hence, for all $0 < \tau < \tau_1$, $\delta > 0$ and $k = 2, 3, \ldots, N$,

$$\begin{aligned}
-\tau d(u_{k-2}^*, D_\tau e_k^*, w_k) &= \tau d(u_{k-2}^*, w_k, D_\tau e_k^*) \le c_d \tau \|u_{k-2}^*\|_1 \|w_k\|_2 \|D_\tau e_k^*\|_0 \\
&\le c_d c_4 \tau \|u_{k-2}^*\|_1 \|D_\tau e_k^*\|_0 = \delta \|D_\tau e_k^*\|_0^2 + \frac{c_d^2 c_4^2}{4\delta}\tau^2 \|u_{k-2}^*\|_1^2, \\
-\tau d(D_\tau u_{k-1}^*, e_k^*, w_k) &= -\tau d(D_\tau u(t_{k-1}), e_k^*, w_k) - \tau d(D_\tau e_{k-1}^*, e_k^*, w_k) \\
&\le c_d \tau \|D_\tau u(t_{k-1})\|_1 \|e_k^*\|_1 \|w_k\|_1 + c_d \|D_\tau e_{k-1}^*\|_0 \|e_k^*\|_1 \|w_k\|_2 \\
&\le c_d c_1 \tau \|D_\tau u(t_{k-1})\|_1 \|w_k\|_1 + c_d c_4 \|D_\tau e_{k-1}^*\|_0 \|e_k^*\|_1 \\
&\le \frac{\delta}{2}\|w_k\|_1^2 + \frac{c_d^2 c_1^2}{2\delta}\tau^2 \|D_\tau u(t_{k-1})\|_1^2 + \delta \|D_\tau e_{k-1}^*\|_0^2 + \frac{c_d^2 c_4^2}{4\delta}\|e_{k-1}^*\|_1^2.
\end{aligned}$$

By (3.24) and (3.25) with $\varphi := w_k$, we have

$$\begin{aligned}
\langle D_\tau R_k^{\text{n.l.}}, w_k \rangle_H &\le \delta(\|D_\tau e_{k-1}^*\|_0^2 + \|D_\tau e_k^*\|_0^2) \\
&\quad + \tilde{c}_\delta \left\{ \|w_k\|_1^2 + \tau^2 \|u_{k-2}^*\|_1^2 + \sum_{i=0}^{1} \left(\|e_{k-i-1}^*\|_1^2 + \tau^2 \|D_\tau u(t_{k-i})\|_1^2 \right) \right\},
\end{aligned}$$

where $\tilde{c}_\delta := \max\{\delta, \delta^{-1}c_d^2(2c_{\max}^2 + 2^{-1}c_1^2 + 4^{-1}c_4^2)\}$. By Lemmas 3.5 and 3.3, there exist two constants $0 < \tau_2 \le \tau_1$ and $c_5 > 0$ such that for all $0 < \tau < \tau_2$,

$$\begin{aligned}
\tau \sum_{k=2}^{N} \|D_\tau e_k\|_0^2 \le c_5 \Bigg(&\|D_\tau e_1\|_{V^*}^2 + \tau \|D_\tau e_1^*\|_0^2 + \tau^2 \|\bar{u}_\tau^*\|_{L^2(H^1)}^2 + \|\bar{e}_\tau^*\|_{L^2(H^1)}^2 \\
&+ \tau^2 \left\| \frac{\partial u}{\partial t} \right\|_{L^2(H^1)}^2 + \tau \sum_{k=2}^{N} (\|D_\tau e_k - D_\tau e_k^*\|_0^2 + \|D_\tau R_k\|_{H^*}^2) \Bigg).
\end{aligned}$$

Hence, by Theorems 2.15, 2.17, Lemmas 3.2, 3.6, and (3.26), it holds that for all $0 < \tau < \tau_2$,

$$\left\| \frac{\partial \hat{e}_\tau}{\partial t} \right\|_{L^2(L^2)}^2 = \tau \|D_\tau e_1\|_0^2 + \tau \sum_{k=2}^{N} \|D_\tau e_k\|_0^2 \le c_7 \tau$$

for a constant $c_7 > 0$, where we have used $\frac{\partial \hat{e}_\tau}{\partial t} = (\overline{D_\tau e})_\tau$ on (t_{k-1}, t_k) for all $k = 1, 2, \ldots, N$ □

Finally, we prove the second inequality of Theorem 2.22.

Proof of the second inequality of Theorem 2.22. By (3.19) and Lemmas 2.2, 2.3, there exists a constant $c_1 > 0$ such that for all $k = 1, 2, \ldots, N$,

$$\begin{aligned}
\|q_k\|_0 &\le c_1 \sup_{0 \ne \varphi \in H} \frac{|(q_k, \operatorname{div} \varphi)|}{\|\varphi\|_1} = c_1 \sup_{0 \ne \varphi \in H} \frac{\left|(D_\tau e_k, \varphi) + a(e_k^*, \varphi) - \langle R_k + R_k^{\text{n.l.}}, \varphi \rangle_H \right|}{\|\varphi\|_1} \\
&\le c_1 \left(\|D_\tau e_k\|_0 + c_a \|e_k^*\|_1 + \|R_k\|_{H^*} + \|R_k^{\text{n.l.}}\|_{H^*} \right).
\end{aligned}$$

By Hypothesis 2.16 and Theorem 2.17, there exist two constants $\tau_1, c_2 > 0$ such that $\|u(t_k)\|_2$, $\tau^{-1/2}\|\bar{e}_\tau^*\|_{L^2(H^1)} \le c_2$ for all $0 < \tau < \tau_1$ and $k = 0, 1, \ldots, N$. By Lemma 3.3, it holds that for all

$0<\tau<\tau_1$, $k=1,2,\dots,N$ and $\varphi\in H$,

$$\begin{aligned}
|\langle R_k^{\text{n.l.}},\varphi\rangle_H| &= |-d(e_{k-1}^*,u(t_k),\varphi)-d(e_{k-1}^*,e_k^*,\varphi)-d(u(t_{k-1}),e_k^*,\varphi)+\tau d(D_\tau u(t_k),u(t_k),\varphi)|\\
&\le c_d\big(\|e_{k-1}^*\|_1\|u(t_k)\|_1+\|e_{k-1}^*\|_1\|e_k^*\|_1+\|u(t_{k-1})\|_1\|e_k^*\|_1+\tau\|D_\tau u(t_k)\|_0\|u(t_k)\|_2\big)\|\varphi\|_1\\
&\le c_dc_2\left(\|e_{k-1}^*\|_1+2\|e_k^*\|_1+\tau\|D_\tau u(t_k)\|_0\right)\|\varphi\|_1\\
&\le c_dc_2\left(\|e_{k-1}^*\|_1+2\|e_k^*\|_1+\sqrt{\tau}\left\|\frac{\partial u}{\partial t}\right\|_{L^2(t_{k-1},t_k,L^2(\Omega)^d)}\right)\|\varphi\|_1,
\end{aligned}$$

where we have used $\|e_k\|_1\le c_2$ for all $k=0,1,\dots,N$. Hence, we have for all $0<\tau<\tau_1$ and $k=1,2,\dots,N$,

$$\|q_k\|_0\le c_3\left(\|D_\tau e_k\|_0+\|e_{k-1}^*\|_1+\|e_k^*\|_1+\sqrt{\tau}\left\|\frac{\partial u}{\partial t}\right\|_{L^2(t_{k-1},t_k,L^2(\Omega)^d)}+\|R_k\|_{H^*}\right)$$

for a constant $c_3>0$. By Lemmas 3.2 and 3.7, there exist three constants $\tau_2,c_4,c_5>0$ such that for all $0<\tau<\tau_2\le\tau_1$,

$$\begin{aligned}
\|\bar P_\tau-P_\tau\|_{L^2(L^2)}^2 &\le c_4\left(\left\|\frac{\partial\hat e_\tau}{\partial t}\right\|_{L^2(L^2)}^2+\|\bar e_\tau^*\|_{L^2(H^1)}^2+\tau^2\left\|\frac{\partial u}{\partial t}\right\|_{L^2(L^2)}^2+\|\bar R_\tau\|_{L^2(H^*)}^2\right)\\
&\le c_4\left(\left\|\frac{\partial\hat e_\tau}{\partial t}\right\|_{L^2(L^2)}^2+c_2^2\tau+\tau^2\left\|\frac{\partial u}{\partial t}\right\|_{L^2(L^2)}^2+\frac{\tau^2}{3}\left\|\frac{\partial^2 u}{\partial t^2}\right\|_{L^2(H^*)}^2\right)\le c_5\tau.
\end{aligned}$$

Therefore, by Lemma 3.3, we conclude the proof:

$$\|P-\bar P_\tau\|_{L^2(L^2)}\le\|P-P_\tau\|_{L^2(L^2)}+\|P_\tau-\bar P_\tau\|_{L^2(L^2)}\le\sqrt{\tau}\left(\left\|\frac{\partial P}{\partial t}\right\|_{L^2(L^2)}+\sqrt{c_5}\right).$$

□

4. Numerical examples

For our simulation, we set $T=1$ and

$$\begin{aligned}
\Omega &= \left\{(r\cos\theta,r\sin\theta)\in\mathbb{R}^2\;\middle|\;r_1<r<r_2,\theta_1<\theta<\theta_2\right\},\\
\Gamma_1 &= \left\{(r\cos\theta,r\sin\theta)\in\mathbb{R}^2\;\middle|\;r\in\{r_1,r_2\},\theta_1<\theta<\theta_2\right\},\\
\Gamma_2 &= \left\{(r\cos\theta,r\sin\theta)\in\mathbb{R}^2\;\middle|\;r_1<r<r_2,\theta\in\{\theta_1,\theta_2\}\right\},
\end{aligned}$$

where $r_1:=2,r_2=3,\theta_1=0,\theta_2:=\pi/2$ (Fig. 1), and define the following constants:

$$p_{\text{in}}:=1,\qquad p_{\text{out}}:=-1,\qquad \alpha:=\frac{p_{\text{in}}-p_{\text{out}}}{\theta_2-\theta_1},$$
$$C:=\frac{1}{2}r_1^2r_2^2\frac{\log\theta_2-\log\theta_1}{r_2^2-r_1^2},\qquad D:=-\frac{1}{2}\frac{r_2^2\log r_2-r_1^2\log r_1}{r_2^2-r_1^2}.$$

The following functions

$$u(x,y,t):=\begin{pmatrix}U(r)e^{-t}\sin\theta\\-U(r)e^{-t}\cos\theta\end{pmatrix},\qquad p(x,y,t):=p_0(\theta)e^{-t},$$

where $(r,\theta)=(r(x,y),\theta(x,y))$ are the polar coordinates and

$$U(r)=\alpha\left(\frac{1}{2}r\log r+\frac{C}{r}+Dr\right),\qquad p_0(\theta)=\frac{p_{\text{in}}(\theta-\theta_1)+p_{\text{out}}(\theta_2-\theta)}{\theta_2-\theta_1},$$

satisfy (1.1) with $\nu = \rho = 1$ and

$$f(x,y,t) := \begin{pmatrix} -\dfrac{U^2(r)}{r}e^{-2t}\cos\theta - U(r)e^{-t}\sin\theta \\ -\dfrac{U^2(r)}{r}e^{-2t}\sin\theta + U(r)e^{-t}\cos\theta \end{pmatrix} = \left\{\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right\}(x,y,t),$$

$$p^b(x,y,t) := p_0(\theta)e^{-t} + \frac{U^2(r)}{2}e^{-2t}, \qquad u_0(x,y) := \begin{pmatrix} U(r)\sin\theta \\ -U(r)\cos\theta \end{pmatrix}.$$

Fig. 2 shows the initial value u_0 of the velocity and the pressure p at $t = 0$.

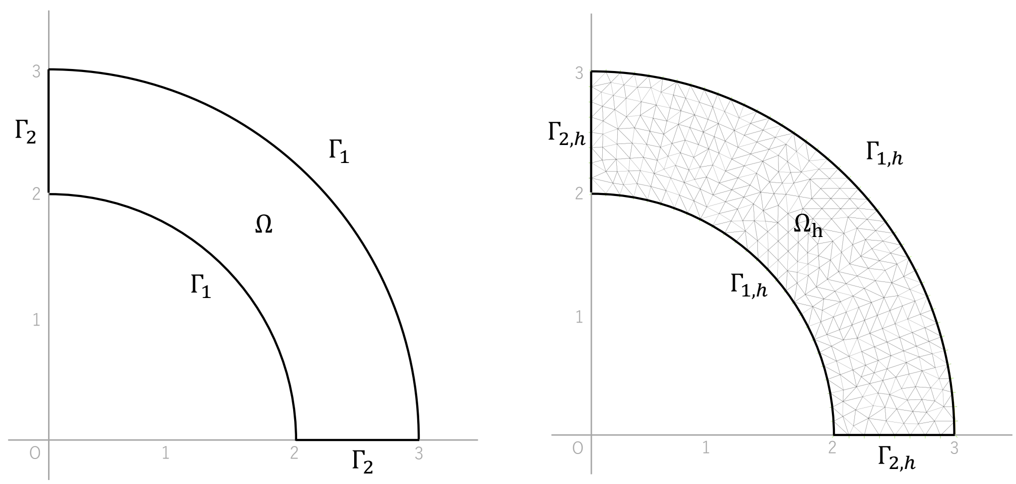

FIGURE 1. The domain Ω with the boundary Γ_1, Γ_2 (left), and $\Omega_h, \Gamma_{1,h}, \Gamma_{2,h}$ with mesh (right).

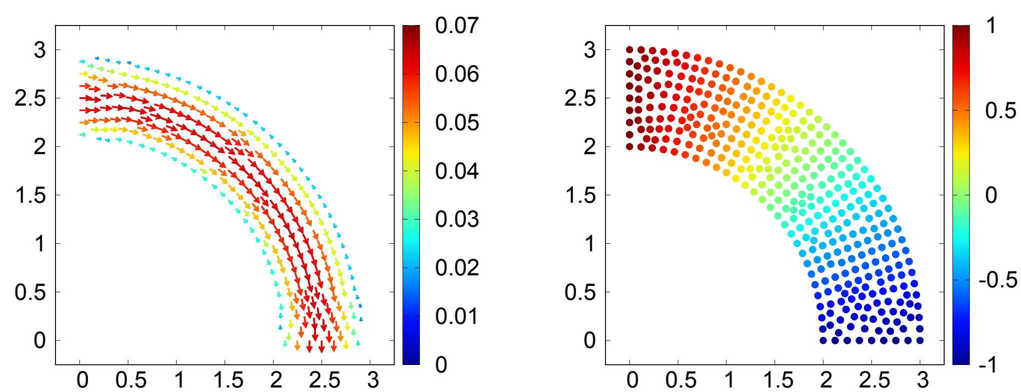

FIGURE 2. The initial value u_0 of the velocity (left) and the pressure p at $t = 0$ (right). In the left figure, the color scale indicates the length of $|u_0(\xi)|$ at each node ξ.

We introduce a domain Ω_h to approximate the domain Ω, with boundary $\partial\Omega_h = \Gamma_{1,h} \cup \Gamma_{2,h}$ (Fig. 1). We also introduce a regular triangulation $\mathcal{T}_h$ to Ω_h, with $h = \max_{K\in\mathcal{T}_h} \operatorname{diam}(K)$ and $\overline{\Omega_h} = \cup_{K\in\mathcal{T}_h}\overline{K}$. To consider the P2 and P1 element approximation for velocity and pressure, respectively, we define the function spaces: for $i = 1, 2$,

$$
\begin{aligned}
X_h^i &:= \left\{ \psi_h \in C(\overline{\Omega_h}) \;\middle|\; \varphi_h|_K \in P_i(K), \forall K \in \mathcal{T}_h \right\}, \\
H_h &:= \left\{ \varphi_h \in (X_h^2)^2 \;\middle|\; \varphi_h = 0 \text{ on } \Gamma_{1,h},\ \varphi_h \times n_h = 0 \text{ on } \Gamma_{2,h} \right\}, \\
Q_h &:= \left\{ \psi_h \in X_h^1 \;\middle|\; \psi_h = 0 \text{ on } \Gamma_{2,h} \right\},
\end{aligned}
$$

where $P_i(K)$ is the set of polynomials of degree i or less on K and n_h is the unit outward normal vector for $\Gamma_{2,h}$. Here, since $\Gamma_{2,h}$ is flat, the normal component of $\varphi_h \in H_h$ is not determined. If $\Gamma_{2,h}$ is not flat, then n_h is discontinuous on $\Gamma_{2,h}$ and $\varphi_h = 0$ on $\Gamma_{2,h}$ (cf. [8]). Let $\Pi_h^i : C(\overline{\Omega_h}) \to X_h^i$ $(i = 1, 2)$ be the Lagrange interpolation operator (on each triangle). By replacing u_{k-1} in the first equation of (PM) with the third equation of (PM) at the previous step (Remark 2.14), we consider the following discrete problem:

Problem 4.1. *For all $k = 1, 2, \ldots, N$, find $(u_k^*, P_k) \in H_h \times X_h^1$ such that $P_k - \Pi_h^1 p^b(t_k) \in Q_h$ and for all $\varphi \in H_h$ and $\psi \in Q_h$,*

$$
\begin{cases}
\dfrac{1}{\tau}(u_k^* - u_{k-1}^*, \varphi) + a(u_k^*, \varphi) + d(u_{k-1}^*, u_k^*, \varphi) + (\nabla P_{k-1}, \varphi) = (f(t_k), \varphi), \\
\tau(\nabla P_k, \nabla \psi) = -(\operatorname{div} u_k^*, \psi),
\end{cases}
\tag{4.27}
$$

where $P_0 := 0$.

For all $k = 1, 2, \ldots, N$, we set $u_k := u_k^* - \tau \nabla P_k$. See [17, 18] for the details on u_k and its divergence.

On a mesh with $h = 2^{-6}$, we solve the problems (4.27) numerically by using the software FreeFEM [20]. We compute the error estimates between the numerical solutions of (4.27) and the interpolation $(\Pi_h^2 u, \Pi_h^1 P)$ of the exact solution (u, P), where $P := p + |u|^2/2$. In Fig. 3, the numerical errors $\|\bar{u}_\tau - \Pi_h^2 u_\tau\|_{L^2(L^2(\Omega_h)^d)}$, $\|\bar{u}_\tau^* - \Pi_h^2 u_\tau\|_{L^2(L^2(\Omega_h)^d)}$, $\|\bar{P}_\tau - \Pi_h^1 P_\tau\|_{L^2(L^2(\Omega_h))}$, and $\|\bar{u}_\tau^* - \Pi_h^2 u_\tau\|_{L^2(H^1(\Omega_h)^d)}$ are presented. One can observe that $\|\bar{u}_\tau - \Pi_h^2 u_\tau\|_{L^2(L^2(\Omega_h)^d)}$ and $\|\bar{u}_\tau^* - \Pi_h^2 u_\tau\|_{L^2(L^2(\Omega_h)^d)}$ are almost of first order in τ and that $\|\bar{P}_\tau - \Pi_h^1 P_\tau\|_{L^2(L^2(\Omega_h))}$ is of 0.5th order in τ, as expected from Theorem 2.22. Furthermore, the error $\|\bar{u}_\tau^* - \Pi_h^2 u_\tau\|_{L^2(H^1(\Omega_h)^d)}$ is almost of first order in τ, which is better than the theoretically predicted rate (Theorem 2.17).

5. Conclusion

We have proposed a new projection method for Navier–Stokes equations (1.1) with a total pressure boundary condition. We have shown the stability of the projection method in Theorem 2.15 and established error estimates for the velocity and the pressure in suitable norms between the solution to (NS) and (PM) in Theorems 2.17 and 2.22. The convergence rates are the same as the case of the usual full-Dirichlet boundary condition for velocity [28, 29]. The traction boundary condition is often used to apply Dirichlet boundary conditions for pressure; however, the convergence rates are worse than our case (Compare [15] and [19]).

The projection method is still evolving, and many high-convergence methods have been proposed [16]. The application of the boundary conditions proposed in this paper to these methods will be a focus of our future works. As another future direction, the case that Γ_2 is not flat in numerical calculations is an important problem (cf. [8]). In addition, since the nonlinear term $(\nabla \times u) \times u$ is different from the standard advection term $(u \cdot \nabla)u$, it cannot be applied to methods using the Lagrangian coordinates, such as the characteristic curve method and particle methods; this problem remains open for further study.

Acknowledgments

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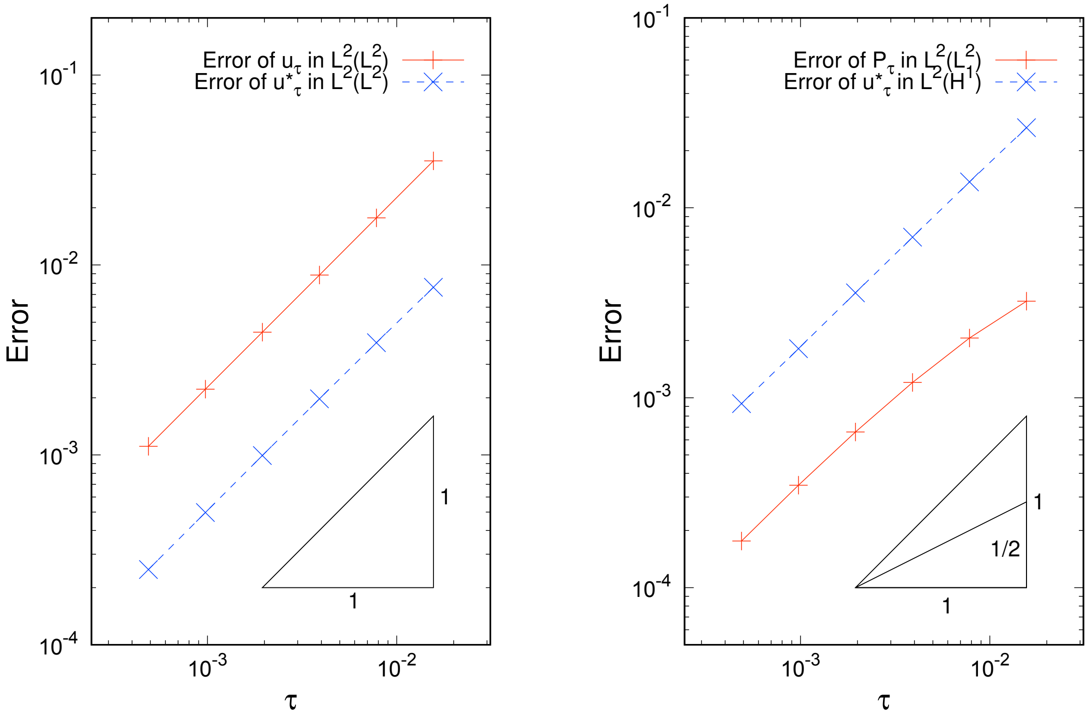

FIGURE 3. The errors of logscale: $\|\bar{u}_\tau - \Pi_h^2 u_\tau\|_{L^2(L^2(\Omega_h)^d)}$, $\|\bar{u}^*_\tau - \Pi_h^2 u_\tau\|_{L^2(L^2(\Omega_h)^d)}$ (left), $\|\bar{P}_\tau - \Pi_h^1 P_\tau\|_{L^2(L^2(\Omega_h))}$, and $\|\bar{u}^*_\tau - \Pi_h^2 u_\tau\|_{L^2(H^1(\Omega_h)^d)}$ (right). The triangles show the slope of $O(\tau)$ and $O(\sqrt{\tau})$.

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Proofs of Lemmas 2.5, 2.7, 3.2, 3.3, and Corollary 2.24

The purpose of this appendix is to provide the proofs of Lemmas 2.5, 2.7, 3.2, 3.3, and Corollary 2.24. The continuity of the operator T follows from Lemmas 2.1, 2.2, and 2.3. By using Lemma 2.3 again, we prove the second inequality of Lemma 2.5.

Proof of Lemma 2.5. By Lemmas 2.1, 2.2 and 2.3, there exists a unique solution $(w, r) \in H \times L^2(\Omega)$ to (2.6) for all $e \in L^2(\Omega)^d$ and T is a continuous operator;

$$\|w\|_1 + \|r\|_0 \leq c_1 \|e\|_{H^*} \leq c_1 \|e\|_0$$

for a constant $c_1 > 0$ independent of e. It is easy to check that T is a linear operator.

Next, we show the second inequality of Lemma 2.5. By the first equation of (2.6), it holds that for all $\varphi \in V$,

$$a(w, \varphi) = (r, \operatorname{div} \varphi) + (e, \varphi) = (e, \varphi).$$

By Lemma 2.3, we have

$$\frac{1}{c_a} \|w\|_1^2 \leq \|w\|_a^2 = a(w, w) = (e, w) \leq \|e\|_{V^*} \|w\|_1,$$

which implies that $\|w\|_1 \leq c_a \|e\|_{V^*}$. On the other hand, by Lemma 2.3, it holds that for all $e \in L^2(\Omega)^d$,

$$\|e\|_{V^*} = \sup_{0 \neq \varphi \in V} \frac{(e, \varphi)}{\|\varphi\|_1} = \sup_{0 \neq \varphi \in V} \frac{a(w, \varphi)}{\|\varphi\|_1} \leq \sup_{0 \neq \varphi \in V} \frac{c_a \|w\|_1 \|\varphi\|_1}{\|\varphi\|_1} = c_a \|w\|_1.$$

$\square$

In order to prove Lemma 2.7 and Corollary 2.24, we define $\tilde{p}_d, \tilde{q}_d$ as

$$\tilde{p}_d := \frac{2p_d}{p_d+2} = \frac{1}{\frac{1}{2}+\frac{1}{p_d}}, \qquad \tilde{q}_d := \frac{\tilde{p}_d}{1-\tilde{p}_d} = \frac{1}{1-\frac{1}{2}-\frac{1}{p_d}}.$$

Here, since $p_2 = 2+\varepsilon$ and $p_3 = 3$, we have $(\tilde{p}_2, \tilde{q}_2) = (\frac{4+2\varepsilon}{4+\varepsilon}, 2+\frac{4}{\varepsilon})$ and $(\tilde{p}_2, \tilde{q}_2) = (\frac{6}{5}, 6)$. By the Sobolev embeddings [9, Theorem III.2.33], it holds that $H^1(\Omega) \subset L^{\tilde{q}_d}(\Omega)$, $H^1(\Omega) \subset L^{p_d}(\Omega)$, $H^2(\Omega) \subset L^\infty(\Omega)$ and the embeddings are continuous.

Proof of Lemma 2.7.

(i) For all $u \in L^{p_d}(\Omega)^d, v, w \in H$, we have

$$\begin{aligned}
|d(u,v,w)| &\le \int_\Omega |u \cdot ((w\cdot\nabla)v - (v\cdot\nabla)w + v \operatorname{div} w - w \operatorname{div} v)|\,dx \\
&\le c_1 \|u\|_{L^{p_d}} (\|w\|_{L^{\tilde{q}_d}} \|\nabla v\|_0 + \|v\|_{L^{\tilde{q}_d}} \|\nabla w\|_0 + \|v\|_{L^{\tilde{q}_d}} \|\operatorname{div} w\|_0 + \|w\|_{L^{\tilde{q}_d}} \|\operatorname{div} v\|_0) \\
&\le \tilde{c}_1 \|u\|_{L^{p_d}} \|v\|_1 \|w\|_1
\end{aligned}$$

for two constants $c_1, \tilde{c}_1 > 0$, which implies the third inequality of Lemma 2.7.

(ii) For all $u \in L^{p_d}(\Omega)^d, v \in H, w \in H \cap H^2(\Omega)^d$, we have

$$\begin{aligned}
|d(u,v,w)| &\le c_2 \|u\|_0 (\|w\|_{L^\infty} \|\nabla v\|_0 + \|v\|_{L^{p_d}} \|\nabla w\|_{L^{\tilde{q}_d}} + \|v\|_{L^{p_d}} \|\operatorname{div} w\|_{L^{\tilde{q}_d}} + \|w\|_{L^\infty} \|\operatorname{div} v\|_0) \\
&\le \tilde{c}_2 \|u\|_0 \|v\|_1 \|w\|_2
\end{aligned}$$

for two constants $c_2, \tilde{c}_2 > 0$.

(iii) For all $u \in H^1(\Omega)^d, v \in H \cap H^2(\Omega)^d, w \in H$, we have

$$|d(u,v,w)| \le \int_\Omega |((\nabla \times u) \times v) \cdot w| dx \le c_3 \|\nabla \times u\|_0 \|v\|_{L^\infty} \|w\|_0 \le \tilde{c}_3 \|u\|_1 \|v\|_2 \|w\|_0$$

for two constants $c_3, \tilde{c}_3 > 0$.

(ix) For all $u \in H^2(\Omega)^d, v, w \in H$, we have

$$|d(u,v,w)| \le c_4 \|\nabla \times u\|_{L^{p_d}} \|v\|_{L^{\tilde{q}_d}} \|w\|_0 \le \tilde{c}_4 \|u\|_2 \|v\|_1 \|w\|_0$$

for two constants $c_4, \tilde{c}_4 > 0$. □

Next, we prove Lemmas 3.2 and 3.3.

Proof of Lemma 3.2. It holds that for all $\varphi \in H$ and $k = 1, 2, \dots, N$,

$$\begin{aligned}
\langle R_k, \varphi \rangle_H &= \tau \int_0^1 \left\langle s \frac{\partial^2 u}{\partial t^2}(t_{k-1} + s\tau), \varphi \right\rangle_H ds \le \tau \int_0^1 s \left\| \frac{\partial^2 u}{\partial t^2}(t_{k-1} + s\tau) \right\|_{H^*} \|\varphi\|_1 ds \\
&\le \sqrt{\frac{\tau}{3}} \|\varphi\|_1 \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^2(t_{k-1}, t_k; H^*)},
\end{aligned}$$

which implies that

$$\|\bar{R}_\tau\|^2_{L^2(H^*)} = \sum_{k=1}^N \tau \left(\sup_{0 \ne \varphi \in H} \frac{\langle R_k, \varphi \rangle_H}{\|\varphi\|_1} \right)^2 \le \frac{1}{3} \tau^2 \left\| \frac{\partial^2 u}{\partial t^2} \right\|^2_{L^2(H^*)}.$$

Next, we show the second inequality of the conclusion. For all $\varphi \in H$ and $k = 2, 3, \ldots, N$, we have

$$\begin{aligned}
\langle D_\tau R_k, \varphi \rangle_H &= \tau \int_0^1 \int_0^1 \left\langle s_1 \frac{\partial^3 u}{\partial t^3}(t_{k-2} + s_1\tau + s_2\tau), \varphi \right\rangle_H ds_1 ds_2 \\
&\le \tau \int_0^1 \int_0^1 s_1 \left\| \frac{\partial^3 u}{\partial t^3}(t_{k-2} + s_1\tau + s_2\tau) \right\|_{H^*} \|\varphi\|_1 ds_1 ds_2 \\
&\le \tau \|\varphi\|_1 \sqrt{\int_0^1 \int_0^1 s_1^2 ds_1 ds_2} \sqrt{\int_0^1 \int_0^1 \left\| \frac{\partial^3 u}{\partial t^3}(t_{k-2} + s_1\tau + s_2\tau) \right\|_{H^*}^2 ds_1 ds_2} \\
&\le \tau \|\varphi\|_1 \sqrt{\frac{1}{3}} \sqrt{\int_{-1}^1 \int_0^2 \left\| \frac{\partial^3 u}{\partial t^3}(t_{k-2} + \tilde{s}_1\tau) \right\|_{H^*}^2 \frac{1}{2} d\tilde{s}_1 d\tilde{s}_2} \\
&= \sqrt{\frac{\tau}{3}} \|\varphi\|_1 \left\| \frac{\partial^3 u}{\partial t^3} \right\|_{L^2(t_{k-2}, t_k; H^*)},
\end{aligned}$$

where we have used the coordinate transformation $(s_1, s_2) \mapsto (\tilde{s}_1, \tilde{s}_2) := (s_1 + s_2, -s_1 + s_2)$. Therefore, we obtain

$$\sum_{k=2}^N \tau \|D_\tau R_k\|_{H^*}^2 \le \sum_{k=2}^N \tau \frac{\tau}{3} \left\| \frac{\partial^3 u}{\partial t^3} \right\|_{L^2(t_{k-2}, t_k; H^*)}^2 \le \frac{2}{3} \tau^2 \left\| \frac{\partial^3 u}{\partial t^3} \right\|_{L^2(H^*)}^2.$$

□

Proof of Lemma 3.3. It holds that for all $k = 1, 2, \ldots, N$ and $t \in [t_{k-1}, t_k]$,

$$\|x(t_k) - x(t)\|_E \le \int_t^{t_k} \left\| \frac{\partial x}{\partial t}(s) \right\|_E ds \le \sqrt{t_k - t} \left\| \frac{\partial x}{\partial t} \right\|_{L^2(t_{k-1}, t_k; E)},$$

which implies that $\|x - x_\tau\|_{L^\infty(E)} \le \sqrt{\tau} \left\| \frac{\partial x}{\partial t} \right\|_{L^2(E)}$ and

$$\|D_\tau x(t_k)\|_E = \frac{1}{\tau} \|x(t_k) - x(t_{k-1})\|_E \le \frac{1}{\sqrt{\tau}} \left\| \frac{\partial x}{\partial t} \right\|_{L^2(t_{k-1}, t_k; E)}.$$

On the other hand, we have

$$\begin{aligned}
\|x - x_\tau\|_{L^2(E)}^2 &= \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \|x(t) - x(t_k)\|_E^2 dt \\
&\le \sum_{k=1}^N \int_{t_{k-1}}^{t_k} (t_k - t) dt \left\| \frac{\partial x}{\partial t} \right\|_{L^2(t_{k-1}, t_k; E)}^2 = \frac{1}{2} \tau^2 \left\| \frac{\partial x}{\partial t} \right\|_{L^2(E)}^2.
\end{aligned}$$

□

We prove Corollary 2.24 by using the boundedness from Theorem 2.15 and the Aubin–Lions compactness lemma.

Proof of Corollary 2.24. By the first and third equations of (PM), it holds that for all $v \in V$ and $k = 1, 2, \ldots, N$,

$$(D_\tau u_k, v) + a(u_k^*, v) + (g_k, v) + (h_k, \nabla v) = \langle f_k, v \rangle_H - (\nabla P_k, v) = \langle f_k, v \rangle_H - \int_{\Gamma_2} p_k^b v \cdot n ds,$$

where g_k and h_k are defined[3] by

$$g_k := (\nabla u_k^*)^T u_{k-1}^* - u_{k-1}^* \mathrm{div}\, u_k^*, \qquad h_k := -u_k^* (u_{k-1}^*)^T,$$

which implies that for all $v \in V$ and $\theta \in C_0^\infty(0,T)$,

$$\int_0^T \left(\left(\frac{\partial \hat{u}_\tau}{\partial t}, v \right) + a(\bar{u}_\tau^*, v) + (\bar{g}_\tau, v) + (\bar{h}_\tau, \nabla v) \right) \theta dt = \int_0^T \left(\langle \bar{f}_\tau, v \rangle_H - \int_{\Gamma_2} \bar{p}_\tau^b v \cdot n ds \right) \theta dt. \tag{A.1}$$

Here, $\bar{f}_\tau \to f$ strongly in $L^2(H^*)$ and $\bar{p}_\tau^b \to p^b$ strongly in $L^2(H^1(\Omega))$ as $\tau \to 0$. By Theorem 2.15 and Lemma 3.1, there exists a constant $c_1 > 0$ such that

$$\|\bar{u}_\tau\|_{L^\infty(L^2)}^2 + \|\bar{u}_\tau^*\|_{L^\infty(L^2)}^2 + \|\bar{u}_k^*\|_{L^2(H^1)}^2 + \tau \left\| \frac{\partial \hat{u}_\tau}{\partial t} \right\|_{L^2(L^2)}^2 + \frac{1}{\tau} \|\bar{u}_\tau - \bar{u}_\tau^*\|_{L^2(L^2)}^2 \le c_1. \tag{A.2}$$

In particular, it holds that

$$\|u_1^*\|_0^2 + \tau \|u_1^*\|_1^2 + \|u_1 - u_0\|_0^2 + \|u_1 - u_1^*\|_0^2 \le c_1, \tag{A.3}$$

which implies that $\|u_1^* - u_0\|_0 \le \|u_1^* - u_1\|_0 + \|u_1 - u_0\|_0 \le 2\sqrt{c_1}$. Furthermore, by the first equation of (PM) and Lemmas 2.3, 2.7, we have

$$\begin{aligned} \|u_1^* - u_0\|_{V^*} &= \sup_{0 \ne v \in H} \frac{\tau}{\|v\|_1} \left| -a(u_1^*, v) - d(u_0, u_1^*, v) + \langle f_1, v \rangle_H \right| \\ &\le c_a \tau \|u_1^*\|_1 + c_d \tau \|u_0\|_{L^{p_d}} \|u^*\|_1 + \tau \|f_1\|_{H^*} \le c_2 \sqrt{\tau}. \end{aligned} \tag{A.4}$$

where $c_2 := \sqrt{c_1}(c_a + c_d \|u_0\|_{L^{p_d}}) + \|f\|_{L^2(H^*)}$. Let $u_0^\circ := u_1^*$, $u_k^\circ := u_k^*$ for all $k = 1, 2, \ldots, N$ and let $\hat{u}_\tau^\circ$ be the piecewise linear interpolant of $(u_k^\circ)_{k=0}^N \subset H$.

From the uniform estimates (A.2), one can show that there exist a sequence $(\tau_k)_{k \in \mathbb{N}}$ and three functions $u \in L^2(H) \cap L^\infty(L^2(\Omega)^d) \cap W^{1,4/p_d}(V^*)$ (in particular, $u \in C([0,T]; V^*)$), $g \in L^{4/p_d}(L^{\tilde{p}_d}(\Omega)^d)$ and $h \in L^{4/p_d}(L^2(\Omega)^{d \times d})$ such that $\tau_k \to 0$ and

$$\begin{aligned} \bar{u}_{\tau_k}^* &\to u \quad \text{weakly in } L^2(H), && \text{(A.5)} \\ & \qquad\ \ \text{strongly in } L^2(L^2(\Omega)^d), && \text{(A.6)} \\ \hat{u}_{\tau_k}^\circ &\to u \quad \text{strongly in } L^2(L^2(\Omega)^d), && \text{(A.7)} \\ & \qquad\ \ \text{strongly in } C([0,T]; V^*), && \text{(A.8)} \\ \hat{u}_{\tau_k} &\to u \quad \text{strongly in } L^2(L^2(\Omega)^d), && \text{(A.9)} \\ & \qquad\ \ \text{weakly in } W^{1,4/p_d}(V^*), && \text{(A.10)} \\ \bar{g}_{\tau_k} &\rightharpoonup g \quad \text{weakly in } L^{4/p_d}(L^{\tilde{p}_d}(\Omega)^d), && \text{(A.11)} \\ \bar{h}_{\tau_k} &\rightharpoonup h \quad \text{weakly in } L^{4/p_d}(L^2(\Omega)^{d \times d}), && \text{(A.12)} \end{aligned}$$

as $k \to \infty$. Here, we note that $\bar{u}_{\tau_k}^*$, $\hat{u}_{\tau_k}^\circ$ and $\hat{u}_{\tau_k}$ possess a common limit function. Indeed, the weak convergence (A.5) of $\bar{u}_\tau^*$ immediately follows from the uniform estimates (A.2). Since we have $1/\tilde{p}_d = 1/2 + 1/p_d$, $p_d/4 = 1/2 + p_d/(2\tilde{q}_d)$, and

$$\|\bar{u}_\tau^*\|_{L^{2\tilde{q}_d/p_d}(L^{p_d})} \le \|\bar{u}_\tau^*\|_{L^2(L^{\tilde{q}_d})}^{p_d/\tilde{q}_d} \|\bar{u}_\tau^*\|_{L^\infty(L^2)}^{1-p_d/\tilde{q}_d} \le c_3 \|\bar{u}_\tau^*\|_{L^2(H^1)}^{p_d/\tilde{q}_d} \|\bar{u}_\tau^*\|_{L^\infty(L^2)}^{1-p_d/\tilde{q}_d}$$

---

[3] Here, it holds that for all $i, j = 1, \ldots, d$ and $k = 1, 2, \ldots, N$,

$$(g_k)_i := \sum_{l=1}^d \frac{\partial (u_k^*)_l}{\partial x_i} (u_{k-1}^*)_l - (u_{k-1}^*)_i \mathrm{div}\, u_k^*, \qquad (h_k)_{ij} := -(u_k^*)_i (u_{k-1}^*)_j.$$

for a constant $c_3 > 0$ (cf. [9, Theorem II.5.5])[4], it holds that

$$\|\bar{g}_\tau\|_{L^{4/p_d}(L^{\tilde{p}_d})} \le \left\{ \tau \sum_{k=1}^{N} \left(\|\nabla u_k^*\|_{L^2} \|u_{k-1}^*\|_{L^{p_d}} + \|u_{k-1}^*\|_{L^{p_d}} \|\operatorname{div} u_k^*\|_{L^2} \right)^{4/p_d} \right\}^{p_d/4}$$

$$\le c_4 \left(\tau \sum_{k=1}^{N} \|u_k^*\|_1^{4/p_d} \|u_{k-1}^*\|_{L^{p_d}}^{4/p_d} \right)^{p_d/4} \le c_4 \left(\tau \sum_{k=1}^{N} \|u_k^*\|_1^2 \right)^{1/2} \left(\tau \sum_{k=1}^{N} \|u_{k-1}^*\|_{L^{p_d}}^{2\tilde{q}_d/p_d} \right)^{p_d/(2\tilde{q}_d)}$$

$$\le c_4 \left(\|\bar{u}_\tau^*\|_{L^2(H^1)}^2 + \|\bar{u}_\tau^*\|_{L^{2\tilde{q}_d/p_d}(L^{p_d})}^2 + \tau^{p_d/\tilde{q}_d} \|u_0\|_{L^{p_d}}^2 \right)$$

$$\le c_4 \left(\|\bar{u}_\tau^*\|_{L^2(H^1)}^2 + c_3 \|\bar{u}_\tau^*\|_{L^2(H^1)}^{2p_d/\tilde{q}_d} \|\bar{u}_\tau^*\|_{L^\infty(L^2)}^{2-2p_d/\tilde{q}_d} + \tau^{p_d/\tilde{q}_d} \|u_0\|_{L^{p_d}}^2 \right),$$

$$\|\bar{h}_\tau\|_{L^{4/p_d}(L^2)} \le \left(\tau \sum_{k=1}^{N} \|u_k^*\|_{L^{\tilde{q}_d}}^{4/p_d} \|u_{k-1}^*\|_{L^{p_d}}^{4/p_d} \right)^{p_d/4} \le c_5 \left(\tau \sum_{k=1}^{N} \|u_k^*\|_1^{4/p_d} \|u_{k-1}^*\|_{L^{p_d}}^{4/p_d} \right)^{p_d/4}$$

$$\le c_5 \left(\|\bar{u}_\tau^*\|_{L^2(H^1)}^2 + c_3 \|\bar{u}_\tau^*\|_{L^2(H^1)}^{2p_d/\tilde{q}_d} \|\bar{u}_\tau^*\|_{L^\infty(L^2)}^{2-2p_d/\tilde{q}_d} + \tau^{p_d/\tilde{q}_d} \|u_0\|_{L^{p_d}}^2 \right)$$

for constants c_4 and c_5. Hence, by (A.2), the weak convergences (A.11) and (A.12) hold. Moreover, since there exists a constant $c_6 > 0$ such that $|(g_k, v)| \le \|g_k\|_{L^{\tilde{p}_d}} \|v\|_{L^{\tilde{q}_d}} \le c_6 \|g_k\|_{L^{\tilde{p}_d}} \|v\|_1$ for all $k = 1, 2, \dots, N$ and $v \in H^1(\Omega)^d$, we have

$$\left\| \frac{\partial \hat{u}_\tau}{\partial t} \right\|_{L^{4/p_d}(V^*)} = \left\{ \int_0^T \left(\sup_{0 \ne v \in V} \frac{1}{\|v\|_1} \left| -a(\bar{u}_\tau^*(t), v) - (\bar{g}_\tau(t), v) - (\bar{h}_\tau(t), \nabla v) \right.\right.\right.$$

$$\left.\left.\left. + \langle f_\tau(t), v \rangle_H - \int_{\Gamma_2} p_\tau^b(t) v \cdot n ds \right| \right)^{4/p_d} dt \right\}^{p_d/4}$$

$$= \left\{ \int_0^T \left(c_a \|\bar{u}_\tau^*(t)\|_1 + c_6 \|\bar{g}_\tau(t)\|_{L^{\tilde{p}_d}} + \|\bar{h}_\tau(t)\|_0 + \|f_\tau(t)\|_{H^*} + \|p_\tau^b(t)\|_1 \right)^{4/p_d} dt \right\}^{p_d/4}$$

$$\le T^{p_d/(2\tilde{q}_d)} (c_a \sqrt{c_1} + \|f\|_{L^2(H^*)} + \|p^b\|_{L^2(H^1)}) + c_6 \|\bar{g}_\tau\|_{L^{4/p_d}(L^{\tilde{p}_d})} + \|\bar{h}_\tau\|_{L^{4/p_d}(L^2)},$$

$$\left\| \frac{\partial \hat{u}_\tau^\circ}{\partial t} - \frac{\partial \hat{u}_\tau}{\partial t} \right\|_{L^{4/p_d}(V^*)} = \left\{ \tau \sum_{k=1}^{N} \left(\sup_{0 \ne v \in V} \frac{|(u_k^\circ - u_{k-1}^\circ - u_k + u_{k-1}, v)|}{\tau \|v\|_1} \right)^{4/p_d} \right\}^{p_d/4}$$

$$\le 2 \left(\tau \sum_{k=1}^{N} \sup_{0 \ne v \in V} \frac{|(\nabla P_k, v)|^{4/p_d}}{\|v\|_1^{4/p_d}} + \tau \|u_1^* - u_0\|_{V^*}^{4/p_d} \right)^{p_d/4}$$

$$\le 2 \left(\tau \sum_{k=1}^{N} \sup_{0 \ne v \in V} \frac{1}{\|v\|_1^{4/p_d}} \left| \int_{\Gamma_2} p^b(t_k) v \cdot n ds \right|^{4/p_d} \right)^{p_d/4} + 2\tau^{p_d/4} \|u_1^* - u_0\|_{V^*}$$

$$\le 2 \left(\tau \sum_{k=1}^{N} \|p^b(t_k)\|_1^{4/p_d} \right)^{p_d/4} + 2c_2 \tau^{p_d/4 + 1/2} \le 2T^{p_d/(2\tilde{q}_d)} (\|p^b\|_{L^2(H^1)} + c_2 T),$$

and $\|\frac{\partial \hat{u}_\tau}{\partial t}\|_{L^{4/p_d}(V^*)}$ and $\|\frac{\partial \hat{u}_\tau^\circ}{\partial t}\|_{L^{4/p_d}(V^*)}$ are also bounded. Hence, (A.10) holds. Furthermore, $\|\hat{u}_\tau^\circ\|_{L^2(H^1)}$ is bounded: by (A.2) and (A.3),

$$\|\hat{u}_\tau^\circ\|_{L^2(H^1)}^2 = \sum_{k=1}^{N} \int_0^1 \|(1-s) u_{k-1}^\circ + s u_k^\circ\|_1^2 \tau ds$$

$$\le \sum_{k=1}^{N} \tau (\|u_{k-1}^\circ\|_1^2 + \|u_k^\circ\|_1^2) \int_0^1 \{(1-s)^2 + s^2\} ds \le \frac{4}{3} \|\bar{u}_\tau^*\|_{L^2(H^1)}^2 + \frac{2c_1}{3} \le 2c_1,$$

[4]Since $p_2 = 2 + \varepsilon$ and $p_3 = 3$, we have $p_2/\tilde{q}_2 = \varepsilon/2$ and $p_3/\tilde{q}_3 = 1/2$.

which implies the strong convergence (A.7) of $\hat{u}_\tau^\circ$ in $L^2(L^2(\Omega)^d)$ from the Aubin–Lions lemma [9, Theorem II.5.16 (i)]. Since we have for all $t \in (t_{k-1}, t_k), k = 1, 2, \ldots, N$,

$$\|\bar{u}_\tau^*(t) - \hat{u}_\tau^\circ(t)\|_0 = \left|\frac{t_k - t}{\tau}\right| \|u_k^\circ - u_{k-1}^\circ\|_0 \leq \|u_k^\circ - u_k\|_0 + \tau\|D_\tau u_k\|_0 + \|u_{k-1} - u_{k-1}^\circ\|_0,$$
$$\|\bar{u}_\tau^*(t) - \hat{u}_\tau(t)\|_0 \leq \|\bar{u}_\tau^*(t) - \bar{u}_\tau(t)\|_0 + \|\bar{u}_\tau(t) - \hat{u}_\tau(t)\|_0 \leq \|u_k^* - u_k\|_0 + \tau\|D_\tau u_k\|_0,$$

the functions $\bar{u}_{\tau_k}^*$, $\hat{u}_{\tau_k}^\circ$ and $\hat{u}_{\tau_k}$ possess a common limit function u, and the strong convergences (A.6) and (A.9) hold: by (A.2) and (A.3),

$$\begin{aligned}\|\bar{u}_\tau^* - \hat{u}_\tau^\circ\|_{L^2(L^2)} &\leq \left(\tau\sum_{k=1}^N (\|u_k^\circ - u_k\|_0 + \tau\|D_\tau u_k\|_0 + \|u_{k-1} - u_{k-1}^\circ\|_0)^2\right)^{1/2} \\ &\leq 2\sqrt{3}\|\bar{u}_\tau^* - \bar{u}_\tau\|_{L^2(L^2)} + \sqrt{3}\tau\left\|\frac{\partial \hat{u}_\tau}{\partial t}\right\|_{L^2(L^2)} + \sqrt{3\tau}\|u_0 - u_1^*\|_0 \leq 5\sqrt{3c_1\tau},\end{aligned}$$

$$\|\bar{u}_\tau^* - \hat{u}_\tau\|_{L^2(L^2)} \leq \sqrt{2}\|\bar{u}_\tau^* - \bar{u}_\tau\|_{L^2(L^2)} + \sqrt{2}\tau\left\|\frac{\partial \hat{u}_\tau}{\partial t}\right\|_{L^2(L^2)} \leq 2\sqrt{2c_1\tau}.$$

It also holds that

$$\|\bar{u}_\tau^* - \hat{u}_\tau^\circ\|_{L^\infty(L^2)} \leq \max_{k=1,2,\ldots,N}(\|u_k^\circ\|_0 + \|u_{k-1}^\circ\|_0) \leq 2\sqrt{c_1}.$$

Since $\|\hat{u}_\tau^\circ\|_{L^\infty(L^2)}$ and $\|\frac{\partial \hat{u}_\tau^\circ}{\partial t}\|_{L^{4/p_d}(V^*)}$ are bounded, we obtain the strong convergence (A.8) of $\hat{u}_\tau^\circ$ in $C([0,T];V^*)$ [9, Theorem II.5.16 (ii)]. In particular, $\hat{u}_\tau^\circ(0)$ converges to $u(0)$ in V^*. On the other hand, by (A.4), $\hat{u}_\tau^\circ(0) = u_1^*$ converges to u_0 in V^*. Through the uniqueness of the limit in V^*, we have indeed obtained that $u(0) = u_0$.

From (A.1) with $\varepsilon := \varepsilon_k$, taking $k \to \infty$, it holds that for all $v \in V$ and $\theta \in C_0^\infty(0,T)$,

$$\int_0^T \left(\left\langle \frac{\partial u}{\partial t}, \theta v\right\rangle_V + a(u, \theta v) + (g, \theta v) + (h, \nabla(\theta v))\right) dt = \int_0^T \left(\langle f, \theta v\rangle_H - \int_{\Gamma_2} p^b \theta v \cdot n ds\right) dt.$$

Next, we show that

$$g = (\nabla u)^T u - u \operatorname{div} u, \qquad h = -u(u)^T. \tag{A.13}$$

We set $\bar{v}_\tau(t) := u_{k-1}^*$ for $t \in (t_{k-1}, t_k], k = 1, 2, \ldots, N$. Then it holds that

$$\|\bar{v}_\tau - \bar{u}_\tau^*\|_{L^2(L^2)} \leq \left(\tau\sum_{k=1}^N (\|u_k^* - u_k\|_0 + \tau\|D_\tau u_k\|_0 + \|u_{k-1} - u_{k-1}^*\|_0)^2\right)^{1/2} \leq 3\sqrt{3c_1\tau},$$

and hence it follows from (A.6) that $\bar{v}_{\tau_k} \to u$ strongly in $L^2(L^2(\Omega)^d)$ as $k \to \infty$. Since $\nabla\bar{u}_\tau^* \rightharpoonup \nabla u$ weakly in $L^2(L^2(\Omega)^{d\times d})$ and $\operatorname{div}\bar{u}_\tau^* \rightharpoonup \operatorname{div} u$ weakly in $L^2(L^2(\Omega))$ as $k \to \infty$, we have

$$\begin{aligned}\bar{g}_\tau = (\nabla\bar{u}_\tau^*)^T\bar{v}_\tau - \bar{v}_\tau \operatorname{div}\bar{u}_\tau^* &\rightharpoonup (\nabla u)^T u - u \operatorname{div} u \quad \text{weakly in } L^1(L^1(\Omega)^d), \\ \bar{h}_\tau = -\bar{u}_\tau^*(\bar{v}_\tau)^T &\to -u(u)^T \quad \text{strongly in } L^1(L^1(\Omega)^{d\times d})\end{aligned}$$

as $k \to \infty$ (cf. [9, Proposition II.2.12]). On the other hand, we also know (A.11) and (A.12). The convergence in these spaces imply the convergence in the distributions sense, therefore (A.13) holds by the uniqueness of the limit in $\mathcal{D}'((0,T)\times\Omega)$. Hence, it holds that for all $v \in V$ and $\theta \in C_0^\infty(0,T)$,

$$\begin{aligned}&\int_0^T \left(\left\langle \frac{\partial u}{\partial t}, v\right\rangle_V + a(u, v) + ((\nabla u)u - u \operatorname{div} u, v) - (u(u)^T, \nabla v)\right)\theta dt \\ &= \int_0^T \left(\langle f, v\rangle_H - \int_{\Gamma_2} p^b v \cdot n ds\right)\theta dt,\end{aligned}$$

which is equivalent to the following

$$\int_0^T \left(\left\langle \frac{\partial u}{\partial t}, v \right\rangle_V + a(u,v) + d(u,u,v) \right) \theta dt = \int_0^T \left(\langle f, v \rangle_H - \int_{\Gamma_2} p^b v \cdot n ds \right) \theta dt.$$

$\square$

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