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CORRECTION

Correction: Random reordering in SOR-type methods

Peter Oswald[1] · **Weiqi Zhou**[2]

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Abstract

My joint paper (Numerische Mathematik 135:1207–1220, 2017. https://doi.org/10.1007/s00211-016-0829-7) with W. Zhou contains two errors which concern the derivation of some auxiliary norm estimates of the lower triangular projection for positive semi-definite Hermitean matrices in dependence on coordinate permutations. These errors are corrected. The main results of Oswald and Zhou (Numerische Mathematik 135:1207–1220, 2017. https://doi.org/10.1007/s00211-016-0829-7) about the convergence behavior of so-called shuffled and preshuffled SOR iterations are not affected.

Keywords SOR method · Kaczmarz method · Random ordering · Triangular truncation · Convergence estimates

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1 Introduction

In [1], the influence of random equation ordering in a linear system $By = b$ on deriving upper bounds for the convergence speed of the classical successive over-relaxation (SOR) method

$$y^{(k+1)} = y^{(k)} + \omega(D + \omega L)^{-1}(b - By^{(k)}), \qquad k = 0, 1, \ldots, \tag{1}$$

The original article can be found online at https://doi.org/10.1007/s00211-016-0829-7.

✉ Peter Oswald
agp.oswald@gmail.com

Weiqi Zhou
weiqizhou@mathematik.uni-marburg.de

1 Institute for Numerical Simulation, University of Bonn, Bonn, Germany

2 FB Mathematics and Informatics, Philipps-University Marburg, Marburg, Germany

was studied. For simplicity, it is assumed that the system is consistent with solution y and that B is a complex $n \times n$ Hermitian positive semi-definite matrix with positive diagonal part D and strictly lower triangular part L. Two strategies of involving permutations of the system were considered. For the so-called *shuffled SOR iteration*, in each step the SOR update formula (1) is applied to a uniformly at random and independently chosen permutation of the linear system, while for the *preshuffled SOR iteration* the iteration (1) is performed for $k = 0, 1, \ldots$ after a single permutation of the system at the beginning. To study the convergence properties of such iterations, 2-norm estimates involving the lower triangular part L_σ of the permuted matrix $B_\sigma = P_\sigma B P_\sigma^*$ played a crucial role, where P_σ denotes the permutation matrix associated with the permutation σ acting according to $(P_\sigma y)_i = y_{\sigma_i}$, $i = 1, \ldots, n$.

The necessary properties were formulated as Theorem 2 and 3 in [1]. Unfortunately, the proof of Theorem 2 uses a wrong formula for $P_\sigma^* L_\sigma L_\sigma^* P_\sigma$. This was pointed out by T. Yilmaz in [2]. In the next section, a proof of Theorem 2 based on the correct formula is provided, including a slight improvement of the involved constants.

Even though the proof of Theorem 3 in [1] is correct, we use the opportunity to give new estimates for the absolute constant in the inequality

$$\inf_\sigma \|L_\sigma\| \le C\|B\|. \tag{2}$$

For the class of all Hermitean matrices (not necessarily positive-definite) we show that taking $C = 245$ is feasible which considerably improves the value $C = C_2 = 2905$ stated in [1]. For positive semi-definite B, one can even take $C = 122{,}3$. These bounds are consequences of recent quantitative improvements of the Anderson paving conjecture and will be shown in Sect. 3. The derivation in [1] that (2) holds for positive semi-definite B with unit diagonal with the smaller value $C = C_1 = 32{,}42$ is based on an flawed application of earlier results on the size of one-sided pavings and thus is not correct. It remains an interesting open question to find more precise bounds for the constant C in (2). For the class of positive semi-definite B with unit diagonal I conjecture that $C = 2/\pi$ is the best possible choice.

2 Correct statement and proof of Theorem 2 from [1]

Theorem 2 in [1] concerns the 2-norm estimate of the matrix

$$E := \frac{1}{n!}\sum_\sigma P_\sigma^* L_\sigma L_\sigma^* P_\sigma, \tag{3}$$

which plays a cruical role in the estimates of the expected squared error of the shuffled SOR iteration, see Theorem 4 a) there. As was mentioned above, its proof uses a wrong formula for the entries of $P_\sigma^* L_\sigma L_\sigma^* P_\sigma$. The correct formula [2] is

$$(P_\sigma^* L_\sigma L_\sigma^* P_\sigma)_{s,t} = \sum_{k=1}^{\min(\sigma_s^{-1},\sigma_t^{-1})-1} H_{s,\sigma_k} H_{\sigma_k,t}, \tag{4}$$

where $H_{i,j}$ are the entries of the Hermitean matrix $H = B - D$, the non-diagonal part of B, and σ^{-1} is the permutation inverse to σ.

To see (4), recall that

$$(L_\sigma)_{i,k} = \begin{cases} H_{\sigma_i,\sigma_k}, & k < i, \\ 0, & k \geq i, \end{cases} \qquad (L^*_\sigma)_{k,j} = (L_\sigma)^*_{j,k} = \begin{cases} H_{\sigma_k,\sigma_j}, & k < j, \\ 0, & k \geq j. \end{cases}$$

Consequently,

$$(L_\sigma L^*_\sigma)_{i,j} = \sum_{k=1}^{\min(i,j)-1} H_{\sigma_i,\sigma_k} H_{\sigma_k,\sigma_j}$$

and, by setting $i = \sigma_s^{-1}$, $j = \sigma_t^{-1}$, we arrive at (4).

Based on (4), we next derive a formula for E in terms of the Hermitean positive-definite matrix H^2, namely,

$$E = \frac{1}{3} H^2 + \frac{1}{6} D_{H^2}, \tag{5}$$

where D_{H^2} is the diagonal part of H^2. Indeed, from (3) and (4) we have

$$\begin{aligned} n! \cdot E_{s,t} &= \sum_\sigma \sum_{k=1}^{\min(\sigma_s^{-1},\sigma_t^{-1})-1} H_{s,\sigma_k} H_{\sigma_k,t} \\ &= \sum_{m=1}^{n} H_{s,m} H_{m,t} \cdot n_{m;s,t}, \end{aligned}$$

where $n_{m;s,t}$ stands for the cardinality of the set of all permutations σ such that, for some $k < \min(\sigma_s^{-1}, \sigma_t^{-1})$, we have $\sigma_k = m$. Equivalently, this is the cardinality of the set of all σ such that $\sigma_m^{-1} < \min(\sigma_s^{-1}, \sigma_t^{-1})$. It is not hard to see that these cardinalities equal

$$n_{m;s,s} = \frac{1}{2} n!, \quad m \neq s, \qquad n_{m;s,t} = \frac{1}{3} n!, \quad m \neq s \neq t \neq m.$$

Indeed, for the case $m \neq t = s$, any σ in the associated set is obtained by first choosing two indices k, i with $k < i$ from $\{1, \ldots, n\}$ and setting $\sigma_m^{-1} = k$, $\sigma_s^{-1} = i$ (this is possible in $n(n-1)/2$ different ways) and then independently assigning the remaining $n-2$ indices arbitrarily (this is possible in $(n-2)!$ different ways). A similar reasoning applies to the case $m \neq s \neq t \neq m$, where one starts with a subset of 3 different indices k, i, j with $k < i < j$, sets

$$\sigma_m^{-1} = k, \quad \sigma_s^{-1} = i, \quad \sigma_t^{-1} = j,$$

or alternatively

$$\sigma_m^{-1} = k, \quad \sigma_t^{-1} = i, \quad \sigma_s^{-1} = j$$

(altogether $n(n-1)(n-2)/3$ different possibilities) and assigns the remaining $n-3$ indices arbitrarily ($(n-3)!$ different possibilities). For index constellations, where $m = s$ or $m = t$, one obviously has $n_{m;s,t} = 0$. With this, we arrive for $s = t$ at

$$E_{s,s} = \frac{1}{2}\sum_{m\neq s} H_{s,m}H_{m,s} = \frac{1}{2}\sum_{m=1}^{n} H_{s,m}H_{m,s} = \frac{1}{2}(H^2)_{s,s},$$

since $H_{m,m} = 0$. Similarly, for $s \neq t$ we have

$$E_{s,t} = \frac{1}{3}\sum_{m\neq s, m\neq t} H_{s,m}H_{m,t} = \frac{1}{3}(H^2)_{s,t}.$$

This establishes the formula (5).

Note that, up to this point, the calculations hold for any Hermitean B. Since H^2 and its diagonal part D_{H^2} are automatically positive semi-definite, we thus get

$$\|E\| \leq \frac{1}{3}\|H^2\| + \frac{1}{6}\|D_{H^2}\| \leq \frac{1}{2}\|H^2\|.$$

Since $\|H\| \leq \|B\| + \|D\| \leq 2\|B\|$ for any B we also have

$$\|E\| \leq \frac{1}{2}\|H\|^2 \leq 2\|B\|^2$$

for all Hermitean B.

If the Hermitean B is positive semi-definite, then $H = B - D$ has norm $\|H\| \leq \|B\|$ since

$$-\|B\|\|x\|^2 \leq -(Dx, x) \leq (Hx, x) \leq (Bx, x) \leq \|B\|\|x\|^2.$$

Thus, in this case $\|E\| \leq \frac{1}{2}\|B\|^2$. If, in addition, B has unit diagonal (i.e., $D = I$) then slightly more precise bounds are possible. Indeed, then

$$\|H^2\| = \lambda_{\max}(H^2) = \max((\lambda_{\max}(B) - 1)^2, \lambda_{\min}(B) - 1)^2) \leq \max((\|B\| - 1)^2, 1).$$

In summary, we have proved the following replacement of Theorem 2 from [1].

Theorem *Let B be an arbitrary Hermitean matrix, and $H = B - D$ its non-diagonal part. Then the matrix E defined in (3) satisfies*

$$\|E\| \leq \frac{1}{2}\|H\|^2 \leq 2\|B\|^2.$$

If, in addition, B is positive semi-definite then

$$\|E\| \leq \frac{1}{2}\|B\|^2.$$

Compared to the statement of Theorem 2 in [1], the constants in these estimates are reduced by a factor of two which also leads to better constants in Theorem 4 a) in [1].

3 New constants in Theorem 3 from [1]

The proof of Theorem 3 in [1], i.e., the proof of (2) with a constant C independent of the size of B, is essentially based on the existence of so-called (k, ϵ)-pavings for Hermitean matrices with zero (or small) diagonal part such as $H = B - D$. We use a consequence of a recent refinement [3] of the original proof of the Anderson paving conjecture. If one carefully follows the proof of Theorem 1.1 in [3, Section 5.2] specialized to the pair $[H, -H]$ (in particular, if one uses the more precise bound at the end of the proof of Theorem 5.6 there) then one sees that for any $\epsilon \in (0, 1)$ there exists a (k, ϵ)-paving of H if $4k^{-1/2} + 2k^{-1} \leq \epsilon$. Equivalently, for any $k \geq 20$ there exists a (k, ϵ_k)-paving for H with

$$\epsilon_k := 4k^{-1/2} + 2k^{-1} < 1.$$

It was shown in the proof of Theorem 3 and in the remarks following it in [1] that this implies the estimate (2) with the constant

$$C = \min_{k \geq 20} \frac{k - 1}{1 - \epsilon_k} < 122{,}3$$

(the minimum is achieved for $k = 43$). Since $\|H\| \leq 2\|B\|$ for general Hermitean B and $\|H\| \leq \|B\|$ for positive semi-definite B, this yields the respective statements about the constant C in (2) in Sect. 1.

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