

# The Effect of Faults on Network Expansion

Amitabha Bagchi<sup>\*</sup>

Ankur Bhargava<sup>†</sup>

Amitabh Chaudhary<sup>\*</sup>

David Eppstein<sup>‡</sup>

Christian Scheideler<sup>§</sup>

## ABSTRACT

In this paper we study the problem of how resilient networks are to node faults. Specifically, we investigate the question of how many faults a network can sustain so that it still contains a large (i.e. linear-sized) connected component that still has approximately the same expansion as the original fault-free network. For this we apply a pruning technique which culls away parts of the faulty network which have poor expansion. This technique can be applied to both adversarial faults and to random faults. For adversarial faults we prove that for every network with expansion  $\alpha$ , a large connected component with basically the same expansion as the original network exists for up to a constant times  $\alpha \cdot n$  faults. This result is tight in the sense that every graph  $G$  of size  $n$  and uniform expansion  $\alpha(\cdot)$ , i.e.  $G$  has an expansion of  $\alpha(n)$  and every subgraph  $G'$  of size  $m$  of  $G$  has an expansion of  $O(\alpha(m))$ , can be broken into sublinear components with  $\omega(\alpha(n) \cdot n)$  faults.

For random faults we observe that the situation is significantly different, because in this case the expansion of a graph only gives a very weak bound on its resilience to random faults. More specifically, there are networks of uniform expansion  $O(\sqrt{n})$  that are resilient against a constant fault probability but there are also networks of uniform expansion  $\Omega(1/\log n)$  that are not resilient against a  $O(1/\log n)$  fault probability. Thus, a different parameter is needed. For this we introduce the *span* of a graph which allows us to determine the maximum fault probability in a much better way than the expansion can. We use the span to show the first known results for the effect of random faults on the expansion of

<sup>\*</sup>Dept. of Computer Science, University of California, Irvine, CA 92697. {bagchi, amic}@ics.uci.edu.

<sup>†</sup>Dept. of Computer Science, Johns Hopkins University, Baltimore, MD 21218. ankur@cs.jhu. Supported by NSF Grant CCR-0311321.

<sup>‡</sup>Dept. of Computer Science, University of California, Irvine, CA 92697. eppstein@ics.uci.edu. Supported by NSF grant CCR-9912338

<sup>§</sup>Dept. of Computer Science, Johns Hopkins University, Baltimore, MD 21218. scheideler@cs.jhu.edu. Supported by NSF grant CCR-0311121 and NSF grant CCR-0311795.

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$d$ -dimensional meshes.

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## 1. INTRODUCTION

Communication in faulty networks is a classical field in network theory. In practice, one cannot expect nodes or communication links to work without complications. Software or hardware faults (or phenomena outside the control of a network operator such as caterpillars) may cause nodes or links to go down. To be able to adapt to faults without a serious degradation of the service, networks and routing protocols have to be set up so that they are fault-tolerant. Fault-tolerant routing has recently attained renewed interest due to the tremendous rise in popularity of mobile ad-hoc networks and peer-to-peer networks. In these networks, faults are actually not an exception but a frequently occurring event: in mobile ad-hoc networks, users may run out of battery power or may move out of reach of others, and in peer-to-peer networks, users may leave without notice.

Central questions in the theoretical area of faulty networks have been:

- How many faults can a network sustain so that the size of its largest connected component is still a constant fraction of the original size?
- How many faults can a network sustain so that it can still emulate its ideal counterpart with constant slowdown?

The first question has been heavily studied in the graph theory community, and the second question has been investigated mostly by the parallel computing community to find out up to which point a faulty parallel computer can still emulate an ideal parallel computer with the same topology with constant slowdown. We refer the reader to [27] for a survey of results in these areas.

## 1.1 Large connected components in faulty networks

We start with an overview of previous results for random faults and afterwards consider adversarial faults.

Given a graph  $G$  and a probability value  $p$ , let  $G^{(p)}$  be the random graph obtained from  $G$  by keeping each edge of  $G$  alive with probability  $p$  (i.e.  $p$  is the *survival probability*). Given a graph  $G$ , let  $\gamma(G) \in [0, 1]$  be the fraction of nodes of  $G$  contained in a largest connected component.

Let  $\mathcal{G} = \{G_n \mid n \in \mathbb{N}\}$  be any family of graphs with parameter  $n$ . Let  $p^*$  be the *critical probability* for the existence of a linear-sized connected component. I.e. for every constant  $\epsilon > 0$  it holds:

1. For every  $p > (1 + \epsilon)p^*$  there exists a constant  $c > 0$  with  $\lim_{n \rightarrow \infty} \Pr[\gamma(G_n^{(p)}) > c] = 1$ .
2. For all constants  $c > 0$  and for all  $p < (1 - \epsilon)p^*$  it holds that  $\lim_{n \rightarrow \infty} \Pr[\gamma(G_n^{(p)}) > c] = 0$ .

Of course, it is not obvious whether critical probabilities exist. However, the results by Erdős and Rényi [10] and its subsequent improvements (e.g. [5, 21]) imply that for the complete graph on  $n$  nodes,  $p^* = 1/(n - 1)$ , and that for a random graph with  $d \cdot n/2$  edges,  $p^* = 1/d$ . For the 2-dimensional  $n \times n$ -mesh, Kesten showed that  $p^* = 1/2$  [16]. Ajtai, Komlós and Szemerédi proved that for the hypercube of dimension  $n$ ,  $p^* = 1/n$  [1]. For the  $n$ -dimensional butterfly network, Karlin, Nelson and Tamaki showed that  $0.337 < p^* < 0.436$  [15]. Leighton and Maggs [17] showed that there is an indirect constant-degree network connecting  $n$  inputs with  $n$  outputs via  $\log n$  levels of  $n$  nodes each, called multi-butterfly, that has the following property: Up to a constant fault probability it is still possible to find  $O(\log n)$  length paths from a constant fraction of the inputs to a constant fraction of the outputs. Subsequently Cole, Maggs and Sitaraman [6] extended this result for the butterfly.

Adversarial fault models have also been investigated. Leighton and Maggs [17] also showed that no matter how an adversary chooses  $f$  nodes to fail, there will be a connected component left in the multi-butterfly with at least  $n - O(f)$  inputs and at least  $n - O(f)$  outputs. (In fact, one can even still route packets between the inputs and outputs in this component in almost the same amount of time steps as in the ideal case.) Subsequently Leighton, Maggs and Sitaraman [19] extended this result for the butterfly.

Upfal [28], following up on work by Dwork et. al. [9] and Alon and Chung [2], showed that there is also a direct constant-degree network on  $n$  nodes, a so-called expander, that has the property: no matter how an adversary chooses  $f$  nodes to fail, there will be a connected component left in it with at least  $n - O(f)$  nodes. Both results are optimal up to constants. Upfal uses a pruning technique to achieve his bound which is similar in spirit to the one we use. Apart from the fact that Upfal gives a polynomial-time algorithm for pruning while we do not, the important difference worth noting is that Upfal's pruning does not guarantee a large component of good expansion. In fact, to the best of our knowledge there is no known constant approximation algorithm to determine the expansion of a graph of unknown topology.

## 1.2 Simulation of fault-free networks by faulty networks

Next we look at the problem of simulating fault-free networks by faulty networks. Consider the situation that there can be up to  $f$  worst-case node faults in the system at any time. One way to check whether the largest remaining component still allows efficient communication is to check whether it is possible to embed

into the largest connected component of a faulty network a fault-free network of the same size and kind. An *embedding* of a graph  $G$  into a graph  $H$  maps the nodes of  $G$  to non-faulty nodes of  $H$  and the edges of  $G$  to non-faulty paths in  $H$ . An embedding is called *static* if the mapping of the nodes and edges is fixed. Both static and dynamic embeddings have been used. A good embedding is one with minimum load, congestion, and dilation, where the *load* of an embedding is the maximum number of nodes of  $G$  that are mapped to any single node of  $H$ , the *congestion* of an embedding is the maximum number of paths that pass through any edge  $e$  of  $H$ , and the *dilation* of an embedding is the length of the longest path. The load, congestion, and dilation of the embedding determine the time required to emulate each step of  $G$  on  $H$ . In fact, Leighton, Maggs, and Rao have shown [18] that if there is an embedding of  $G$  into  $H$  with load  $\ell$ , congestion  $c$ , and dilation  $d$ , then  $H$  can emulate any communication step (and also computation step) on  $G$  with slowdown  $O(\ell + c + d)$ .

When demanding a constant slowdown, only a few results are known so far. In the case of worst-case faults, it was shown by Leighton, Maggs and Sitaraman (using dynamic embedding strategies) that an  $n$ -input butterfly with  $n^{1-\epsilon}$  worst-case faults (for any constant  $\epsilon$ ) can still emulate a fault-free butterfly of the same size with only constant slowdown [19]. Furthermore, Cole, Maggs and Sitaraman showed that an  $n \times n$  mesh can sustain up to  $n^{1-\epsilon}$  worst-case faults and still emulate a fault-free mesh of the same size with (amortized) constant slowdown [7]. It seems that also the  $n$ -node hypercube can even achieve a constant slowdown for  $n^{1-\epsilon}$  worst-case faults, but so far only partial answers have been obtained [19].

Random faults have also been studied. For example, Håstad, Leighton and Newman [12] showed that if each edge of the hypercube fails independently with any constant probability  $p < 1$ , then the functioning parts of the hypercube can be reconfigured to simulate the original hypercube with constant slowdown. Leighton, Maggs and Sitaraman [19] showed that a butterfly network whose nodes fail with some constant probability  $p$  can still emulate a fault-free butterfly of the same size with slowdown  $2^{O(\log^* n)}$ . Interestingly, in the conference version of [7], Cole, Maggs and Sitaraman claim that an  $n \times n$  mesh in which each node is faulty independently with a constant fault probability is able to emulate a fault-free mesh with a constant slowdown [8]. The proof of this claim, which is stronger than the theorem we prove about the  $n \times n$  mesh in this paper, is omitted in [8] and has not appeared elsewhere to the best of our knowledge.

For a list of further references concerning embeddings of fault-free into faulty networks see the paper by Leighton, Maggs and Sitaraman [19].

## 1.3 Our approach

The two common approaches – connectivity and emulation of fault-free by faulty networks – are too extreme for many practical applications. Knowing how long a network is still connected may not be very useful, because in extreme cases (just a single line connects one half to the other) the speed of communication may be reduced to a crawl, making it useless for applications that need a fast interaction or a large bandwidth such as interactive gaming or video conferences. On the other hand, emulating a fault free network on a faulty network is like using a giant hammer to crack a lesser nut, so to speak. Emulation may not be needed when all we want is reduced congestion or good expansion.

Applications in ad-hoc networks or peer-to-peer systems usually do not care about how a network is connected, concerning themselves instead with whether it still provides sufficient bandwidth and ensures sufficiently small delays. In this scenario a more rele-

vant question is:

*How many faults can a network sustain so that it still contains a network of at least a constant fraction of its original size that still has approximately the same expansion?*

Knowing an answer to this question would have many useful consequences for distributed data management, routing, and distributed computing. Research on load balancing has shown that if the expansion basically stays the same, the ability of a network to balance single-commodity or multi-commodity load basically stays the same, and this ability can be exploited through simple local algorithms [11, 3]. Also, the ability of a network to route information is preserved because it is closely related to its expansion [26]. Furthermore, as long as the original network still has a large connected component of almost the same expansion, one can still achieve almost everywhere agreement which is an important prerequisite for fundamental primitives such as atomic broadcast, Byzantine agreement, and clock synchronization [9, 28, 4].

Many different fault models have been studied in the literature: faults may be permanent or transient, nodes and/or edges may break down, and faults may happen at random or may be caused by an adversary or attacker. The former faults are called *random faults*, and the latter faults are called *adversarial faults*. We will concentrate on situations in which there are static node faults, i.e. nodes either break down randomly or due to some adversary. For adversarial faults, we will consider the node expansion of a graph, and for random faults we will use the edge expansion of a graph.

Given a graph  $G = (V, E)$  and a subset  $U \subseteq V$ , the (*node*) *expansion* of  $U$  is defined as

$$\alpha(U) = \frac{|\Gamma(U)|}{|U|}$$

where  $\Gamma(U)$  is the set of nodes in  $V \setminus U$  that have an edge from  $U$  and  $|S|$  denotes the size of set  $S$ . The (*node*) *expansion* of  $G$  is defined as  $\alpha = \min_{U, |U| \leq |V|/2} \alpha(U)$ .

Similarly, the edge expansion of  $G$  is defined as:

$$\alpha_e = \min_{U \subseteq V} \left\{ \frac{|(U, V \setminus U)|}{\min\{|U|, |V \setminus U|\}} \right\}$$

where  $(U, V \setminus U)$  denotes the set of edges with one endpoint in  $U$  and the other in  $V \setminus U$ .

## 1.4 Our main results

### *Adversarial faults*

We give general upper and lower bounds for the number of node faults a graph can sustain so that it still has a large component with basically the same expansion, where the bounds are tight up to a constant factor. More specifically, we show that the number of adversarial node faults a graph with node expansion  $\alpha$  and  $n$  nodes can sustain, with only a constant factor decrease in its expansion, is a constant times  $\alpha \cdot n$ . For graphs  $G$  of size  $n$  and uniform expansion  $\alpha(\cdot)$ , i.e.  $G$  has an expansion of  $\alpha(n)$  and every subgraph  $G'$  of size  $m$  of  $G$  has an expansion of  $O(\alpha(m))$ , this result is best possible up to constant factors.

### *Random faults*

We also study random faults. Our main contribution here is to suggest a new parameter for their study, which may be of independent interest.

Consider a graph  $G = (V, E)$ . Let  $U \subseteq V$  be any subset of nodes.  $U$  is defined to be *compact* if and only if  $U$  and  $V \setminus U$  are connected in  $G$ . Let  $\mathcal{U}$  be the set of all compact sets of  $G$ . Let  $P(U)$  be the smallest tree in  $G$  which connects every node in  $\Gamma(U)$  (i.e. it essentially spans the boundary of  $U$ ). Note that the set of nodes in  $P(U)$  need not be from  $U$  alone or from  $V \setminus U$  alone. Then the *span* of a graph is defined as:

$$\sigma = \max_{U \in \mathcal{U}} \left\{ \frac{|P(U)|}{|\Gamma(U)|} \right\} \quad (1)$$

The span helps us characterize the resilience of the expansion to random faults. We show that a graph with maximum degree  $\delta$  and span  $\sigma$  can tolerate a fault probability up to a constant times  $\frac{1}{\delta\sigma}$  and still retain an expansion within a factor of  $\delta$  of its original expansion.

We also show that the  $d$ -dimensional meshes have constant span. The proof of this theorem is of independent value as it establishes an interesting property of the  $d$ -dimensional mesh: The boundary of any set of connected vertices in the  $d$ -dimensional mesh, whose complement is also connected, can be spanned by a tree of size at most twice the size of the boundary.

## 1.5 Outline of the paper

The rest of the paper is organized as follows: In Section 2 we consider adversarial faults, and in Section 3 we consider random faults. The paper ends in Section 4 with a discussion of how our results are related to previous research and some open problems.

## 2. ADVERSARIAL FAULTS

In this section we prove the existence of a large connected component with good expansion in a graph with faulty nodes. We assume that a malicious adversary decides which nodes are faulty. More formally, we are given a network  $G = (V, E)$  with  $n$  nodes and vertex expansion  $\alpha$ . An adversary gives us a faulty version of this network, called  $G_f$ , with  $f$  faulty nodes removed. We will show that there exists a subnetwork of  $G_f$  called  $H$  which has  $\Theta(n)$  nodes and has an expansion of  $\Theta(\alpha)$  provided that the adversary is given no more than  $O(\alpha \cdot n)$  faults.

We cannot argue that the expansion of  $G_f$  is no more than a constant factor less than  $\alpha$  for the simple reason that the adversary can create bottlenecks in the network. However, we describe a way to find a large connected component of  $G_f$  with the required properties using an algorithm called *Prune* described in Figure 1. Note that the running time of *Prune* is not necessarily polynomial, nor are we claiming it is. *Prune* simply helps us prove an existential result.

Before we get to the algorithm we need to introduce some notation. We define  $\Gamma(S)$  to be the set of nodes in the neighbourhood of a subnetwork  $S$ . The algorithm generates a sequence of graphs  $G_0$  to  $G_m$ . We now present the algorithm and state the main theorem of this subsection.

**THEOREM 2.1.** *Given a network  $G$  with  $n$  nodes, node expansion  $\alpha$  and  $f$  faulty nodes chosen by an adversary, for any constant  $k$  such that  $k \geq 2$  and  $\frac{k \cdot f}{\alpha} \leq \frac{n}{4}$ ,  $\text{Prune}(1 - \frac{1}{k})$  returns a subnetwork  $H$  of at least size  $n - \frac{f \cdot k}{\alpha}$  with expansion  $(1 - \frac{1}{k}) \cdot \alpha$ .*

**PROOF.** Denote  $G_f \setminus H$  as  $S$ .  $S$  is thus the union of all the regions culled by *Prune*. To prove the result we will first show that the size of  $S$  is bounded by  $\frac{k \cdot f}{\alpha}$ . To show this we will use the fact that the number of faults required to cull a region is proportional to the size of the region. To demonstrate that we need the following lemma.

**Algorithm Prune( $\epsilon$ )**

- 1:  $G_0 \leftarrow G_f; i \leftarrow 0$
- 2: **while**  $\exists S_i \subseteq G_i$  such that  $|\Gamma(S_i)| \leq \alpha \cdot \epsilon \cdot |S_i|$  and  $|S_i| \leq |G_i|/2$
- 3:      $G_{i+1} \leftarrow G_i \setminus S_i$
- 4:      $i \leftarrow i + 1$
- 5: **end while**
- 6:  $H \leftarrow G_i; m \leftarrow i$

**Figure 1: The pruning algorithm**

LEMMA 2.2.

$$\left| \Gamma\left(\bigcup_{0 \leq i \leq j} S_i\right) \right| \leq \sum_{0 \leq i \leq j} |\Gamma(S_i)| \leq \alpha \cdot \left(1 - \frac{1}{k}\right) \cdot \left| \bigcup_{0 \leq i \leq j} S_i \right|.$$

PROOF. Consider the first inequality. Obviously, any node that lies in the neighborhood of  $\bigcup_i S_i$  must lie in the neighborhood of some  $S_i$ . Therefore  $\Gamma(\bigcup_i S_i) \subseteq \bigcup_i \Gamma(S_i)$ . Hence the first inequality. Each set  $S_i$  that is culled by *Prune*( $1 - \frac{1}{k}$ ) has the property that  $|\Gamma(S_i)| \leq \alpha \cdot (1 - \frac{1}{k}) \cdot |S_i|$ . Since the sets  $S_i$  are disjoint,  $\sum_i |S_i| = |\bigcup_i S_i|$ . Hence the second inequality.  $\square$

We will show that  $S \leq \frac{k \cdot f}{\alpha}$  by contradiction. Let, if possible,  $S > \frac{k \cdot f}{\alpha}$ . Since at every iteration of the algorithm we pick an  $S_i$  which is the smaller side of the cut we have found, each  $S_i$  is at most  $n/2$  in size. Now, since  $\frac{k \cdot f}{\alpha} \leq \frac{n}{4}$ , there is a  $j$  such that either  $\frac{k \cdot f}{\alpha} < \left| \bigcup_{0 \leq i \leq j} S_i \right| \leq n/2$  or  $S_j$  such that  $\frac{k \cdot f}{\alpha} < |S_j| \leq n/2$ . So we can always choose an  $S' \subseteq S$  such that  $\frac{k \cdot f}{\alpha} < |S'| \leq n/2$ . In either case, from Lemma 2.2, we have:

$$|\Gamma(S')| \leq \alpha \cdot \left(1 - \frac{1}{k}\right) \cdot |S'|.$$

We know that in  $G$ ,  $|\Gamma(S')|$  is at least  $\alpha \cdot |S'|$ . Hence, the number of faulty nodes in  $S'$ 's neighborhood must be at least  $\alpha(1 - (1 - \frac{1}{k})) \cdot |S'|$  i.e. greater than  $\alpha \cdot \frac{1}{k} \cdot \frac{k \cdot f}{\alpha}$  i.e. greater than  $f$ . Since the total number of faults allowed to the adversary is at most this number, we have a contradiction. Hence,  $H$  is at least  $n - \frac{k \cdot f}{\alpha}$  in size and has expansion at least  $(1 - \frac{1}{k}) \cdot \alpha$ .  $\square$

The result given in Theorem 2.1 is the best possible up to constant factors. To prove this we will first show that for every  $\alpha > 0$  smaller than some constant there is an infinite family of graphs which disintegrate into sublinear components on removing some  $c \cdot \alpha \cdot n$  vertices where  $n$  is the number of nodes in the given graph and  $c$  is some constant. Then we show that Theorem 2.1 is also the best possible up to constant factors for arbitrary graphs of uniform expansion.

**THEOREM 2.3.** *There exists a constant  $\beta$  such that, given any  $\alpha < \beta$ , there is an infinite family of graphs with expansion  $\alpha$  for which there is an adversarial selection of  $c \cdot \alpha \cdot n$  faulty nodes causing the graph to break into sublinear components, where  $n$  is the number of nodes in the graph and  $c$  is an appropriately chosen constant.*

PROOF. To construct this family of graphs let us consider  $G(n)$  to be an infinite family of expander graphs with constant expansion  $\beta$  and constant degree  $\delta$ .

For each  $G \in G(n)$ , construct a graph,  $H$ , which is a copy of  $G$  with each edge replaced by a chain of  $k$  nodes, where  $k$  is even. Then  $H$  has  $\frac{\delta \cdot n \cdot k}{2} + n = O(k \cdot n)$  nodes.

CLAIM 2.4. *Graph  $H$  has expansion  $\Theta(\frac{1}{k})$ .*

PROOF. Take any subset  $U$  of nodes in  $H$  representing original nodes in  $G$  and let  $U'$  be the set resulting from  $U$  by adding the  $k/2$  nearest nodes of each chain a node in  $U$  is connected to. Then  $|U'| = (\frac{\delta \cdot k}{2} + 1) \cdot |U|$  but  $|\Gamma(U')| = |\Gamma(U)| \leq \delta \cdot |U|$ . Hence,

$$\alpha(U') = \frac{|\Gamma(U')|}{|U'|} \leq \frac{2}{k} \cdot |U'|$$

completing the proof of the claim.  $\square$

Now, from each chain of  $k$  nodes we remove the central node. Each component remaining has  $\delta \cdot \frac{k}{2}$  nodes left, i.e. a sublinear number, and the total number of nodes removed is  $\frac{\delta}{2} \cdot n$ , which is  $\frac{1}{k}$  times the number of nodes in the graph.  $\square$

Recall that a graph  $G$  of size  $n$  is of uniform expansion  $\alpha(\cdot)$  if the expansion of  $G$  is  $\alpha(n)$  and every subgraph  $G'$  of size  $m$  of  $G$  has an expansion of  $O(\alpha(m))$ . This is the case for all well-known classes of graphs. Consider, for example, the  $m \times m$ -mesh with  $n = m^2$  nodes and let  $\alpha(m) = \sqrt{m}$ . Its expansion approximately  $\sqrt{n}$ , and every subgraph of that mesh of size  $m$  has an expansion of  $O(\sqrt{m})$ . Hence, it has a uniform expansion.

**THEOREM 2.5.** *For every connected graph of size  $n$  and uniform expansion  $\alpha(x)$  there is an adversarial selection of  $\omega(\alpha(n) \cdot n)$  faulty nodes that causes the graph to break into sublinear components.*

PROOF. Let  $G = (V, E)$  be any graph of uniform expansion  $\alpha(x)$  that consists of  $n$  nodes. Then there must be a set  $U_1 \subseteq V$ ,  $|U_1| \leq n/2$ , so that  $|\Gamma(U_1)| \leq \alpha(n) \cdot |U_1|$ . Removing  $\Gamma(U_1)$  leaves  $G$  with a set  $\mathcal{V}_1 = \{V', V''\}$  of two node sets,  $V' = U_1$  and  $V'' = V \setminus (U_1 \cup \Gamma(U_1))$ . Let  $V_1$  be a set in  $\mathcal{V}_1$  of maximum size. It follows from the uniformity of  $G$  that there must be a set  $U_2 \subseteq V_1$ ,  $|U_2| \leq |V_1|/2$ , so that  $|\Gamma(U_2)|$  w.r.t.  $G(V_1)$  is  $O(\alpha(|V_1|)) \cdot |U_2|$ . Removing  $U_2$  results in a new set  $\mathcal{V}_2$  of sets of nodes in which  $V_1$  is replaced by  $U_2$  and  $V_1 \setminus (U_2 \cup \Gamma(U_2))$ . We continue to take a node set  $V_i$  of largest size out of  $\mathcal{V}_i$  and remove nodes at the minimum expansion part in  $G(V_i)$  until there is no subset in  $\mathcal{V}_i$  left of size at least  $\epsilon n$ .

Our goal is to show that this process only removes  $O(\frac{\log(1/\epsilon)}{\epsilon} \cdot \alpha(n) \cdot n)$  nodes from  $G$ . If this is true, the theorem would follow immediately. We prove the bound with a charging strategy: Each time a set  $V_i$  is selected from  $\mathcal{V}_i$ , we charge all nodes in  $\Gamma(U_{i+1})$  taken away from  $V_i$  to the nodes in  $U_{i+1}$ . Since

$$|\Gamma(U_{i+1})| = O(\alpha(\epsilon n)) \cdot |U_{i+1}| = O\left(\frac{\alpha(n)}{\epsilon} \cdot |U_{i+1}|\right)$$

for any  $\alpha(x) \geq 1/x$ , this means that every node in  $U_{i+1}$  is charged with a value of  $O(\epsilon^{-1} \cdot \alpha(n))$ . Every node can be charged at most  $\log(1/\epsilon)$  times because each time a node is charged, it ends up in a node set  $U_{i+1}$  that is at most half as large as  $V_i$ , and we stop splitting a node set once it is of size less than  $\epsilon n$ . Hence, at the end, every node in  $V$  is charged with a value of  $O(\frac{\log(1/\epsilon)}{\epsilon} \cdot \alpha(n))$ . Summing up over all nodes, the total charge is

$$O\left(\frac{\log(1/\epsilon)}{\epsilon} \cdot \alpha(n) \cdot n\right),$$

which represents the number of nodes that have been removed from the graph.  $\square$

### 3. RANDOM FAULTS

We now direct our attention to the case of random faults. We assume that each node in the network can independently become faulty with a given probability  $p$ .

#### 3.1 Random faults aren't (always) easier to handle

Intuitively it appears that in general this situation might be easier to handle since there is no malicious adversarial intent behind the distribution of node failures. But, in general this does not seem to be true. We begin this section by showing that there are families of graphs for which a fault probability of  $\Theta(\alpha)$  causes the graph to disintegrate into sublinear fragments, where  $\alpha$  is the node expansion of the graph. In other words, in these graphs  $\Theta(\alpha n)$  random node failures can be catastrophic: they don't even allow us to find a linear sized connected component, hence making it impossible to find a linear sized connected component with good expansion.

To construct this family of graphs we begin with an infinite family of constant degree expander graphs with a constant node expansion  $\beta$  and maximum degree  $\delta$ . We denote this family as  $G(n)$ .

**THEOREM 3.1.** *Given any  $\alpha < \beta$ , there exists an infinite family of graphs with node expansion  $\alpha$  for which a fault probability of  $\frac{3 \log \delta}{\beta} \cdot \alpha$  causes the graph to disintegrate.*

**PROOF.** We use the family of graphs constructed in the proof of Theorem 2.3, i.e. let  $G(n)$  be an infinite family of constant degree expander graphs with constant expansion  $\beta$  and degree  $\delta$ . Construct a graph,  $H$ , which is a copy of  $G$  with each edge replaced by a chain of  $k$  nodes. Graph  $H$  has  $O(k \cdot n)$  nodes. From Claim 2.4 we know that  $H$  has expansion  $\Theta(\frac{1}{k})$ . Exercise 5.7 of [23] gives us the following important property of  $H$ :

**CLAIM 3.2.** *The number of connected subgraphs of  $H$  with  $r$  vertices from  $G$  in them is at most  $n \cdot \delta^{2r}$ .*

**PROOF.** Any connected subgraph of size  $r$  can be spanned by a tree with  $r - 1$  edges. This tree can be traversed by an Eulerian tour in which each edge is used at most twice. Hence the subgraph is represented by a walk along the graph of length at most  $2r$  vertices from  $G$ . Since the root can be one of  $n$  vertices, the result follows.  $\square$

Let the failure probability of the nodes in  $H$  be  $p = \frac{4 \ln \delta}{k}$ . Consider any subgraph of  $H$  with  $r = \ln n$  vertices from  $G$ . The total number of nodes in this subgraph is at most  $\delta \cdot k \cdot r$  and at least  $k \cdot r$ . Hence, this particular subgraph survives in  $H$  with probability at most  $(1 - p)^{k \cdot r} \leq e^{-k \cdot r \cdot p}$ . By Claim 3.2 there are no more than  $n \cdot \delta^{2r}$  such components in  $H$ . Hence, the probability that such a subgraph survives is at most  $n \cdot \delta^{2r} \cdot e^{-k \cdot r \cdot p} = n^{1 - 2 \ln \delta} \leq \frac{1}{n}$ . Since with high probability there can be no connected subgraph with size  $\Theta(\delta \cdot k \ln n)$  in  $H$  which has  $k \cdot n$  vertices and  $\delta$  is a constant, we conclude that  $H$  breaks down into sublinear components with high probability.

In the above construction, set  $k = \lceil \frac{\beta}{\alpha} \rceil$  for a given  $\alpha < \beta$  and the theorem follows.  $\square$

However it isn't as if the expansion of the graph is a critical point for all graphs. There are several important classes of graphs which can sustain a much higher fault probability and still yield a linear sized connected component with good expansion.

#### 3.2 Extracting a subnetwork of size $\Theta(n)$ and edge expansion $\Theta(\alpha_e)$

We are given a network  $G = (V, E)$  with  $n$  nodes, edge expansion  $\alpha_e$  and graph span  $\sigma$ . Let us call the faulty version of this network  $G_f$ . We want to find a network  $H \subseteq G_f$  of size  $\Theta(n)$  with edge expansion  $\Theta(\alpha_e)$ . Let  $\mathcal{U}$  be the set of all compact sets of  $G$ . Note that a set is compact if both it and its complement are connected. We will use the notion of *edge expansion* in this section.

**LEMMA 3.3.** *If  $S \subset G$  is connected and  $|S| < n/2$  then there exists a compact set  $K_G(S)$  in  $G$  whose edge expansion is no more than  $S$ 's edge expansion.*

**PROOF.** If  $S \in \mathcal{U}$  then  $K_G(S)$  is simply  $S$ . If  $S \notin \mathcal{U}$ ,  $G \setminus S$  is not connected. Let  $\mathcal{C}(S)$  be the set of maximal connected subgraphs of  $G \setminus S$ . Let  $\Gamma_e(\cdot)$  be the set of edges leaving a set. It is clear that  $\mathcal{C}(S) \subset \mathcal{U}$  (if not then they are not maximal). We consider two cases.

*Case 1:* There is a  $C \in \mathcal{C}(S)$  with  $|C| \geq n/2$ . Then  $G \setminus C \in \mathcal{U}$ ,  $S \subseteq G \setminus C$ ,  $|G \setminus C| < n/2$ , and  $\Gamma_e(G \setminus C) \subseteq \Gamma_e(S)$ . Hence,  $G \setminus C$  has an edge expansion less than  $S$ 's edge expansion. So,  $K_G(S) = G \setminus C$ .

*Case 2:* For all  $C \in \mathcal{C}(S)$ ,  $|C| < n/2$ . If any of the connected components in  $\mathcal{C}(S)$  has an edge expansion less than  $S$ 's then let that component be  $K_G(S)$ . If not, then all components  $C_i \in \mathcal{C}(S)$  have an edge expansion strictly larger than  $S$ 's, i.e. for all  $i$ ,  $\frac{\Gamma_e(C_i)}{|C_i|} > \frac{\Gamma_e(S)}{|S|}$ . But,  $\Gamma_e(\cup_i C_i) = \Gamma_e(S)$ . Hence,  $|S| > |G \setminus S|$ , which is a contradiction. Therefore, one of the  $C_i$ 's must have an edge expansion less than or equal to  $S$ 's edge expansion.  $\square$

**Algorithm Prune2( $\epsilon$ )**

- 1:  $G_0 \leftarrow G_f$ ;  $i \leftarrow 0$
- 2: **while**  $\exists (S_i, G_i \setminus S_i)$  in  $G_i$  s.t.  $|(S_i, G_i \setminus S_i)| \leq \alpha_e \cdot \epsilon \cdot |S_i|$  and  $|S_i| \leq |G_i|/2$  and  $S_i$  is connected
- 3:  $K_i \leftarrow K_{G_i}(S_i)$
- 4:  $G_{i+1} \leftarrow G_i \setminus K_i$
- 5:  $i \leftarrow i + 1$
- 6: **end while**
- 7:  $H \leftarrow G_i$

**Figure 2: The pruning algorithm**

We use notation from algorithm *Prune2* in the proof and statement of theorem 3.4.

**THEOREM 3.4.** *Prune2( $\epsilon$ ) returns a subnetwork  $H$  of size  $|H| \geq n/2$  with edge expansion  $\epsilon \cdot \alpha_e$  with high probability, provided that edge expansion,  $\alpha_e \geq \frac{6\delta^2 \cdot \log^3 n}{n}$ , fault probability,  $p \leq \frac{1}{2e \cdot \delta^{4\sigma}}$  and degradation in expansion,  $\epsilon \leq \frac{1}{2\delta}$ .*

**PROOF.** Let  $\mathcal{T} = G_f \setminus H$ . Hence  $\mathcal{T}$  is the union of all the culled regions. To prove the result we will show that with high probability the size of  $\mathcal{T}$  is not more than  $n/2$ . Let  $\{T_1, T_2, \dots, T_l\}$  be maximal connected components of  $\mathcal{T}$ .

**CLAIM 3.5.**  $\forall T_i \in \mathcal{T}$ ,  $T_i$  is compact in  $G_f$ .

PROOF. Suppose  $T_i$  is not compact in  $G_f$ . Select the largest  $j$  such that  $T_i$  is not compact in  $G_j$  and  $T_i \subseteq G_j$ . (i.e. no part of  $T_i$  has been culled yet, which means that  $G_{j+1}$  is well-defined.) Let us consider two cases:

*Case 1:  $T_i \subseteq G_{j+1}$*

This means that  $T_i$  must be compact in  $G_{j+1}$  else  $j$  could have been one higher. So, we have 3 components in  $G_j$ , namely:  $K_j$ ,  $T_i$  and  $G_{j+1} \setminus T_i$ . Since  $T_i$  is noncompact in  $G_j$ , the neighborhood of  $K_j$  in  $G_j$  is wholly in  $T_i$ . Since  $K_j$  is disjoint with  $T_i$ ,  $T_i$  is not maximal. Contradiction.

*Case 2:  $T_i \not\subseteq G_{j+1}$*

This means that  $T_i$  and  $K_j$  are not disjoint. Since  $K_j$  is a culled set it must be wholly inside  $T_i$ , else  $T_i$  is not maximal.  $T_i$  is not compact in  $G_j$ , so  $T_i \setminus K_j$  is not compact in  $G_{j+1}$ . We know that  $T_i \setminus K_j$  will not be in  $H$ . Hence, all but one connected component (the one that contains  $H$ ) in  $G_{j+1} \setminus T_i$  must belong to  $\mathcal{T}$ . Hence  $T_i$  is not maximal. Contradiction.  $\square$

Let  $\Gamma(\cdot)$  and  $\Gamma^f(\cdot)$  denote the node neighbourhoods in the faultless graph and the faulty graph respectively. It is easy to see the following inequalities:  $|\Gamma(T_i)| \geq \frac{\alpha_e |T_i|}{\delta}$ , and  $|\Gamma^f(T_i)| \leq \alpha_e \epsilon |T_i|$ . These two inequalities imply that  $|\Gamma^f(T_i)| \leq \epsilon \delta |\Gamma(T_i)|$ . Note that any set  $T_i$  was culled by *prune2* because its edge neighbourhood fell by a factor of more than  $\epsilon$ .

The probability that the neighbourhood of some connected set  $T_i$  in the faulty graph went down from  $\Gamma(T_i)$  to  $\Gamma^f(T_i)$  is (for the sake of brevity,  $\Delta := |\Gamma(T_i)| - |\Gamma^f(T_i)|$ ):

$$\begin{aligned} \left( \frac{|\Gamma(T_i)|}{|\Gamma^f(T_i)|} \right) \cdot p^\Delta &\leq \left( \frac{ep|\Gamma(T_i)|}{\Delta} \right)^\Delta \\ &\leq \left( \frac{ep}{1-\epsilon\delta} \right)^{(1-\epsilon\delta)|\Gamma(T_i)|} \end{aligned} \quad (2)$$

Note that this is valid under the condition that  $ep + \epsilon\delta < 1$ . It turns out that we have flexibility in bounding these two terms. We want to set  $\epsilon\delta$  closest to 1 so that degradation in expansion is minimal. Therefore, if the following inequalities hold:

$$\epsilon\delta \leq \frac{1}{2}, \quad ep \leq \frac{1}{2\delta^{4\sigma}},$$

then the probability that  $T_i$  is culled by *prune2* is at most  $\delta^{-3\sigma|\Gamma(T_i)|}$  (this is an upperbound on the RHS in 2).

$$\Pr[T_i \text{ is culled}] \leq \delta^{-3\sigma|\Gamma(T_i)|}$$

We enumerate two cases on the size of the neighbourhood of  $T_i$ s. In case 1 we argue that a  $T_i$  with a large neighbourhood is unlikely with high probability. In case 2 we show that if all  $T_i$ s have small neighbourhoods then it is unlikely that  $\sum_i |T_i|$  is more than  $\frac{n}{2}$  with high probability. So, in case 2 assume that  $|\bigcup_{i=1}^l T_i| \geq n/2$ . Let  $k = 3 \log_\delta n$  in the following cases:

*Case 1:  $\exists i, |\Gamma(T_i)| \geq k$ .*

We know from before that the probability that a given compact subgraph  $T_i$  is culled is at most  $\delta^{-3\sigma|\Gamma(T_i)|}$ . We multiply this probability with the number of ways of choosing such a subgraph. This gives us the probability that there is a  $T_i$  with such a large neighbourhood. Each compact subgraph has its corresponding perimeter. Therefore, the number of compact subgraphs with boundary  $|\Gamma(T_i)|$  is at most the number of  $\sigma \cdot |\Gamma(T_i)|$  sized spanning trees in the graph. This is at most  $n \cdot \delta^{2\sigma|\Gamma(T_i)|}$ . Note that by definition,

$\sigma \geq 1$ . Hence,

$$\begin{aligned} \Pr[\exists T_i, |\Gamma(T_i)| > k] &\leq \sum_{t=k}^n n \cdot \delta^{2\sigma \cdot t} \cdot \delta^{-3\sigma \cdot t} \\ &\leq n^2 \cdot \delta^{-k} \leq \frac{1}{n} \end{aligned}$$

*Case 2:  $\forall i, |\Gamma(T_i)| < k$ .*

$$\Pr[T_i \text{ is culled}] \leq \delta^{-3\sigma|\Gamma(T_i)|} \leq \delta^{-3}$$

$T_i$ s are disjoint by definition. Some  $T_i$  and  $T_j$  might share a bad node in their neighbourhood leading to a dependency between them. But we do know that since the perimeter of each  $T_i$  is at most  $k-1$ , the maximum degree of the dependency graph between the  $T_i$ s is  $\delta \cdot (k-1)$ . Hence the dependency graph can be coloured with  $\delta \cdot (k-1) + 1 \leq \delta \cdot k$  colours.

We know that  $|\bigcup_{i=1}^l T_i| \geq n/2$ . Hence there has to be a colour class in the colouring of the dependency graph, let us call it  $\mathcal{C}$ , such that the  $T_i$ s in that colour class contain at least  $\frac{n}{2 \cdot \delta \cdot k}$  nodes.

$|T_i| \leq \frac{k \cdot \delta}{\alpha_e}$ . Hence, the number of distinct  $T_i$ s in  $\mathcal{C}$  has to be at least  $\frac{n \cdot \alpha_e}{2 \cdot \delta^2 \cdot k^2}$ . We know that the  $T_i$ s in  $\mathcal{C}$  are independent of each other. We set a bound on  $\alpha_e$  such that this probability becomes small. Let  $\alpha_e \geq \frac{2\delta^2 \cdot k^3}{9n}$ .

$$\begin{aligned} \Pr[\forall T_i : T_i \text{ is bad}] &\leq \Pr[\forall T_i \in \mathcal{C} : T_i \text{ is bad}] \\ &\leq \delta^{-\frac{3\alpha_e \cdot n}{2 \cdot \delta \cdot k^2}} \leq \delta^{-k/3} \leq \frac{1}{n} \end{aligned}$$

$$\Pr[\text{nodes pruned} \geq n/2] \leq \Pr[\text{Case 1}] + \Pr[\text{Case 2}] \leq \frac{2}{n}$$

$\square$

### 3.3 Span of the mesh

**THEOREM 3.6.** *The  $d$ -dimensional mesh has span 2.*

PROOF. Consider a compact set  $S$  in the  $d$ -dimensional mesh  $M$ . Let  $B$  be the boundary nodes  $\Gamma(S)$ . We place virtual edges between nodes in  $B$ . Two distinct nodes  $u = (u_0, \dots, u_{d-1})$  and  $v = (v_0, \dots, v_{d-1})$  have a virtual edge between them if  $|v_i - u_i| = 0$  for at least  $d-2$  of its dimensions and  $|v_i - u_i| \leq 1$  for the rest. Call the set of such virtual edges  $E_v$ . In Lemma 3.7, stated below, we claim that the graph  $(B, E_v)$  is connected. Therefore, we can find a spanning tree for  $B$  which has exactly  $|B| - 1$  virtual edges. Since each edge in  $E_v$  can be simulated by exactly 2 edges of  $M$ , we can say that there is a spanning tree in  $M$  for the nodes of  $B$  with at most  $2 \cdot (|B| - 1)$  edges.  $\square$

**LEMMA 3.7.** *Let  $S \subset Z^d$  be a finite compact set, let  $B$  be the boundary nodes  $\Gamma(S)$ , and let  $E_v$  be the set of virtual edges. Then the graph  $(B, E_v)$  is connected.*

PROOF. We will show that for any two points  $u$  and  $v$  in  $B$ , there is a path in  $E_v$  connecting the two; if this can be done for every two points, then  $B$  is connected as we hope to prove.

Our proof uses some basic and standard homology theory of cell complexes, which can be found in any introductory topology text; for instance, see [13]. Specifically, we use the  $Z_2$  homology of  $d$ -dimensional Euclidean space  $R^d$ . We partition  $R^d$  into a complex of unit hypercube cells having the points of  $Z^d$  as their vertices. Each  $d$ -dimensional unit hypercube cell has as its boundary a set

of  $2d(d-1)$ -dimensional unit hypercube facets, again having  $Z^d$  as vertices, and so on. In this complex, a  $k$ -chain is defined to be any finite set of  $k$ -dimensional unit hypercubes having points of  $Z^d$  as vertices. The boundary of a  $k$ -chain  $C$  is the symmetric difference of the boundaries of its hypercubes; that is, it is the set of  $(k-1)$ -dimensional hypercubes that are on the boundary of an odd number of the  $k$ -dimensional hypercubes in  $C$ . A  $k$ -cycle is defined to be a  $k$ -chain that has an empty boundary, and a  $k$ -boundary is defined to be a  $k$ -chain that is the boundary of some  $(k+1)$ -chain. For quite general classes of cell complexes in more complicated topological spaces than  $R^d$ , every  $k$ -boundary is a  $k$ -cycle, but in  $R^d$ , the reverse is also known to be true: every  $k$ -cycle is a  $k$ -boundary.

Now, given  $u$  and  $v$ , since  $S$  is connected we can find a path  $p_1$  connecting  $u$  to  $v$  by a sequence of adjacent points in  $S$ . We also find an edge  $e_1$  connecting  $u$  to an adjacent point of  $Z^d$  outside  $S$ , an edge  $e_2$  connecting  $v$  to an adjacent point of  $Z^d$  outside  $S$ , and a path  $p_2$  connecting these two exterior points by a sequence of adjacent points outside  $S$  (since the complement of  $S$  is connected). The union of  $p_1$ ,  $p_2$ , and  $\{e_1, e_2\}$  forms a 1-chain in the cubical complex described above. Moreover, this is a 1-cycle, because it has degree two at every vertex it touches. Therefore, it is the boundary of a 2-chain  $C$ ; that is,  $C$  is a set of squares and  $p_1 \cup p_2 \cup \{e_1, e_2\}$  is the set of edges in the cubical complex that touch odd numbers of squares in  $C$ .

Next, let  $U$  be the subset of  $R^d$  formed by a union of axis-aligned unit hypercubes, one for each member of  $S$ , and having that member as its centroid; note that these hypercubes do not have integer vertices. Let  $B$  be the boundary facets of  $U$ ;  $B$  consists of a collection of  $(d-1)$ -dimensional unit hypercubes that again do not have integer vertices. Finally, let  $G = B \cap C$ .

Whenever a square  $s$  of  $C$  and a  $(d-1)$ -dimensional hypercube  $h$  of  $G$  meet, they do so in a line segment of length  $1/2$ , that connects the centroid of  $h$  (where it is crossed by one edge of the square) to the centroid of one of its boundary  $(d-2)$ -dimensional hypercubes. Thus  $G$ , the union of these line segments, can be viewed as a graph that connects vertices at these points. The degree of a vertex at the centroid of  $h$  is equal to the number of squares of  $C$  that touch that point, and the degree of the other vertices can only be two or four depending on which of the four vertices of the square defining the vertex is interior to  $U$ .

Since the boundary of  $C$  crosses  $B$  only on the two edges  $e_1$  and  $e_2$ , these two crossing points have odd degree and all the other vertices of  $G$  have even degree. Any connected component of any graph must have an even number of odd-degree vertices, so the two odd vertices  $e_1 \cap B$  and  $e_2 \cap B$  must belong to the same component and can be connected by a path  $p_3$  in  $G$ .

Each length- $1/2$  segment of  $p_3$  belongs to the boundary of a single hypercube in  $U$ , which has as its centroid a point of  $B$ . Let  $p_4$  be the sequence of centroids corresponding to the sequence of edges in  $p_3$ . Then  $p_4$  starts at  $u$ , and ends at  $v$ . Further, at each step from one edge in  $p_4$ , either the current point in  $B$  does not change, or it changes from one point in  $B$  to an adjacent point (when the corresponding pair of edges in  $p_4$  form a  $180^\circ$  angle on two adjoining hypercubes), or it changes from one point in  $B$  to a point at distance  $\sqrt{2}$  away (when the corresponding edges in  $p_4$  form a  $270^\circ$  angle across a concavity on the boundary of  $U$ ).

So, we have constructed a path in  $E_v$  between an arbitrarily chosen pair of points  $u, v$  in  $B$ , and therefore the graph  $(B, E_v)$  is connected.  $\square$

Theorem 3.6 implies that the  $d$ -dimensional mesh can sustain a fault probability inversely polynomial in  $d$  and still have a large component whose expansion is no more than a factor of  $d$  worse

than the original.

## 4. CONCLUSION

In this paper we presented a general technique for determining the robustness of the expansion of different networks both for adversarial and random faults. For random faults we have come up with a new parameter, the span, which allows us to prove a strong result regarding the robustness of high dimensional meshes. Among other things, this result can provide useful insights into the working of peer-to-peer networks like CAN [25] which behaves like a  $d$ -dimensional mesh in its steady state. Basically we have shown that CAN can tolerate a fault probability which is inversely polynomial in its dimension without losing too much in its expansion properties.

For the 2-dimensional mesh our result is related to the line of research followed by Raghavan [24], Kakkamanis et. al. [14] and Mathies [22] who show that despite a constant fault probability (of as high as 0.4) a mesh with random failures can emulate a fault free mesh using paths with stretch factor at most  $O(\log n)$ . Since the distance of nodes in a graph of expansion  $\alpha$  is  $O(\alpha^{-1} \log n)$  [20], our technique gives essentially the same result albeit with a lower fault probability. Additionally for meshes of constant dimension greater than 2 our results imply a  $O(\log n)$  dilation for path lengths, and hence a way to generalize these earlier results to higher dimensions.

The strength of our technique is that it is able to yield results for the 2-dimensional mesh which are comparable to previous results while giving new results for higher dimensional meshes and providing a general method suitable for analyzing any network whose span can be estimated.

## Open problems

We conjecture that the butterfly, shuffle-exchange, and deBruijn network all have a span of  $O(1)$ , which means that they can tolerate a constant fault probability. Though the span may provide tight results for these networks, the exponential dependency of the fault probability on the span does not really give useful results if the span is beyond  $\log n$ . Hence, either a better dependency result is needed or a parameter better than the span is needed. Clearly, as mentioned in the introduction, having a parameter that can accurately describe the fault tolerance of graphs w.r.t. expansion under random faults would be very useful for many applications.

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