Network design with edge-connectivity and degree constraints*

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Abstract

We consider the following network design problem; Given a vertex set V with a metric cost c on V, an integer $k \geq 1$, and a degree specification b, find a minimum cost k-edge-connected multigraph on V under the constraint that the degree of each vertex $v \in V$ is equal to $b(v)$. This problem generalizes metric TSP. In this paper, we propose that the problem admits a ρ-approximation algorithm if $b(v) \geq 2$, $v \in V$, where $\rho = 2.5$ if k is even, and $\rho = 2.5 + 1.5/k$ if k is odd. We also prove that the digraph version of this problem admits a 2.5-approximation algorithm and discuss some generalization of metric TSP.
Keywords: approximation algorithm, degree constraint, edge-connectivity, (m, n)-VRP, TSP, vehicle routing problem

1 Introduction

It is a main concern in the field of network design to construct a graph of the least cost which satisfies some connectivity requirement. Actually many results on this topic have been obtained so far. In this paper, we consider a network design problem that asks to find a minimum cost k-edge-connected multipraph on a metric edge cost under degree specification. This provides a natural and flexible framework for treating many network design problems. For example, it generalizes the vehicle routing problem with m vehicles (m-VRP) [4, 8], which will be introduced below, and hence contains a well-known metric traveling salesperson problem (TSP), which has already been applied to numerous practical problems [9].

Let $\mathbb{Z}_+$ and $\mathbb{Q}_+$ denote the sets of non-negative integers and non-negative rational numbers, respectively. Let $G = (V, E)$ be a multigraph with a vertex set V and an edge set E, where a multigraph may have some parallel edges but is not allowed to have any loops. For two vertices u and v, an edge joining u and v is denoted by uv. Since we consider multigraphs in this paper, we distinguish two parallel edges $e_1 = uv$ and $e_2 = uv$, which may be simply denoted by uv and uv. For a non-empty vertex set $X \subset V$,

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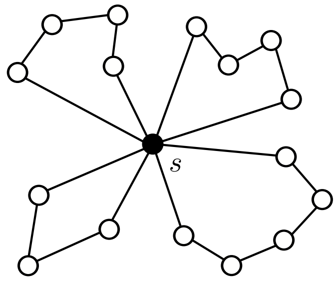

Figure 1: A solution for 4-VRP

$d(X;G)$ (or $d(X)$) denotes the number of edges whose one end vertex is in X and the other is in $V-X$. In particular $d(v;G)$ (or $d(v)$) denotes the degree of vertex v in G. The edge-connectivity $\lambda(u,v;G)$ (or $\lambda(u,v)$) between u and v is the maximum number of edge-disjoint paths between them in G. The edge-connectivity $\lambda(G)$ of G is defined as $\min_{u,v\in V}\lambda(u,v;G)$. If $\lambda(G)\geq k$ for some $k\in\mathbb{Z}_+$, then G is called *k-edge-connected*. For a function $r:\binom{V}{2}\rightarrow\mathbb{Z}_+$, G is called *r-edge-connected* if $\lambda(u,v;G)\geq r(u,v)$ for every $u,v\in V$. Edge cost $c:\binom{V}{2}\rightarrow\mathbb{Q}_+$ is called *metric* if it obeys the triangle inequality, i.e., $c(uv)+c(vw)\geq c(uw)$ for every $u,v,w\in V$.

For a degree specification $b:V\rightarrow\mathbb{Z}_+$, a multigraph G with $d(v;G)=b(v)$ for all $v\in V$ is called a *perfect b-matching*. In this paper, we focus on the following network design problem.

k-edge-connected multigraph with degree specification (k-ECMDS):
A vertex set V, a metric edge cost $c:\binom{V}{2}\rightarrow\mathbb{Q}_+$, a degree specification $b:V\rightarrow\mathbb{Z}_+$, and a positive integer k are given. We are asked to find a minimum cost perfect b-matching $G=(V,E)$ of edge-connectivity k. □

In this paper, we suppose that $b(v)\geq 2$ for all $v\in V$ unless stated otherwise, and propose approximation algorithms to k-ECMDS in this case.

Problem k-ECMDS is a generalization of m-VRP, which asks to find a minimum cost set of m cycles, each containing a designated initial city s, such that each of the other cities is covered by exactly one cycle (see Fig. 1). Observe that this problem is 2-ECMDS where $b(s)=2m$ for the initial city $s\in V$ and $b(v)=2$ for every $v\in V-s$. If $m=1$, then m-VRP is exactly TSP. Since TSP is known to be NP-hard [12] even if a given cost is metric (metric TSP), k-ECMDS is also NP-hard. If a given cost is not metric, TSP cannot be approximated unless P = NP [12]. For m-VRP, there is a 2-approximation algorithm based on the primal-dual method [8].

It is well studied to find a minimum cost multigraph either with k-edge-connectivity or with degree specification. It is known that finding a minimum cost k-edge-connected graph is NP-hard since it is equivalent to metric TSP when $k=2$ and a given edge cost is metric. On the other hand, it is known that a minimum cost perfect b-matching can be constructed in polynomial time (for example, see [11]). As a prior result on problems equipped with both edge-connectivity requirements and degree constraints, Frank [2] showed that it is polynomially solvable to find a minimum cost r-edge-connected multigraph G with

$\ell(v) \leq d(v;G) \leq u(v)$, $v \in V$ for degree lower and upper bounds $\ell, u : V \to \mathbb{Z}_+$ and a metric edge cost c such that $c(uv)$ is defined by $w(u)+w(v)$ for some weight $w : V \to \mathbb{Q}_+$ (in particular, $c(uv) = 1$ for every $uv \in \binom{V}{2}$). Recently Fukunaga and Nagamochi [5] presented approximation algorithms for a network design problem with a general metric edge cost and some degree bounds; For example, they presented a $(2+1/\lfloor \min_{u,v\in V} r(u,v)/2\rfloor)$-approximation algorithm for constructing a minimum cost r-edge-connected multigraph that meets a local-edge-connectivity requirement r with $r(u,v) \geq 2$, $u,v \in V$ under a uniform degree upper bound. Afterwards Fukunaga and Nagamochi [6] gave a 3-approximation algorithm for the case where $r(u,v) \in \{1,2\}$ for every $u,v \in V$ and $\ell(v) = u(v)$ for each $v \in V$. In this paper, we extend the 3-approximation result [6] to k-ECMDS. Concretely, we prove that k-ECMDS is ρ-approximable if $b(v) \geq 2$, $v \in V$, where $\rho = 2.5$ if k is even and $\rho = 2.5 + 1.5/k$ if k is odd. Moreover, we show that this factor can be improved when a degree specification is uniform. To design our algorithms for k-ECMDS, we take a similar approach with famous 2- and 1.5-approximation algorithms for metric TSP.

Furthermore, we also generalize k-ECMDS to a network design problem in digraphs. We denote an arc (i.e., a directed edge) from a vertex u to another vertex v by uv. Two arcs from u to v are called *parallel*. Let $D = (V, A)$ be a multi-digraph, where a multi-digraph may have some parallel arcs but is not allowed to have any loops. For an ordered pair of vertices u and v, $\lambda(u,v;D)$ (or $\lambda(u,v)$) denotes the arc-connectivity from u to v, i.e., the maximum number of arc-disjoint paths from u to v in D. The arc-connectivity $\lambda(D)$ of D is defined as $\min_{u,v\in V} \lambda(u,v;D)$. If $\lambda(D) \geq k$ for some $k \in \mathbb{Z}_+$, D is called *k-arc-connected*. Moreover, $d^-(v;D)$ (or $d^-(v)$) and $d^+(v;D)$ (or $d^+(v)$) denote in- and out-degree of vertex v in digraph D, respectively. Arc cost $c : V \times V \to \mathbb{Q}_+$ is called *symmetric* if $c(uv) = c(vu)$ for every $u, v \in V$, and *metric* if it obeys the triangle inequality, i.e., $c(uv) + c(vz) \geq c(uz)$ for every $u, v, z \in V$.

We call a multi-digraph D with $d^-(v;D) = b^-(v)$ and $d^+(v;D) = b^+(v)$ for all $v \in V$ *perfect (b^-, b^+)-matching* for in- and out-degree specifications $b^-, b^+ : V \to \mathbb{Z}_+$. A minimum cost perfect (b^-, b^+)-matching can be found by computing a minimum cost perfect b-matching in a bipartite graph. The digraph version of the problem is described as follows.

k-arc-connected multi-digraph with degree specification (k-ACMDS):
A vertex set V, a symmetric metric arc cost $c : V \times V \to \mathbb{Q}_+$, in- and out-degree specifications $b^-, b^+ : V \to \mathbb{Z}_+$, and a positive integer k are given. We are asked to find a minimum cost perfect (b^-, b^+)-matching $D = (V, A)$ of arc-connectivity k. $\square$

We also introduce a problem (m,n)-vehicle routing problem ((m,n)-VRP), which generalizes m-VRP so that each of the other cities than a special city is visited by exactly n of the m cycles. This problem is not contained in k-ECMDS. However, we show that our algorithm for k-ECMDS also delivers a 2.5-approximate solution to (m,n)-VRP. Moreover, we improve this algorithm to an $(1.5 + \frac{m-n}{m})$-approximation algorithm.

This paper is organized as follows. Section 2 presents an algorithm for k-ECMDS. Section 3 provides a 2.5-approximation algorithm for k-ACMDS problem. Section 4 improves the approximation factors of these algorithms assuming that a degree specification is uniform. Section 5 shows how to apply our algorithm for k-ECMDS to (m, n)-VRP. Section 6 makes some concluding remarks.

2 Algorithm for k-ECMDS

This section describes an approximation algorithm for k-ECMDS. Before describing the algorithm, we consider how to check the feasibility of a given instance.

2.1 Feasibility

For some degree specification b, there is no perfect b-matching. The following theorem shows provides a necessary and sufficient condition for a degree specification to admit a perfect b-matching. Note that $b(v)$ can be 1 in this theorem.

Theorem 1 *Let V be a vertex set with $|V| \geq 2$ and $b : V \to \mathbb{Z}_+$ be a degree specification. Then there exists a perfect b-matching if and only if $\sum_{v \in V} b(v)$ is even and $b(v) \leq \sum_{u \in V - v} b(u)$ for each $v \in V$.*

Proof: The necessity is trivial. We show the sufficiency by constructing a perfect b-matching. We let $V = \{v_1, \ldots, v_n\}$ and $B = \sum_{\ell=1}^{n} b(v_\ell)/2$. For $j = 1, \ldots, B$, we define i_j as the minimum integer such that $\sum_{\ell=1}^{i_j} b(v_\ell) \geq j$, and i'_j as the minimum integer such that $\sum_{\ell=1}^{i'_j} b(v_\ell) \geq B + j$. Notice that $\sum_{\ell=1}^{i_j - 1} b(v_\ell) < j$ holds by the definition if $i_j \geq 2$. Then we can see that $i_j \neq i'_j$ since otherwise we would have $b(v_{i_j}) = \sum_{\ell=1}^{i_j} b(v_\ell) - \sum_{\ell=1}^{i_j-1} b(v_\ell) > (B + j) - j = B$ if $i_j \geq 2$ and $b(v_{i_j}) \geq B + j > B$ otherwise, which contradicts to the assumption.

Let $M = \{e_j = v_{i_j} v_{i'_j} \mid j = 1, \ldots, B\}$. Then M contains no loop by $i_j \neq i'_j$. Moreover G_M is a perfect b-matching since $|\{j \mid i_j = \ell \text{ or } i'_j = \ell\}| = b(v_i)$, as required. $\square$

Theorem 1 does not mention the edge-connectivity. For existence of connected perfect b-matchings, we additionally need the condition that $\sum_{v \in V} b(v) \geq 2(|V| - 1)$ [6]. This is always satisfied if $b(v) \geq 2$, $v \in V$, which we assume for 1-ECMDS. For $k \geq 2$, the conditions in Theorem 1 and $b(v) \geq k$, $v \in V$ are sufficient for the existence of k-edge-connected perfect b-matchings as our algorithm will construct such b-matchings under the conditions.

2.2 Algorithm

Now we describe our algorithm to k-ECMDS. Let (V, b, c, k) be an instance of k-ECMDS. The conditions appeared in Theorem 1 and $b(v) \geq k$ for all $v \in V$ can be verified in polynomial time, where they are apparently necessary for an instance to have k-edge-connected perfect b-matchings. Hence our algorithm checks them, and if some of them are violated, it outputs message "INFEASIBLE". In the following, we suppose the existence of perfect b-matchings with $b(v) \geq k$ for all $v \in V$. If $2 \leq |V| \leq 3$, then every perfect b-matching is k-edge-connected because any non-empty vertex set $X \subset V$ is $\{v\}$ or $V - \{v\}$

for some $v \in V$, and then $d(X) = d(v) \geq k$. Hence we can assume without loss of generality that $|V| \geq 4$.

For an edge set F on V, we denote graph (V, F) by G_F. Let M be a minimum cost edge set such that G_M is a perfect b-matching. In addition, let H be an edge set of a Hamiltonian cycle spanning V constructed by the 1.5-approximation algorithm for TSP due to Christofides [12].

Initialization: After testing the feasibility of a given instance, our algorithm first prepares M and $k' = \lceil k/2 \rceil$ copies $H_1, \ldots, H_{k'}$ of H. Let E denote the union $M \cup H_1 \cup \cdots \cup H_{k'}$ of them. Notice that G_E is $2k'$-edge-connected by the existence of edge-disjoint k' Hamiltonian cycles. We call a vertex v in a handling graph G an *excess vertex* if $d(v; G) > b(v)$ (otherwise a *non-excess vertex*). In G_E, all vertices are excess vertices since $d(v; G_E) = b(v) + 2k'$. In the following steps, the algorithm reduces the degree of excess vertices until no excess vertex exists while generating no loops and keeping k-edge-connectivity (Notice that $k < 2k'$ if k is odd). This is achieved by two phases, Phase 1 and Phase 2, as follows.

Phase 1: In this phase, we modify only edges in M while keeping edges in $H_1, \ldots, H_{k'}$ unchanged. We define the following two operations on an excess vertex $v \in V$.

Operation 1: If v has two incident edges xv and yv in M with $x \neq y$, replace xv and yv by new edge xy.

Operation 2: If v has two parallel edges uv in M with $d(u) > b(u)$, remove those edges.

Phase 1 repeats Operations 1 and 2 until none of them is executable. For avoiding ambiguity, we let M' denote M after executing Phase 1, and M denote the original set in what follows. Moreover, let $E' = M' \cup H_1 \cup \cdots \cup H_{k'}$. Note that $d(v) - b(v)$ is always a non-negative even integer throughout (and after) these operations because $d(v; G_E) - b(v) = 2k'$ and each operation decreases the degree of a vertex by 2. If no excess vertex remains in $G_{E'}$, then we are done. We consider the case in which there remain some excess vertices, and show some properties on M' before describing Phase 2.

Claim 1 *Every excess vertex in $G_{E'}$ has at least one incident edge in M' and its neighbors in $G_{M'}$ are unique.*

Proof: Since $d(v; G_{E'}) - b(v)$ is a positive even integer for an excess vertex v in $G_{E'}$, it holds $d(v; G_{M'}) = d(v; G_{E'}) - d(v; G_{H_1 \cup \cdots \cup H_{k'}}) \geq (b(v) + 2) - 2k' > 0$, Hence v has at least one incident edges in M'. If neighbors of v in $G_{M'}$ are not unique, Operation 1 can be applied to v. □

For an excess vertex v in $G_{E'}$, let $n(v)$ denote the unique neighbor of v in $G_{M'}$. If $n(v)$ is also an excess vertex in $G_{E'}$, we call the pair $\{v, n(v)\}$ by a *strict pair.*

Claim 2 *Let $\{v, n(v)\}$ be a strict pair. Then $d(v; G_{M'}) = d(n(v); G_{M'}) = 1$, k is odd, and $b(v) = b(n(v)) = k$.*

Proof: By Claim 1, $d(v; G_{M'}) = d(n(v); G_{M'})$. If $d(v; G_{M'}) = d(n(v); G_{M'}) > 1$, Operation 2 can be applied to v and $n(v)$, a contradiction. Hence $d(v; G_{M'}) = d(n(v); G_{M'}) = 1$ holds. Let $u \in \{v, n(v)\}$. Then it holds that $d(u; G_{E'}) = d(u; G_{H_1 \cup \cdots \cup H_{k'}}) + d(u; G_{M'}) = 2k' + 1 = 2\lceil k/2 \rceil + 1$. Since $d(u; G_{E'}) - b(u)$ is even, $b(u)$ must be odd. This fact and $d(u, G_{E'}) > b(u) \geq k$ indicates that $b(u) = k$ and k is odd. □

By definition, the existence of excess vertices which are in no strict pairs indicate that of some non-excess vertices. Upon completion of Phase 1, let N denote the set of non-excess vertices in $G_{E'}$, and S denote the set of strict pairs in $G_{E'}$. If $N = \emptyset$, all excess vertices are in some strict pairs. By Claim 2, k is an odd integer in this case, and furthermore $k \geq 3$ by the assumption that $b(v) \geq 2$, $v \in V$ if $k = 1$. From this fact and $|V| \geq 4$, $N = \emptyset$ implies that at least two strict pairs exist (i.e., $|S| \geq 2$).

Phase 2: Now we describe Phase 2. First, we deal with a special case in which V consists of only two strict pairs.

Claim 3 *If V consists of two strict pairs after Phase 1, we can transform $G_{E'}$ into a k-edge-connected perfect b-matching without increasing the cost.*

Proof: Let $V = \{u, v, w, z\}$ and $H = \{uv, vw, wz, zu\}$. Now $E' = M' \cup H_1 \cup \cdots \cup H_{k'}$ ($k \geq 2$). Then either $M' = \{uv, wz\}$ (or $\{vw, zu\}$) or $M' = \{uw, vz\}$ holds. In both cases, we replace $M' \cup H_1 \cup H_2$ by $E'' = \{uv, vw, wz, zu, uw, vz\}$ (see Fig. 2). Then, we can see that $d(v; G_{E''}) = 3$ for all $v \in V$ and $G_{E''}$ is 3-edge-connected. Since $d(v; G_{H_i}) = 2$ for $v \in V, i = 3, \ldots, k'$ and G_{H_i} is 2-edge-connected for $i = 3, \ldots, k'$, it holds that $d(v; G_{E'' \cup H_3 \cup \cdots \cup H_{k'}}) = 3 + 2(k' - 2) = k = b(v)$ for $v \in V$ and the edge-connectivity of $G_{E'' \cup H_3 \cup \cdots \cup H_{k'}}$ is $3 + 2(k' - 2) = k$ (The existence of strict pair implies that k is odd by Claim 2.).

Hence it suffices to show that $c(E'') \leq c(M') + c(H_1) + c(H_2)$. If $M' = \{uw, vz\}$ (or $\{vw, zu\}$), then it is obvious since $E'' = M' \cup H_1 \subseteq M' \cup H_1 \cup H_2$. Let us consider the other case, i.e., $M' = \{uv, wz\}$. From $M' \cup H_1 \cup H_2$, remove $\{uv, uv\}$, replace $\{wz, zu\}$ by $\{wu\}$, and replace $\{vw, wz\}$ by $\{vz\}$. Then the edge set becomes E'' without increasing edge cost, as required. □

In the following, we assume that $|S| \geq 3$ when $N = \emptyset$. In this case, Phase 2 modifies only edges in H_i, $i = 1, \ldots, k'$ while keeping the edges in M' unchanged. Let $V(H_i)$ denote the set of vertices spanned by H_i. We define *detaching v from cycle H_i* to be an operation that replaces the pair $\{uv, vw\} \subseteq H_i$ of edges incident to v by a new edge uw. Note that this decreases $d(v)$ by 2, but H_i remains a cycle on $V(H_i) := V(H_i) - \{v\}$. For each excess vertex v in $G_{E'}$, Phase 2 reduces $d(v)$ to $b(v)$ by detaching v from $(d(v; G_{E'}) - b(v))/2$ cycles in $H_1, \ldots, H_{k'}$. We notice that $(d(v; G_{E'}) - b(v))/2 \leq k'$ by $d(v; G_{E'}) - b(v) \leq d(v; G_E) - b(v) = 2k'$. One important point is to keep $|V(H_i)| \geq 2$ for each $i = 1, \ldots, k'$ during Phase 2. In other words, we always select H_i with $|V(H_i)| \geq 3$ to detach an excess vertex. This is necessary because, if we detach a vertex from H_i with $V(H_i) = 2$, then H_i becomes a loop. In addition, we detach the two excess vertices u and v in a strict pair from different cycles in $H_1, \ldots, H_{k'}$, respectively. This is in order to maintain the k-edge-connectivity of $G_{E'}$ as will be explained below.

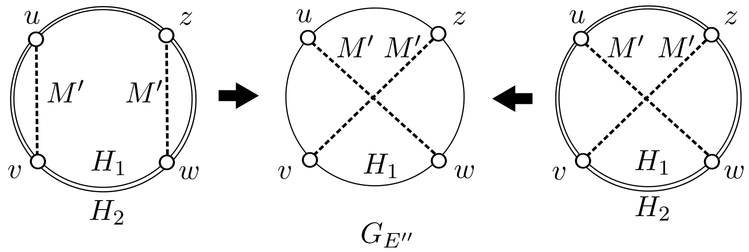

Figure 2: Operations when V consists of two strict pairs

Claim 4 *It is possible to decrease the degree of each excess vertex v in $G_{E'}$ to $b(v)$ by detaching from some cycles in $H_1, \ldots, H_{k'}$ so that $|V(H_i)|$ remains at least 2 for $i = 1, \ldots, k'$ and the two excess vertices in each strict pair are detached from H_i and H_j with $i \neq j$, respectively.*

Proof: First, let us consider the case of $S \neq \emptyset$. Recall $k \geq 3$ and $k' = \lceil k/2 \rceil \geq 2$ in this case. For each strict pair $\{u, v\} \in S$, we detach u and v from different cycles in $H_1, \ldots, H_{k'}$. On the other hand, we detach excess vertex z from arbitrary $(d(z; G_{E'}) - b(z))/2$ cycles. After this, each of $H_1, \ldots, H_{k'}$ is incident to at least one vertex of any strict pair in S in addition to all non-excess vertices in N. By the relation between $|S|$ and $|N|$ we explained in the above, it holds that $|V(H_i)| \geq |S| + |N| \geq 2$ for each $i = 1, \ldots, k'$, as required.

Next, let us consider the case of $S = \emptyset$. As explained in the above, $|N| \geq 1$ holds for this case. If $|N| \geq 2$, the claim is obvious since each of $H_1, \ldots, H_{k'}$ is always incident to all vertices in N. Hence suppose that $|N| = 1$, and let x be the unique non-excess vertex in N. Then all edges in M' are incident to x, since otherwise $S = \emptyset$ implies that Operation 1 or 2 would be applicable to some vertex in $V - x$. In other words, $b(x) = d(x; G_{E'}) = |M'| + 2k'$ holds before Phase 2. Moreover $\sum_{v \in V - x} b(v) \geq b(x)$ also holds by the assumption that perfect b-matchings exist. Now assume that we have converted some excess vertices in $G_{E'}$ into non-excess vertices by detaching them from some of $H_1, \ldots, H_{k'}$ while keeping $|V(H_i)| \geq 2$, $i = 1, \ldots, k'$, and yet an excess vertex $y \in V - x$ remains. Hence $\sum_{v \in V} d(v) > \sum_{v \in V} b(v)$. Then there remains a cycle H_i with $|V(H_i)| > 2$ because

$$
\begin{aligned}
2 \sum_{1 \leq i \leq k'} |V(H_i)| = \sum_{v \in V} d(v; G_{H_1 \cup \cdots \cup H_{k'}}) &= \sum_{v \in V} d(v) - 2|M'| \\
&> \sum_{v \in V - \{x\}} b(v) + b(x) - 2|M'| \geq 2(b(x) - |M'|) \geq 4k'.
\end{aligned}
$$

Therefore we can detach an excess vertex y from such H_i as long as such a vertex exists. This implies that the claim holds also for $|N| = 1$. □

In the following, we let H'_i denote H_i after Phase 2, and H_i denote the original Hamiltonian cycle for $i = 1, \ldots, k'$. Moreover let $E'' = M' \cup H'_1 \cup \cdots \cup H'_{k'}$. The algorithm outputs $G_{E''}$. The entire algorithm is described as follows.

Algorithm UNDIRECT(k)

Input: A vertex set V, a degree specification $b : V \to \mathbb{Z}_+$, a metric edge cost $c : V \to \mathbb{Q}_+$, and a positive integer k

Output: A k-edge-connected perfect b-matching or "INFEASIBLE"

if $\sum_{v \in V} b(v)$ is odd, $\exists v : b(v) > \sum_{u \in V - v} b(u)$ or $k > b(v)$ **then**
    Output "INFEASIBLE" and halt
end if;
Compute a minimum cost perfect b-matching G_M;
if $|V| \leq 3$ **then**
    Output G_M and halt
end if;
Compute a Hamiltonian cycle G_H on V by Christofides' algorithm;
$k' := \lceil k/2 \rceil$; Let $H_1, \ldots, H_{k'}$ be k' copies of H;

\# Phase 1
$M' := M$;
while Operation 1 or 2 is applicable to a vertex $v \in V$
with $d(v; G_{M' \cup H_1 \cup \cdots \cup H_{k'}}) > b(v)$ **do**
    if $\exists \{xv, vy\} \subseteq M'$ such that $x \neq y$ **then**
        $M' := (M' - \{xv, vy\}) \cup \{xy\}$ \# Operation 1
    else
        if $\exists \{xv, vx\} \subseteq M'$ such that $d(x; G_{M' \cup H_1 \cup \cdots \cup H_{k'}}) > b(x)$ **then**
            $M' := M' - \{xv, vx\}$ \# Operation 2
        end if
    end if
end while;

\# Phase 2
if V consists of two strict pairs **then**
    Rename vertices so that $H = \{uv, vw, wz, zu\}$;
    $H'_2 := \emptyset$; $M' := \{uw, vz\}$;
    Output $G_{M' \cup H'_1 \cup \cdots \cup H'_{k'}}$ and halt
end if;
$H'_i := H_i$ for each $i = 1, \ldots, k'$;
while $\exists v \in V$ with $d(v; G_{M' \cup H'_1 \cup \cdots \cup H'_{k'}}) > b(v)$ **do**
    if v and $n(v)$ forms a strict pair **then**
        Detach v from H'_i and $n(v)$ from H'_j, where $i \neq j$
    else
        Detach v from H'_i with $V(H'_i) > 2$
    end if
end while;
$E'' := M' \cup H'_1 \cup \cdots \cup H'_{k'}$;
Output $G_{E''}$

Claim 5 *$G_{E''}$ is a k-edge-connected perfect b-matching.*

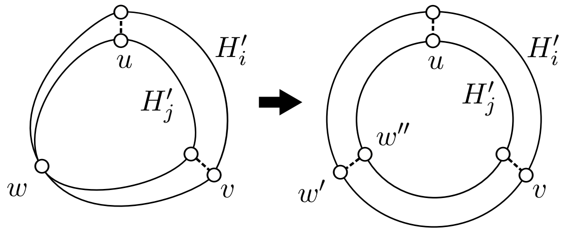

Figure 3: Reduction to the case of $V(H'_i) \cap V(H'_j) = \emptyset$

Proof: We have already seen the case in which V consists of two strict pairs. Hence we suppose the other case in the following. Moreover we have already observed that $d(v; G_{E''}) = b(v)$ holds for each $v \in V$. Furthermore $G_{E''}$ is loopless since G_E is loopless and no operations in the algorithm generate loops. Hence we prove the k-edge-connectivity of $G_{E''}$ below.

Let $u, v \in V$. (i) First suppose that u and v are in some (possibly different) strict pairs in $G_{E'}$. Moreover, let $u \notin V(H'_i)$ and $v \notin V(H'_j)$ (hence $u \in V(H'_{i'})$ for $i' \neq i$ and $v \in V(H'_{j'})$ for $j' \neq j$). For each $\ell \in \{1, \ldots, k'\} - \{i, j\}$, $\lambda(u, v; G_{H'_\ell}) = 2$ holds because $u, v \in V(H'_\ell)$. If $i = j$, $\lambda(u, v; G_{H'_i \cup M'}) = 1$ holds because $d(u; G_{M'}) = d(v; G_{M'}) = 1$ and $n(u), n(v) \in V(H'_i)$. Then it holds that $\lambda(u, v; G_{E''}) = 2(k'-1)+1 = k$ in this case (Recall that the existence of strict pairs implies that k is odd by Claim 2). Hence we let $i \neq j$, and show that $\lambda(u, v; G_{H'_i \cup H'_j \cup M'}) \geq 3$ from now on, from which $\lambda(u, v; G_{E''}) \geq 2(k'-2)+3 = k$ can be derived.

Let N and S denote the sets of non-excess vertices and strict pairs in $G_{E'}$ after Phase 1, respectively. Suppose that $V(H'_i) \cap V(H'_j) = \emptyset$. In this case, it can be seen that $N = \emptyset$, and hence $|S| \geq 3$ by the assumption about the relation between N and S. Since at least one vertex of each strict pair is spanned by each cycle in $H'_1, \ldots, H'_{k'}$, we can see that M' contains at least three vertex-disjoint edges that join vertices in $V(H'_i)$ and in $V(H'_j)$, two of which are u and v. This indicates that $\lambda(u, v; G_{H'_i \cup H'_j \cup M'}) \geq 3$ holds (see the graph of Figure 3 (b)).

Let us consider the case of $V(H'_i) \cap V(H'_j) \neq \emptyset$ in the next. By the existence of u and v, $|S| \geq 1$ holds. If u and v forms a strict pair (i.e., $uv \in M'$), $\lambda(u, v; G_{M'}) = 1$ holds. Since $V(H'_i) \cap V(H'_j) \neq \emptyset$ implies $\lambda(G_{H'_i \cup H'_j}) \geq 2$, we see that $\lambda(u, v; G_{H'_i \cup H'_j \cup M'}) \geq 3$ in this case. Thus let u and v belong to different strict pairs (i.e., $|S| \geq 2$). Then there exists two vertex-disjoint edges in M' joins vertices in $V(H'_i)$ and in $V(H'_j)$ (see Figure 3 (a)). If we split each vertex $w \in V(H'_i) \cap V(H'_j)$ into two vertices w' and w'' so that H'_i and H'_j are vertex-disjoint cycles, and add new edges $w'w''$ joining those two split vertices to M', then we can reduce this case to the case of $V(H'_i) \cap V(H'_j) = \emptyset$, in which $\lambda(u, v; G_{H'_i \cup H'_j \cup M'}) \geq 3$ has already been observed in the above (see Figure 3). Accordingly, we have $\lambda(u, v; G_{H'_i \cup H'_j \cup M'}) \geq 3$ if u and v are in some strict pairs, as required.

(ii) In the next, let u and v be not in any strict pairs. For $z \in \{u, v\}$, let $n'(z)$ denote z itself if $z \in N$, and $n(z)$ otherwise. Notice that $n'(z) \in N$ for any $z \in \{u, v\}$, i.e., it is spanned by $H'_1, \ldots, H'_{k'}$. If $z \in \{u, v\}$ is not spanned by $p > 0$ cycles in $H'_1, \ldots, H'_{k'}$ (and hence z is an excess vertex in $G_{E'}$), then z has at least $k - 2(k' - p)$ incident edges in M'

because $d(z; G_{M'}) = b(z) - d(z; G_{H'_1 \cup \cdots \cup H'_{k'}}) \geq k - 2(k' - p)$. Hence $\lambda(z, n'(z); G_{E''}) \geq 2(k' - p) + k - 2(k' - p) = k$ holds for each $z \in \{u, v\}$, where we define $\lambda(z, z; G_{E''}) = +\infty$. Moreover it is obvious that $\lambda(n'(u), n'(v); G_{E''}) \geq 2k'$. Therefore, it holds that

$$\lambda(u, v; G_{E''}) \geq \min\{\lambda(u, n'(u); G_{E''}), \lambda(n'(u), n'(v); G_{E''}), \lambda(n'(v), v; G_{E''})\} \geq k.$$

(iii) Finally, let us consider the remaining case, i.e., u is in a strict pair and v is a vertex which is not in any strict pair. Let us define $n'(v)$ as in the above. Then $\lambda(v, n'(v); G_{E''}) \geq k$ holds. Without loss of generality, let u be detached from H'_1, and spanned by $H'_2, \ldots, H'_{k'}$. Since $un(u) \in M'$ and $n(u), n'(v) \in V(H'_1)$, it holds that $\lambda(u, n(u); G_{M' \cup H'_1}) = 1$, and $\lambda(n(u), n'(v); G_{M' \cup H'_1}) \geq 2$. Then,

$$\begin{aligned}\lambda(u, n'(v); G_{E''}) \geq \min\{&\lambda(u, n(u); G_{M' \cup H'_1}), \lambda(n(u), n'(v); G_{M' \cup H'_1})\} \\ &+ \lambda(u, n'(v); G_{H'_2 \cup \cdots \cup H'_{k'}}) \geq 1 + 2(k' - 1) = 2k' - 1 = k.\end{aligned}$$

Therefore,

$$\lambda(u, v; G_{E''}) \geq \min\{\lambda(u, n'(v); G_{E''}), \lambda(v, n'(v); G_{E''})\} \geq k,$$

holds, as required. □

Let us consider the cost of the graph $G_{E''}$. The following theorem on the Christofides' algorithm gives us an upper bound on $c(H)$. Here, we let $\delta(U)$ denote the set of edges whose one end vertex is in U and the other is in $V - U$ for nonempty $U \subset V$.

Theorem 2 ([7, 13]) *Let*

$$\begin{array}{lll} OPT_{TSP} = \min & \sum_{e \in E} c(e) x(e) & \\ \textit{subject to} & \sum_{e \in \delta(U)} x(e) \geq 2 & \textit{for each nonempty } U \subset V, \\ & x(e) \geq 0 & \textit{for each } e \in E. \end{array}$$

Christofides' algorithm for TSP always outputs a solution of cost at most $1.5 OPT_{TSP}$.
□

Claim 6 $c(E'')$ *is at most* $1 + 3\lceil k/2 \rceil / k$ *times the optimal cost of* k*-ECMDS.*

Proof: No operation in Phases 1 and 2 increases the cost of the graph since the edge cost is metric. Hence it suffices to show that $c(M \cup H_1 \cup \cdots \cup H_{k'})$ is at most $(1 + 3\lceil k/2 \rceil / k) \cdot c(G)$, where G denotes an optimal solution of k-ECMDS. Since G is a perfect b-matching, $c(M) \leq c(G)$ obviously holds. Thus it suffices to show that $c(H_i) \leq 3c(G)/k$ for $1 \leq i \leq k'$, from which the claim follows.

Let $x_G : \binom{V}{2} \to \mathbb{Z}_+$ be the function such that $x_G(uv)$ denotes the number of edges joining u and v in G. Since G is k-edge-connected, $\sum_{e \in \delta(U)} x_G(e) \geq k$ holds for every nonempty $U \subset V$. Hence $2x_G/k$ is feasible for the linear programming in Theorem 2, which means that $\mathrm{OPT}_{TSP} \leq 2c(G)/k$. By Theorem 2, $c(H_i) \leq 1.5\mathrm{OPT}_{TSP}$. Therefore we have $c(H_i) \leq 3c(G)/k$, as required. □

Claims 5 and 6 establish the next.

Theorem 3 *Algorithm* UNDIRECT(k) *is a ρ-approximation algorithm for k-ECMDS, where $\rho = 2.5$ if k is even and $\rho = 2.5 + 1.5/k$ if k is odd.* □

Algorithm UNDIRECT(k) always outputs a solution for $k \geq 2$ as long as there exists a perfect b-matching and $b(v) \geq k$ for all $v \in V$. This fact and Theorem 1 imply the following corollary.

Corollary 1 *For $k \geq 2$, there exists a k-edge-connected perfect b-matching if and only if $\sum_{v \in V} b(v)$ is even and $k \leq b(v) \leq \sum_{u \in V-v} b(u)$ for all $v \in V$.* □

We close this section with a few remarks. The operations in Phases 1 and 2 are equivalent to a graph transformation called *splitting*, followed by removing generated loops if any. There are many results on the conditions for splitting to maintain the edge-connectivity [3, 10]. However, the splittings in these results may generate loops. Hence algorithm UNDIRECT(k) needs to specify a sequence of splitting so that removing loops does not make the degrees lower than the degree specification.

One may consider that a perfect $(b - 2k')$-matching is more appropriate than a perfect b-matching as a building block of our algorithm, since there is no excess vertex for the union of a perfect $(b - 2k')$-matching and k' Hamiltonian cycles. However, there is a degree specification b that admits a perfect b-matching, and no perfect $(b - 2k')$-matching. Furthermore, even if there exits a perfect $(b - 2k')$-matching, the minimum cost of the perfect $(b - 2k')$-matching may not be a lower bound on the optimal cost of k-ECMDS. Therefore we do not use a perfect $(b - 2k')$-matching in general case. In Section 4, we show that a perfect $(b - 2k')$-matching always exist and its cost can be estimated when a degree specification b is uniform.

3 Algorithm for k-ACMDS

This section shows that k-ACMDS is 2.5-approximable. The algorithm for k-ACMDS can be designed analogously with that for k-ECMDS. Before describing the algorithm, we consider the feasibility of k-ACMDS.

3.1 Feasibility

Frobenius' classic theorem (see [11] for example) tells the relation-ship between the existence of perfect bipartite matchings and the minimum size of vertex covers in bipartite graphs.

Theorem 4 (Frobenius) *A bipartite graph G has a perfect matching if and only if each vertex cover has size at least $|V(G)|/2$.* □

From this, we can immediately derive a condition for a digraph to have a perfect (b^-, b^+)-matching.

Theorem 5 *Let V be a vertex set, and $b^-, b^+ : V \to \mathbb{Z}_+$ be in- and out- degree specifications, respectively. There exists a perfect (b^-, b^+)-matching if and only if $\sum_{v \in V} b^-(v) =$*

$\sum_{v\in V} b^+(v)$, $b^-(v) \le \sum_{u\in V-v} b^+(u)$ *for each* $v \in V$, *and* $b^+(v) \le \sum_{u\in V-v} b^-(u)$ *for each* $v \in V$.

Proof: The necessity is obvious. Hence we consider the sufficiency in the following. For each $v \in V$, prepare two vertex sets V_v^- and V_v^+ corresponding to v such that $|V_v^-| = b^-(v)$ and $|V_v^+| = b^+(v)$. Furthermore, let $V^- = \cup_{v\in V} V_v^-$, $V^+ = \cup_{v\in V} V_v^+$, and $E = \{u^-v^+ \mid u^- \in V_u^-, v^+ \in V_v^+, u \neq v\}$. Then a perfect matching in a bipartite graph (V^-, V^+, E) corresponds to a perfect (b^-, b^+)-matching on V. So by Theorem 4, it suffices to show that each vertex cover of (V^-, V^+, E) has size at least $(|V^-| + |V^+|)/2$.

To the contrary, let us suppose that there exists a vertex cover $C \subset V^- \cup V^+$ of (V^-, V^+, E) such that $|C| < (|V^-|+|V^+|)/2$ under the assumption in this theorem. Since $|V^-| = \sum_{v\in V} b^-(v) = \sum_{v\in V} b^+(v) = |V^+|$, it holds that $|C| < |V^-| = |V^+|$. This implies the existence of vertices $x \in V^- - C$ and $y \in V^+ - C$. Let x correspond to $u \in V$ (i.e., $x \in V_u^-$) and y correspond to $v \in V$ (i.e., $y \in V_v^+$). If $u \neq v$, there exists an edge $xy \in E$, which is not covered by any vertices in C, a contradiction. Hence $u = v$ holds. Then $\cup_{z\in V-v}(V_z^- \cup V_z^+) \subseteq C$ holds. This implies that $|C| \ge \sum_{z\in V-v} |V_z^-| + \sum_{z\in V-v} |V_z^+|$. Then it holds that

$$\begin{aligned}(\sum_{v\in V} b^-(v) + \sum_{v\in V} b^+(v))/2 &= (|V^-| + |V^+|)/2 > |C| \\ &\ge \sum_{z\in V-v} |V_z^-| + \sum_{z\in V-v} |V_z^+| = \sum_{z\in V-v} b^-(z) + \sum_{z\in V-v} b^+(z),\end{aligned}$$

implying $b^-(v) + b^+(v) > \sum_{z\in V-v} b^-(z) + \sum_{z\in V-v} b^+(z)$. However, this indicates that at least $b^-(v) > \sum_{z\in V-v} b^-(z)$ or $b^+(v) > \sum_{z\in V-v} b^+(z)$ holds, contradicting to the assumption. $\square$

Notice that the proof of Theorem 5 indicates the reduction of the minimum cost perfect (b^-, b^+)-matching problem to the minimum cost perfect b-matching problem in an undirected bipartite graph.

3.2 Algorithm

We are ready to explain the algorithm for k-ACMDS. In the following, we assume that $b^-(v), b^+(v) \ge k$ for each $v \in V$ and a perfect (b^-, b^+)-matching exists.

Let M be a minimum cost perfect (b^-, b^+)-matching and H be a directed Hamiltonian cycle constructed by Christofides' algorithm for the edge cost obtained from c by ignoring the direction of arcs (Recall that c is symmetric). Moreover let $H_1, \ldots, H_k$ be k copies of H, $A = M \cup H_1 \cup \cdots \cup H_k$, and D_F denote the digraph (V, F) for an arc set F. A vertex $v \in V$ is called an *excess vertex* if $d^-(v) > b^-(v)$ or $d^+(v) > b^+(v)$ (otherwise v is called a *non-excess vertex*). Notice that $d^-(v; D_A) - b^-(v) = d^+(v; D_A) - b^+(v)$. This condition is maintained throughout the algorithm, i.e., $d^-(v) > b^-(v)$ is equivalent to $d^+(v) > b^+(v)$. Our algorithm for k-ACMDS decreases the degree of excess vertices as k-ECMDS. One difference between algorithms for k-ECMDS and for k-ACMDS is the definition of Operations 1 and 2. These will be executed for a pair of arcs entering and leaving the same vertex as follows.

Operation 1: If an excess vertex v has two incident arcs xv and vy in M with $x \neq y$, replace xv and vy by new edge $xy \in M$.

Operation 2: If an excess vertex v has two arcs uv and vu in M with $d^-(u) > b^-(u)$ (and $d^+(v) > b^+(v)$), remove these arcs.

Phase 1 of our algorithm modifies edges in M by repeating Operations 1 and 2 until none of them is executable. We let M' denote M after Phase 1, and M denote the original set in the following. Moreover let $A' = M' \cup H_1 \cup \cdots \cup H_k$, and N denote the set of non-excess vertices in $D_{A'}$. Note that the number of arcs in M' entering (resp., leaving) each excess vertices v in $D_{A'}$ has $d^-(v; D_{A'}) - k \geq d^-(v; D_{A'}) - b^-(v)$ (resp., $d^-(v; D_{A'}) - b^-(v) > d^+(v; D_{A'}) - b^+(v)$) arcs. The other end vertex of them is unique and in N (i.e., a non-excess vertex in $D_{A'}$) since otherwise Operation 1 or 2 can be applied to v. This situation is simpler than after Phase 2 of UNDIRECT(k) since no correspondence of strict pairs exists. Notice that $N \neq \emptyset$ always holds here.

Phase 2 of our algorithm for k-ACMDS modifies edges in $H_1, \ldots, H_k$ so as to decrease the degrees of all excess vertices as in UNDIRECT(k). We repeat *detaching* each excess vertex from some of $H_1, \ldots, H_k$, where detaching a vertex v from H_i is defined as an operation that replaces the pair $\{uv, vw\} \subseteq H_i$ of arcs entering and leaving v by new arc uw. We can prove that it is possible to detach excess vertices from Hamiltonian cycles while keeping $V(H_i) \geq 2$ for $1 \leq i \leq k$ as in UNDIRECT(k).

Claim 7 *It is possible to decrease the degree of each excess vertex v to $b(v)$ by detaching v from some cycles in $H_1, \ldots, H_k$ so that $|V(H_i)|$ remains at least two for all $i = 1, \ldots, k$.*

Proof: Recall that $N \neq \emptyset$. If $|N| \geq 2$, the claim is obvious since each of $H_1, \cdots, H_k$ is incident to all vertices in N. Hence suppose that $|N| = 1$, and let x be the unique vertex in N. Then all arcs in M' are incident to x since otherwise Operation 1 or 2 would be applicable to some vertex in $V - x$. In other words, it hold $|M'| = d^-(x; D_{M'}) + d^+(v; D_{M'}) = b^-(x) + b^+(x) - 2k$. Recall that $\sum_{v \in V - x} b^+(v) \geq b^-(x)$ and $\sum_{v \in V - x} b^-(v) \geq b^+(x)$ hold by the assumption that perfect (b^-, b^+)-matchings exist. Now assume that we have converted some excess vertices in $D_{A'}$ into non-excess vertices by detaching them from some of $H_1, \ldots, H_k$ while keeping $|V(H_i)| \geq 2$, $i = 1, \ldots, k$, and yet an excess vertex $y \in V - x$ remains. Then there remains a cycles H_i with $|V(H_i)| > 2$ because

$$\begin{aligned}\sum_{1 \leq i \leq k} |V(H_i)| &= \sum_{v \in V} d^-(v; D_{H_1 \cup \cdots \cup H_k}) = \sum_{v \in V} d^-(v; D_{E'}) - |M'| \\ &> \sum_{v \in V - \{x\}} b^-(v) + d^-(x; D_{E'}) - |M'| \geq b^+(x) + b^-(x) - |M'| \geq 2k.\end{aligned}$$

Hence we can detach y from such H_i, implying the claim also for $|N| = 1$. □

In the following, we let H'_i denote H_i after Phase 2, and H_i denote the original Hamiltonian cycle for $i = 1, \ldots, k$ in order to avoid the ambiguity. Moreover let $A'' = M' \cup H'_1 \cup \cdots \cup H'_k$. Our algorithm outputs $D_{A''}$ as a solution.

Algorithm DIRECT(k)

Input: A vertex set V, in- and out-degree specification $b^-, b^+ : V \to \mathbb{Z}_+$, a symmetric metric arc cost $c : V \times V \to \mathbb{Q}_+$, and a positive integer k

Output: A k-arc-connected perfect (b^-, b^+)-matching or "INFEASIBLE"

if $\sum_{v\in V} b^-(v) \neq \sum_{v\in V} b^+(v)$, $\exists v : b^-(v) > \sum_{u\in V-v} b^+(u)$, $\exists v : b^+(v) > \sum_{u\in V-v} b^-(u)$, $\exists v : k > b^-(v)$, or $\exists v : k > b^+(v)$ **then**
    Output "INFEASIBLE" and halt
end if;
Compute a minimum cost perfect (b^-, b^+)-matching D_M;
Compute a Hamiltonian cycle D_H on V by Christofides' algorithm; Let $H_1, \ldots, H_k$ be k copies of H;

Phase 1
$M' := M$;
while Operation 1 or 2 is applicable to a vertex $v \in V$ with $d^-(v; D_{M'\cup H_1\cup\cdots\cup H_k}) > b^-(v)$ **do**
    if $\exists\{xv, vy\} \subseteq M'$ such that $x \neq y$ **then**
        $M' := (M' - \{xv, vy\}) \cup \{xy\}$ # Operation 1
    else if $\exists\{xv, vx\} \subseteq M'$ such that $d^-(x; D_{M'\cup H_1\cup\cdots\cup H_k}) > b^-(x)$ **then**
        $M' := M' - \{xv, vx\}$ # Operation 2
    end if
end while;

Phase 2
$H'_i := H_i$ for each $i = 1, \ldots, k$;
while $\exists v \in V$ with $d^-(v; D_{M'\cup H'_1\cup\cdots\cup H'_k}) > b^-(v)$ **do**
    Detach v from H'_i with $V(H'_i) > 2$
end while;
$A'' := M' \cup H'_1 \cup \cdots \cup H'_k$;
Output $D_{A''}$

Let OPT denote the optimal cost of k-ACMDS. We can show that $D_{A''}$ is k-arc-connected, $c(M) \leq \mathrm{OPT}$ and $c(H_i) \leq 1.5\mathrm{OPT}/k$ for $1 \leq i \leq k$, similarly for UNDIRECT(k) although we leave the proof to the readers. As a conclusion, we have the following theorem.

Theorem 6 *Algorithm* DIRECT(k) *is a* 2.5*-approximation algorithm for* k-ACMDS. □

Algorithm DIRECT(k) always outputs a solution when there exists a perfect (b^-, b^+)-matching and $b^-(v) \geq k$, $b^+(v) \geq k$ for all $v \in V$. This fact and Theorem 5 implies the following corollary.

Corollary 2 *For $k \geq 1$, there exists a k-arc-connected perfect (b^-, b^+)-matching if and only if $\sum_{v\in V} b^-(v) = \sum_{v\in V} b^+(v)$, $k \leq b^-(v) \leq \sum_{u\in V-v} b^+(u)$ for each $v \in V$, and $k \leq b^+(v) \leq \sum_{u\in V-v} b^-(u)$ for each $v \in V$.* □

4 Uniform degree specification

In this section, we show that the approximation factor of our algorithms can be improved when $b(v) = \ell$ in k-ECMDS or $b^-(v) = b^+(v) = \ell$ in k-ACMDS for all $v \in V$ with some integer $\ell \geq k$.

We call a perfect b-matching (resp., a perfect (b^-, b^+)-matching) M *ℓ-regular* if $b(v) = \ell$ (resp., $b^-(v) = b^+(v) = \ell$) for all $v \in V$.

Lemma 1 *Assume that $b^-(v) = b^+(v) = \ell$ for all $v \in V$ and an ℓ-regular digraph exists. Let OPT denote the optimal cost of k-ACMDS. Then there exists an $(\ell-m)$-regular digraph D_R with $c(R) \leq \frac{\ell-m}{\ell} OPT$ for an arbitrary non-negative integer $m \leq \ell$.*

Proof: Let A denote an optimal arc set of k-ACMDS. As seen in Section 3, digraph D_A corresponds to the bipartite undirected graph (V^-, V^+, E), which is a ℓ-regular. A theorem derived from Frobenius' theorem tells that every ℓ-regular bipartite graph can be decomposed into ℓ graphs each of which is 1-regular [11]. Let R be the set of arcs corresponding to edges in least cost $\ell - m$ graphs of them. Then R is $(\ell - m)$-regular and $c(R) \leq \frac{\ell-m}{\ell} c(A)$, as required. □

The union of an $(\ell-k)$-regular digraph and k Hamiltonian cycles are obviously feasible to k-ACMDS if $b^-(v) = b^+(v) = \ell$, $v \in V$. Therefore we can derive the following theorem.

Theorem 7 *If $b^-(v) = b^+(v) = \ell$ for all $v \in V$, then k-ACMDS is approximable within a factor of $1.5 + \frac{\ell-k}{\ell}$.* □

Next, we consider k-ECMDS.

Lemma 2 *Assume that $b(v) = \ell$ for all $v \in V$ and an ℓ-regular graph exists. Let OPT denote the optimal cost of k-ECMDS. Then there exists an $(\ell - 2m)$-regular graph G_R such that $c(R) \leq \frac{\ell-2m}{\ell} OPT$ if ℓ is even, and $c(R) \leq (\frac{\ell-2m-1}{\ell} + \frac{1}{k}) OPT$ if ℓ is odd for an arbitrary non-negative integer m with $2m \leq \ell$.*

Proof: Let E denote an optimal edge set of k-ECMDS. First suppose that ℓ is even. Then E can be oriented into an arc set A such that D_A is $\ell/2$-regular. Let c' be an arc cost on A naturally defined from c (i.e., $c'(a) = c(e)$ if $a \in A$ corresponds to $e \in E$). As in the proof of Lemma 1, we can obtain an $(\ell/2 - m)$-regular digraph R' with $c'(R') \leq \frac{\ell/2-m}{\ell/2} c'(A)$. Let R be an edge set corresponding to R'. Then clearly G_R is $(\ell - 2m)$-regular and $c(R) \leq \frac{\ell/2-m}{\ell/2} c(E)$, as required.

Next, suppose that ℓ is odd. Let $2E$ denote the edge set obtained by duplicating each edge in E. Then G_{2E} is 2ℓ-regular. By the above argument about the case of ℓ is even, we can obtain an $(\ell-2m-1)$-regular graph G_F such that $c(F) \leq \frac{\ell-2m-1}{2\ell} c(2E) = \frac{\ell-2m-1}{\ell} c(E)$ (Notice that $\ell - 2m - 1$ is even). Let M be a minimum cost 1-regular graph. Notice that such M exists since $|V|$ is even by the existence of an ℓ-regular graph with odd ℓ. Since the minimum cost of Hamiltonian cycles spanning all vertices is at most $2c(E)/k$ as shown in the proof of Claim 6, we can see that $c(M) \leq c(E)/k$. Let $R = F \cup M$. Then G_R is $(\ell - 2m)$-regular and $c(R) = c(F) + c(M) \leq (\frac{\ell-2m-1}{\ell} + \frac{1}{k}) c(E)$, as required. □

Let $k' = \lceil k/2 \rceil$. The union of an $(\ell - 2k')$-regular graph and $2k'$ Hamiltonian cycles are obviously feasible to k-ECMDS if $b(v) = \ell$, $v \in V$. Therefore we can derive the following theorem.

Theorem 8 *If $b(v) = \ell$ for all $v \in V$, then k-ECMDS is approximable within a factor of $\frac{\ell - 2k'}{\ell} + 3\frac{k'}{k}$ if ℓ is even, and $\frac{(\ell-2k'-1)}{\ell} + \frac{1+3k'}{k}$ if ℓ is odd, where $k' = \lceil k/2 \rceil$.* □

Recall that metric TSP can be formulated as k-ECMDS with $b(v) = 2$, $v \in V$ and $k = 2$. Theorem 8 indicates that this case can be approximated within 1.5 as Christofides' algorithm.

5 Application for (m, n)-VRP

In this section, we consider the problem (m, n)-VRP. The formal definition of this problem is as follows. An instance of (m, n)-VRP consists of a vertex set V containing a special vertex s, a metric edge cost $c : \binom{V}{2} \to \mathbb{Q}_+$, and two non-negative integers m and n. The objective is to find a minimum cost set of m cycles, each containing s, such that each vertex in $V - s$ is contained in exactly n of those cycles. We can assume without loss of generality that $n \le m \le n(|V| - 1)$ since otherwise the instance is clearly infeasible.

An example of applying the (m, n)-VRP is the schedule of garbage collection. Let us consider the case in which a garbage collecting truck must visit each city on n of 5 weekdays in a week. A solution of $(5, n)$-VRP gives a schedule of this truck minimizing total length of routes.

Each solution to (m, n)-VRP is obviously feasible to $2n$-ECMDS with $b(s) = 2m$ and $b(v) = 2n$ for $v \in V - s$ (Hence the optimal value of $2n$-ECMDS with such b is at most that of (m, n)-VRP). However, the opposite direction does not hold as an example in Figure 5. Nevertheless we can see that algorithm UNDIRECT($2n$) outputs a feasible solution for (m, n)-VRP.

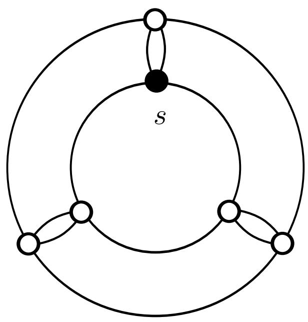

Figure 4: A solution to 4-ECMDS with $b(v) = 4$, $v \in V$, that is not feasible to $(2, 2)$-VRP

Theorem 9 *Let $b(s) = 2m$, $b(v) = 2n$ for each $v \in V - s$ and $k = 2n$. Then algorithm* UNDIRECT(k) *outputs a 2.5-approximate solution to (m, n)-VRP.*

Proof: The solution given by algorithm UNDIRECT(k) consists of edge set M' and cycles $H'_1, \ldots, H'_n$. In what follows, we see that this solution is feasible to (m, n)-VRP.

Let us consider the moment after Phase 1, and define E', M' and $H'_1, \ldots, H'_{k'}$ as in Section 2. Since $k = 2n$ is even, there exists no strict pair. Hence at least one end vertex of each edge in M' is a non-excess vertex. Let v be such a vertex. Then $b(v) = d(v; G_{E'}) > d(v; G_{H_1 \cup \cdots \cup H_n}) = 2n$ (Recall that each non-excess vertex is covered by all of $H_1, \ldots, H_n$). However, a vertex of degree more than $2n$ is only s since $b(u) = 2n$ for each $u \in V - s$. Hence we can see that (i) s is a non-excess vertex after Phase 1, and (ii) one end vertex of each in M' is s. Condition (i) implies that each of $H'_1, \ldots, H'_n$ covers s. Condition (ii) indicates that edges between s and a vertex $v \in V - s$ forms $d(v; M')/2$ cycles whose vertex sets are $\{s, v\}$ because $d(v; M')$ is even. Therefore, combining the fact that $d(v; G_{M' \cup H'_1 \cup \cdots \cup H'_n}) = b(v)$ for all $v \in V$, these shows that UNDIRECT(k) outputs a feasible solution to (m, n)-VRP. $\Box$

The approximation factor can be improved as follows.

Theorem 10 *Problem (m, n)-VRP can be approximated within a factor of* $1.5 + \frac{m-n}{m}$.

Proof: Let $b(s) = 2m$, $b(v) = 2n$ for each $v \in V - s$ and $k = 2n$. Moreover, let E be an optimal solution for (m, n)-VRP, and F be the set of edges contained by $m - n$ cycles in G_E of least cost. Then it holds that $d(s; G_F) = 2m - 2n$ and $d(v; G_F) \leq 2n$ for $v \in V - s$. Besides this, we have $c(F) \leq \frac{m-n}{m} c(E)$ by the definition of F.

Now we let $V - s = \{v_1, \ldots, v_{|V|-1}\}$ so that $c(sv_1) \leq c(sv_2) \leq \cdots \leq c(sv_{|V|-1})$. Moreover we define R as an edge set which consists of $2n$ edges sv_i for each $i = 1, \ldots, p$ and $2m - 2n(p + 1)$ edges sv_{p+1}, where $p = \lfloor (m - n)/n \rfloor$. Then it is clear that R is a minimum cost edge set such that $d(s; G_R) = 2np + 2m - 2n(p + 1) = 2m - 2n$ and $d(v; G_R) \leq 2n$ for all $v \in V - s$. This implies that $c(R) \leq c(F) \leq \frac{m-n}{m} c(E)$.

By using R instead of M in UNDIRECT(k), we can obtain a feasible solution to k-ECMDS. As in Theorem 9, this solution is also feasible to (m, n)-VRP. Moreover the cost of the solution is at most $c(H_1) + \cdots + c(H_{k'}) + c(R) \leq (1.5 + \frac{m-n}{m}) c(E)$, which completes the proof. $\Box$

6 Concluding Remarks

We note that some cases of k-ECMDS/k-ACMDS remain open. One is 1-ECMDS with $b(v) = 1$ for some $v \in V$. Our algorithm cannot deal with this case, because detaching the vertices in a strict pair from the same Hamiltonian cycle in Phase 2 may lose the connectivity. Also a key problem for approximating 1-ECMDS would be to find a minimum cost spanning tree such that $d(v) \leq b(v)$, $v \in V$ for a given $b : V \to \mathbb{Z}_+$. However, no constant factor approximation algorithm is known to this problem if $b(v) = 1$ for some $v \in V$, although it can be approximated within a constant factor of 2 if $b(v) \geq 2$ for all $v \in V$ [1]. Another interesting open problem is a generalization of k-ECMDS (resp., k-ACMDS) in which the k-edge-connectivity (resp., k-arc-connectivity) requirement is replaced by a local-edge-connectivity requirement.

It is also valuable to characterize the feasible solutions to (m, n)-VRP. In Section 5, we noted that specifying the edge-connectivity and the degree of each vertex is not enough for this although our algorithm always outputs a feasible solution to (m, n)-VRP. Moreover, it is interesting to study a further generalizatoin of (m, n)-VRP in which the number $b(v)/2$ of cycles containing each vertex v is not uniform.

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