On the Complexity of Matroid Isomorphism Problem

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Abstract

We study the complexity of testing if two given matroids are isomorphic. The problem is easily seen to be in Σ_2^p. In the case of linear matroids, which are represented over polynomially growing fields, we note that the problem is unlikely to be Σ_2^p-complete and is coNP-hard. We show that when the rank of the matroid is bounded by a constant, linear matroid isomorphism, and matroid isomorphism are both polynomial time many-one equivalent to graph isomorphism.

We give a polynomial time Turing reduction from graphic matroid isomorphism problem to the graph isomorphism problem. Using this, we are able to show that graphic matroid isomorphism testing for planar graphs can be done in deterministic polynomial time. We then give a polynomial time many-one reduction from bounded rank matroid isomorphism problem to graphic matroid isomorphism, thus showing that all the above problems are polynomial time equivalent.

Further, for linear and graphic matroids, we prove that the automorphism problem is polynomial time equivalent to the corresponding isomorphism problems. In addition, we give a polynomial time membership test algorithm for the automorphism group of a graphic matroid.

1   Introduction

Isomorphism problems over various mathematical structures have been a source of intriguing problems in complexity theory (see [AT05]). The most important problem of this domain is the well-known graph isomorphism problem. Though the complexity characterization of the general version of this problem is still unknown, there have been various interesting special cases of the problem which are known to have polynomial time algorithms [BGM82, Luk80]. In this paper we talk about isomorphism problem associated with matroids.

A matroid M is a combinatorial object defined over a finite set S (of size m) called the *ground set*, equipped with a non-empty family $\mathcal{I}$ of subsets of S (containing the empty subset)

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which is closed under taking of subsets and satisfies the *exchange axiom* : for any $I_1, I_2 \in \mathcal{I}$ such that $|I_1| > |I_2|$, $\exists x \in I_1 \setminus I_2$, $I_2 \cup \{x\} \in \mathcal{I}$. The sets in $\mathcal{I}$ are called *independent sets*. The rank of the matroid is the size of the maximal independent set. This provides useful abstractions of many concepts in combinatorics and linear algebra [Whi35]. The theory of matroids is a well studied area of combinatorics [Oxl92]. We study the problem of testing isomorphism between two given matroids.

Two matroids M_1 and M_2 are said to be isomorphic if there is a bijection between the elements of the ground set which maps independent sets to independent sets, (or equivalently circuits to circuits, or bases to bases, see section 2). Quite naturally, the representation of the input matroids is important in deciding the complexity of the algorithmic problem.

There are several equivalent representations of a matroid. For example, enumerating the maximal independent sets (called bases) or the minimal dependent sets (called circuits) also defines the matroid. These representations, although can be exponential in the size of the ground set, indeed exist for every matroid, by definition. With this enumerative representation, Mayhew [May08] studied the matroid isomorphism problem, and shows that the problem is equivalent to graph isomorphism problem. However, a natural question is whether the problem is difficult when the representation of the matroid is more implicit?. In a black-box setting, one can also consider the input representation in the form of an oracle or a black-box, where the oracle answers whether a given set is independent or not.

More implicit (and efficient) representation of matroids have been studied. One natural way is to identify the given matroid with matroids defined over combinatorial or algebraic objects which have implicit descriptions. A general framework in this direction is the representation of a matroid over a field. A matroid $M = (S, \mathcal{I})$ of rank r is said to be *representable* over a field $\mathbb{F}$ if there is a map, $\phi : S \to \mathbb{F}^r$ such that, $\forall A \subseteq S$, $A \in \mathcal{I} \iff \phi(A)$ is linearly independent over $\mathbb{F}^r$ as a vector space. However, there are matroids which do not admit linear representations over any field. (For example, the Vamós Matroid, See Proposition 6.1.10, [Oxl92].). In contrast, there are matroids (called regular matroids) which admit linear representations over all fields.

Another natural representation for a matroid is over graphs. For any graph X, we can associate a matroid $M(X)$ as follows: the set of edges of X is the ground set, and the acyclic subgraphs of the given graph form the independent sets. A matroid M is called a *graphic matroid* (also called polygon matroid or cyclic matroid) if it is isomorphic to $M(X)$ for some graph X. It is known that graphic matroids are linear. Indeed, the incidence matrix of the graph will give a representation over $\mathbb{F}_2$. There are linear matroids which are not graphic. (See [Oxl92] for more details.)

The above definitions themselves highlight the importance of testing isomorphism between two given matroids. We study the isomorphism problem for the case of linear matroids (Linear Matroid Isomorphism problem (LMI) and graphic matroids (Graphic Matroid Isomorphism problem (GMI)).

From a complexity perspective, the general case of the problem is in Σ_2^p. However, it is not even clear a priori if the problem is in NP even in the above restricted cases where there are implicit representations. But we note that for the case of graphic matroids the

problem admits an NP algorithm. Hence an intriguing question is about the comparison of this problem to the well studied graph isomorphism problem.

At an intuitive level, in the graph isomorphism problem we ask for a map between the vertices that preserves the adjacency relations, whereas in the case of graphic matroid isomorphism, we ask for maps between the edges such that the set of cycles (or spanning trees) in the graph are preserved. As an example, in the case of trees, any permutation gives a 2-isomorphism, where as computing the isomorphism between trees is known to be Ł-complete. This indicates that the reduction between the problems cannot be obtained by a local replacement of edges with gadgets, and has to consider the global structure.

An important result in this direction, due to Whitney (see [Whi32]), says that in the case of 3-connected graphs, the graphs are isomorphic if and only if the corresponding matroids are isomorphic (see section 5). Thus the problem of testing isomorphism of graphs and the corresponding graphic matroids are equivalent for the case of 3-connected graphs are equivalent. Despite this similarity between the problems, to the best of our knowledge, there has not been a systematic study of GMI and its relationships to graph isomorphism problem (GI). This immediately gives a motivation to study the isomorphism problem for 3-connected graphs. In particular, from the recent results on graph isomorphism problem for these classes of graphs [DLN08, TW08], it follows that graphic matroid isomorphism problem for 3-connected planar graphs Ł-complete.

In this context we study the general, linear and graphic matroid isomorphism problems. Our main contributions in the paper are as follows:

- Matroid isomorphism problem is easily seen to be in Σ_2^p. In the case of linear matroids where the field is also a part of the input we observe that the problem is coNP-hard (Proposition 3.4), and is unlikely to be Σ_2^p-complete (Proposition 3.2). We also observe that when the rank of the matroid is bounded, linear matroid isomorphism, and matroid isomorphism are both equivalent to GI (Theorem 3.5)[1]

- We develop tools to handle colouring of ground set elements in the context of isomorphism problem. We show that coloured version of the linear matroid isomorphism and graphic matroid isomorphisms are as hard as the general version (Lemma 4.2, 4.1). As an immediate application of this, we show that the automorphism problems for graphic matroids and linear matroids are polynomial time Turing equivalent to the corresponding isomorphism problems. In this context, we also give a polynomial time membership test algorithm for the automorphism group of a graphic matroid (Theorem 6.5).

- We give a polynomial time Turing reduction from graphic matroid isomorphism problem to the graph isomorphism problem by developing an edge colouring scheme which algorithmically uses a decomposition given by [HT73] (and [CE80]) and reduce the

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[1]We note that, although not explicitly stated, the equivalence of bounded rank matroid isomorphism and and graph isomorphism also follows from the results of Mayhew [May08]. However, it is not immediately clear if the GI-hard instances are linearly representable. Our proofs are different and extends this to linear matroids.

graphic matroid (Theorem 5.3). Our reduction, in particular implies that the graphic matroid isomorphism testing for planar graphs can be done in deterministic polynomial time (Corollary 5.9).

- Finally, we give a reduction from bounded rank matroid isomorphism problem to graphic matroid isomorphism (Theorem 5.11), thus showing that all the above problems are poly-time equivalent.

Table 1 below summarizes the complexity of matroid isomorphism problem under various input representations.

Repn. of M_1, M_2	Complexity Bounds for MI
List of Ind. sets	GI-complete [May08]
Linear	GI-hard, coNP-hard ([Hli07, OW02]).
Linear (bounded rank)	GI complete
Graphic	Turing equivalent to GI
Planar	P
Planar 3-connected	L-complete

Table 1: Complexity of MI under various input representations

2 Notations and Preliminaries

All the complexity classes used here are standard and we refer the reader to any standard text book (for e.g. see [Gol08]). Now we collect some basic definitions on matroids (see also [Oxl92]). Formally, a matroid M is a pair $(S, \mathcal{I})$, where S is a finite set called the ground set of size m and $\mathcal{I}$ is a collection of subsets of S such that: (1) the empty set ϕ, is in $\mathcal{I}$. (2) If $I_1 \in I$ and $I_2 \subset I_1$, then $I_2 \in \mathcal{I}$. (3) If $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$, then $\exists x \in I_2 \setminus I_1$ such that $I_1 \cup \{x\}$ is in $\mathcal{I}$.

The RANK function of a matroid is a map rank: $2^S \to \mathbb{N}$, is defined for a $T \subseteq S$, as the maximum size of any element of $\mathcal{I}$ that is contained in T. The rank of the matroid is the maximum value of this function. A *circuit* is a minimal dependent set. Spanning sets are subsets of S which contains at least one basis as its subset. Notice that a set $X \subseteq S$ is spanning if and only if $rank(X) = rank(S)$. Moreover, X is a basis set if and only if it is a minimal spanning set. For any $F \subseteq S$, $cl(F) = \{x \in S : rank(F \cup x) = rank(F)\}$. A set $F \subseteq S$ is a flat if $cl(F) = F$. *Hyperplanes* are flats which are of rank $r - 1$, where $r = \text{RANK}(S)$. $X \subseteq S$ is a hyperplane if and only if it is a maximal non-spanning set.

An isomorphism between two matroids M_1 and M_2 is a bijection $\phi : S_1 \to S_2$ such that $\forall C \subseteq S_1 : C \in \mathcal{C}_1 \iff \phi(C) \in \mathcal{C}_2$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ are the family of circuits of the matroids M_1 and M_2 respectively. Now we state the computational problems more precisely.

Problem 1 (MATROID ISOMORPHISM(MI)). *Given two matroids M_1 and M_2 as their independent set oracles, does there exist an isomorphism between the two matroids?*

Given a matrix A over a field $\mathbb{F}$, we can define a matroid $M[A]$ with columns of A as the ground set and linearly independent columns as the independent sets of $M[A]$. A matroid $M = (E, \mathcal{I})$ with rank= r is said to be representable over $\mathbb{F}$, if there is a map $\Phi : E \to \mathbb{F}^r$ such that $I \in \mathcal{I} \iff \Phi(I)$ is linearly independent in $\mathbb{F}^n$. *Linear matroids* are matroids representable over fields. Without loss of generality we can assume that the representation is of the form of a matrix where the columns of the matrix correspond to the ground set elements. We assume that the field on which the matroid is represented is also a part of the input, also that the field has at least m elements and at most $poly(m)$ elements, where $m = poly(n)$.

Problem 2 (LINEAR MATROID ISOMORPHISM(LMI)). *Given two matrices A and B over a given field $\mathbb{F}$ does there exist an isomorphism between the two linear matroids represented by them?.*

As mentioned in the introduction, given a graph $X = (V, E)$ ($|V| = n, |E| = m$), a classical way to associate a matroid $M(X)$ with X is to treat E as ground set elements, the bases of $M(X)$ are spanning forests of X. Equivalently circuits of $M(X)$ are simple cycles in X. A matroid M is called *graphic* iff $\exists X$ such that $M = M(X)$.

Evidently, adding vertices to a graph G with no incident edges will not alter the matroid of the graph. Without loss of generality we can assume that G does not have self-loops.

Problem 3 (GRAPHIC MATROID ISOMORPHISM(GMI)). *Given two graphs X_1 and X_2 does there exist an isomorphism between $M(X_1)$ and $M(X_2)$?.*

Another associated terminology in the literature is about 2-ismorphism. Two graphs X_1 and X_2 are said to 2-*isomorphic* (denoted by $X_1 \cong_2 X_2$) if their corresponding graphic matroids are isomorphic. Thus the above problem asks to test if two given graphs are 2-isomorphic.

In a rather surprising result, Whitney [Whi33] came up with a combinatorial characterisation of 2-isomorpic graphs. We briefly describe it here. Whitney defined the following operations.

- *Vertex Identification:* Let v and v' be vertices of distinct components of X. We modify X by identifying v and v' as a new vertex $\bar{v}$.

- *Vertex Cleaving:* This is the reverse operation of vertex identification so that a graph can only be cleft at the a cut-vertex or at a vertex incident with a loop.

- *Twisting:* Suppose that the graph X is obtained from the disjoint graphs X_1 and X_2 by identifying vertices u_1 of X_1 and u_2 of X_2 as the vertex u of X, identifying vertices v_1 of X_1 and v_2 of X_2 as the vertex v of X. In a *twisting* of X about $\{u, v\}$, we identify, instead u_1 with v_2 and u_2 with v_1 to get a new graph X'.

Theorem 2.1 (Whitney's 2-ismorphism theorem). *([Whi33], see also [Oxl92]) Let X_1 and X_2 be two graphs having no isolated vertices. Then $M(X_1)$ and $M(X_2)$ are isomorphic if and only if X_1 can be transformed to a graph isomorphic to X_2 by a sequence of operations of vertex identification, cleaving and/or twisting.*

The graphic matroids of planar graphs are called *planar matroids*. We now define the corresponding isomorphism problem for graphic matroids,

Problem 4 (PLANAR MATROID ISOMORPHISM(PMI)). *Given two planar graphs X_1 and X_2 does there exist an isomorphism between their graphic matroids ?.*

As a basic complexity bound, it is easy to see that MI $\in \Sigma_2^p$. Indeed, the algorithm will existentially guess a bijection $\sigma : S_1 \to S_2$ and universally verify if for every subset $C \subseteq S_1$, $C \in \mathcal{C}_1 \iff \sigma(C) \in \mathcal{C}_2$ using the independent set oracle.

3 Linear Matroid Isomorphism

In this section we present some observations and results on LINEAR MATROID ISOMORPHISM. Some of these follow easily from the techniques in the literature. We make them explicit in a form that is relevant to the problem that we are considering.

We first observe that using the arguments similar to that of [KST93] one can show $\overline{\text{LMI}} \in \mathsf{BP}.\Sigma_2^\mathsf{P}$ (Notice that an obvious upper bound for this problem is Π_2). We include some details of this here while we observe some points about the proof.

Proposition 3.1. $\overline{\text{LMI}} \in \mathsf{BP}.\Sigma_2^\mathsf{P}$

Proof. Let M_1 and M_2 be the given linear matroids having m columns each. We proceed as in [KST93], for the case of GI. To give a $\mathsf{BP}.\Sigma_2^\mathsf{P}$ algorithm for $\overline{\text{LMI}}$, define the following set:

$$N(M_1, M_2) = \{(N, \phi) : (N \cong M_1) \vee (N \cong M_2) \wedge \phi \in Aut(N)\}$$

where $Aut(H)$ contains all the permutations (bijections) which are isomorphisms of matroid N to itself. The key property that is used in [KST93] has the following easy counterpart in our context.

For any matroid M on a ground set of size m, if $Aut(M)$ denote the automorphism group of M, $\#M$ denotes the number of different matroids isomorphic to M, $|Aut(M)| * (\#M) = |S_m|$.

$$M_1 \cong M_2 \implies |N(M_1, M_2)| = m!$$
$$M_1 \not\cong M_2 \implies |N(M_1, M_2)| = 2.m!$$

As in [KST93], we can amplify this gap and then using a good hash family and utilise the gap to distinguish between the two cases. In the final protocol (before amplifying) the verifier chooses a hash function and sends it to the prover, the prover returns a tuple (N, ϕ) along with a proof that this belongs to $N(M_1, M_2)$. (Notice that this will not work over very

large fields, especially over infinite fields.) Verifier checks this claim along with the hash value of the tuple. This can be done in Σ_2^p. Hence the entire algorithm gives an upper bound of $\mathsf{BP}.\exists.\Sigma_2^p = \mathsf{BP}.\Sigma_2^p$, and thus the result follows. $\qquad\square$

Now, we know that [Sch99], if $\Pi_2^p \subseteq \mathsf{BP}.\Sigma_2^p$ then $\mathsf{PH} = \mathsf{BP}.\Sigma_2^p = \Sigma_3^p$. Thus we get the following:

Theorem 3.2. $\mathrm{LMI} \in \Sigma_2^p$. *In addition,* LMI *is* Σ_2^{P}-*hard* $\implies \mathsf{PH} = \Sigma_3^{\mathsf{P}}$.

We notice that a special case of this is already known to be coNP-hard. A matroid of rank k is said to be *uniform* if all subsets of size at most k are independent. Testing if a given linear matroid of rank k is uniform is known to be coNP-complete [OW02]. We denote them by $U_{k,m}$, the uniform matroid whose ground set is of m elements. Now notice that the above result is equivalent to checking if the given linear matroid of rank k is isomorphic to $U_{k,m}$. To complete the argument, we use a folklore result that $U_{k,m}$ is representable over any field $\mathbb{F}$ which has at least m non-zero elements. We give some details here since we have not seen an explicit description of this in the literature.

Claim 3.3. *Let* $|\mathbb{F}| > m$, $U_{k,m}$ *has a representation over* $\mathbb{F}$.

Proof. Let $\{\alpha_1, \ldots, \alpha_m\}$ be distinct elements of $\mathbb{F}$, and $\{s_1, \ldots, s_m\}$ be elements of the ground set of $U_{k,m}$ Assign the vector $(1, \alpha_i, \alpha_i^2, \ldots, \alpha_i^{k-1}) \in \mathbb{F}^k$ to the element s_i. Any k subset of these vectors forms a Vandermonde matrix, and hence linearly independent. Any larger set is dependent since the vectors are in $\mathbb{F}^k$. $\qquad\square$

This gives us the following proposition.

Proposition 3.4. LMI *is* coNP-*hard.*

The above proposition also holds when the representation is over infinite fields. In this case, the proposition also more directly follows from a result of Hlinený [Hli07], where it is shown that the problem of testing if a spike (a special kind of matroids) represented by a matrix over $\mathbb{Q}$ is the free spike is coNP complete. He also derives a linear representation for spikes.

Now we look at bounded rank variant of the problem. We denote by LMI_b (MI_b), the restriction of LMI (MI) for which the input matrices have rank bounded by b. In the following we use the following construction due to Babai [Bab78] to prove $\mathrm{LMI}_b \equiv_m^p \mathrm{GI}$.

Given a graph $X = (V, E)$ ($3 \le k \le d$, where, d is the minimum vertex degree of X), define a matroid $M = St_k(X)$ of rank k with the ground set as E as follows: every subset of $k - 1$ edges is independent in M and every subset of E with k edges is independent if and only if they do not share a common vertex. Babai proved that $Aut(X) \cong Aut(St_k(X))$ and also gave a linear representation for $St_k(X)$ (Lemma 2.1 in [Bab78]) for all k in the above range.

Theorem 3.5. *For any constant* $b \ge 3$, $\mathrm{LMI}_b \equiv_m^p \mathrm{GI}$.

Proof. GI $\leq_m^p$ LMI$_b$: Let $X_1 = (V_1, E_1)$ and $X_2 = (V_2, E_2)$ be the given GI instance. We can assume that the minimum degree of the graph is at least 3 since otherwise we can attach cliques of size $n + 1$ at every vertex. We note that from Babai's proof we can derive the following stronger conclusion.

Lemma 3.6. $X_1 \cong X_2 \iff \forall\, k \in [3, d], \; St_k(X_1) \cong St_k(X_2)$

Proof. Suppose $X_1 \cong X_2$ via a bijection $\pi : V_1 \to V_2$. (The following proof works for any $k \in [3, d]$.) Let $\sigma : E_1 \to E_2$ be the map induced by π. That is $\sigma(\{u, v\}) = \{\pi(u), \pi(v)\}$. Consider an independent set $I \subseteq E_1$ in $St_k(X_1)$. If $|I| \leq k-1$ then $|\sigma(I)| \leq k-1$ and hence $\sigma(I)$ is independent in $St_k(X_2)$. If $|I| = k$, and let $\sigma(I)$ be dependent. This means that the edges in $\sigma(I)$ share a common vertex w in X_2. Since π is an isomorphism which induces σ, $\pi^{-1}(w)$ must be shared by all edges in I. Thus I is independent if and only if $\sigma(I)$ is independent. Suppose $St_k(X_1) \cong St_k(X_2)$ via a bijection $\sigma : E_1 \to E_2$. By definition, any subset $H \subseteq E_1$ is a hyperplane of $St_k(X_1)$ if and only if $\sigma(H)$ is a hyperplane of $St_k(X_2)$. Now we use the following claim which follows from [Bab78].

Claim 3.7 ([Bab78]). *For any graph X, any dependent hyperplane in $St_k(X)$ is a maximal set of edges which share a common vertex (forms a star) in X, and these are the only dependent hyperplanes.*

Now we define the graph isomorphism $\pi : V_1 \to V_2$ as follows. For any vertex v, look at the star $E_1(v)$ rooted at v, we know that $\sigma(E_1(v)) = E_2(v')$ for some v'. Now set $\pi(v) = v'$. From the above claim, π is an isomorphism. $\qquad\square$

It remains to show that representation for $St_k(X)$ ($X = (V, E)$) can be computed in polynomial time. We choose $k = 3$ (by the above proof, $\exists k$ and $\forall k$ in the Lemma 3.6 are equivalent). Now we show that the representation of $St_k(X)$ given in [Bab78] is computable in polynomial time. The representation of $St_k(X)$ is over a field $\mathbb{F}$ such that $|\mathbb{F}| \geq |V|^{2k-1}$. For $e = \{u, v\} \in E$ assign a vector $b_e = [1, (x_u + x_v), (x_u x_v), y_{e,1}, \ldots, y_{e,k-3}] \in \mathbb{F}^k$, where x_u, x_v and $y_{e,i}$ are distinct unknowns. To represent $St_k(X)$ we need to ensure that the k-subsets of the columns corresponding to a basis form a linearly independent set, and all the remaining k-subsets form a dependent set. Babai [Bab78] showed that by the above careful choice of b_e, it will be sufficient to ensure only the independence condition. He also proved the existence of a choice of values for the variables which achieves this if $|\mathbb{F}| \geq |V|^{2k-1}$.

We make this constructive. As k is a constant, the number of bases is bounded by $\mathsf{poly}(m)$. We can greedily choose the value for each variable at every step, such that on assigning this value, the resulting set of constant ($k \times k$) size matrices are non-singular. Since there exists a solution, this algorithm will always find one. Thus we can compute a representation for $St_k(X)$ in polynomial time.

LMI$_b \leq_m^p$ GI: Let $A_{k \times m}$ and $B_{k \times m}$ be two matrices of rank b at the input. Now define the following bipartite graph $X_A = (U_A, V_A, E_A)$ (similarly for X_B), where U_A has a vertex for each column of A, and V_A has a vertex for each maximal independent set of A (Notice that there are at most $\binom{m}{b} = O(m^b)$ of them) and $\forall i \in U_A, I \in V_A, \{i, I\} \in E_A \iff i \in I$. Now we claim that $M(A) \cong M(B) \iff X_A \cong X_B$ where the isomorphism maps V_A to

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V_B, and which is reducible to GI. It is easy to see that the matroid isomorphism can be recovered from the map between the sets. □

Observe that the reduction $\text{LMI}_b \leq_m^p \text{GI}$ can be done even if the input representation is an independent set oracle. This gives the following corollary.

Corollary 3.8. $\text{LMI}_b \equiv_m^p \text{MI}_b \equiv_m^p \text{GI}$.

4   Isomorphism Problem of Coloured Matroids

Vertex or edge colouring is a classical tool used extensively in proving various results in graph isomorphism problem. We develop similar techniques for matroid isomorphism problems too.

An edge-k-colouring of a graph $X = (V, E)$ is a function $f : E \to \{1, \ldots, k\}$. Given two coloured graphs $X_1 = (V_1, E_1, f_1)$ and $X_2 = (V_2, E_2, f_2)$, the COLOURED-GMI asks for an isomorphism which preserves the colours of the edges. Not surprisingly, we can prove the following.

Lemma 4.1. COLOURED-GMI *is* AC^0 *many-one reducible to* GMI.

Proof. Let $X_1 = (V_1, E_1, f_1)$ and $X_2 = (V_2, E_2, f_2)$, be the two k-coloured graphs at the input, with $n = |V_1| = |V_2|$. For every edge $e = (u, v) \in E_1$ (respectively E_2), add a path $P_e = \{(u, v_{e,1}), (v_{e,1}, v_{e,2}), \ldots, (v_{e,n+f_1(e)}, v)\}$ of length $n + f_1(e)$ (respectively $n + f_2(e)$)Where $v_{e,1}, \ldots v_{e,n+f_1(e)}$ are new vertices. Let X_1' and X_2' be the two new graphs thus obtained. By definition, any 2-isomorphism between X_1' and X_2' can only map cycles of equal length to themselves. There are no simple cycles of length more than n in the original graphs. Thus, given any 2-isomorphism between X_1' and X_2', we can recover a 2-isomorphism between X_1 and X_2 which preserves the colouring and vice versa. □

Now we generalize the above construction to the case of linear matroid isomorphism. COLOURED-LMI denotes the variant of LMI where the inputs are the linear matroids M_1 and M_2 along with colour functions $c_i : \{1, \ldots, m\} \to \mathbb{N}, i \in \{1, 2\}$. The problem is to test if there is an isomorphism between M_1 and M_2 which preserves the colours of the column indices. We have,

Lemma 4.2. COLOURED-LMI *is* AC^0 *many-one reducible to* LMI.

Proof. Let M_1 and M_2 be two coloured linear matroids represented over a field $\mathbb{F}$. We illustrate the reduction where only one column index of M_1 (resp. M_2) is coloured. Without loss of generality, we assume that there are no two vectors in M_1 (resp.M_2) which are scalar multiples of each other.

We transform M_1 and M_2 to get two matroids M_1' and M_2'. In the transformation, we add more columns to the matrix (vectors to the ground set) and create dependency relations in such a way that any isomorphism between the matroids must map these new vectors in M_1 to the corresponding ones M_2.

We describe this transformation in a generic way for a matroid M. Let $\{e_1, \ldots, e_m\}$ be the column vectors of M, where $e_i \in \mathbb{F}^n$. Let $e = e_1$ be the coloured vector in M.

Choose $m' > m$, we construct $\ell = m + m'$ vectors $f_1, \ldots f_\ell \in \mathbb{F}^{n+m'}$ as the columns of the following $(n + m') \times \ell$ matrix. The ith column of the matrix represents f_i.

$$
\left[
\begin{array}{cccc|cccccc}
e_{11} & e_{21} & \cdots & e_{m1} & e_{11} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
e_{12} & e_{22} & \cdots & e_{m2} & 0 & e_{12} & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
e_{1m} & e_{2m} & \cdots & e_{mm} & 0 & 0 & \cdots & e_{1m} & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & 0 & 1 & -1 & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & 0 & 1 & -1 & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \ddots & \cdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & & 0 & 1 & -1 \\
0 & 0 & \cdots & 0 & -1 & 0 & \cdots & & 0 & 0 & 1 \\
\end{array}
\right]
$$

where -1 denotes the additive inverse of 1 in $\mathbb{F}$. Denote the above matrix as $M' = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

Let $S = \{f_{m+1}, \ldots, f_{m+m'}\}$. We observe the following:

1. Columns of B generate e_1. Since C is a 0-matrix $f_1 \in Span(S)$.

2. Columns of D are minimal dependent. Any proper subset of columns of D will split the $1, -1$ pair in at least a row and hence will be independent.

3. S is linearly independent. Suppose not. Let $\sum_{i=m}^{m+m'} \alpha_i f_i = 0$. Restricting this to the columns of B gives that $\alpha_j = 0$ for first j such that $e_{1j} \neq 0$. Thus this gives a linearly dependent proper subset of columns of B, and contradicts the above observation.

4. If for any $f \notin S$, $f = \sum_{f_i \in S} \alpha_i f_i$, then α_i's must be the same.

Now we claim that the newly added columns respect the circuit structure involving e_1. Let $\mathcal{C}$ and $\mathcal{C}'$ denote the set of circuits of M and M' respectively.

Claim 4.3.

$$
\{e_1, e_{i_2}, \ldots, e_{i_k}\} \in \mathcal{C} \quad \Longleftrightarrow \quad \begin{aligned} &\{f_1, f_{i_2} \ldots, f_{i_k}\} \in \mathcal{C}' \text{ and} \\ &\{f_{i_2}, \ldots, f_{i_k}, f_{m+1}, \ldots, f_{m+m'}\} \in \mathcal{C}' \end{aligned}
$$

Proof. Suppose $c = \{e_1, e_{i_2}, \ldots, e_{i_k}\}$ is a circuit in M. Then clearly $\{f_1, f_{i_2}, \ldots, f_{i_k}\}$ is a cycle, since they are nothing but vectors in c extended with 0s. Since $\{f_{i_2}, \ldots, f_{i_k}\}$ and $\{f_{m+1}, \ldots, f_{m+m'}\}$ both generate f_1, the set $F = \{f_{i_2}, \ldots, f_{i_k}, f_{m+1}, \ldots, f_{m+m'}\}$ is a linearly dependent set. Now we argue that F is a minimal dependent set, and hence is a circuit. Denote by G the set $\{f_{i_2}, \ldots, f_{i_k}\}$.

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Suppose not, let $F' \subset F$ be linearly dependent. Since S is linearly independent (property 3 above), we note that $F' \nsubseteq \{f_{m+1}, \ldots, f_{m+m'}\}$. Therefore, $f_{i_j} \in F'$ for some $0 \le j \le k$. Since F' is dependent, express f_j in terms of the other elements in F':

$$f_j = \sum_{g \in G} \gamma_g g + \sum_{s \in S} \delta_s s$$

Since G is linearly independent, at least one of the δ_s should be non-zero. Restrict this to the matrices C and D. This gives a non-trivial dependent proper subset of D and hence a contradiction. □

From the above two observations and the fact that there is no other column in M which is a multiple of e, the set $f(e) = \{f_1, f_{m+1}, \ldots, f_{m+m'}\}$ is a unique circuit of length $m' + 1$ in M', where e is column which is coloured.

Now we argue about the isomorphism between M'_1 and M'_2 obtained from the above operation, and there is a unique circuit of length $m' + 1 > m$ in both M'_1 and M'_2 corresponding to two vectors $e \in M_1$ and $e' \in M_2$. Hence any matroid isomorphism should map these sets to each other. From such an isomorphism, we can recover the a matroid isomorphism between M_1 and M_2 that maps between e and e', thus preserving the colours. Indeed, if there is a matroid isomorphism between M_1 and M_2, that can easily be extended to M'_1 and M'_2.

For the general case, let k be the number of different colour classes and c_i denote the size of the ith colour class. Then for each vector e in the color class i, we add $l_i = m + m' + i$ many new vectors, which also increases the dimension of the space by l_i. Thus the total number of vectors in the new matroid is $\sum c_i (l_i) \le m^3$. Similarly, the dimension of the space is bounded by m^3. This completes the proof of Lemma 4.2.

□

5 Graphic Matroid Isomorphism

In this section we study GMI. Unlike in the case of the graph isomorphism problem, an NP upper bound is not so obvious for GMI. We start with the discussion of an NP upper bound for GMI.

As stated in Theorem 2.1, Whitney gave an exact characterization of when two graphs are 2-isomorphic, in terms of three operations; twisting, cleaving and identification. Note that it is sufficient to find 2-isomorphisms between 2-connected components of X_1 and X_2. In fact, any *matching* between the sets of 2-connected components whose edges connect 2-isomorphic components will serve the purpose. This is because, any 2-isomorphism preserves simple cycles, and any simple cycle of a graph is always within a 2-connected component. Hence we can assume that both the input graphs are 2-connected and in the case of 2-connected graphs, twist is the only possible operation.

The set of separating pairs does not change under a twist operation. Despite the fact that the twist operations need not commute, Truemper [Tru80] gave the following bound.

Lemma 5.1 ([Tru80]). *Let X be a 2-connected graph of n vertices, and let Y be a graph 2-isomorphic to X, then: X can be transformed to graph X' isomorphic to Y through a sequence at most $n - 2$ twists.*

Using this lemma we get an NP upper bound for GMI. Given two graphs, X_1 and X_2, the NP machine just guesses the sequence of $n - 2$ separating pairs which corresponding to the 2-isomorphism. For each pair, guess the cut w.r.t which the twist operation is to be done, and apply each of them in sequence to the graph X_1 to obtain a graph X_1'. Now ask if $X_1' \cong X_2'$. This gives an upper bound of $\exists.\text{GI} \subseteq \text{NP}$. Thus we have,

Proposition 5.2. GMI *is in* NP.

This can also be seen as an NP-reduction from GMI to GI. Now we will give a deterministic reduction from GMI to GI. Although, this does not improve the NP upper bound, it implies that it is unlikely that GMI is hard for NP (Using methods similar to that of Proposition 3.2, one can also directly prove that if GMI is NP-hard, then PH collapses to the second level).
Now we state the main result of the paper:

Theorem 5.3. GMI $\leq_T^p$ GI

Let us first look into the case of 3-connected graphs. A *separating pair* is a pair of vertices whose deletion leaves the graph disconnected. A 3-connected graph is a connected graph which does not have any separating pairs. Whitney ([Whi32]) proved the following equivalence,

Theorem 5.4 (Whitney, [Whi32]). *X_1 and X_2 be 3-connected graphs, $X_1 \cong_2 X_2 \iff X_1 \cong X_2$.*

Before giving a formal proof of Theorem 5.3, we describe the idea roughly here:

Basic Idea: Let X_1 and X_2 be the given graphs. From the above discussion, we can assume that the given graph is 2-connected.

In [HT73], Hopcroft and Tarjan proved that every 2-connected graph can be decomposed uniquely into a tree of 3-connected components, bonds or polygons.[2] Moreover, [HT73] showed that this decomposition can be computed in polynomial time. The idea is to then find the isomorphism classes of these 3-connected components using queries to GI (see theorem 5.4), and then colour the tree nodes with the corresponding isomorphism class, and then compute a coloured tree isomorphism between the two trees produced from the two graphs.

---

[2]Cunningham et al. [CE80] shows that any graphic matroid $M(X)$ is isomorphic to $M(X_1) \oplus M(X_2) \ldots \oplus M(X_k)/\{e_1, e_2, \ldots, e_k\}$, where $M(X_1), \ldots, M(X_k)$ are 3-connected components, bonds or polygons of $M(X)$ and $e_1, \ldots, e_k$ are the virtual edges. However, it is unclear if this can be turned into a reduction from GMI to GI using edge/vertex colouring.

A first mind block is that these isomorphisms between the 3-connected components need not map separating pairs to separating pairs. We overcome this by colouring the separating pairs (in fact the edge between them), with a canonical label of the two sub trees which the corresponding edge connects. To support this, we observe the following. There may be many isomorphisms between two 3-connected components which preserves the colours of the separating pairs. However, the order in which the vertices are mapped within a separating pair is irrelevant, since any order will be canonical up to a twist operation with respect to the separating pair.

So with the new colouring, the isomorphism between 3-connected components maps a separating pair to a separating pair, if and only if the two pairs of sub trees are isomorphic. However, even if this is the case, the coloured sub trees need not be isomorphic. This creates a simultaneity problem of colouring of the 3-connected components and the tree nodes and thus a second mind block.

We overcome this by colouring again using the code for coloured sub trees, and then finding the new isomorphism classes between the 3-connected components. This process is iterated till the colours stabilize on the tree as well as on the individual separating pairs (since there are only linear number of 3-connected components). Once this is ensured, we can recover the 2-isomorphism of the original graph by weaving the isomorphism of the 3-connected components guided by the tree adjacency relationship. In addition, if two 3-connected components are indeed isomorphic in the correctly aligned way, the above colouring scheme, at any point, does not distinguish between them.

Now we convert this idea into an algorithm and a formal proof.

Breaking into Tree of 3-connected components: We use the algorithm of Hopcroft and Tarjan [HT73] to compute the set of 3-connected components of a 2-connected graph in polynomial time. We will now describe some details of the algorithm which we will exploit.

Let $X(V, E)$ be a 2-connected graph. Let Y be a connected component of $X \setminus \{a, b\}$, where a, b is a separating pair. X is an *excisable* component w.r.t $\{a, b\}$ if $X \setminus Y$ has at least 2 edges and is 2-connected. The operation of excising Y from X results in two graphs: $C_1 = X \setminus Y$ plus a *virtual edge* joining (a, b), and $C_2 =$ the induced subgraph on $X \cup \{a, b\}$ plus a *virtual edge* joining (a, b). This operation may introduce multiple edges.

The decomposition of X into its 3-connected components is achieved by the repeated application of the excising operation (we call the corresponding separating pairs as *excised pairs*) until all the resulting graphs are free of excisable components. This decomposition is represented by a graph G_X with the 3-connected components of X as its vertices and two components are adjacent in G_X if and only if they share a virtual edge. In the above explanation, the graph G_X need not be a tree as the components which share a separating pair will form a clique.

To make it a tree, [HT73] introduces another component corresponding to the virtual edges thus identifying all the virtual edges created in the same excising operation with each other.

Instead, we do a surgery on the original graph X and the graph G_X. We add an edge between all the *excised pairs* (excised while obtaining G_X) to get graph X'. Notice that,

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following the same series of decomposition gives a new graph T_X which is the same as G_X except that the cliques are replaced by star centered at a newly introduced vertex (component) corresponding to the newly introduced excised edges in X'. The newly introduced edges form a 3-connected component themselves with one virtual edge corresponding to each edge of the clique they replace.

We list down the properties of the tree T_X for further reference. (1) For every node in $t \in T_X$, there is exactly one 3-connected component in X'. We denote this by c_t. (2) For every edge $e = (u, v) \in T_X$, there are exactly two virtual edges, one each in the 3-connected components c_u and c_v. We call these virtual edges as the *twin edges* of each other. (3) For any given graph X, T_X is unique up to isomorphism (since G_X is unique [HT73]). In addition, T_X can be obtained from G_X in polynomial time.

In the following claim, we prove that this surgery in the graphs does not affect the existence of 2-isomorphisms.

Claim 5.5. $X_1 \cong_2 X_2 \iff X_1' \cong_2 X_2'$.

Proof. Suppose $X_1 \cong_2 X_2$, via a bijection $\phi : E_1 \to E_2$. This induces a map ψ between the sets of 3-connected components of X_1 and X_2. By theorem 5.4, for every 3-connected component c of X_1, $c \cong \psi(c)$ (via say τ_c; when c is clear from the context we refer to it as τ).

We claim that ψ is an isomorphism between G_1 and G_2. To see this, consider an edge $e = (u, v) \in T_1$. This corresponds to two 3-connected components c_u and c_v of X_1 which share a separating pair s_1. The 3-connected components $\psi(c_u)$ and $\psi(c_v)$ must share a separating pair say s_2; otherwise, the cycles spanning across c_u and c_v will not be preserved by ϕ which contradicts the fact that ϕ is a 2-isomorphism. Hence $(\psi(c_u), \psi(c_v))$ correspond to an edge in G_2. Therefore, ψ is an isomorphism between G_1 and G_2. In fact, this also gives an isomorphism between T_1 and T_2, which in turn gives a map between the excised pairs of X_1 and X_2. To define the 2-isomorphism between X_1' and X_2', we extend the map ψ to the excised edges.

To argue the reverse direction, let $X_1' \cong_2 X_2'$ via ψ. In a very similar way, this gives an isomorphism between T_1 and T_2. The edge map of this isomorphism gives the map between the excised pairs. Restricting ψ to the edges of X_1 gives the required 2-isomorphism between X_1 and X_2. This is because, the cycles of $X_1(X_2)$ are anyway contained in X_1' (X_2'), and the excised pairs does not interfere in the mapping. $\square$

Thus it is sufficient to give an algorithm to test if $X_1' \cong_2 X_2'$, which we describe as follows.

INPUT: 2-connected graphs X_1' and X_2' and tree of 3-connected components T_1 and T_2.
OUTPUT: YES if $X_1' \cong_2 X_2'$, and NO otherwise.
ALGORITHM:
Notation: CODE(T) denotes the canonical label[3] for a tree T.

---

[3] When T is coloured, CODE(T) is the code of the tree obtained after attaching the necessary gadgets to the coloured nodes. Notice that even after colouring, the graph is still a tree. In addition, for any T, CODE(T) can be computed in L [Lin92].

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1. Initialize $T_1' = T_1$, $T_2' = T_2$.

2. REPEAT

   (a) Set $T_1 = T_1'$, $T_2 = T_2'$.

   (b) For each edge $e = (u, v) \in T_i$, $i \in \{1, 2\}$:

   Let $T_i(e, u)$ and $T_i(e, v)$ be subtrees of T_i obtained by deleting the edge e, containing u and v respectively.

   Colour virtual edges corresponding to the separating pairs in the components c_u and c_v with the set $\{\text{CODE}(T_i(e, u)), \text{CODE}(T_i(e, v))\}$. From now on, c_t denotes the coloured 3-connected component corresponding to node $t \in T_1 \cup T_2$.

   (c) Let S_1 and S_2 be the set of coloured 3-connected components of X_1' and X_2' and let $S = S_1 \cup S_2$. Using queries to GI (see observation 5.8) find out the isomorphism classes in S. Let $C_1, \ldots, C_q$ denote the isomorphism classes.

   (d) Colour each node $t \in T_i$, $i \in \{1, 2\}$, with colour ℓ if $c_t \in C_\ell$. (This gives two coloured trees T_1' and T_2'.)

   UNTIL $(\text{CODE}(T_i) \neq \text{CODE}(T_i'), \forall i \in \{1, 2\})$

3. Check if $T_1' \cong T_2'$ preserving the colours. Answer YES if $T_1' \cong T_2'$, and NO otherwise.

First we prove that the algorithm terminates in linear number of iterations of the repeat-until loop. Let q_i denote the number of isomorphism classes of the set of the coloured 3-connected components after the ith iteration. We claim that, if the termination condition is not satisfied, then $|q_i| > |q_{i-1}|$. To see this, suppose the termination is not satisfied. This means that the coloured tree T_1' is different from T_1. This can happen only when the colour of a 3-connected component c_v, $v \in T_1 \cup T_2$ changes. In addition, this can only increase the isomorphism classes. Thus $|q_i| > |q_{i-1}|$. Since q can be at most $2n$, this shows that the algorithm exits the loop after at most $2n$ steps.

Now we prove the correctness of the algorithm. We follow the notation described in the algorithm.

Lemma 5.6. $X_1' \cong_2 X_2'. \iff T_1' \cong T_2'.$

Proof. ($\Rightarrow$) Suppose $X_1' \cong_2 X_2'$, via a bijection $\phi : E_1 \to E_2$. This induces a map ψ between the sets of 3-connected components of X_1' and X_2'. By theorem 5.4, for every 3-connected component c of X_1', $c \cong \psi(c)$ (via say τ_c; when c is clear from the context we refer to it as τ).

We claim that ψ is an isomorphism between T_1 and T_2. To see this, consider an edge $e = (u, v) \in T_1$. This corresponds to two 3-connected components c_u and c_v of X_1' which share a separating pair s_1. The 3-connected components $\psi(c_u)$ and $\psi(c_v)$ must share a separating pair say s_2; otherwise, the cycles spanning across c_u and c_v will not be preserved by ϕ which contradicts the fact that ϕ is a 2-isomorphism. Hence $(\psi(c_u), \psi(c_v))$ correspond

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to an edge in T_2. Therefore, ψ is an isomorphism between T_1 and T_2. So in what follows, we interchangeably use ψ to be a map between the set of 3-connected components as well as between the vertices of the tree. Note that ψ also induces (and hence denotes) a map between the edges of T_1 and T_2.

Now we prove that ψ preserves the colours attached to T_1 and T_2 after all iterations of the repeat-until loop in step 2. To simplify the argument, we do it for the first iteration and the same can be carried forward for any number of iterations. Let T_1' and T_2' be the coloured trees obtained after the first iteration. We argue that ψ itself is an isomorphism between T_1' and T_2'.

To this end, we prove that for any vertex u in T_1, $c_u \cong \psi(c_u)$ even after colouring as in step 2b. That is, the map preserves the colouring of the virtual edges in step 2b.

Consider any virtual edge f_1 in c_u, we know that $f_2 = \tau(f_1)$ is a virtual edge in $\psi(c_u)$. Let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ be the tree edges in T_1 and T_2 corresponding to f_1 and f_2 respectively. We know that, $e_1 = \psi(e_2)$. Since $T_1 \cong T_2$ via ψ, we have

$$\{\text{CODE}(T_1(e_1, u_1)), \text{CODE}(T_1(e_1, v_1))\} = \{\text{CODE}(T_2(e_2, u_2)), \text{CODE}(T_2(e_2, v_2))\}.$$

Thus, in Step 2b, the virtual edges f_1 and f_2 get the same colour. Therefore, c_u and $\psi(c_u)$ belong to the same colour class after step 2b. Hence ψ is an isomorphism between T_1' and T_2'.

($\Leftarrow$) First, we recall some definitions needed in the proof. A *center* of a tree T is defined as a vertex v such that $\max_{u \in T} d(u, v)$ is minimized at v, where $d(u, v)$ is the number of edges in the unique path from u to v. It is known [Har69] that every tree T has a center consisting of a single vertex or a pair of adjacent vertices. The minimum achieved at the center is called the *height* of the tree, denoted by $ht(T)$.

Claim 5.7. *Let ψ be a colour preserving isomorphism between T_1' and T_2', and χ_t is an isomorphism between the 3-connected components c_t and $c_{\psi(t)}$. Then, $X_1' \cong_2 X_2'$ via a map σ such that $\forall t \in T_1'$, $\forall e \in c_t \cap E_1 : \sigma(e) = \chi_t(e)$ where E_1 is the set of edges in X_1'.*

Proof. The proof is by induction on height of the trees $h = ht(T_1') = ht(T_2')$, where the height (and center) is computed with respect to the underlying tree ignoring colours on the vertices.

Base case is when $h = 0$; that is, T_1' and T_2' have just one node (3-connected component) without any virtual edges. Simply define $\sigma = \chi$. By Theorem 5.4, this gives the required 2-isomorphism.

Suppose that if $h = ht(T_1') = ht(T_2') < k$, the above claim is true. For the induction step, suppose further that $T_1' \cong T_2'$ via ψ, and $ht(T_1') = ht(T_2') = k$. Notice that ψ should map the center(s) of T_1 to that of T_2. We consider two cases (we present one case here, and the other in the appendix).

In the first case, T_1' and T_2' have unique centers α and β. It is clear that $\psi(\alpha) = \beta$. Let c_1 and c_2 be the corresponding coloured (as in step 2b) 3-connected components. Therefore, there is a colour preserving isomorphism $\chi = \chi_\alpha$ between c_α and c_β. Let $f_1, \ldots f_k$ be the

virtual edges in c_α corresponding to the tree edges $e_1 = (\alpha, v_1), \ldots, e_k = (\alpha, v_k)$ where $v_1, \ldots, v_k$ are neighbors of α in T_1'. Denote $\psi(e_i)$ by e_i', and $\psi(v_i)$ by v_i'.

Observe that only virtual edges are coloured in the 3-connected components in step 2b while determining their isomorphism classes. Therefore, for each i, $\chi(f_i)$ will be a virtual edge in c_β, and in addition, with the same colour as f_i. That is,

$$\{\text{CODE}(T_1(e_i, \alpha)), \text{CODE}(T_1(e_i, v_i))\} = \{\text{CODE}(T_2(e_i', \beta)), \text{CODE}(T_2(e_i', v_i')))\}.$$

Since α and β are the centers of T_1' and T_2', it must be the case that in the above set equality, $\text{CODE}(T_1(e_i, v_i)) = \text{CODE}(T_2(e_i', v_i'))$. From the termination condition of the algorithm, this implies that $\text{CODE}(T_1'(e_i, v_i)) = \text{CODE}(T_2'(e_i', v_i'))$. Hence, $T_1'(e_i, v_i) \cong T_2'(e_i', v_i')$. In addition, $ht(v_i) = ht(v_i') < k$. Let X_{f_i}' and $X_{\chi(f_i)}'$ denote the subgraphs of X_1' and X_2' corresponding to $T_1'(e_i, v_i)$ and $T_2'(e_i', v_i')$ respectively. By induction hypothesis, the graphs X_{f_i}' and $X_{\chi(f_i)}'$ are 2-isomorphic via σ_i which agrees with the corresponding χ_t for $t \in T_1'(e_i, v_i)$. Define π_i as a map between the set of all edges, such that it agrees with σ_i on all edges of $X_{f(i)}'$ and with χ_t (for $t \in T_1'(e_i, v_i)$) on the coloured virtual edges.

We claim that π_i must map the twin-edge of f_i to twin-edge of $\tau(f_i)$. Suppose not. By the property of the colouring, this implies that there is a subtree of $T_1'(e_i, v_i)$ isomorphic to $T_1' \setminus T_1'(e_i, v_i)$. This contradicts the assumption that c_α is the center of T_1'.

For each edge $e \in E_1$, define $\sigma(e)$ to be $\chi(e)$ when $e \in c_\alpha$ and to be $\pi_i(e)$ when $e \in E_{f_i}$ (edges of X_{f_i}). From the above argument, $\chi = \chi_\alpha$ and σ_i indeed agrees on where it maps f_i to. This ensures that every cycle passing through the separating pairs of c_α gets preserved. Thus σ is a 2-isomorphism between X_1' and X_2'.

For case 2, let T_1' and T_2' have two centers (α_1, α_2) and (β_1, β_2) respectively. It is clear that $\psi(\{\alpha_1, \alpha_2\}) = \{\beta_1, \beta_2\}$. Without loss of generality, we assume that $\psi(\alpha_1) = \beta_1$, $\psi(\alpha_2) = \beta_2$. Therefore, there are colour preserving isomorphisms χ_1 from c_{α_1} to c_{β_1} and χ_2 from c_{α_2} and c_{β_2}. Define $\chi(e)$ as follows:

$$\chi(e) = \begin{cases} \chi_1(e) & e \in c_{\alpha_1} \\ \chi_2(e) & e \in c_{\alpha_2} \end{cases}$$

$$c_\alpha = \cup_i c_{\alpha_i}, \quad c_\beta = \cup_i c_{\beta_i}$$

With this notation, we can appeal to the proof in the case 1, and construct the 2-isomorphism σ between X_1' and X_2'. $\qquad\square$

This completes the proof of correctness of the algorithm (Lemma 5.6). $\qquad\square$

To complete the proof of Theorem 5.3, we need the following observation,

Observation 5.8. COLOURED-GMI *for 3-connected graphs reduces to* GI.

Observing that the above construction does not use non-planar gadgets, we get the following.

Corollary 5.9. *Given two planar matroids, $M(X_1)$ and $M(X_2)$, testing if $M(X_1) \cong M(X_2)$ can be performed in* P.

Now we give a polynomial time many-one reduction from MI_b to GMI. Let M_1 and M_2 be two matroids of rank b over the ground set S_1 and S_2. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively denote the set of cycles of M_1 and M_2. Note that $|\mathcal{C}_1|, |\mathcal{C}_2| \leq m^{b+1}$.

Define graphs $X_1 = (V_1, E_1)$ (respectively for $X_2 = (V_2, E_2)$) as follows. For each circuit $c = \{s_{i_1}, \ldots, s_{i_\ell}\} \subseteq S_1$ in M_1, X_1 contains a simple cycle $\{e(c, s_{i_1}), \ldots, e(c, s_{i_\ell})\}$. Now pairwise interconnect all the endpoints of the edges corresponding to each of the ground set elements (these edges form a clique), and colour these edges as RED and the remaining edges as BLUE. Now we claim the following.

Lemma 5.10. $M_1 \cong M_2$ *if and only if* $X_1 \cong_2 X_2$.

Proof. Suppose $M_1 \cong M_2$, via a map $\phi : S_1 \to S_2$. This gives a map ψ between the BLUE edges of the graphs X_1 and X_2 which preserves BLUE cycles. Now we extend this to the RED edges. Take a RED edge r, there are two BLUE edges e_1 and e_2 which share an endpoint with r. We know that e_1 and e_2 are corresponding to the same ground set element (say s). Thus $\psi(e_1)$ and $\psi(e_2)$ correspond to the same ground set element $\phi(s)$, and hence shares a RED edge in X_2. Thus ψ can be extended to preserve the RED edges. Hence $X_1 \cong_2 X_2$.

Conversely, suppose $X_1 \cong_2 X_2$ via $\psi : E_1 \to E_2$. Define $\phi : S_1 \to S_2$ as follows: For $s \in S_1$ let R_s denote the clique in X_1 corresponding to s. R_s is either a single blue edge or a clique on at least 4 vertices (in the latter case it is 3-connected). Thus ψ should map R_s to $R'_{s'}$ for some s' in S_2.. Define $\phi(s) = s'$. Now we argue that ψ is an isomorphism between M_1 and M_2. Let $c = \{s_1, \ldots, s_\ell\} \subseteq S_1$ be a cycle in M_1.

$$
\begin{aligned}
c \in \mathcal{C}_1 \quad &\Longleftrightarrow \quad \bigcap_i \psi(R_{s_i}) \text{ is a BLUE cycle in } X_1 \\
&\Longleftrightarrow \quad \bigcap_i \psi(R'_{s'_i}) \text{ is a BLUE cycle in } X_2 \\
&\Longleftrightarrow \quad \phi(c) \in \mathcal{C}_2
\end{aligned}
$$

$\square$

From the above construction, we have the following theorem.

Theorem 5.11. $MI_b \leq^p_m GMI$.

Thus we have,

Theorem 5.12. $GI \equiv^p_T GMI \equiv^p_T MI_b \equiv^p_T LMI_b$

6 Matroid Automorphism Problem

With any isomorphism problem, there is an associated automorphism problem i.e, to find a generating set for the automorphism group of the underlying object. Relating the isomorphism problem to the corresponding automorphism problem gives access to algebraic tools associated with the automorphism groups. In the case of graphs, studying automorphism problem has been fruitful.(e.g. see [Luk80, BGM82, AK02].) In this section we turn our attention to Matroid automorphism problem.

An automorphism of a matroid $M = (S, \mathcal{C})$ (where S is the ground set and $\mathcal{C}$ is the set of circuits) is a permutation ϕ of elements of S such that $\forall C \subseteq S$, $C \in \mathcal{C} \iff \phi(C) \in \mathcal{C}$. $Aut(M)$ denotes the group of automorphisms of the matroid M. When the matroid is graphic we denote by $Aut(X)$ and $Aut(M_X)$ the automorphism group of the graph and the graphic matroid respectively.

To begin with, we note that given a graph X, and a permutation $\pi \in S_m$, it is not clear apriori how to check if $\pi \in Aut(M_X)$ efficiently. This is because we need to ensure that π preserves all the simple cycles, and there could be exponentially many of them. Note that such a membership test (given a $\pi \in S_n$) for $Aut(X)$ can be done easily by testing whether π preserves all the edges. We provide an efficient test for this problem.

We use the notion of a cycle bases of X. A *cycle basis* of a graph X is a minimal set of cycles $\mathcal{B}$ of X such that every cycle in X can be written as a linear combination (viewing every cycle as a vector in $\mathbb{F}_2^m$) of the cycles in $\mathcal{B}$. Let $\mathscr{B}$ denote the set of all cycle basis of the graph X.

Lemma 6.1. *Let $\pi \in S_n$, $\exists \mathcal{B} \in \mathscr{B} : \pi(\mathcal{B}) \in \mathscr{B} \implies \forall \mathcal{B} \in \mathscr{B} : \pi(\mathcal{B}) \in \mathscr{B}$*

Proof. Let $\mathcal{B} = \{b_1, \dots b_\ell\} \in \mathscr{B}$ such that $\pi(\mathcal{B}) = \{\pi(b_1), \dots, \pi(b_\ell)\}$ is a cycle basis. Now consider any other cycle basis $\mathcal{B}' = \{b_1', \dots, b_k'\} \in \mathscr{B}$. Thus, $b_i = \sum_j \alpha_j b_j'$. This implies,

$$\pi(b_i) = \sum_j \alpha_j \pi(b_j').$$

Thus, $\pi(B') = \{\pi(b_1'), \dots, \pi(b_\ell')\}$ forms a cycle basis. $\qquad\square$

Lemma 6.2. *Let $\pi \in S_m$, and let $\mathcal{B} \in \mathscr{B}$, then $\pi \in Aut(M_X) \iff \pi(\mathcal{B}) \in \mathscr{B}$.*

Proof. Let $\mathcal{B} = \{b_1, \dots, b_\ell\}$ be the given cycle basis.

For the forward direction, suppose $\pi \in Aut(M_X)$. That is, $C \subseteq E$ is a cycle in X if and only if $\pi(C)$ is also a cycle in X. Let C be any cycle in X, and let $D = \pi^{-1}(C)$. Since $\mathcal{B} \in \mathscr{B}$, we can write, $D = \sum_i \alpha_i b_i$, and hence $C = \sum_i \alpha_i \pi(b_i)$. Hence $\pi(\mathcal{B})$ forms a cycle basis for X.

For the reverse direction, suppose $\pi(\mathcal{B})$ is a cycle basis of X. Let C be any cycle in X. We can write $C = \sum_i \alpha_i b_i$. Hence, $\pi(C) = \sum_i \alpha_i \pi(b_i)$. As π is a bijection, we have $\pi(b_i \cap b_j) = \pi(b_i) \cap \pi(b_j)$. Thus $\pi(C)$ is also a cycle in X. Since π extends to a permutation on the set of cycles, we get that C is a cycle if and only if $\pi(C)$ is a cycle. $\qquad\square$

Using Lemmas 6.1 and 6.2 it follows that, given a permutation π, to test if $\pi \in Aut(M_X)$ it suffices to check if for a cycle basis $\mathcal{B}$ of X, $\pi(\mathcal{B})$ is also a cycle basis. Given a graph X a cycle basis $\mathcal{B}$ can be computed in polynomial time (see e.g, [Hor87]). Now it suffices to show:

Lemma 6.3. *Given a permutation $\pi \in S_m$, and a cycle basis $\mathcal{B} \in \mathcal{B}$, testing whether $\pi(\mathcal{B})$ is a cycle basis, can be done in polynomial time.*

Proof. To check if $\pi(\mathcal{B})$ is a cycle basis, it is sufficient to verify that every cycle in $\mathcal{B} = \{b_1, \ldots, b_\ell\}$ can be written as a $\mathbb{F}_2$-linear combination of the cycles in $\mathcal{B}' = \{b'_1, \ldots, b'_\ell\} = \pi(\mathcal{B})$. This can be done as follows.

For $b_i \in \mathcal{B}$, let $\pi(b_i) = b'_i$. View b_i and b'_i as vectors in $\mathbb{F}_2^m$. Let b_{ij} (resp. b'_{ij}) denote the jth component of b_i (resp. b'_i). Construct the set of linear equations, $b'_{ij} = \sum_{b_k \in \mathcal{B}} x_{ik} b_{kj}$ where x_{ik} are unknowns. There are exactly ℓ b'_is and each of them gives rise to exactly m equations like this. This gives a system I of ℓm linear equations in ℓ^2 unknowns such that, $\pi(B)$ is a cycle basis if and only if I has a non-trivial solution. This test can indeed be done in polynomial time. $\square$

This gives us the following:

Theorem 6.4. *Given any $\pi \in S_m$, the membership test for π in $Aut(M_X)$ is in P.*

Notice that similar arguments can also give another proof of Proposition 5.2. As in the case of graphs, we can define automorphism problems for matroids.

MATROID AUTOMORPHISM(MA)*: Given a matroid M as independent set oracle, compute a generating set for $Aut(M)$.*

We define GMA and LMA as the corresponding automorphism problems for graphic and linear matroids, when the input is a graph and matrix respectively. Using the colouring techniques from Section 4, we prove the following equivalence.

Theorem 6.5. LMI $\equiv_T^p$ LMA, *and* GMI $\equiv_T^p$ GMA.

Proof. This proof follows a standard idea due to Luks [Luk93]. We argue the forward direction as follows. Given two matrices M_1 and M_2, form the new matrix M as,

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$$

Now using queries to LMA construct the generating set of $Aut(M)$. Check if at least one of the generators map the columns in M corresponding to columns of M_1 to those corresponding to the columns of M_2.

To see the other direction, we use the colouring idea, and the rest of the details is standard. The idea is to find the orbits of each element of the ground set as follows: For every element of $e \in S$, for each $f \in S$, colour e and f by the same colour to obtain coloured matroids M_1 and M_2. Now by querying to LMI we can check if f is in the orbit of e. Thus we can construct the orbit structure of $Aut(M)$ and hence compute a generating set.

Using similar methods we can prove GMI $\equiv_T^p$ GMA. $\square$

7 Conclusion and Open Problems

We studied the matroid isomorphism problem under various input representations and restriction on the rank of the matroid. We proved that graph isomorphism, graphic matroid isomorphism and bounded rank version of matroid isomorphism are all polynomial time equivalent.

In addition, we find it interesting that in the bounded rank case, MI_b and LMI_b are equivalent, though there exist matroids of bounded rank which are not representable over any field. Some of the open questions that we see are as follows:

- Our results provide new possibilities to attack the graph isomorphism problem. For example, it will be interesting to prove a coNP upper bound for LMI_b. Note that this will imply that GI $\in$ NP $\cap$ coNP. Similarly, are there special cases of GMI (other than what is translated from the bounds for GI) which can be solved in polynomial time?

- The representations of the matroid in the definition of LMI is over fields of size at least m and at most $\mathsf{poly}(m)$, where m is the size of the ground set of the matroid. This is critically needed for the observation of coNP-hardness. One could ask if the problem is easier over fixed finite fields independent on the input. However, we note that, by our results, it follows that this problem over $\mathbb{F}_2$ is already hard for GI. It will still be interesting to give a better (than the trivial Σ_2) upper bound for linear matroids represented over fixed finite fields (even for $\mathbb{F}_2$).

- Can we use the colouring technique of linear matroid isomorphism to reduce the general instances of linear matroid isomorphism to isomorphism testing of "simpler components" of the matroid?

- Can we make the reduction from GMI to GI many-one? Can we improve the complexity of this reduction in the general case?

8 Acknowledgements

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A   Proof of Claim 3.7

Claim 3.7: *[Bab78] For any graph X, any hyperplane of $St_k(X)$ is a maximal set edges which share a common vertex (forms a star) in X, and these are the only hyperplanes.*

Proof. To see the first part, note that since k is at most the min-degree of the graph, any maximal set of edges which forms a star is a hyperplane. To argue that these are the only hyperplanes, suppose H is a hyperplane whose edges do not share a common vertex in X. Since any independent set cannot be a hyperplane, H has size at least k, and the rank of H by definition is $k-1$. Take any k sized subset S of H. It has to be a star (say at vertex v), since otherwise H will have rank k. Since H by itself is not a star, there is an edge e which does not have v as its end point. But then, H contains a k sized subset which is not a star, and hence independent. This is a contradiction. $\square$

B   3-connected Coloured-GMI

By combining Lemma 4.1 with Theorem 5.4, we prove the following corollary which we present although we do not need it explicitly in the paper.

Corollary B.1. *Let X_1 and X_2 be two 3-connected graphs with given edge colourings, testing if there is a colour preserving 2-isomorphism between X_1 and X_2, can be reduced to GI.*

Proof. We follow the notation in the proof of Lemma 4.1. Let X_1' and X_2' be the two graphs obtained from X_1 and X_2 by attaching the colouring gadgets. Now we claim that $X_1' \cong_2 X_2' \iff X_1' \cong X_2'$.

One direction is easy to see. $X_1' \cong X_2' \Rightarrow X_1' \cong_2 X_2'$. To see the other direction, suppose $X_1' \cong_2 X_2'$ via σ. By proof of Lemma 4.1 this induces a 2-isomorphism between X_1 and X_2, which in turn induces an isomorphism τ between X_1 and X_2. Observe that τ also preserves colours on the edges of X_1 and X_2.

Let π be the map between the vertices of X_1' and X_2' induced by τ. Note that π is not defined on the new vertices introduced by the colouring gadgets. However, π already preserves all the edges between X_1' and X_2' except the newly introduced edges.

Since τ preserves edge colours in X_1, any coloured edge $e \in X_1$, the paths P_e and $P_{\tau(e)}$, introduced while constructing X_1' and X_2' are of the same length. Hence the vertex map π can be extended to the vertices in these paths to an edge preserving vertex map between X_1' and X_2'. $\square$

C   A hardness result for GMI

We show that known hardness results of GI ([Tor04]) can be adapted to the case of GMI. This is subsumed by the many-one reduction from GI to GMI. But we state this observation here anyway.

Theorem C.1. GMI *is* NL-*hard under* AC^0 *many one reductions.*

The hardness is proved using ideas similar to that in [Tor04]. We include the details of this modified part of the proof.

Consider the graph $X(k) = (V, E)$ where,

$$
\begin{aligned}
V &= \{x_0, \ldots, x_{k-1}, y_0, \ldots, y_{k-1}, z_0, \ldots, z_{k-1}\} \\
&\quad \cup \{u_{i,j} \mid 0 \le i, j \le k-1\} \\
E &= \{(x_i, u_{i,j}) \mid 0 \le i, j \le k-1\} \\
&\quad \cup \{(y_j, u_{i,j}) \mid 0 \le i, j \le k-1\} \\
&\quad \cup \{(u_{i,j}, z_{i\oplus j}) \mid 0 \le i, j \le k-1\}
\end{aligned}
$$

The following is easy to verify for the above graph.

Observation C.2. *For $k \ge 3$, the graph $X(k)$ is 3-connected.*

Toran [Tor04] argued that for any $a, b \in \{0, \ldots, k-1\}$ the graph $X(k)$ has a unique automorphism which maps vertices $x_i \to x_{a\oplus i}$ and $y_i \to y_{b\oplus i}$, which also maps $z_i \to z_{i\oplus a\oplus b}$. Combining this with Proposition 5.4, and the above observation, we get the following lemma.

Lemma C.3. *For any $k \ge 3$, given any $a, b \in \{0, \ldots, k-1\}$, there exists a unique automorphism for the matroid $M_{X(k)}$ which maps*

$$
\begin{aligned}
(x_i, u_{i,j}) &\to (x_{i\oplus a}, u_{(i\oplus a, j)}) \\
(y_j, u_{i,j}) &\to (y_{i\oplus b}, u_{(i, j\oplus a)}) \\
(u_{i,j}, z_{i\oplus j}) &\to (u_{i\oplus a, j\oplus b}, z_{i\oplus j\oplus a\oplus b})
\end{aligned}
$$

Lemma C.4. *For $k \ge 3$, GMI is* Mod$_k$L-*hard under* AC^0 *many-one reductions.*

Proof. Let C be a circuit with Mod$_k$ gates for $k \ge 3$. Without loss of generality, we assume that each gate is of fan-in 2. We construct a new graph X as follows : For each gate g in C, X contains a copy of the graph $X(k)$, denoted by X_g. Colour each edge of X_g with colour g (see Lemma 4.1). If gate g has h_1 and h_2 as its inputs identify $x_0, \ldots x_{k-1}$ and $y_0, \ldots, y_{k-1}$ of X_g with $z_0, \ldots, z_{k-1}$ of X_{h_1} and X_{h_2} respectively. Let r be the root gate of C, $g_1, \ldots g_m$ be the gates which receive the inputs directly, then the following claim can be verified by induction on the circuit structure.

Claim C.5. *C evaluates to $\ell \in \{0, \ldots, k-1\}$ on input $a_1, \ldots, a_n$ if and only if M_X has an automorphism which maps, for the gate g_i receiving the input a_j as its 'x' input, in X_{g_i}*

$$
(x_i, u_{i,t}) \to (x_{i\oplus a_j}, u_{(i\oplus a_j, t)})
$$

for all $0 \le t \le k-1$, and in X_r, for all $a \oplus b = \ell$, $j \oplus t = i$,

$$
(u_{t,j}, z_i) \to (u_{t\oplus a, j\oplus b}, z_{i\oplus \ell})
$$

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$\square$

Proof. (of theorem C.1) Using ideas from [Tor04], it is easy to see that we can write an NL computation as a series $\mathsf{Mod}_k\mathsf{L}$ computations, for $3 \le k \le 2n$. Thus, the theorem follows. $\square$

The following corollary is immediate as in (Theorem 4.4, [Tor04]), using Chinese remaindering, and the above reduction.

Corollary C.6. *Every* GapL-*function is* AC^0 *many-one reducible to* GMI.

D Testing Uniformity of Matroids

A problem of testing if a given matroid is uniform, is clearly as a special case of the matroid isomorphism problem, where one of the matroid is uniform. The problem is known to be coNP-complete as shown below.

Proposition D.1. *Given a representable matroid of rank k (input is a $k \times n$ matrix A), the problem of checking if the matroid is uniform is* coNP-*complete. In other words, testing if the matroid M represented by columns of A is isomorphic to $U_{k,n}$ is* coNP-*complete.*

Proof. This proof is due to Oxley and Welsh [OW02]. We make it more explicit. A set of n points in d dimensions is *linearly degenerate* if there is a set of d points which is linearly dependent. In other words, the set of n points is said to be in *general position* if all subsets of size d are linearly independent. Khanchiyan [Kha95] proved that given n points testing if they are in general position is NP-hard. Now notice that this exactly answers the question of testing if the matroid is uniform. $\square$

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