GENERALIZED SEMIMAGIC SQUARES FOR DIGITAL HALFTONING

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Abstract. Completing Aronov et al.'s study on zero-discrepancy matrices for digital halftoning, we determine all (m, n, k, l) for which it is possible to put mn consecutive integers on an $m \times n$ board (with wrap-around) so that each $k \times l$ region has the same sum. For one of the cases where this is impossible, we give a heuristic method to find a matrix with small discrepancy.

A *semimagic square* is a square matrix whose entries are consecutive integers and which has equal row and column sums. One way to generalize this millennia-old concept is to specify the sums on regions other than rows and columns. Ingenious constructions of squares satisfying various sum constraints have been described by many professional and amateur mathematicians. While most of them are interested in adding more and more constraints to make their squares impressive, one can generally consider sum conditions on any set of regions.

Aronov et al. [1] took up this problem for square regions: is there an $n \times n$ matrix with entries 0, …, $n^2 - 1$ such that every $k \times k$ region has the same sum? It is amusing to note that this variant of the classical problem is motivated by an engineering question of finding good dither matrices for *digital halftoning*, a method to approximate a continuous-tone image by a binary image for printing (see their paper for details). They showed [1, Theorem 1], using what they call *constant-gap matrices*, that the answer is yes if k and n are even or if n is an integer power of k, and no if k and n are relatively prime or if k is odd and n is even. We will solve this problem completely by determining all (n, k) for which such matrices exist (Section 1). Our construction of the matrices is much simpler even for the cases that have already been settled positively. We also give counterexamples to Asano et al.'s conjecture on the smallest possible discrepancy when n is odd and $k = 2$ (Section 2).

Definitions. For a positive integer N, we write $[N] = \{0, 1, \dots, N-1\}$. The remainder when an integer x is divided by N belongs to $[N]$ and is denoted by $x \bmod N$.

We consider the slightly generalized setting where the matrices and regions are rectangles instead of squares. Let m and n be positive integers. For an $m \times n$ matrix D and index $(i, j) \in [m] \times [n]$, we denote the (i, j)th entry of D by $D(i, j)$. Any set $R \subseteq [m] \times [n]$ of indices is called a *region*. The sum of the numbers on R is denoted by $D(R) = \sum_{(i,j) \in R} D(i, j)$. The *discrepancy* of D with respect to a set $\mathscr{R}$ of regions is the difference between the maximum and minimum $D(R)$ as R varies in $\mathscr{R}$. When it is zero, D is said to be *$\mathscr{R}$-uniform*.

The translate of R by $(a, b) \in \mathbf{Z}^2$ is denoted by

$$(1) \qquad R + (a, b) = \left\{ \big((i + a) \bmod m, (j + b) \bmod n\big) : (i, j) \in R \right\} \subseteq [m] \times [n].$$

The set of all translates of R is denoted by $\overline{R} = \left\{ R + (a, b) : (a, b) \in \mathbf{Z}^2 \right\}$.

14	1	21	0	18
16	13	9	22	4
5	17	12	7	19
20	2	15	11	8
6	24	3	23	10

FIGURE 1. This 5×5 table D has discrepancy 8 with respect to $\overline{[2]\times[2]}$, because $44 \le D(R) \le 52$ for every 2×2 region R.

By an $m \times n$ *table* we mean an $m \times n$ matrix in which each element of $[mn]$ appears exactly once. We are interested in tables with small (or zero) discrepancy with respect to $\overline{[k]\times[l]}$, the set of all k-by-l rectangles (Figure 1).

1. Zero discrepancy

The greatest common divisor of positive integers x and y is denoted by $\gcd(x, y)$. The goal of this section is to show the following:

Theorem 1. *Let m, n, k, l be positive integers with $k < m$ and $l < n$. Let $k' = \gcd(k, m)$ and $l' = \gcd(l, n)$. Then there exists a $\overline{[k]\times[l]}$-uniform $m \times n$ table if and only if k' and l' are greater than 1 and $k'l'(mn - 1)$ is even.*

This is an immediate consequence of the following Lemmas 2 and 3.

Lemma 2. *A $\overline{[k]\times[l]}$-uniform $m \times n$ matrix is $\overline{[\gcd(k,m)]\times[\gcd(l,n)]}$-uniform.*

Proof. Let D be a $\overline{[k]\times[l]}$-uniform $m \times n$ matrix. We will show that D is $\overline{[k']\times[l]}$-uniform, where $k' = \gcd(k, m)$. We get the conclusion of the lemma by repeating the same argument with rows and columns switched.

For each $(i, j) \in [m] \times [n]$, the regions $[k'] \times [l] + (i, j)$ and $[k'] \times [l] + (i + k, j)$ have the same sum on D, because each of them combined with $[k - k'] \times [l] + (i + k', j)$ makes a $k \times l$ rectangle. Thus for each $(i, j) \in [m] \times [n]$, the rectangles

$$[k'] \times [l] + (i + qk, j), \qquad q \in [m/k'], \tag{2}$$

all have the same sum on D. Since $k' = \gcd(k, m)$, these m/k' rectangles cover the strip $[m] \times [l] + (0, j)$ without overlap. Hence,

$$\begin{aligned} \frac{m}{k'} \cdot D\big([k'] \times [l] + (i, j)\big) &= \sum_{q\in[m/k']} D\big([k'] \times [l] + (i + qk, j)\big) \\ &= D([m] \times [l] + (0, j)) = \frac{1}{k} \sum_{r\in[m]} D([k] \times [l] + (r, j)). \end{aligned} \tag{3}$$

Since the rightmost side is a constant independent of (i, j) by $\overline{[k]\times[l]}$-uniformity, so is the leftmost side. Thus D is $\overline{[k']\times[l]}$-uniform. □

Lemma 3. *Let m and n be positive integers, and let $k < m$ and $l < n$ be their positive divisors, respectively. Then there exists a $\overline{[k]\times[l]}$-uniform $m \times n$ table if and only if k and l are greater than 1 and $kl(mn - 1)$ is even.*

0	1	2	3	4	5	6
6	5	4	3	2	1	0

$(k, l, n) = (2, 1, 7)$

0	1	2	3	4	5	6
3	4	5	6	0	1	2
6	4	2	0	5	3	1

$(k, l, n) = (3, 1, 7)$

0	4	1	5	2	6	3	7
0	4	1	5	2	6	3	7
7	6	5	4	3	2	1	0

$(k, l, n) = (3, 2, 8)$

FIGURE 2. Examples of matrices of Lemma 4.

One direction is a simple generalization of [1, Theorem 1 (b, c)]:

Proof of the "only if" part of Lemma 3. Let D be a $\overline{[k] \times [l]}$-uniform $m \times n$ table. It is easy to see that $D(R) = kl(mn-1)/2$ for each $R \in \overline{[k] \times [l]}$. Since $D(R)$ must be an integer, the second claim follows. For the first claim, assume $k = 1$ for contradiction (the case $l = 1$ is similar). Then $D([1] \times [l]) = D([1] \times [l] + (0, 1))$ and hence $D(0, 0) = D(0, l)$, contradicting the assumption that D is a table. $\square$

For the converse, we use the building blocks provided by the following lemma:

Lemma 4. *Let $k > 1$ and $l > 0$ be integers and let n be a positive multiple of l. If $kl(n-1)$ is even, then there exists a $\overline{[k] \times [l]}$-uniform $k \times n$ matrix in which each row is a permutation of $[n]$.*

Proof. A $\overline{[k] \times [l]}$-uniform $k \times n$ matrix and a $\overline{[k'] \times [l]}$-uniform $k' \times n$ matrix stacked vertically make a $\overline{[k + k'] \times [l]}$-uniform $(k + k') \times n$ matrix. Also, a $\overline{[k] \times [l]}$-uniform matrix is $\overline{[k] \times [l']}$-uniform for any multiple l' of l. Therefore, it suffices to construct the desired matrix P for the cases $(k, l) = (2, 1)$, $(3, 1)$ and $(3, 2)$ (Figure 2). If $(k, l) = (2, 1)$, let

$$P(0, j) = j, \qquad P(1, j) = n - 1 - j. \tag{4}$$

If $(k, l) = (3, 1)$, then n is odd by the assumption; let

$$P(0, j) = j, \quad P(1, j) = \left(j + \frac{n-1}{2}\right) \bmod n, \quad P(2, j) = (-2j - 1) \bmod n. \tag{5}$$

If $(k, l) = (3, 2)$, let

$$P(0, j) = P(1, j) = \left\lfloor \frac{j}{2} \right\rfloor + \frac{n}{2}(j \bmod 2), \qquad P(2, j) = n - 1 - j. \tag{6}$$

It is easy to verify that P is $\overline{[k] \times [l]}$-uniform in each case. $\square$

Proof of the "if" part of Lemma 3. We may assume without loss of generality that $l(mn-1)$ is even. In this case, both $kl(n-1)$ and $l(m/k-1)$ are even, so by Lemma 4, there are a $\overline{[k] \times [l]}$-uniform $k \times n$ matrix P whose rows are permutations of $[n]$, and a $\overline{[l] \times [1]}$-uniform $l \times (m/k)$ matrix Q whose rows are permutations of $[m/k]$. Define an $m \times l$ matrix T by

$$T(a, j) = Q(j, \lfloor a/k \rfloor)k + (a \bmod k). \tag{7}$$

Then T is $\overline{[k] \times [l]}$-uniform and its columns are permutations of $[m]$. Define an $m \times n$ matrix D by

$$D(a, b) = P(a \bmod k, b)m + T(a, b \bmod l) \tag{8}$$

$$D = \begin{bmatrix} P \\ P \\ P \end{bmatrix} \times 9 + \begin{bmatrix} T & T & T & T \end{bmatrix}$$

$$= \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0&4&1&5&2&6&3&7\\\hline 0&4&1&5&2&6&3&7\\\hline 7&6&5&4&3&2&1&0\\\hline 0&4&1&5&2&6&3&7\\\hline 0&4&1&5&2&6&3&7\\\hline 7&6&5&4&3&2&1&0\\\hline 0&4&1&5&2&6&3&7\\\hline 0&4&1&5&2&6&3&7\\\hline 7&6&5&4&3&2&1&0\\\hline \end{array} \times 9 + \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0&6&0&6&0&6&0&6\\\hline 1&7&1&7&1&7&1&7\\\hline 2&8&2&8&2&8&2&8\\\hline 3&3&3&3&3&3&3&3\\\hline 4&4&4&4&4&4&4&4\\\hline 5&5&5&5&5&5&5&5\\\hline 6&0&6&0&6&0&6&0\\\hline 7&1&7&1&7&1&7&1\\\hline 8&2&8&2&8&2&8&2\\\hline \end{array} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0&42&9&51&18&60&27&69\\\hline 1&43&10&52&19&61&28&70\\\hline 65&62&47&44&29&26&11&8\\\hline 3&39&12&48&21&57&30&66\\\hline 4&40&13&49&22&58&31&67\\\hline 68&59&50&41&32&23&14&5\\\hline 6&36&15&45&24&54&33&63\\\hline 7&37&16&46&25&55&34&64\\\hline 71&56&53&38&35&20&17&2\\\hline \end{array}$$

FIGURE 3. Construction of D for $(k, l, m, n) = (3, 2, 9, 8)$.

(Figure 3). Since P and T are $\overline{[k] \times [l]}$-uniform, so is D. To see that D is a table, suppose that $D(a, b) = D(a', b')$. By (7) and (8) we see that

$$\begin{cases} P(a \bmod k, b) = P(a' \bmod k, b'), \\ Q(b \bmod l, \lfloor a/k \rfloor) = Q(b' \bmod l, \lfloor a'/k \rfloor), \\ a \bmod k = a' \bmod k. \end{cases} \tag{9}$$

Since P's rows are permutations, the first and the third equation imply that $b = b'$. Since Q's rows are permutations, this and the second equation imply that $a = a'$. $\square$

In the above, we constructed the uniform table as a linear combination of two uniform matrices with smaller entries. This idea is due to Euler [3] who gave a construction of a semimagic square (that is, a $(\overline{[1] \times [n]} \cup \overline{[n] \times [1]})$-uniform $n \times n$ table) from a pair of special $(\overline{[1] \times [n]} \cup \overline{[n] \times [1]})$-uniform matrices called *Latin squares*.

2. Finding low-discrepancy tables by ranking

In this section, we confine ourselves, as Asano et al. [2] did, to the case where $k = l = 2$ and $m = n$. Theorem 1 states that in this case a uniform table exists if and only if n is even. For odd n's, they construct a table with discrepancy $2n$, and conjecture that it is the smallest possible. This is refuted by our Figures 1 and 4. Figure 1 was discovered by an exhaustive search. We describe briefly how Figure 4 was obtained.

Define $f \colon [0, 1]^2 \to \mathbf{R}$ by $f(x, y) = g(x) + g(y)$, where

$$g(x) = \begin{cases} 1 - (4x - 1)^2 & \text{if } x \le 1/2, \\ -1 + (4x - 3)^2 & \text{if } x \ge 1/2 \end{cases} \tag{10}$$

(Figure 5). Let α, $\beta \in [0, 1]$ and define $s \colon [n]^2 \to [0, 1]^2$ by

$$s(i, j) = \left(\frac{i + \alpha}{n}, \frac{j + \beta}{n} \right). \tag{11}$$

433	523	439	519	445	511	453	507	460	497	465	490	472	486	478	480	482	476	487	470	492	462	500	458	508	450	515	442	520	437	525
616	348	603	362	592	378	576	393	556	411	543	427	524	449	494	479	471	505	441	529	422	547	403	564	387	581	374	594	357	607	345
283	675	298	655	312	634	337	614	356	590	383	560	416	531	451	483	498	436	538	405	570	377	595	350	622	327	643	306	662	292	678
727	241	716	253	692	278	669	305	637	340	604	375	569	418	522	475	454	535	401	580	361	615	329	646	300	677	273	701	249	721	238
198	762	206	747	225	720	255	686	293	649	328	608	376	559	428	488	517	413	578	358	625	320	660	279	695	248	729	218	751	201	765
797	166	786	180	769	207	733	245	698	282	652	330	602	385	539	469	434	558	371	621	315	670	270	706	233	746	199	775	173	790	163
132	823	141	813	160	782	196	745	237	703	284	648	342	588	414	491	530	388	606	324	668	269	714	223	755	185	792	156	818	137	827
856	109	848	120	826	150	793	191	749	239	696	294	635	359	553	464	425	583	341	653	275	712	221	764	174	802	140	831	116	852	107
82	876	88	862	113	833	145	796	192	744	247	684	307	610	397	496	541	370	633	291	704	226	761	172	810	131	843	102	867	84	879
898	61	893	78	869	103	836	147	791	197	732	258	667	339	572	461	417	598	310	685	242	752	177	808	130	851	94	880	73	895	58
43	916	50	900	71	872	104	832	152	781	210	718	280	632	381	504	550	347	658	259	736	193	798	136	849	91	884	65	906	48	919
933	30	924	46	903	72	868	114	824	161	768	229	688	317	586	456	402	623	290	715	209	783	148	839	100	883	59	912	37	928	28
14	941	25	927	47	899	79	861	123	811	181	742	256	651	369	509	561	331	682	234	766	162	825	111	871	68	911	36	934	20	943
953	10	944	26	923	51	892	89	846	144	784	208	713	299	597	448	394	638	267	734	190	804	126	859	80	901	42	930	18	950	8
2	955	11	940	31	915	64	873	112	821	169	757	244	671	354	514	571	314	699	215	778	153	838	95	889	54	921	24	946	6	958
960	3	952	17	932	44	896	83	853	134	794	200	725	289	609	444	384	647	257	743	179	814	117	866	70	909	34	939	12	956	0
1	957	9	942	29	917	60	875	110	822	167	759	243	673	351	516	574	311	702	214	779	149	840	93	890	52	925	22	949	4	959
954	7	947	21	926	49	894	87	850	139	788	205	717	297	599	446	389	642	264	737	186	809	124	863	76	904	39	936	15	951	5
13	945	19	931	41	905	75	864	118	815	175	748	251	656	366	510	563	326	687	228	770	158	829	105	878	62	914	32	937	16	948
938	27	929	38	910	67	874	106	830	157	774	222	694	309	591	455	400	626	285	722	203	789	142	845	90	887	56	918	33	935	23
35	920	45	907	66	882	99	841	146	787	202	726	276	639	379	506	551	344	666	252	741	184	806	128	855	85	891	57	913	40	922
908	55	897	69	881	97	847	135	799	189	739	250	672	334	575	459	415	605	304	691	232	760	170	817	121	857	86	886	63	902	53
74	885	81	870	101	844	133	805	178	753	236	690	301	618	392	499	544	365	641	281	710	216	772	164	819	122	854	92	877	77	888
865	98	858	115	834	138	801	176	756	227	705	287	644	355	557	463	423	587	335	661	265	724	212	773	165	816	129	842	108	860	96
119	835	127	820	151	795	183	754	224	711	274	657	333	593	410	493	532	382	613	313	676	260	728	213	771	171	803	143	828	125	837
812	155	800	168	780	195	750	231	707	272	665	323	611	380	545	468	432	566	360	628	302	679	261	723	219	758	187	785	159	807	154
182	776	194	763	211	731	246	700	277	664	321	620	367	567	424	489	521	406	584	349	630	303	674	268	708	235	740	204	767	188	777
738	220	730	240	709	263	681	295	650	325	619	364	577	408	527	474	447	542	395	589	352	627	316	659	286	689	254	719	230	735	217
262	693	271	680	296	654	318	629	343	601	372	573	404	536	443	485	503	430	548	396	582	363	612	336	640	308	663	288	683	266	697
645	322	631	338	617	353	596	373	579	391	554	419	533	438	502	477	466	513	431	540	409	562	386	585	368	600	346	624	332	636	319
390	565	399	552	412	546	421	534	429	526	440	512	457	495	473	481	484	467	501	452	518	435	528	426	537	420	549	407	555	398	568

FIGURE 4. A 31×31 table whose discrepancy with respect to $\overline{[2] \times [2]}$ is 27.

Let H be the $n \times n$ table whose (i,j)th entry is the rank of $f(s(i,j))$ (with some tie-breaking rule):

$$H(i,j) = \Bigl|\Bigl\{\, (i',j') \in [n]^2 : f\bigl(s(i',j')\bigr) < f\bigl(s(i,j)\bigr) \text{ or } \Bigl(f\bigl(s(i',j')\bigr) = f\bigl(s(i,j)\bigr) \text{ and } ni'+j' < ni+j\Bigr) \,\Bigr\}\Bigr|. \tag{12}$$

Finally, define the desired matrix D by

$$D\bigl((i+j) \bmod n, (i-j) \bmod n\bigr) = H(i,j). \tag{13}$$

Figure 4 was obtained by this method with $n = 31$ and $(\alpha, \beta) = (0.286, 0)$.

To see intuitively why D has small discrepancy, note that a 2×2 region in D corresponds to the region in H (or its translate) shown in Figure 6. These four cells are mapped by s to two nearby points $(x \pm \varepsilon, y)$ and another two points $(x+1/2, y+1/2 \pm \varepsilon)$ (the coordinates are modulo 1). Since $f(x,y) = -f(x+1/2, y+1/2)$, the sum of the values of f at these four points is almost zero. Thus, assuming that taking the ranks does not distort the distribution of values too much, we can expect that D has low discrepancy. We add the displacement (α, β) in (11) in order to reduce the chance of ties in the ranking which seem to work adversely.

As Aronov et al. [1] point out, our problem is analogous to a common situation in discrete geometry where we try to arrange discrete objects so that they look close to some "balanced" continuous distribution. The constraint peculiar to our problem is

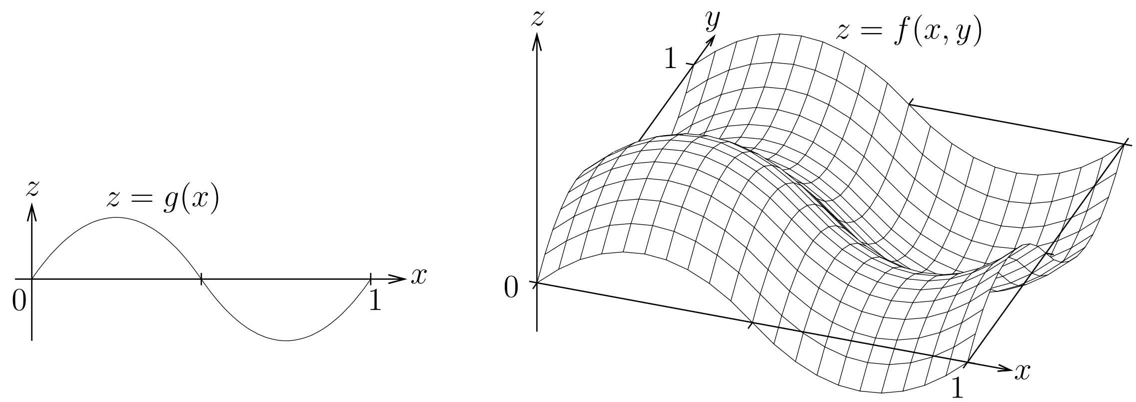

FIGURE 5. The functions g and f.

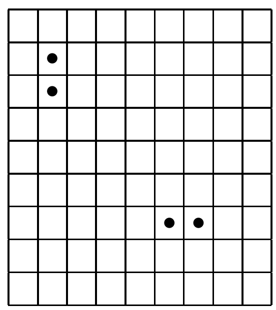

FIGURE 6. A region in H corresponding to a 2×2 square in D (for $n = 9$).

that we have to use each number in $[mn]$ exactly once. The ranking technique used here may be applicable to other problems with this constraint. However, analyzing its performance seems to be hard: although our computer experiment for several n's suggests that the above method achieves sublinear 2×2 discrepancy, we have no proof yet.

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