# HOLANT PROBLEMS FOR REGULAR GRAPHS WITH COMPLEX EDGE FUNCTIONS

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ABSTRACT. We prove a complexity dichotomy theorem for Holant Problems on 3-regular graphs with an arbitrary complex-valued edge function. Three new techniques are introduced: (1) higher dimensional iterations in interpolation; (2) Eigenvalue Shifted Pairs, which allow us to prove that a pair of combinatorial gadgets in combination succeed in proving #P-hardness; and (3) algebraic symmetrization, which significantly lowers the symbolic complexity of the proof for computational complexity. With holographic reductions the classification theorem also applies to problems beyond the basic model.

## 1. Introduction

In this paper we consider the following subclass of Holant Problems [5, 6]. An input regular graph G=(V,E) is given, where every  $e\in E$  is labeled with a (symmetric) edge function g. The function g takes 0-1 inputs from its incident nodes and outputs arbitrary values in  $\mathbb{C}$ . The problem is to compute the quantity  $\operatorname{Holant}(G)=\sum_{\sigma:V\to\{0,1\}}\prod_{\{u,v\}\in E}g(\{\sigma(u),\sigma(v)\}).$ 

Holant Problems are a natural class of counting problems. As introduced in [5, 6], the general Holant Problem framework can encode all Counting Constraint Satisfaction Problems (#CSP). This includes special cases such as weighted VERTEX COVER, GRAPH COLORINGS, MATCHINGS, and PERFECT MATCHINGS. The subclass of Holant Problems in this paper can also be considered as (weighted) H-homomorphism (or H-coloring) problems [2, 3, 7, 8, 9, 10] with an arbitrary  $2\times 2$  symmetric complex matrix H, however restricted

to regular graphs G as input. E.g., VERTEX COVER is the case when  $H = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ . When

the matrix H is a 0-1 matrix, it is called unweighted. Dichotomy theorems (i.e., the problem is either in P or #P-hard, depending on H) for unweighted H-homomorphisms with undirected graphs H and directed acyclic graphs H are given in [8] and [7] respectively. A dichotomy theorem for any symmetric matrix H with non-negative real entries is proved

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in [2]. Goldberg et al. [9] proved a dichotomy theorem for all real symmetric matrices H. Finally, Cai, Chen, and Lu have proved a dichotomy theorem for all complex symmetric matrices H [3].

The crucial difference between Holant Problems and #CSP is that in #CSP, EQUALITY functions of arbitrary arity are presumed to be present. In terms of H-homomorphism problems, this means that the input graph is allowed to have vertices of arbitrarily high degrees. This may appear to be a minor distinction; in fact it has a major impact on complexity. It turns out that if Equality gates of arbitrary arity are freely available in possible inputs then it is technically easier to prove #P-hardness. Proofs of previous dichotomy theorems make extensive use of constructions called thickening and stretching. These constructions require the availability of Equality gates of arbitrary arity (equivalently, vertices of arbitrarily high degrees) to carry out. Proving #P-hardness becomes more challenging in the degree restricted case. Furthermore there are indeed cases within this class of counting problems where the problem is #P-hard for general graphs, but solvable in P when restricted to 3-regular graphs.

We denote the (symmetric) edge function g by [x,y,z], where x=g(0,0), y=g(0,1)=g(1,0) and z=g(1,1). Functions will also be called gates or signatures. (For VERTEX COVER, the function corresponding to H is the OR gate, and is denoted by the signature [0,1,1].) In this paper we give a dichotomy theorem for the complexity of Holant Problems on 3-regular graphs with arbitrary signature g=[x,y,z], where  $x,y,z\in\mathbb{C}$ . First, if y=0, the Holant Problem is easily solvable in P. Assuming  $y\neq 0$  we may normalize g and assume g=1. Our main theorem is as follows:

**Theorem 1.1.** Suppose  $a, b \in \mathbb{C}$ , and let X = ab,  $Z = (\frac{a^3 + b^3}{2})^2$ . Then the Holant Problem on 3-regular graphs with g = [a, 1, b] is #P-hard except in the following cases, for which the problem is in P.

- (1) X = 1.
- (2) X = Z = 0.
- (3) X = -1 and Z = 0.
- (4) X = -1 and Z = -1.

If we restrict the input to planar 3-regular graphs, then these four categories are solvable in P. as well as a fifth category  $X^3 = Z$ . The problem remains #P-hard in all other cases.  $^1$ 

These results can be extended to k-regular graphs (we detail how this is accomplished in a forthcoming work). One can also use holographic reductions [16] to extend this theorem to more general Holant Problems.

In order to achieve this result, some new proof techniques are introduced. To discuss this we first take a look at some previous results. Valiant [14, 15] introduced the powerful technique of *interpolation*, which was further developed by many others. In [5] a dichotomy theorem is proved for the case when g is a Boolean function. The technique from [5] is to provide certain algebraic criteria which ensure that *interpolation* succeeds, and then apply these criteria to prove that (a large number yet) finitely many individual problems are #Phard. This involves (a small number of) gadget constructions, and the algebraic criteria are powerful enough to show that they succeed in each case. Nonetheless this involves a

<sup>&</sup>lt;sup>1</sup>Technically, computational complexity involving complex or real numbers should, in the Turing model, be restricted to computable numbers. In other models such as the Blum-Shub-Smale model [1] no such restrictions are needed. Our results are not sensitive to the exact model of computation.

case-by-case verification. In [6] this theorem is extended to all real-valued a and b, and we have to deal with infinitely many problems. So instead of focusing on one problem, we devised (a large number of) recursive gadgets and analyzed the regions of  $(a, b) \in \mathbb{R}^2$  where they fail to prove #P-hardness. The algebraic criteria from [5] are not suitable (Galois theoretic) for general a and b, and so we formulated weaker but simpler criteria. Using these criteria, the analysis of the failure set becomes expressible as containment of semi-algebraic sets. As semi-algebraic sets are decidable, this offers the ultimate possibility that if we found enough gadgets to prove #P-hardness, then there is a computational proof (of computational intractability) in a finite number of steps. However this turned out to be a tremendous undertaking in symbolic computation, and many additional ideas were needed to finally carry out this plan. In particular, it would seem hopeless to extend that approach to all complex a and b.

In this paper, we introduce three new ideas. (1) We introduce a method to construct gadgets that carry out iterations at a higher dimension, and then collapse to a lower dimension for the purpose of constructing unary signatures. This involves a starter gadget, a recursive iteration gadget, and a finisher gadget. We prove a lemma that guarantees that among polynomially many iterations, some subset of them satisfies properties sufficient for interpolation to succeed (it may not be known a priori which subset worked, but that does not matter). (2) Eigenvalue Shifted Pairs are coupled pairs of gadgets whose transition matrices differ by  $\delta I$  where  $\delta \neq 0$ . They have shifted eigenvalues, and by analyzing their failure conditions, we can show that except on very rare points, one or the other gadget succeeds. (3) Algebraic symmetrization. We derive a new expression of the Holant polynomial over 3-regular graphs, with a crucially reduced degree. This simplification of the Holant and related polynomials condenses the problem of proving #P-hardness to the point where all remaining cases can be handled by symbolic computation. We also use the same expression to prove tractability.

The rest of this paper is organized as follows. In Section 2 we discuss notation and background information. In Section 3 we cover interpolation techniques, including how to collapse higher dimensional iterations to interpolate unary signatures. In Section 4 we show how to perform algebraic symmetrization of the Holant, and introduce Eigenvalue Shifted Pairs (ESP) of gadgets. Then we combine the new techniques to prove Theorem 1.1.

## 2. Notations and Background

We state the counting framework more formally. A signature  $grid \Omega = (G, \mathcal{F}, \pi)$  consists of a labeled graph G = (V, E) where  $\pi$  labels each vertex  $v \in V$  with a function  $f_v \in \mathcal{F}$ . We consider all edge assignments  $\xi : E \to \{0,1\}$ ;  $f_v$  takes inputs from its incident edges E(v) at v and outputs values in  $\mathbb{C}$ . The counting problem on the instance  $\Omega$  is to compute<sup>2</sup>

$$\operatorname{Holant}_{\Omega} = \sum_{\xi} \prod_{v \in V} f_v(\xi \mid_{E(v)}).$$

Suppose G is a bipartite graph (U, V, E) such that each  $u \in U$  has degree 2. Furthermore suppose each  $v \in V$  is labeled by an EQUALITY gate  $=_k$  where  $k = \deg(v)$ . Then any non-zero term in  $\operatorname{Holant}_{\Omega}$  corresponds to a 0-1 assignment  $\sigma: V \to \{0,1\}$ . In fact, we can merge the two incident edges at  $u \in U$  into one edge  $e_u$ , and label this edge  $e_u$  by the function  $f_u$ . This gives an edge-labeled graph (V, E') where  $E' = \{e_u : u \in U\}$ . For an edge-labeled

<sup>&</sup>lt;sup>2</sup>The term Holant was first introduced by Valiant in [16] to denote a related exponential sum.

graph (V, E') where  $e \in E'$  has label  $g_e$ ,  $\operatorname{Holant}_{\Omega} = \sum_{\sigma:V \to \{0,1\}} \prod_{e=(v,w) \in E'} g_e(\sigma(v), \sigma(w))$ . If each  $g_e$  is the same function g (but assignments  $\sigma: V \to [q]$  take values in a finite set [q]) this is exactly the H-coloring problem (for undirected graphs g is a symmetric function). In particular, if (U, V, E) is a (2, k)-regular bipartite graph, equivalently G' = (V, E') is a k-regular graph, then this is the H-coloring problem restricted to k-regular graphs. In this paper we will discuss 3-regular graphs, where each  $g_e$  is the same symmetric complex-valued function. We also remark that for general bipartite graphs (U, V, E), giving EQUALITY (of various arities) to all vertices on one side V defines  $\# \operatorname{CSP}$  as a special case of Holant Problems. But whether EQUALITY of various arities are present has a major impact on complexity, thus Holant Problems are a refinement of  $\# \operatorname{CSP}$ .

A symmetric function  $g:\{0,1\}^k\to\mathbb{C}$  can be denoted as  $[g_0,g_1,\ldots,g_k]$ , where  $g_i$  is the value of g on inputs of Hamming weight i. They are also called signatures. Frequently we will revert back to the bipartite view: for (2,3)-regular bipartite graphs (U,V,E), if every  $u\in U$  is labeled  $g=[g_0,g_1,g_2]$  and every  $v\in V$  is labeled  $r=[r_0,r_1,r_2,r_3]$ , then we also use  $\#[g_0,g_1,g_2]\mid [r_0,r_1,r_2,r_3]$  to denote the Holant Problem. Note that [1,0,1] and [1,0,0,1] are Equality gates  $=_2$  and  $=_3$  respectively, and the main dichotomy theorem in this paper is about  $\#[x,y,z]\mid [1,0,0,1]$ , for all  $x,y,z\in\mathbb{C}$ . We will also denote  $\mathrm{Hol}(a,b)=\#[a,1,b]\mid [1,0,0,1]$ . More generally, If  $\mathcal G$  and  $\mathcal R$  are sets of signatures, and vertices of U (resp. V) are labeled by signatures from  $\mathcal G$  (resp.  $\mathcal R$ ), then we also use  $\#\mathcal G\mid \mathcal R$  to denote the bipartite Holant Problem. Signatures in  $\mathcal G$  are called generators and signatures in  $\mathcal R$  are called recognizers. This notation is particularly convenient when we perform holographic transformations. Throughout this paper, all (2,3)-regular bipartite graphs are arranged with generators on the degree 2 side and recognizers on the degree 3 side.

We use Arg to denote the principal value of the complex argument; i.e.,  $\operatorname{Arg}(c) \in (-\pi, \pi]$  for all nonzero  $c \in \mathbb{C}$ .

#### 2.1. $\mathcal{F}$ -Gate

Any signature from  $\mathcal{F}$  is available at a vertex as part of an input graph. Instead of a single vertex, we can use graph fragments to generalize this notion. An  $\mathcal{F}$ -gate  $\Gamma$  is a pair  $(H, \mathcal{F})$ , where H = (V, E, D) is a graph with some dangling edges D (Figure 1 contains some examples). Other than these dangling edges, an  $\mathcal{F}$ -gate is the same as a signature grid. The role of dangling edges is similar to that of external nodes in Valiant's notion [17], however we allow more than one dangling edge for a node. In H = (V, E, D) each node is assigned a function in  $\mathcal{F}$  (we do not consider "dangling" leaf nodes at the end of a dangling edge among these), E are the regular edges, and D are the dangling edges. Then we can define a function for this  $\mathcal{F}$ -gate  $\Gamma = (H, \mathcal{F})$ ,

$$\Gamma(y_1, y_2, \dots, y_q) = \sum_{(x_1, x_2, \dots, x_p) \in \{0, 1\}^p} H(x_1, x_2, \dots, x_p, y_1, y_2, \dots, y_q),$$

where p = |E|, q = |D|,  $(y_1, y_2, ..., y_q) \in \{0, 1\}^q$  denotes an assignment on the dangling edges, and  $H(x_1, x_2, ..., x_p, y_1, y_2, ..., y_q)$  denotes the value of the partial signature grid on an assignment of all edges, i.e., the product of evaluations at every vertex of H, for  $(x_1, x_2, ..., x_p, y_1, y_2, ..., y_q) \in \{0, 1\}^{p+q}$ . We will also call this function the signature of the  $\mathcal{F}$ -gate  $\Gamma$ . An  $\mathcal{F}$ -gate can be used in a signature grid as if it is just a single node with the

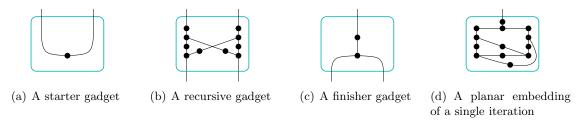


Figure 1: Examples of binary starter, recursive, and finisher gadgets

same signature. We note that even for a very simple signature set  $\mathcal{F}$ , the signatures for all  $\mathcal{F}$ -gates can be quite complicated and expressive. Matchgate signatures are an example [17].

The dangling edges of an  $\mathcal{F}$ -gate are considered as input or output variables. Any m-input n-output  $\mathcal{F}$ -gate can be viewed as a  $2^n$  by  $2^m$  matrix M which transforms arity-m signatures into arity-n signatures (this is true even if m or n are 0). Our construction will transform symmetric signatures to symmetric signatures. This implies that there exists an equivalent n+1 by m+1 matrix  $\widetilde{M}$  which operates directly on column vectors written in symmetric signature notation. We will henceforth identify the matrix  $\widetilde{M}$  with the  $\mathcal{F}$ -gate itself. The constructions in this paper are based upon three different types of bipartite  $\mathcal{F}$ -gates which we call starter gadgets, recursive gadgets, and finisher gadgets. An arity-r starter gadget is an  $\mathcal{F}$ -gate with no input but r output edges. If an  $\mathcal{F}$ -gate has r input and r output edges then it is called an arity-r recursive gadget. Finally, an  $\mathcal{F}$ -gate is an arity-r finisher gadget if it has r input edges 1 output edge. As a matter of convention, we consider any dangling edge incident with a generator as an output edge and any dangling edge incident with a recognizer as an input edge; see Figure 1.

# 3. Interpolation Techniques

#### 3.1. Binary recursive construction

In this section, we develop our new technique of higher dimensional iterations for interpolation of unary signatures.

**Lemma 3.1.** Suppose  $M \in \mathbb{C}^{3\times 3}$  is a nonsingular matrix,  $s \in \mathbb{C}^3$  is a nonzero vector, and for all integers  $k \geq 1$ , s is not a column eigenvector of  $M^k$ . Let  $F_i \in \mathbb{C}^{2\times 3}$  be three matrices, where  $\operatorname{rank}(F_i) = 2$  for  $1 \leq i \leq 3$ , and the intersection of the row spaces of  $F_i$  is trivial  $\{0\}$ . Then for every n, there exists some  $F \in \{F_i : 1 \leq i \leq 3\}$ , and some  $S \subseteq \{FM^ks : 0 \leq k \leq n^3\}$ , such that  $|S| \geq n$  and vectors in S are pairwise linearly independent.

Proof. Let  $k > j \ge 0$  be integers. Then  $M^ks$  and  $M^js$  are nonzero and also linearly independent, since otherwise s is an eigenvector of  $M^{k-j}$ . Let  $N = [M^js, M^ks] \in \mathbb{C}^{3\times 2}$ , then  $\operatorname{rank}(N) = 2$ , and  $\ker(N^{\mathrm{T}})$  is a 1-dimensional linear subspace. It follows that there exists an  $F \in \{F_i : 1 \le i \le 3\}$  such that the row space of F does not contain  $\ker(N^{\mathrm{T}})$ , and hence has trivial intersection with  $\ker(N^{\mathrm{T}})$ . In other words,  $\ker(N^{\mathrm{T}}F^{\mathrm{T}}) = \{0\}$ . We conclude that  $FN \in \mathbb{C}^{2\times 2}$  has rank 2, and  $FM^js$  and  $FM^ks$  are linearly independent.

Each  $F_i$ , where  $1 \leq i \leq 3$ , defines a coloring of the set  $K = \{0, 1, \ldots, n^3\}$  as follows: color  $k \in K$  with the linear subspace spanned by  $F_iM^ks$ . Thus,  $F_i$  defines an equivalence relation  $\approx_i$  where  $k \approx_i k'$  iff they receive the same color. Assume for a contradiction that for each  $F_i$ , where  $1 \leq i \leq 3$ , there are not n pairwise linearly independent vectors among  $\{F_iM^ks: k \in K\}$ . Then, including possibly the 0-dimensional space  $\{0\}$ , there can be at most n distinct colors assigned by  $F_i$ . By the pigeonhole principle, some k and k' with  $0 \leq k < k' \leq n^3$  must receive the same color for all  $F_i$ , where  $1 \leq i \leq 3$ . This is a contradiction and we are done.

The next lemma says that under suitable conditions we can construct all unary signatures [x, y]. The method will be interpolation at a higher dimensional iteration, and finishing up with a suitable *finisher* gadget. The crucial new technique here is that when iterating at a higher dimension, we can guarantee the existence of *one* finisher gadget that succeeds on polynomially many steps, which results in overall success. Different finisher gadgets may work for different initial signatures and different input size n, but these need not be known in advance and have no impact on the final success of the reduction.

**Lemma 3.2.** Suppose that the following gadgets can be built using complex-valued signatures from a finite generator set  $\mathcal{G}$  and a finite recognizer set  $\mathcal{R}$ .

- (1) A binary starter gadget with nonzero signature  $[z_0, z_1, z_2]$ .
- (2) A binary recursive gadget with nonsingular recurrence matrix M, for which  $[z_0, z_1, z_2]^T$  is not a column eigenvector of  $M^k$  for any positive integer k.
- (3) Three binary finisher gadgets with rank 2 matrices  $F_1, F_2, F_3 \in \mathbb{C}^{2\times 3}$ , where the intersection of the row spaces of  $F_1$ ,  $F_2$ , and  $F_3$  is the zero vector.

Then for any  $x, y \in \mathbb{C}$ ,  $\#\mathcal{G} \cup \{[x, y]\} \mid \mathcal{R} \leq_T \#\mathcal{G} \mid \mathcal{R}$ .

*Proof.* The construction begins with the binary starter gadget with signature  $[z_0, z_1, z_2]$ , which we call  $N_0$ . Let  $\mathcal{F} = \mathcal{G} \cup \mathcal{R}$ . Recursively,  $\mathcal{F}$ -gate  $N_{k+1}$  is defined to be  $N_k$  connected to the binary recursive gadget in such a way that the input edges of the binary recursive gadget are merged with the output edges of  $N_k$ . Then  $\mathcal{F}$ -gate  $G_k$  is defined to be  $N_k$ connected to one of the finisher gadgets, with the input edges of the finisher gadget merged with the output edges of  $N_k$  (see Figure 1(d)). Herein we analyze the construction with respect to a given bipartite signature grid  $\Omega$  for the Holant Problem  $\#\mathcal{G} \cup \{[x,y]\} \mid \mathcal{R}$ , with underlying graph G = (V, E). Let  $Q \subseteq V$  be the set of vertices with [x, y] signatures, and let n = |Q|. By Lemma 3.1 fix j so that at least n + 2 of the first  $(n + 2)^3 + 1$  vectors of the form  $F_j M^k[z_0, z_1, z_2]^T$  are pairwise linearly independent. We use finisher gadget  $F_j$ in the recursive construction, so that the signature of  $G_k$  is  $F_iM^k[z_0,z_1,z_2]^T$ , which we denote by  $[X_k, Y_k]$ . We note that there exists a subset S of these signatures for which each  $Y_k$  is nonzero and |S| = n + 1. We will argue using only the existence of S, so there is no need to algorithmically "find" such a set, and for that matter, one can try out all three finisher gadgets without any need to determine which finisher gadget is "the correct one" beforehand. If we replace every element of Q with a copy of  $G_k$ , we obtain an instance of  $\#\mathcal{G} \mid \mathcal{R}$  (note that the correct bipartite signature structure is preserved), and we denote this new signature grid by  $\Omega_k$ . Then

$$\operatorname{Holant}_{\Omega_k} = \sum_{0 \le i \le n} c_i X_k^i Y_k^{n-i}$$

where  $c_i = \sum_{\sigma \in J_i} \prod_{v \in V \setminus Q} f_v(\sigma|_{E(v)})$ ,  $J_i$  is the set of  $\{0,1\}$  edge assignments where the number of 0s assigned to the edges incident to the copies of  $G_k$  is i,  $f_v$  is the signature at v,

and E(v) is the set of edges incident to v. The important point is that the  $c_i$  values do not depend on  $X_k$  or  $Y_k$ . Since each signature grid  $\Omega_k$  is an instance of  $\#\mathcal{G} \mid \mathcal{R}$ , Holant $\Omega_k$  can be solved exactly using the oracle. Carrying out this process for every  $k \in \{0, 1, \ldots, (n+2)^3\}$ , we arrive at a linear system where the  $c_i$  values are the unknowns.

$$\begin{bmatrix} \operatorname{Holant}_{\Omega_0} \\ \operatorname{Holant}_{\Omega_1} \\ \vdots \\ \operatorname{Holant}_{\Omega_{(n+2)^3}} \end{bmatrix} = \begin{bmatrix} X_0^0 Y_0^n & X_0^1 Y_0^{n-1} & \cdots & X_0^n Y_0^0 \\ X_1^0 Y_1^n & X_1^1 Y_1^{n-1} & \cdots & X_1^n Y_1^0 \\ \vdots & \vdots & \ddots & \vdots \\ X_{(n+2)^3}^0 Y_{(n+2)^3}^n & X_{(n+2)^3}^1 Y_{(n+2)^3}^{n-1} & \cdots & X_{(n+2)^3}^n Y_{(n+2)^3}^0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

For  $0 \le i \le n$ , let  $k_i$  such that  $S = \{[X_{k_0}, Y_{k_0}], [X_{k_1}, Y_{k_1}], \dots, [X_{k_n}, Y_{k_n}]\}$ , and let  $[x_i, y_i] = [X_{k_i}, Y_{k_i}]$ . Then we have a subsystem

$$\begin{bmatrix} y_0^{-n} \cdot \operatorname{Holant}_{\Omega_{k_0}} \\ y_1^{-n} \cdot \operatorname{Holant}_{\Omega_{k_1}} \\ \vdots \\ y_n^{-n} \cdot \operatorname{Holant}_{\Omega_{k_n}} \end{bmatrix} = \begin{bmatrix} x_0^0 y_0^0 & x_0^1 y_0^{-1} & \cdots & x_0^n y_0^{-n} \\ x_1^0 y_1^0 & x_1^1 y_1^{-1} & \cdots & x_1^n y_1^{-n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^0 y_n^0 & x_n^1 y_n^{-1} & \cdots & x_n^n y_n^{-n} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

The matrix above has entry  $(x_r/y_r)^c$  at index (r,c). Due to pairwise linear independence of  $[x_r,y_r]$ ,  $x_r/y_r$  is pairwise distinct for  $0 \le r \le n$ . Hence this is a Vandermonde system of full rank. Therefore the initial feasible linear system has full rank and we can solve it for the  $c_i$  values. With these values in hand, we can calculate  $\operatorname{Holant}_{\Omega} = \sum_{0 \le i \le n} c_i x^i y^{n-i}$  directly, completing the reduction.

The ability to simulate all unary signatures will allow us to prove #P-hardness. The next lemma says that, if  $\mathcal{R}$  contains the Equality gate  $=_3$ , then other than on a 1-dimensional curve ab=1 and an isolated point (a,b)=(0,0), the ability to simulate unary signatures gives a reduction from Vertex Cover. Note that counting Vertex Cover on 3-regular graphs is just #[0,1,1] | [1,0,0,1]. Xia et al. showed that this is #P-hard even when the input is restricted to 3-regular planar graphs [18]. We will see shortly that on the curve ab=1 and at (a,b)=(0,0), the problem  $\operatorname{Hol}(a,b)$  is tractable.

**Lemma 3.3.** Suppose that  $(a,b) \in \mathbb{C}^2 - \{(a,b) : ab = 1\} - \{(0,0)\}$  and let  $\mathcal{G}$  and  $\mathcal{R}$  be finite signature sets where  $[a,1,b] \in \mathcal{G}$  and  $[1,0,0,1] \in \mathcal{R}$ . Further assume that  $\#\mathcal{G} \cup \{[x_i,y_i] : 0 \le i < m\} \mid \mathcal{R} \le_T \#\mathcal{G} \mid \mathcal{R}$  for any  $x_i,y_i \in \mathbb{C}$  and  $m \in \mathbb{Z}^+$ . Then  $\#\mathcal{G} \cup \{[0,1,1]\} \mid \mathcal{R} \le_T \#\mathcal{G} \mid \mathcal{R}$ , and  $\#\mathcal{G} \mid \mathcal{R}$  is #P-hard.

*Proof.* Assume  $ab \neq 1$  and  $(a,b) \neq (0,0)$ . Since Hol(0,1) (which is the same as  $\#[0,1,1] \mid [1,0,0,1]$ , or counting vertex covers on 3-regular graphs) is #P-hard, we only need to show how to simulate the generator signature [0,1,1]. We split this into three cases, and use a chain of three reductions, each involving a gadget in Figure 2 (each gadget has [1,0,0,1] assigned to the degree 3 vertices and [a,1,b] assigned to the degree 2 vertices).

- (1)  $ab \neq 0$  and  $ab \neq -1$
- (2) ab = 0
- (3) ab = -1

If  $ab \neq 0$  and  $ab \neq -1$ , then we use Gadget 3, and we set its unary signatures to be  $\theta = [(ab+1)/(1-ab), -a^2(ab+1)/(1-ab)], \ \gamma = [-a^{-2}, b^{-1}(1+ab)^{-1}], \ \text{and} \ \rho = [-b/(ab-1), a/(ab-1)].$  Calculating the resulting signature of Gadget 3, we find that it is [0,1,1] as desired.

If ab=0 then assume without loss of generality that a=0 and  $b\neq 0$ . This time we use Gadget 1, setting  $\theta=[b,b^{-1}]$ . Then Gadget 1 simulates a  $[b^{-1},1,2b]$  generator signature, but since this signature fits the criteria of case 1 above, we are done by reduction from that case.

Similarly, if ab = -1, then Gadget 2 exhibits a generator signature of the form [0, 1, 5/(2a)] under the signatures  $\theta = [1/(6a), -a/24]$  and  $\gamma = [-3/a, a]$ . Since 5/(2a) is nonzero, we are done by reduction from case 2.

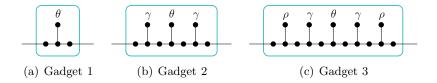


Figure 2: Gadgets used to simulate the [0,1,1] signature

It will be helpful to have conditions that are easier to check than those in Lemma 3.2. To this end, we establish condition 2 in terms of eigenvalues, and we build general-purpose finisher gadgets to eliminate condition 3. Let  $M_4$ ,  $M_5$ , and F be the recurrence matrices for Gadget 4, Gadget 5, and the simplest possible binary finisher gadget (each built using generator signature [a, 1, b] and recognizer signature [1, 0, 0, 1]; see Figures 3(a), 3(b), and 1(c)). Provided that  $ab \neq 1$  and  $a^3 \neq b^3$ , it turns out that the finisher gadget sets  $\{F, FM_4, FM_4^2\}$  and  $\{F, FM_4, FM_5\}$  satisfy condition 3 of Lemma 3.2 when  $ab \neq 0$  and ab = 0, respectively. Together with Lemma 3.3, these observations yield the following.

**Theorem 3.4.** If the following gadgets can be built using generator [a, 1, b] and recognizer [1, 0, 0, 1] where  $a, b \in \mathbb{C}$ ,  $ab \neq 1$ , and  $a^3 \neq b^3$ , then the problem  $\operatorname{Hol}(a, b)$  is #P-hard.

- (1) A binary recursive gadget with nonsingular recurrence matrix M which has eigenvalues  $\alpha$  and  $\beta$  such that  $\frac{\alpha}{\beta}$  is not a root of unity.
- (2) A binary starter gadget with signature s which is not orthogonal to any row eigenvector of M.

*Proof.* First we show how to build general-purpose binary finisher gadgets for the main construction using the assumed generator and recognizer, starting first with the case where  $ab \neq 0$ . Using the simplest possible choice for a finisher gadget F (Figure 1(c)), we get  $F = \begin{bmatrix} a & 0 & 1 \\ 1 & 0 & b \end{bmatrix}$ . Let  $M_4$  be the recurrence matrix for binary recursive Gadget 4 (Figure 3(a)), and we calculate that

$$M_4 = \begin{bmatrix} a^3 & 2a & b \\ a^2 & ab+1 & b^2 \\ a & 2b & b^3 \end{bmatrix}.$$

We build two more finisher gadgets F' and F'' using Gadget 4 so that  $F' = FM_4$  and  $F'' = FM_4^2$ . Since F and  $M_4$  both have full rank (note  $\det(M_4) = ab(ab-1)^3$ ), it follows that F' and F'' also have full rank. Now we will show that the row spaces of F, F' and F'' have trivial intersection, and it suffices to verify that the cross products of the row vectors of F, F', and F'' (denoted respectively by v, v', and v'') are linearly independent. (To see this, suppose u is a complex vector in the intersection of the row spaces of  $F_1$ ,

 $F_1'$ , and  $F_1''$ . Then v, v', and v'' are all orthogonal to u, but since v, v', and v'' are linearly independent, they span the conjugate vector  $\overline{u}$  which is then also orthogonal to u. This means  $|u|^2 = u\overline{u} = 0$ , and that u = 0.) The cross products of the row vectors of F, F', and F'' are [0, 1-ab, 0],  $(ab-1)^2[2b^2, -ab(ab+1), 2a^2]$ , and  $(ab-1)^3[2b(a^2b^3+a^2+ab^2+b^4), -ab(a^3b^3+2a^3+2a^2b^2+ab+2b^3), 2a(a^4+a^3b^2+a^2b+b^2)]$  respectively. Then to see that these 3 vectors are linearly independent, it suffices to verify that the matrix  $\begin{bmatrix} 2b^2 & 2a^2 \\ 2b(a^2b^3+a^2+ab^2+b^4) & 2a(a^4+a^3b^2+a^2b+b^2) \end{bmatrix}$  is nonsingular. Since  $a\neq 0$ ,  $b\neq 0$ , and  $ab\neq 1$ , we just check  $\det\begin{bmatrix} b & a \\ a^2b^3+a^2+ab^2+b^4 & a^4+a^3b^2+a^2b+b^2 \end{bmatrix}=(ab-1)(a^3-b^3)\neq 0$ , so the row spaces of F, F', and F'' have trivial intersection when  $ab\neq 0$ .

Now suppose ab = 0. Since  $a^3 \neq b^3$ , by symmetry, if ab = 0 we may assume without loss of generality that  $a \neq 0$  and b = 0. Let  $M_5$  be the recurrence matrix for binary recursive Gadget 5 (Figure 3(b)).

$$M_5 = \begin{bmatrix} a^6 + 2a^3 + 1 & 2a^4 + 2a & a^2 \\ a^5 + a^2 & 2a^3 + 1 & a \\ a^4 & 2a^2 & 1 \end{bmatrix}.$$

Composing F with  $M_5$ , we get a finisher gadget with matrix  $FM_5$ , which has full rank since F has full rank and  $\det(M_5) = 1$ . It is also straightforward to see that F' has full rank, as  $F' = \begin{bmatrix} a^4 + a & 2a^2 & 0 \\ a^3 & 2a & 0 \end{bmatrix}$ . The cross products of the rows of F, F', and  $FM_5$  are [0,1,0],  $[0,0,2a^2]$ , and  $[-2a,2a^3+1,-2a^2(1+a)(a^2-a+1)]$  respectively. Then the matrix of cross products is clearly nonsingular, and we conclude that for any  $a,b \in \mathbb{C}$ , we have 3 finisher gadgets satisfying item 3 of Lemma 3.2 unless ab=1 or  $a^3=b^3$ .

Now we want to show that s is not a column eigenvector of  $M^k$  for any positive integer k (note that s is nonzero by assumption). Writing out the Jordan Normal Form for M, we

have 
$$M^k s = T^{-1} D^k T s$$
, where  $D^k$  has the form  $\begin{bmatrix} \alpha^k & 0 & 0 \\ 0 & \beta^k & 0 \\ 0 & * & * \end{bmatrix}$ , and where  $\alpha$  and  $\beta$  are

eigenvalues of M for which  $\frac{\alpha}{\beta}$  is not a root of unity. Let t = Ts and write  $t = [c, d, e]^T$ . By hypothesis, s is not orthogonal to the first two rows of T, thus  $c, d \neq 0$ . If s were an eigenvector of  $M^k$  for some positive integer k, then  $T^{-1}D^kTs = M^ks = \lambda s$  for some nonzero complex value  $\lambda$ , and  $D^kt = T(\lambda s) = \lambda t$ . But then  $c\alpha^k = \lambda c$  and  $d\beta^k = \lambda d$ , which means  $\frac{\alpha^k}{\beta^k} = 1$ , contradicting the fact that  $\frac{\alpha}{\beta}$  is not a root of unity.

We have now met all the criteria for Lemma 3.2, so the reduction  $\#\mathcal{S} \cup \{[a,1,b],[x,y]\} \mid [1,0,0,1] \leq_T \#\mathcal{S} \cup \{[a,1,b]\} \mid [1,0,0,1] \text{ holds for any } x,y \in \mathbb{C} \text{ and any finite signature set } \mathcal{S}.$  By Lemma 3.3 the problem Hol(a,b) is #P-hard.

# 3.2. Unary recursive construction

Now we consider the unary case. The following lemma arrives from [13] and is stated explicitly in [6]. It can be viewed as a unary version of Lemma 3.2 without finisher gadgets.

**Lemma 3.5.** Suppose there is a unary recursive gadget with nonsingular matrix M and a unary starter gadget with nonzero signature vector s. If the ratio of the eigenvalues of M is

not a root of unity and s is not a column eigenvector of M, then these gadgets can be used to interpolate all unary signatures.

Surprisingly, a set of general-purpose starter gadgets can be made for this construction as long as  $ab \neq 1$  and  $a^3 \neq b^3$ , so we refine this lemma by eliminating the starter gadget requirement. The starter gadgets are Fs,  $FM_4s$ , and  $FM_6s$  where  $M_6$  is Gadget 6 and s is the single-vertex starter gadget (see Figures 3(c) and 1(a)).

**Theorem 3.6.** Suppose there is a unary recursive gadget with nonsingular matrix M, and the ratio of the eigenvalues of M is not a root of unity. Then for any  $a, b \in \mathbb{C}$  where  $ab \neq 1$  and  $a^3 \neq b^3$ , there is a starter gadget built using generator [a, 1, b] and recognizer [1, 0, 0, 1] for which the resulting construction can be used to interpolate all unary signatures.

*Proof.* Let  $M_4$ ,  $M_6$ , F, and s be Gadget 4, Gadget 6, the binary finisher gadget in Figure 1(c), and single-vertex binary starter gadget (Figure 1(a)), respectively. Note  $s = [a, 1, b]^{\mathrm{T}}$  and

$$M_6 = \begin{bmatrix} a^3 + 1 & 0 & a^2 + b \\ a^2 + b & 0 & a + b^2 \\ a + b^2 & 0 & b^3 + 1 \end{bmatrix}.$$

Using Mathematica<sup>TM</sup>, we verify that the block matrices  $[FM_4s \ Fs]$ ,  $[FM_6s \ Fs]$ , and  $[FM_4s \ FM_6s]$  are all nonsingular provided that  $ab \neq 1$  and  $a^3 \neq b^3$ , so the vectors Fs,  $FM_4s$ , and  $FM_6s$  are pairwise linearly independent. If the ratio of the eigenvalues of M is not a root of unity, then the eigenvalues of M are distinct, each eigenvalue corresponds to an eigenspace of dimension 1, and at least one element of  $\{FM_4s, FM_6s, Fs\}$  is not a column eigenvector of M. The corresponding starter gadget can be used with M in a recursive construction and the result follows from Lemma 3.5.

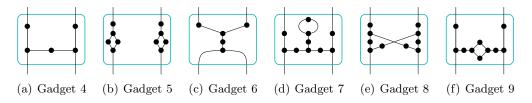


Figure 3: Binary recursive gadgets

## 4. Complex Signatures

Now we aim to characterize  $\operatorname{Hol}(a,b)$  where  $a,b \in \mathbb{C}$ . The next lemma introduces the technique of algebraic symmetrization. We show that over 3-regular graphs, the Holant value is expressible as an integer polynomial P(X,Y), where X=ab and  $Y=a^3+b^3$ . This change of variable, from (a,b) to (X,Y), is crucial in two ways. First, it allows us to derive tractability results easily, drawing connections between problems that may appear unrelated, and the tractability of one implies the other. Second, it facilitates the proof of hardness for those (a,b) where the problem is indeed #P-hard by reducing the degree of the polynomials involved. Once this transformation is made, four binary recursive gadgets

easily cover all of the #P-hard problems where X and Y are real-valued, with a straightforward symbolic computation using CylindricalDecomposition in Mathematica<sup>TM</sup>. All gadget constructions in this section use [a, 1, b] and [1, 0, 0, 1] signatures exclusively, and we henceforth denote X = ab and  $Y = a^3 + b^3$  for the remainder of this paper.

**Lemma 4.1.** Let G be a 3-regular graph. Then there exists a polynomial  $P(\cdot, \cdot)$  with two variables and integer coefficients such that for any signature grid  $\Omega$  having underlying graph G and every edge labeled [a, 1, b], the Holant value is  $\operatorname{Holant}_{\Omega} = P(ab, a^3 + b^3)$ .

*Proof.* Consider any  $\{0,1\}$  vertex assignment  $\sigma$  with a non-zero valuation. If  $\sigma'$  is the complement assignment switching all 0's and 1's in  $\sigma$ , then for  $\sigma$  and  $\sigma'$ , we have the sum of valuations  $a^ib^j + a^jb^i$  for some i and j. Here i (resp. j) is the number of edges connecting two degree 3 vertices both assigned 0 (resp. 1) by  $\sigma$ . We note that  $a^ib^j + a^jb^i = (ab)^{\min(i,j)}(a^{|i-j|} + b^{|i-j|})$ .

We prove  $i \equiv j \pmod 3$  inductively. For the all-0 assignment, this is clear since every edge contributes a factor a and the number of edges is divisible by 3 for a 3-regular graph. Now starting from any assignment  $\sigma$ , if we switch the assignment on one vertex from 0 to 1, it is easy to verify that it changes the valuation from  $a^ib^j$  to  $a^{i'}b^{j'}$ , where i-j=i'-j'+3. As every  $\{0,1\}$  assignment is obtainable from the all-0 assignment by a sequence of switches, the conclusion  $i \equiv j \pmod 3$  follows.

Now  $a^ib^j + a^jb^i = (ab)^{\min(i,j)}(a^{3k} + b^{3k})$ , for some  $k \ge 0$  and a simple induction  $a^{3(k+1)} + b^{3(k+1)} = (a^{3k} + b^{3k})(a^3 + b^3) - (ab)^3(a^{3(k-1)} + b^{3(k-1)})$  shows that the Holant is a polynomial  $P(ab, a^3 + b^3)$  with integer coefficients.

Corollary 4.2. If X = -1 and  $Y \in \{0, \pm 2i\}$ , then  $\operatorname{Hol}(a, b)$  is in P.

Proof. The problems  $\operatorname{Hol}(1,-1)$ ,  $\operatorname{Hol}(-\mathfrak{i},-\mathfrak{i})$ , and  $\operatorname{Hol}(\mathfrak{i},\mathfrak{i})$  are all solvable in P (these fall within the families  $\mathcal{F}_1$ ,  $\mathcal{F}_2$ , and  $\mathcal{F}_3$  in [3]); X=-1 for each, whereas the value of Y for these problems is 0, 2 $\mathfrak{i}$ , and  $-2\mathfrak{i}$  respectively. Since the value of any 3-regular signature grid is completely determined by X, Y, and the polynomial  $P(\cdot,\cdot)$  (which in turn depends only on the underlying graph G), any a and b such that ab=-1 and  $a^3+b^3\in\{0,\pm 2\mathfrak{i}\}$  (i.e. ab=-1 and  $a^{12}=1$ ) is computable in polynomial time.

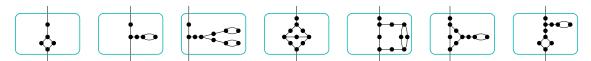
We now list all of the cases where Hol(a, b) is computable in polynomial time.

**Theorem 4.3.** If any of the following four conditions is true, then Hol(a,b) is solvable in P:

- (1) X = 1,
- (2) X = Y = 0.
- (3) X = -1 and  $Y \in \{0, \pm 2i\}$
- (4)  $4X^3 = Y^2$  and the input is restricted to planar graphs.

*Proof.* If X=1 then the signature [a,1,b] is degenerate and the Holant can be computed in polynomial time. If X=Y=0 then a=b=0, and a 2-coloring algorithm can be employed on the edges. If X=-1 and  $Y \in \{0,\pm 2i\}$  then we are done by Corollary 4.2. If we restrict the input to planar graphs and  $4X^3=Y^2$  (equivalently,  $a^3=b^3$ ), holographic algorithms can be applied [4].

Our main task in this paper is to prove that all remaining problems are #P-hard. The following two lemmas provide sufficient conditions to satisfy the eigenvalue requirement of the recursive constructions.



(a) Gadget 10 (b) Gadget 11 (c) Gadget 12 (d) Gadget 13 (e) Gadget 14 (f) Gadget 15 (g) Gadget 16

Figure 4: Unary recursive gadgets

**Lemma 4.4.** If both roots of the complex polynomial  $x^2 + Bx + C$  have the same norm, then  $B|C| = \overline{B}C$  and  $B^2\overline{C} = \overline{B}^2C$ . If further  $B \neq 0$  and  $C \neq 0$ , then  $Arg(B^2) = Arg(C)$ .

Proof. If the roots have equal norm, then for some  $a,b\in\mathbb{C}$  and nonnegative  $r\in\mathbb{R}$  and we can write  $x^2+Bx+C=(x-ra)(x-rb)$ , where |a|=|b|=1, so  $B|C|=-r(a+b)r^2=-r(a^{-1}+b^{-1})r^2ab=\overline{B}C$ . Squaring both sides and dividing by C, we have  $B^2\overline{C}=\overline{B}^2C$  (note this is justified since this equality still holds when C=0). Multiplying  $B|C|=\overline{B}C$  by B we get  $B^2|C|=|B^2|C$ , and if B and C are both nonzero then  $\frac{B^2}{|B^2|}=\frac{C}{|C|}$ , that is,  $\operatorname{Arg}(B^2)=\operatorname{Arg}(C)$ .

**Lemma 4.5.** If all roots of the complex polynomial  $x^3 + Bx^2 + Cx + D$  have the same norm, then  $C|C|^2 = \overline{B}|B|^2D$ .

*Proof.* If the roots have equal norm, then for some  $a, b, c \in \mathbb{C}$  and nonnegative  $r \in \mathbb{R}$  we can write  $x^3 + Bx^2 + Cx + D = (x - ra)(x - rb)(x - rc)$ , where |a| = |b| = |c| = 1, so B = -r(a+b+c),  $C = r^2(ab+bc+ca)$ , and  $D = -r^3abc$ . Then

 $C|C|^2 = r^2(ab + bc + ca)r^4|ab + bc + ca|^2 = r(\overline{a+b+c})r^2|a+b+c|^2r^3abc = \overline{B}|B|^2D,$  where we used the fact that  $|ab+bc+ca| = |ab+bc+ca| \cdot |a^{-1}b^{-1}c^{-1}| = |a^{-1}+b^{-1}+c^{-1}| = |\overline{a+b+c}| = |a+b+c|.$ 

Now we introduce a powerful new technique called *Eigenvalue Shifted Pairs*.

**Definition 4.6.** A pair of nonsingular square matrices M and M' is called an *Eigenvalue Shifted Pair (ESP)* if  $M' = M + \delta I$  for some non-zero  $\delta \in \mathbb{C}$ , and M has distinct eigenvalues.

Clearly for such a pair, M' also has distinct eigenvalues. The recurrence matrices of Gadgets 10 and 11 (Figure 4) differ only by ab-1 along the diagonal, and form an Eigenvalue Shifted Pair for nearly all  $a, b \in \mathbb{C}$ . We will make significant use of such Eigenvalue Shifted Pairs, but first we state a technical lemma.

**Lemma 4.7.** Suppose  $\alpha, \beta, \delta \in \mathbb{C}$ ,  $|\alpha| = |\beta|$ ,  $\alpha \neq \beta$ ,  $\delta \neq 0$ , and  $|\alpha + \delta| = |\beta + \delta|$ . Then there exists  $r, s \in \mathbb{R}$  such that  $r\delta = \alpha + \beta$  and  $s\delta^2 = \alpha\beta$ .

*Proof.* After a rotation in the complex plane, we can assume  $\alpha = \overline{\beta}$ , and then since  $\alpha + \beta$ ,  $\alpha\beta \in \mathbb{R}$  we just need to prove  $\delta \in \mathbb{R}$ . Then  $(\alpha + \delta)\overline{(\alpha + \delta)} = |\alpha + \delta|^2 = |\beta + \delta|^2 = (\beta + \delta)\overline{(\beta + \delta)} = (\overline{\alpha} + \delta)(\alpha + \overline{\delta})$  and we distribute to get  $\alpha\overline{\alpha} + \delta\overline{\delta} + \alpha\overline{\delta} + \overline{\alpha}\delta = \alpha\overline{\alpha} + \delta\overline{\delta} + \overline{\alpha}\overline{\delta} + \alpha\delta$ . Canceling repeated terms and factoring, we have  $(\overline{\alpha} - \alpha)(\overline{\delta} - \delta) = 0$ , and since  $\alpha \neq \beta = \overline{\alpha}$  we know  $\overline{\delta} = \delta$  therefore  $\delta \in \mathbb{R}$ .

**Corollary 4.8.** Let M and M' be an Eigenvalue Shifted Pair of 2 by 2 matrices. If both M and M' have eigenvalues of equal norm, then there exists  $r, s \in \mathbb{R}$  such that  $\operatorname{tr}(M) = r\delta$  (possibly 0) and  $\det(M) = s\delta^2$ .

*Proof.* Let  $\alpha$  and  $\beta$  be the eigenvalues of M, so  $\alpha + \delta$  and  $\beta + \delta$  are the eigenvalues of M'. Suppose that  $|\alpha| = |\beta|$  and  $|\alpha + \delta| = |\beta + \delta|$ . Then by Lemma 4.7, there exists  $r, s \in \mathbb{R}$  such that  $\operatorname{tr}(M) = \alpha + \beta = r\delta$  and  $\det(M) = \alpha\beta = s\delta^2$ .

We now apply an ESP to prove that most settings of Hol(a, b) are #P-hard.

**Lemma 4.9.** Suppose  $X \neq \pm 1$ ,  $X^2 + X + Y \neq 0$ , and  $4(X - 1)^2(X + 1) \neq (Y + 2)^2$ . Then either unary Gadget 10 or unary Gadget 11 has nonzero eigenvalues with distinct norm, unless X and Y are both real numbers.

Proof. Gadgets 10 and 11 have  $M_{10} = \begin{bmatrix} a^3+1 & a+b^2 \\ a^2+b & b^3+1 \end{bmatrix}$  and  $M_{11} = \begin{bmatrix} a^3+ab & a+b^2 \\ a^2+b & ab+b^3 \end{bmatrix}$  as their recurrence matrices, so  $M_{11} = M_{10} + (X-1)I$ , and the eigenvalue shift is nonzero. Checking the determinants,  $\det(M_{10}) = (X-1)^2(X+1) \neq 0$  and  $\det(M_{11}) = (X-1)(X^2+X+Y) \neq 0$ . Also,  $\operatorname{tr}(M_{10})^2 - 4\det(M_{10}) = (Y+2)^2 - 4(X-1)^2(X+1) \neq 0$ , so the eigenvalues of  $M_{10}$  are distinct. Therefore by Corollary 4.8, either  $M_{10}$  or  $M_{11}$  has nonzero eigenvalues of distinct norm unless  $\operatorname{tr}(M_{10}) = r(X-1)$  and  $\det(M_{10}) = s(X-1)^2$  for some  $r, s \in \mathbb{R}$ . Then we would have  $(X-1)^2(X+1) = s(X-1)^2$  so  $X = s-1 \in \mathbb{R}$  and Y+2=r(X-1) so  $Y=r(X-1)-2 \in \mathbb{R}$ .

Now we will deal with the following exceptional cases from Lemma 4.9 (X=1 is tractable by Theorem 4.3).

- 0.  $X \in \mathbb{R}$  and  $Y \in \mathbb{R}$
- 1.  $X^2 + X + Y = 0$
- 2. X = -1
- 3.  $4(X-1)^2(X+1) = (Y+2)^2$

The case where X and Y are both real is dealt with using the tools developed in Section 3, and some symbolic computation. This includes the case where a and b are both real as a subcase. When a and b are both real, a dichotomy theorem for the complexity of  $\operatorname{Hol}(a,b)$  has been proved in [6] with a significant effort. With the new tools developed, we offer a simpler proof. This also covers some cases where a or b is complex. Working with real-valued X and Y is a significant advantage, since the failure condition given by Lemma 4.5 is simplified by the disappearance of norms and conjugates. This brings the problem of proving #P-hardness within reach of symbolic computation via cylindrical decomposition. We apply Theorem 3.4 to Gadgets 4, 7, 8, and 9 (Figure 3) together with a starter gadget (Figure 1(a)) to prove that these problems are #P-hard. Conditions 1 and 2 of Theorem 3.4 are encoded directly into a query for CYLINDRICALDECOMPOSITION in Mathematica<sup>TM</sup>, but first we give a lemma to show how to encode condition 2 of Lemma 3.4.

**Lemma 4.10.** Suppose  $M \in \mathbb{C}^{n \times n}$  and  $s \in \mathbb{C}^{n \times 1}$ . If  $det([s, Ms, M^2s, \dots, M^{n-1}s]) \neq 0$  then s is not orthogonal to any row eigenvector of M.

*Proof.* Suppose s is orthogonal to a row eigenvector v of M with eigenvalue  $\lambda$ . Then  $v[s, Ms, ..., M^{n-1}s] = 0$ , since  $vM^is = \lambda^i vs = 0$ . Since  $v \neq 0$  this is a contradiction.

**Theorem 4.11.** Suppose  $a, b \in \mathbb{C}$ ,  $X, Y \in \mathbb{R}$ ,  $X \neq 1$ ,  $4X^3 \neq Y^2$ , and it is not the case that both X = -1 and Y = 0. Then the problem Hol(a, b) is #P-hard.

*Proof.* We will use binary recursive Gadgets 4, 7, 8, and 9 together with the single-vertex starter gadget given in Figure 1(a) (denote the respective matrices by  $M_4$ ,  $M_7$ ,  $M_8$ ,  $M_9$ , and s). Calculating the recurrence matrices of these gadgets, we get

$$M_{7} = \begin{bmatrix} a^{6} + a^{4}b + a^{3} + a^{2}b^{2} & 2a^{4} + 4a^{2}b + 2ab^{3} & a^{2} + ab^{2} + b^{4} + b \\ a^{5} + a^{3}b + a^{2} + ab^{2} & a^{4}b + a^{3} + 2a^{2}b^{2} + ab^{4} + 2ab + b^{3} & a^{2}b + ab^{3} + b^{5} + b^{2} \\ a^{4} + a^{2}b + a + b^{2} & 2a^{3}b + 4ab^{2} + 2b^{4} & a^{2}b^{2} + ab^{4} + b^{6} + b^{3} \end{bmatrix},$$

$$M_{8} = \begin{bmatrix} a^{6} + 2a^{3} + 1 & 2a^{4} + 4a^{2}b + 2b^{2} & a^{2} + 2ab^{2} + b^{4} \\ a^{5} + a^{3}b + a^{2} + b & 2a^{3} + 2a^{2}b^{2} + 2ab + 2b^{3} & ab^{3} + a + b^{5} + b^{2} \\ a^{4} + 2a^{2}b + b^{2} & 2a^{2} + 4ab^{2} + 2b^{4} & b^{6} + 2b^{3} + 1 \end{bmatrix},$$

$$M_{9} = \begin{bmatrix} a^{6} + 2a^{3} + a^{2}b^{2} & 2a^{4} + 2a^{2}b + 2ab^{3} + 2a & a^{2} + b^{4} + 2b \\ a^{5} + 2a^{2} + ab^{2} & a^{4}b + a^{3} + a^{2}b^{2} + ab^{4} + 2ab + b^{3} + 1 & a^{2}b + b^{5} + 2b^{2} \\ a^{4} + 2a + b^{2} & 2a^{3}b + 2ab^{2} + 2b^{4} + 2b & a^{2}b^{2} + b^{6} + 2b^{3} \end{bmatrix}.$$

Calculating the characteristic polynomials  $x^3 + Bx^2 + Cx + D$  of Gadgets 4, 7, 8, and 9, we get

$$B_4 = -(X+Y+1)$$

$$C_4 = (X^2+X+Y)(X-1)$$

$$D_4 = -X(X-1)^3$$

$$B_7 = -(-2X^3+4X^2+2XY+2X+Y^2+2Y)$$

$$C_7 = (X-1) \cdot (X^5-4X^4-X^3Y+6X^3+7X^2Y+4X^2+4XY^2+5XY+X+Y^3+2Y^2+Y)$$

$$D_7 = -(X-1)^3(2X+Y)(X^4-X^3+X^2Y+3X^2+2XY+X+Y^2+Y)$$

$$B_8 = -(-2X^3+2X^2+2X+Y^2+4Y+2)$$

$$C_8 = (X-1)^2(X^4-2X^3+2X^2+4XY+6X+2Y^2+4Y+1)$$

$$D_8 = -2(X-1)^6X(X+1)$$

$$B_9 = -(3X^2+XY+2X+Y^2+3Y+1)$$

$$C_9 = (X-1) \cdot (X^5-3X^4-2X^3Y-X^3+4X^2Y+7X^2+2XY^2+6XY+4X+Y^3+4Y^2+4Y)$$

$$D_9 = -(X-1)^3(X+Y+1)(X^4-2X^3+X^2+2XY+4X+Y^2+2Y)$$

Suppose  $X \neq 1$ ,  $4X^3 \neq Y^2$  (equivalently,  $a^3 \neq b^3$ ), and it is not the case that both X = -1 and Y = 0. For any real-valued setting of X and Y compatible with these constraints, we will see that at least one of these four binary recursive gadgets satisfies the requirements of Theorem 3.4 (the only exception is (X,Y) = (0,-1), but by Lemma 4.1 any such problem is equivalent to  $\operatorname{Hol}(0,-1)$  which is known to be  $\#\operatorname{P-hard}[6,11]$ ). To verify that Gadget j satisfies condition 1 of Theorem 3.4, we apply Lemma 4.5 and check that  $D_j(B_j^3D_j-C_j^3) \neq 0$  (note that the norm and conjugate disappear from the test since we are only considering real valued X and Y). By Lemma 4.10, Gadget 4 satisfies condition 2 because  $\det[s, M_4s, M_4^2s] = (X-1)^4(b^3-a^3) \neq 0$ . However,  $\det[s, M_7s, M_7^2s] = (X-1)^5(b^3-a^3)(X^2+X+Y)(X+Y+1)$ ,  $\det[s, M_8s, M_8^2s] = (X-1)^5(b^3-a^3)(X^2Y+4X^2+2XY+Y^2+Y)$ , and  $\det[s, M_9s, M_9^2s] = (X-1)^5(b^3-a^3)(X^2Y+4X^2+2XY+Y^2+Y)$ ,

 $(X-1)^6(b^3-a^3)(X+1)(Y+2)$ , so these are zero for some settings of X and Y. We summarize the essential observations in terms of (X,Y) coordinates as follows.

$$X = 1 \iff ab = 1$$

$$4X^{3} = Y^{2} \iff a^{3} = b^{3}$$

$$X = 0 \land Y = -1 \iff (a = 0 \land b^{3} = -1) \lor (a^{3} = -1 \land b = 0)$$

$$X = -1 \land Y = 0 \iff a^{6} = 1 \land ab = -1$$

$$D_{4}(B_{4}^{3}D_{4} - C_{4}^{3}) \neq 0 \implies \text{Gadget 4 fulfills Theorem 3.4}$$

$$D_{7}(B_{7}^{3}D_{7} - C_{7}^{3})(X^{2} + X + Y)(X + Y + 1) \neq 0 \implies \text{Gadget 7 fulfills Theorem 3.4}$$

$$D_{8}(B_{8}^{3}D_{8} - C_{8}^{3})(X^{2}Y + 4X^{2} + 2XY + Y^{2} + Y) \neq 0 \implies \text{Gadget 8 fulfills Theorem 3.4}$$

$$D_{9}(B_{9}^{3}D_{9} - C_{9}^{3})(X + 1)(Y + 2) \neq 0 \implies \text{Gadget 9 fulfills Theorem 3.4}$$

If we can verify that at least one of the 8 conditions on the left hand side holds for any real-valued setting of X and Y then we are done. Note that a disjunction of the left hand sides is a semi-algebraic set, and as such, is decidable by Tarski's Theorem [12]. Using symbolic computation via the CYLINDRICALDECOMPOSITION function from Mathematica<sup>TM</sup>, we verify that for any  $X, Y \in \mathbb{R}$ , at least one of the eight conditions above is true.

Now we can assume that  $X \notin \mathbb{R}$  or  $Y \notin \mathbb{R}$ , and we deal with the remaining three conditions. Note that if  $X^2 + X + Y = 0$  then  $X \in \mathbb{R}$  implies  $Y \in \mathbb{R}$ . So in the following lemma, the assumption that X and Y are not both real numbers amounts to  $X \notin \mathbb{R}$ .

**Lemma 4.12.** If  $X^2 + X + Y = 0$  and  $X \notin \mathbb{R}$  then the recurrence matrix of unary Gadget 12 has nonzero eigenvalues with distinct norm.

*Proof.* Let  $M_{12}$  be the recurrence matrix for unary Gadget 12.

$$M_{12} = \begin{bmatrix} a^6 + 2a^4b + a^3 + 3a^2b^2 + ab^4 & a^4 + 3a^2b + 2ab^3 + b^5 + b^2 \\ a^5 + 2a^3b + a^2 + 3ab^2 + b^4 & a^4b + 3a^2b^2 + 2ab^4 + b^6 + b^3 \end{bmatrix}$$

Then the determinant is the polynomial  $X^6 - 6X^5 - X^4Y + 16X^4 + 11X^3Y - 10X^3 + 5X^2Y^2 - 7X^2Y - X^2 + XY^3 - 4XY^2 - 3XY - Y^3 - Y^2$ . Amazingly, with the condition  $X^2 + X + Y = 0$ , this polynomial factors into  $-X^2(X-1)^5$ . Similarly, the trace, which is  $-2X^3 + 6X^2 + 3XY + Y^2 + Y$ , also factors into  $X(X-1)^3$ . Since  $\det(M_{12}) \neq 0$ ,  $\operatorname{tr}(M_{12}) \neq 0$ , and  $(1-X)\det(M_{12}) = \operatorname{tr}(M_{12})^2$ , we know  $\operatorname{Arg}(\det(M_{12})) \neq \operatorname{Arg}(\operatorname{tr}(M_{12})^2)$  and conclude by Lemma 4.4 that the eigenvalues of  $M_{12}$  (which are nonzero) have distinct norm.

Similarly, Gadgets 11 and 13 can be used to deal with the X = -1 condition. Recall that any setting of a and b such that X = -1 and  $Y = \pm 2i$  is tractable by Theorem 4.3.

**Lemma 4.13.** If X = -1,  $Y \neq \pm 2i$ , and  $Y \notin \mathbb{R}$ , then either Gadget 11 or Gadget 13 has a recurrence matrix with nonzero eigenvalues with distinct norm.

Proof. Suppose  $|Y| \neq 2$ ,  $Y \notin \mathbb{R}$ , and let  $M_{11}$  be the recurrence matrix for unary Gadget 11. Well,  $\det(M_{11}) = -2Y \neq 0$  and  $\operatorname{tr}(M_{11}) = Y - 2$ , so  $\overline{\operatorname{tr}(M_{11})} \cdot \det(M_{11}) - \operatorname{tr}(M_{11}) \cdot |\det(M_{11})| = -(\overline{Y} - 2)(2Y) - (Y - 2) \cdot |-2Y| = 4Y - 2|Y|^2 - 2Y \cdot |Y| + 4|Y| = -2(|Y| - 2)(|Y| + Y) \neq 0$ . Thus  $\overline{\operatorname{tr}(M_{11})} \cdot \det(M_{11}) \neq \operatorname{tr}(M_{11}) \cdot |\det(M_{11})|$  and by Lemma 4.4,  $M_{11}$  has (nonzero) eigenvalues with distinct norm.

Now suppose |Y| = 2, but  $Y \neq \pm 2i$  and  $Y \notin \mathbb{R}$ . Let  $M_{13}$  be the recurrence matrix for unary Gadget 13.

$$M_{13} = \begin{bmatrix} a^6 + 3a^3 + 3ab + b^3 & a^4 + 2a^2b + ab^3 + a + b^5 + 2b^2 \\ a^5 + a^3b + 2a^2 + 2ab^2 + b^4 + b & a^3 + 3ab + b^6 + 3b^3 \end{bmatrix}.$$

Then  $det(M_{13}) = -16Y \neq 0$ . Using the substitution  $\overline{Y} = 4/Y$ ,

$$\operatorname{tr}(M_{13})^2 \overline{\det(M_{13})} -$$

Hence  $\operatorname{tr}(M_{13})^2 \overline{\det(M_{13})} \neq \overline{\operatorname{tr}(M_{13})}^2 \det(M_{13})$  and the eigenvalues of  $M_{13}$  (which are nonzero) have distinct norm by Lemma 4.4.

The condition  $4(X-1)^2(X+1) = (Y+2)^2$  is somewhat resilient to individual unary recursive gadgets, but by using a second Eigenvalue Shifted Pair, we can reduce it to simpler conditions.

**Lemma 4.14.** Suppose  $4(X-1)^2(X+1)=(Y+2)^2$ . Then either unary Gadget 13 or unary Gadget 14 has nonzero eigenvalues with distinct norm, unless either  $X^3+2X^2+X+2Y=0$ , or  $X^3+4X^2+2Y-1=0$ , or both  $X,Y\in\mathbb{R}$ .

*Proof.* Assume that  $X^3 + 2X^2 + X + 2Y \neq 0$ ,  $X^3 + 4X^2 + 2Y - 1 \neq 0$ , and it is not the case that both  $X, Y \in \mathbb{R}$ . Note that  $X \notin \{0, 1\}$  since otherwise  $Y \in \mathbb{R}$  and we know that X and Y are not both real. The recurrence matrix for Gadget 14 is

$$M_{14} = \begin{bmatrix} a^6 + 3a^3 + a^2b^2 + ab + b^3 + 1 & a^4 + 2a^2b + ab^3 + a + b^5 + 2b^2 \\ a^5 + a^3b + 2a^2 + 2ab^2 + b^4 + b & a^3 + a^2b^2 + ab + b^6 + 3b^3 + 1 \end{bmatrix}$$

so  $M_{14} = M_{13} + (X-1)^2 I$ , and the eigenvalue shift is nonzero. Now,  $\det(M_{13}) = (X-1)^3 (X^3 + 2X^2 + X + 2Y) \neq 0$  and note that  $\operatorname{tr}(M_{13}) = -2X^3 + 6X + Y^2 + 4Y$  simplifies to  $\operatorname{tr}(M_{13}) = -2X^3 + 6X + Y^2 + 4Y - (Y+2)^2 + 4(X-1)^2 (X+1) = 2X(X-1)^2$  using the fact that  $4(X-1)^2 (X+1) = (Y+2)^2$ .

Similarly,  $\det(M_{14}) = \det(M_{14}) + (X-1)^2(4(X-1)^2(X+1) - (Y+2)^2) = (X-1)^3(X^3 + 4X^2 + 2Y - 1) \neq 0$ . Furthermore  $\operatorname{tr}[M_{13}]^2 - 4\det(M_{13}) = 4X^2(X-1)^4 - 4(X-1)^3(X^3 + 2X^2 + X + 2Y) = -4(X-1)^3(3X^2 + X + 2Y)$ . If this is zero, then substituting  $Y = (-3X^2 - X)/2$  into  $(Y+2)^2 - 4(X-1)^2(X+1) = 0$  we get  $X(X-1)^2(9X+8) = 0$  and  $X \in \mathbb{R}$ , with  $Y \in \mathbb{R}$  as a direct consequence. Corollary 4.8 implies that either Gadget 13 or Gadget 14 has nonzero eigenvalues of distinct norm, unless  $\operatorname{tr}(M_{13}) = r(X-1)^2$  and  $\det(M_{13}) = s(X-1)^4$  for some  $r, s \in \mathbb{R}$ . But then  $2X(X-1)^2 = r(X-1)^2$  hence  $X = r/2 \in \mathbb{R}$ , and  $(X-1)^3(X^3 + 2X^2 + X + 2Y) = s(X-1)^4$  hence  $Y = (-X^3 - 2X^2 - X + s(X-1))/2 \in \mathbb{R}$ . A contradiction.

Now we take advantage of another interesting coincidence; two gadgets with recurrence matrices that have identical trace.

**Lemma 4.15.** If  $X^2 + X + Y \neq 0$ ,  $4(X - 1)^2(X + 1) = (Y + 2)^2$ , and either  $X^3 + 2X^2 + X + 2Y = 0$  or  $X^3 + 4X^2 + 2Y - 1 = 0$ , then the recurrence matrix of unary Gadget 15 or unary Gadget 16 has nonzero eigenvalues with distinct norm, unless both  $X, Y \in \mathbb{R}$ .

*Proof.* The recurrence matrices for Gadget 15 and Gadget 16 are

$$M_{15} = \begin{bmatrix} a^6 + a^4b + 2a^3 + a^2b^2 + 2ab + b^3 & a^4 + 3a^2b + 2ab^3 + b^5 + b^2 \\ a^5 + 2a^3b + a^2 + 3ab^2 + b^4 & a^3 + a^2b^2 + ab^4 + 2ab + b^6 + 2b^3 \end{bmatrix},$$

$$M_{16} = \begin{bmatrix} a^6 + a^4b + 2a^3 + a^2b^2 + 2ab + b^3 & a^4 + a^3b^2 + a^2b + 2ab^3 + a + b^5 + b^2 \\ a^5 + 2a^3b + a^2b^3 + a^2 + ab^2 + b^4 + b & a^3 + a^2b^2 + ab^4 + 2ab + b^6 + 2b^3 \end{bmatrix}.$$

Let  $T = X^3 + 2X^2 + X + 2Y$ ,  $U = X^3 + 4X^2 + 2Y - 1$ , and let R denote  $(Y + 2)^2 - 4(X - 1)^2(X + 1)$ . Note that regardless of whether T = 0 or U = 0,  $X \in \mathbb{R}$  implies  $Y \in \mathbb{R}$ , so we will assume  $X \notin \mathbb{R}$ . The main diagonals of  $M_{15}$  and  $M_{16}$  are identical, so  $\operatorname{tr}(M_{15}) = \operatorname{tr}(M_{16})$ . Furthermore, if T = 0 then  $\operatorname{tr}(M_{15}) = \operatorname{tr}(M_{15}) - R - (X - 1)T/2 = -X(X - 1)^3/2 \neq 0$ . If U = 0 then  $\operatorname{tr}(M_{15}) = \operatorname{tr}(M_{15}) - R - (X - 1)U/2 = -(X - 1)(X^3 - 1)/2$ , and we claim this is nonzero as well. Otherwise,  $X^3 = 1$  then since U = 0,  $Y = -2X^2$  and using  $(Y + 2)^2 = 4(X - 1)^2(X + 1)$  we get  $(X^2 - 1)^2 = (X - 1)^2(X + 1)$  i.e.  $(X - 1)^2(X + 1)^2 = (X - 1)^2(X + 1)$  together with  $X \notin \mathbb{R}$  we get a contradiction. Next,  $\operatorname{det}(M_{16}) = (X - 1)^3(X + 1)(X^2 + X + Y)$  and  $\operatorname{det}(M_{15}) = \operatorname{det}(M_{15}) - R(X - 1)^2 = (X - 1)^3(X + 4)(X^2 + X + Y)$ , so these are both nonzero. If both  $M_{15}$  and  $M_{16}$  have eigenvalues with equal norm, then applying Lemma 4.4 twice,  $\operatorname{Arg}(\operatorname{det}(M_{15})) = \operatorname{Arg}(\operatorname{tr}(M_{15})^2) = \operatorname{Arg}(\operatorname{tr}(M_{16})^2) = \operatorname{Arg}(\operatorname{det}(M_{16}))$ . However, this would imply  $\operatorname{Arg}(X + 4) = \operatorname{Arg}(X + 1)$  and  $X \in \mathbb{R}$ , so we conclude that either  $M_{15}$  or  $M_{16}$  has nonzero eigenvalues with distinct norm.

Now we sum up the result of these lemmas.

**Theorem 4.16.** Suppose  $a, b \in \mathbb{C}$  such that  $X \neq 1$ ,  $4X^3 \neq Y^2$ , and  $(X, Y) \neq (-1, 0)$ . Then the problem  $\operatorname{Hol}(a, b)$  is  $\#\operatorname{P-hard}$ .

*Proof.* Under these assumptions, if X and Y are both real then Hol(a,b) is #P-hard by Lemma 4.11, so assume either X or Y is not real. For any such a and b, we know by Lemma 4.9 that either Gadget 10 or 11 has a recurrence matrix with nonzero eigenvalues of distinct norm, except in the following cases, where we will use other gadgets to fill this requirement.

- (1)  $X^2 + X + Y = 0$ .
- (2) X = -1.
- (3)  $4(X-1)^2(X+1) = (Y+2)^2$ .

If  $X^2 + X + Y = 0$  then  $X \notin \mathbb{R}$ , lest X and Y be real, so Lemma 4.12 implies that unary Gadget 12 has a recurrence matrix of the required form. If X = -1, then Lemma 4.13 indicates that either Gadget 11 or Gadget 13 satisfies the requirement, unless  $Y = \pm 2i$ . Now we may assume  $X^2 + X + Y \neq 0$ , so by Lemmas 4.14 and 4.15 if  $4(X - 1)^2(X + 1) = (Y + 2)^2$  then either unary Gadget 13, 14, 15, or 16 has a suitable recurrence matrix. In any case, we have a unary recursive gadget whose recurrence matrix has nonzero eigenvalues of distinct norm. Hence we are done by Theorem 3.6 and Lemma 3.3.

Recall Vertex Cover is #P-hard on 3-regular planar graphs, and note that all gadgets discussed are planar (in the case of Gadget 8, each iteration can be redrawn in a planar way by "going around" the previous iterations; see Figure 1(d)). Thus, all of the hardness results proved so far still apply even when the input graphs are restricted to planar graphs. There are, however, some problems that are #P-hard in general, yet polynomial time computable when the input is restricted to planar graphs. This class of problems corresponds exactly with the problems we still need to resolve at this point, i.e. when  $4X^2 = Y^3$  but  $X \notin \{0,\pm 1\}$ . The relevant interpolation results can be obtained with Gadget 4 and holographic reductions, using a technique demonstrated in [5].

**Lemma 4.17.** The problem  $\#[a,1,a] \mid [1,0,0,1]$  is #P-hard, unless  $a \in \{0,\pm 1,\pm \mathfrak{i}\}$ , in which case it is in P.

Proof. If  $a \in \mathbb{R}$  then this is already known [6], and a polynomial time algorithm for  $a=\pm i$  is in [3]. Now assume  $a \notin \mathbb{R}$  and  $a \neq \pm i$ . Since these problems have an extra degree of symmetry, we use a 2 by 2 recurrence matrix to describe the recursive construction which consists of a single-vertex starter gadget (Figure 1(a)) followed by some number of applications of binary recursive Gadget 4 (no finisher gadget is used here). That is, if  $\mathcal{F}$ -gate  $N_i$  has signature  $[a_i,b_i,a_i]$ , then the signature of  $N_{i+1}$  is given by  $[a_{i+1},b_{i+1},a_{i+1}]$  where  $[a_{i+1},b_{i+1}]^T=M[a_i,b_i]^T$ , and  $M=\begin{bmatrix}a(a^2+1)&2a\\2a^2&a^2+1\end{bmatrix}$ . Now,  $\det(M)=a(a-1)^2(a+1)^2$  and  $\operatorname{tr}(M)=(a+1)(a^2+1)$  are both nonzero under our assumptions. It can be verified (using the RESOLVE function of Mathematica<sup>TM</sup>) that  $\operatorname{tr}(M)|\det(M)|\neq \overline{\operatorname{tr}(M)}\det(M)$  provided that  $a\notin\mathbb{R}$ , so by Lemma 4.4, the eigenvalues of M have distinct norm. Also,  $M\begin{bmatrix}a\\1\end{bmatrix}=(a+1)\begin{bmatrix}a(a^2-a+2)\\(2a^2-a+1)\end{bmatrix}$ , so the starter gadget is not an eigenvector of M. We conclude by an analogous version of Lemma 3.5 that we can interpolate all signatures of the form [c,1,c], and since  $\#[0,1,0]\mid[0,1,1,0]$  is known to be #P-hard [5] and equivalent to  $\#[-\omega(\omega^2-2)/3,1,-\omega(\omega^2-2)/3]\mid[1,0,0,1]$  by holographic reduction (where  $\omega$  is the principal  $12^{\rm th}$  root of unity), we conclude that  $\#[a,1,a]\mid[1,0,0,1]$  is #P-hard.

**Lemma 4.18.** If  $4X^3 = Y^2$ , then  $\operatorname{Hol}(a,b)$  is  $\#\operatorname{P-hard}$  unless  $X \in \{0,\pm 1\}$ , in which case it is in P.

Proof. If X=0 then X=Y=0 and the problem is in P by Theorem 4.3. Otherwise,  $X\neq 0$ , let  $\omega=ba^{-1}$ , and applying a holographic reduction to  $\#[a,1,b]\mid [1,0,0,1]$  under the basis  $\begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix}$  we see that the problem  $\#[a,1,b]\mid [1,0,0,1]$  is equivalent to  $\#[\omega^2a,1,\omega b]\mid [1,0,0,1]$ , because  $\omega^3=b^3a^{-3}=1$ . Since  $\omega^2a=\omega b$ , we can apply Lemma 4.17 and the problem  $\#[a,1,b]\mid [1,0,0,1]$  is in P if  $ab=\omega^2a\cdot\omega b=\pm 1$  and #P-hard otherwise.

Given this, we have the following result.

**Theorem 4.19.** The problem  $\operatorname{Hol}(a,b)$  is #P-hard for all  $a,b \in \mathbb{C}$  except in the following cases, for which the problem is in P.

- (1) X = 1
- (2) X = Y = 0
- (3) X = -1 and  $Y \in \{0, \pm 2i\}$

If we restrict the input to planar graphs, then these three categories are tractable in P, as well as a fourth category  $4X^3 = Y^2$ , and the problem remains #P-hard in all other cases.

A simple coordinate change from (X,Y) to  $(X,(\frac{Y}{2})^2)$  translates this into Theorem 1.1.

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