

# On the complexity of automatic complexity

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## Abstract

Generalizing the notion of automatic complexity of individual words due to Shallit and Wang, we define the automatic complexity  $A(E)$  of an equivalence relation  $E$  on a finite set  $S$  of words.

We prove that the problem of determining whether  $A(E)$  equals the number  $|E|$  of equivalence classes of  $E$  is NP-complete. The problem of determining whether  $A(E) = |E| + k$  for a fixed  $k \geq 1$  is complete for the second level of the Boolean hierarchy for NP, i.e.,  $\text{BH}_2$ -complete.

Let  $L$  be the language consisting of all words of maximal non-deterministic automatic complexity. We characterize the complexity of infinite subsets of  $L$  by showing that they can be co-context-free but not context-free, i.e.,  $L$  is CFL-immune, but not coCFL-immune.

We show that for each  $\varepsilon > 0$ ,  $L_\varepsilon \notin \text{coCFL}$ , where  $L_\varepsilon$  is the set of all words whose deterministic automatic complexity  $A(x)$  satisfies  $A(x) \geq |x|^{1/2-\varepsilon}$ .

## 1 Introduction

Automatic complexity was introduced by Shallit and Wang [10] as a way to retain some of the power of Kolmogorov complexity while obtaining a computable notion. They raised the question whether the automatic complexity of a string (which we shall call a *word*)  $x$  is in fact polynomial-time computable as a function of  $|x|$ , the length of  $x$ . We give partial negative results for that question in two ways. (Our results also partially address Allender's question [1, Open Question 3.8] whether there is evidence that automatic complexity is computationally intractable.)

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- For nondeterministic automatic complexity, introduced by Hyde and Kjos-Hanssen [9], there is a natural notion of maximally complex words. We show that the language  $L$  consisting of all such words is not context-free, by virtue of being CFL-immune. This result appears to be at the right level of the complexity hierarchy, insofar as we also show that  $L$  is not coCFL-immune. While we do not know whether  $L \in \text{coCFL}$ , a related language consisting of “somewhat complex” words is shown to be non-coCFL.
- We generalize automatic complexity to a more general notion of automatic complexity of equivalence relations on words, and show that is not polynomial-time computable. In particular, we show that the set of minimally complex equivalence relations is NP-complete and the set of equivalence relations whose complexity is exactly a constant  $k$  above the minimum is  $\text{BH}_2$ -complete.

In the past, Gold [8] and Angluin [2] established NP-completeness for related problems. Heggernes et al. [7] considered parametrized complexity variations, such as fixing the number of states at two ( $|Q| = 2$ ) and increasing the alphabet size.

As an illustration of the power and computability of automatic complexity, we have created the following web service. To find the complexity of, say, the word 01011010, and an illustration of any automaton used in the associated proof, go to

<http://math.hawaii.edu/wordpress/bjoern/complexity-of-01011010/>

Alternatively, play the COMPLEXITY GUESSING GAME at:

<http://math.hawaii.edu/wordpress/bjoern/complexity-guessing-game/>

## 2 The set of maximally complex words is CFL-immune but not coCFL-immune

**Definition 1** ([9, 10]). *Let  $x$  be a finite word. The nondeterministic automatic complexity  $A_N(x)$  of  $x$  is the minimum number of states of a nondeterministic finite automaton that accepts  $x$ , and does not accept any other word of length  $|x|$ , and accepts  $x$  via only one computation path.*

*The (deterministic) automatic complexity  $A(x)$  of  $x$  is the minimum number of states of a deterministic finite automaton that accepts  $x$ , and does not accept any other word of length  $|x|$ .*

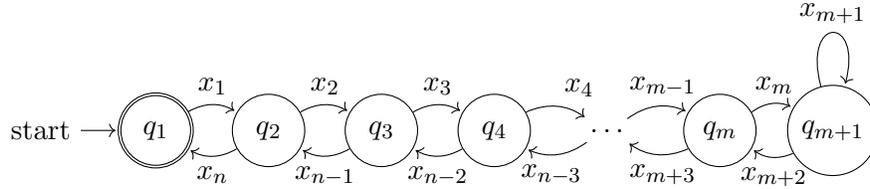


Figure 1: A nondeterministic finite automaton that only accepts one word  $x = x_1x_2x_3x_4 \cdots x_n$  of length  $n = 2m + 1$ .

**Theorem 2** (Hyde [9]). *For a word  $x$  of length  $n$ ,*

$$A_N(x) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

An idea of the proof of Theorem 2 is given in Figure 1.

**Definition 3.** *Let  $b(n) = \lfloor \frac{n}{2} \rfloor + 1$  be the canonical upper bound for  $A_N$  from Theorem 2. Let  $L_k = \{x \in \{0, 1, \dots, k-1\}^* : A_N(x) = b(n)\}$ . Any  $x \in L_k$  is called a maximally complex word.*

**Remark 4.**  *$L_3$  is known to be infinite (see Theorem 9) but we do not know whether  $L_2$  is infinite.*

**Lemma 5.** *Let  $x_0, y_0, a, b$  be positive integers with  $a$  and  $b$  relatively prime,  $x_0 < b$ , and  $y_0 < a$ . Then the equation*

$$ax + by = ax_0 + by_0 \tag{1}$$

*has a unique solution  $(x, y)$  in nonnegative integers.*

*Proof.* Equation (1) implies

$$a(x - x_0) \equiv 0 \pmod{b}.$$

Since  $a$  and  $b$  are relatively prime it follows that  $x - x_0 \equiv 0 \pmod{b}$ . Thus  $x = x_0 + nb$  for some  $n \in \mathbb{Z}$ . If  $n < 0$  then  $x \leq x_0 - b < 0$ , which contradicts the requirement that  $x \geq 0$ . If  $n > 0$  then using  $y \geq 0$ ,

$$ax + by \geq a(x_0 + b) + b(0) > ax_0 + by_0$$

contradicting (1). Thus  $n = 0$  and the only solution is  $x = x_0$ . □

**Definition 6.** For any collection of languages  $\mathcal{M}$ , a language  $L$  is  $\mathcal{M}$ -immune if it is infinite and contains no infinite subset in  $\mathcal{M}$ . We say that  $L \in \text{co}\mathcal{M}$  if the complement of  $L$  belongs to  $\mathcal{M}$ . Let  $\text{CFL}$  be the class of all context-free languages.

**Theorem 7** (Pumping lemma for CFL [3]). *If a language  $L$  is context-free, then there exists some integer  $p \geq 1$  (a pumping length) such that every word  $s$  in  $L$  with  $|s| \geq p$  can be written as  $s = uvwxy$  where*

1.  $|vwx| \leq p$ ,
2.  $|vx| \geq 1$ , and
3.  $uv^Nwx^Ny$  is in  $L$  for all  $N \geq 0$ .

**Definition 8.** Let  $\Sigma$  be a finite alphabet. A function  $\pi : \Sigma^* \rightarrow \Sigma^*$  is a homomorphism if it respects concatenation: for all  $x, y$ ,

$$\pi(xy) = \pi(x)\pi(y).$$

**Theorem 9.**  $L_3$  is not  $\text{coCFL}$ -immune.

*Proof.* Let  $\mathbf{t}$  be an infinite square-free word over  $\{0, 1, 2\}$  generated by, and a fixed point of, a homomorphism. Such a  $\mathbf{t}$  was constructed by Thue [12]. Let

$$\text{Pref}(\mathbf{t}) = \{x : x \text{ is a prefix of } \mathbf{t}\}.$$

By Berstel [4, Theorem on page 7],  $\text{Pref}(\mathbf{t}) \in \text{coCFL}$ . Since by [9, Theorem 18] every square-free word over  $\{0, 1, 2\}$  belongs to  $L_3$ , we also have  $\text{Pref}(\mathbf{t}) \subseteq L_3$ .  $\square$

**Theorem 10.**  $L_3$  is  $\text{CFL}$ -immune.

*Proof.* Since  $L_3$  is not  $\text{coCFL}$ -immune (Theorem 9), in particular  $L_3$  is infinite. Suppose  $L_3$  has an infinite subset  $K \in \text{CFL}$ . By the pumping lemma (Theorem 7) there is a ‘‘pumping length’’  $p$  such that any word  $X \in K$  of length at least  $p$  can be written as  $X = uvwxy$ , where  $|vx| \geq 1$  and

$$X_N := uv^Nwx^Ny \in K \subseteq L_3 \quad \text{for all } N \geq 0.$$

We denote the length of  $X_N$  by  $n_N$ . Since  $L_3$  is infinite, there exists at least one such word  $X$ .

*Case 1:*  $|v| \neq |x|$ . Let us first assume  $|v| > |x|$ . In particular,  $\varepsilon := \frac{|v|}{|v|+|x|} - \frac{1}{2} > 0$ . Consider an automaton which loops at each occurrence of

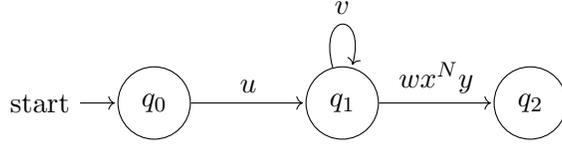


Figure 2: Schematic of the automaton for Case 1 of the proof of Theorem 10. The number of states in the actual automaton is  $|uvw x^N y|$ .

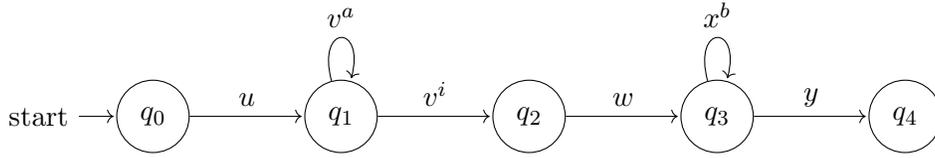


Figure 3: Schematic of the automaton for Case 2 of the proof of Theorem 10. The number of states in the actual automaton is  $|uv^a v^i w x^b y| - 1$ .

$v$  and otherwise proceeds to the right (Figure 2). Let  $N$  be so large that  $|v^N| \geq (\frac{1}{2} + \frac{\varepsilon}{2}) n_N$  and  $\frac{|v|}{n_N} \leq \frac{\varepsilon}{2}$ . Then

$$\begin{aligned} A_N(X_N) &\leq |uwx^N y| + |v| = n_N - |v^N| + |v| \leq n_N - \left(\frac{1}{2} + \frac{\varepsilon}{2}\right) n_N + |v| \\ &= \left(\frac{1}{2} - \frac{\varepsilon}{2}\right) n_N + |v| \leq \frac{n_N}{2}. \end{aligned}$$

and so  $X_N \notin L_3$ .

The case where  $|x| > |v|$  is quite identical. The remainder of the proof concerns Case 2.

*Case 2:*  $d := |v| = |x| > 0$ . By Lemma 5, for any positive integer  $i$ , the equation  $ar + bs = i(a + b)$  has only the solution  $r = s = i$  provided that  $a$  and  $b$  are relatively prime and both  $a$  and  $b$  are greater than  $i$ . In particular, this holds for any  $a$  and  $b$  with  $a > i$  and  $b = a + 1$ .

We construct an automaton  $M$  as follows (Figure 3). We put one loop of length  $ad$  and later one of length  $bd$ , and add  $id$  additional straggling states after the smaller loop of length  $ad$ . There are no loops apart from that.

Now for the analysis. Let  $N = bi$ . Each of the loops of  $M$  will be traversed  $i$  times during the processing of the word

$$X_N = uv^{bi} wx^{bi} y.$$

Let  $U = |u| + |w| + |y|$ . Let us compare

$$|X_N| = n_N = U + 2bdi$$

to the number of states of  $M$ ,

$$q = U + bd + ad + id - 1 = U + 2bd + (i - 1)d - 1.$$

(Note that when  $i = 1$ ,  $q = (n_N + 1) - 2$  as expected as there are 2 repetitions of states.) In order to show  $X_N \notin L_3$  we need  $q < \lfloor n_N/2 \rfloor + 1$ . To that end it suffices to have  $q < n_N/2$ , i.e.,

$$bd + ad + id + U - 1 = (2b - 1 + i)d + U - 1 < \frac{1}{2}(2bdi + U) = bdi + \frac{U}{2}.$$

Equivalently,

$$i - 1 + \frac{U}{2d} - \frac{1}{d} < (i - 2)b.$$

Choose  $i = 3$ ; then the inequality will hold for all sufficiently large  $b$ . Thus,  $M$  witnesses that  $X_N \notin L_3$ , in contradiction to the pumping lemma (Theorem 7).  $\square$

### 3 Somewhat simple words do not form a CFL

Let RE denote the collection of recursively enumerable (or if you prefer, computably enumerable) languages. Recall that  $L_3$  is the set of maximally complex words for nondeterministic automatic complexity over the alphabet  $\{0, 1, 2\}$ . Let  $C$  denote plain Kolmogorov complexity and let

$$R = \{x : C(x) \geq |x|\}$$

be the corresponding set of random words. We have seen (Theorem 9 and Theorem 10) that  $L_3$  is CFL-immune but not coCFL-immune. This is a pleasant analogue of the classical fact that  $R$  is RE-immune [6, Section 3.1] but not coRE-immune. Indeed,  $R \in \text{coRE}$ , but we conjecture that the analogous statement  $L_3 \in \text{coCFL}$  fails.

**Conjecture 11.**  $L_3 \notin \text{coCFL}$ .

We shall confirm a variant of Conjecture 11 in Theorem 15. To that end, we need a couple of lemmas.

**Lemma 12.** *Let  $A$  denote deterministic automatic complexity. Then  $A(y) \leq A(xyz)$  for all words  $x, y, z$ .*

*Proof.* Given an automaton witnessing  $A(xyz)$ , we merely change the initial and final states to obtain an automaton witnessing an upper bound on  $A(y)$ .  $\square$

**Lemma 13.** *Let  $A$  denote deterministic automatic complexity. Let  $\pi : \{0, 1\}^* \rightarrow \{0, 1\}^*$  be an injective homomorphism with  $|\pi(0)| = |\pi(1)|$ . Then  $A(x) \leq A(\pi(x))$  for each word  $x$ .*

*Proof.* Let  $M$  be a witnessing automaton for  $A(\pi(x))$ , with transition function  $\delta$ . We can now make an automaton  $M'$  with the same states as  $M$  (and the same initial and final states) that uniquely accepts  $x$  among words of length  $|x|$  as follows. Throw out all the edges of  $M$ . Put an edge labeled  $i$  from  $q_1$  to  $q_2$  in  $M'$  if  $\delta(q_1, \pi(i)) = q_2$  in  $M$ .

It is clear that  $M'$  accepts  $x$ . We now turn to uniqueness.

Suppose  $|y| = |x|$  and  $M'$  accepts  $y$ . Since  $|\pi(0)| = |\pi(1)|$ ,  $|\pi(y)| = |\pi(x)|$ . But  $M$  only accepts one word of length  $|\pi(x)|$ , so it must be that  $\pi(y) = \pi(x)$ . Since  $\pi$  is injective, it follows that  $y = x$ .  $\square$

**Theorem 14** (Shallit and Wang [10, Theorems 10 and 12]). *Let  $A$  denote deterministic automatic complexity. There is a constant  $n_0$  such that for  $n \geq n_0$ ,*

$$\sqrt{n} - 1 \leq A(0^n 1^n) \leq 6\sqrt{n} + 1.$$

**Theorem 15.** *Let  $\varepsilon > 0$ ,  $f(x) = x^{1/2-\varepsilon}$ , and*

$$S = \{x \in \{0, 1\}^* : A(x) < f(|x|)\}.$$

*Then  $S \notin \text{CFL}$ .*

*Proof.* We assume  $S \in \text{CFL}$  and derive a contradiction using the pumping lemma (Theorem 7). Let  $p$  be any sufficiently large pumping length. (The meaning of “sufficiently large” is determined below.) We shall build our unpumpable word as  $s = r^k$  where

$$r = 0^p 1^p$$

and  $k$  is sufficiently large relative to  $p$ . Consider any decomposition of  $s$  as  $s = uvwxy$  where  $|vwx| \leq p$  and  $|vx| \geq 1$ . The main idea of the proof is the combination of the following two facts.

- $r$  is the shortest contiguous subword  $R$  of  $s$  such that the simplicity of  $s$  comes from repeating  $R$ .
- the “pumpable part”  $vwx$  of  $s$  is shorter than  $r$ .

This means that pumping cannot help but increase the complexity. Thus by choosing  $k$  wisely we will have  $X_1 \in S$  and  $X_N := uv^N wx^N y \notin S$  for some  $N$ , which will be a contradiction.

The details are as follows.

- Case 1:  $vx$  is all 0s. (We omit the case when  $vx$  is all 1s as the proof is identical.) Then since  $|vwx| \leq p$  and  $|vx| \geq 1$ , the word  $vwx$  is also all 0s. Let  $N = pk$ . We have  $0^N = (0^p 0^p)^{k/2}$ , and  $v^N w x^N$  contains  $0^N$  as a contiguous subword. Now either at least half of the occurrences of  $1^p$  as contiguous subwords of  $X_N$  are in  $u$ , or at least half of them are in  $y$ . Hence  $X_N$  contains a contiguous subword of the form either  $(0^p 0^p)^{k/2} (1^p 0^p)^{k/2}$  or  $(0^p 1^p)^{k/2} (0^p 0^p)^{k/2}$ . Then using the inequality  $1 \leq |vx| \leq p$ ,

$$3pk = n_1 + pk = n_1 + N \leq n_N \quad (2)$$

and

$$n_N = |uv^N w x^N y| \leq n_1 + Np = n_1 + p^2 k = 2pk + p^2 k. \quad (3)$$

Now,

$$\begin{aligned} A(X_N) &\geq A(0^{k/2} 1^{k/2}) && \text{by Lemma 12 and Lemma 13} \\ &\geq \sqrt{k/2} - 1 && \text{by Theorem 14} \\ &\geq \sqrt{n_N / (4p + 2p^2)} - 1 && \text{by (3)} \\ &\geq f(n_N) \end{aligned}$$

provided

$$(f(n_N) + 1)^2 / n_N \leq \frac{1}{4p + 2p^2}.$$

Since  $f$  is monotonically increasing, by (3) it suffices that

$$\frac{(f(2pk + p^2 k) + 1)^2}{n_N} \leq \frac{1}{4p + 2p^2}.$$

For this, by (2) it suffices that

$$\frac{(f(2pk + p^2 k) + 1)^2}{3pk} \leq \frac{1}{4p + 2p^2}.$$

In other words,

$$\frac{(f((2p + p^2)k) + 1)^2}{k} \leq \frac{3p}{4p + 2p^2} = \frac{3}{4 + 2p}.$$

This is true for large enough  $k$ , since

$$f(x) = o(\sqrt{x}).$$

In order to guarantee that  $A(X_1) < f(n_1)$ , we require

$$A(X_1) \leq 2p + 1 < f(n_1) = f(2pk).$$

Since  $f(x) \rightarrow \infty$  as  $x \rightarrow \infty$ , this holds by taking  $k$  large enough.

- Case 2:  $vx$  contains both 0s and 1s. Then  $uv^Nwx^Ny$  contains a contiguous subword of the form  $0^N1^N$  (using the fact that the blocks  $0^p1^p$  in  $s$  are longer than the pumping length). The analysis is then similar to Case 1 (but without using Lemma 13).

□

## 4 Automatic complexity of equivalence relations

We now go higher in the complexity-theoretic hierarchy, from CFL to NP. We shall not be able to determine the NP-completeness, or lack thereof, of problems like “is  $A(x) \leq c$ ?” Nevertheless, we obtain results for a generalization of automatic complexity.

**Definition 16.** *Given a deterministic finite automaton (DFA)*

$$M = (Q, \Sigma, \delta, q_0, F),$$

*an equivalence relation  $D$  on  $Q$  induces an equivalence relation  $E$  on a subset  $S$  of  $\{0, 1\}^*$  if*

$$E = \{(x, y) \in S^2 \mid (\delta(q_0, x), \delta(q_0, y)) \in D\}.$$

*A deterministic finite automaton  $M = (Q, \Sigma, \delta, q_0, F)$  coheres with  $E$  if there is an equivalence relation  $D$  on  $Q$  such that  $D$  induces  $E$ .*

In words, if  $D$  induces  $E$  then two words  $x, y \in S$  are  $E$ -equivalent iff  $M$  ends in  $D$ -equivalent states on input  $x$  and on input  $y$ .

Note that the set of final states  $F$  is irrelevant in Definition 16.

**Definition 17.** *The automatic complexity  $A(E)$  of an equivalence relation  $E$  is the least number of states of a DFA that coheres with  $E$ .*

**Remark 18.** *Automatic complexity of a word  $x$  (Shallit and Wang [10]) is a special case of automatic complexity of equivalence relations. Namely, the two equivalence classes are  $\{x\}$  and  $\{y : |y| = |x|, y \neq x\}$ .*

## 4.1 Complexity of equivalence relations is $\text{BH}_2$ -complete

As usual, let us say that a Boolean formula is *CNF* if it is in conjunctive normal form, i.e., it is a conjunction of clauses, each of which is a disjunction of literals.

**Definition 19** (Encoding of literals). *For a variable  $x_j$ , we denote the negation of  $x_j$  by  $\overline{x_j}$ . We define*

$$\begin{aligned}\neg^0 x_j &= x_j, \\ \neg^1 x_j &= \overline{x_j}.\end{aligned}$$

For a literal  $l = \neg^b x_j$ , where  $b \in \{0, 1\}$ , we define the encoding word

$$t(l) = 1^j 0 b 0.$$

**Definition 20** (inspired by [11, Victor Kuncak's solution to Exercise 7.36]). *Let  $\phi$  be a CNF formula with  $m$  clauses. Let*

$$Q_m = \{q_0, q_1, \dots, q_m, h, v_t, v_f, l_t, l_f, r, s\}$$

be a set of cardinality  $m + 8$ . For each  $\sigma \in \{0, 1\}^*$  and  $q \in Q_m$ ,

$$\sigma \rightarrow q$$

is the ordered pair  $\langle \sigma, q \rangle$ .<sup>1</sup> Let  $S$  be the following set, where  $1^0 = 0^0 = \lambda$ , the empty word.

$$\begin{aligned}S &= \{1^{m+1} \rightarrow q_0, 0 \rightarrow h, \\ &\quad 00 \rightarrow v_t, 01 \rightarrow v_f, 000 \rightarrow l_t, 001 \rightarrow l_f, 010 \rightarrow l_f, 011 \rightarrow l_t, \\ &\quad 0^2 00 \rightarrow s, 0^2 01 \rightarrow r, 0^2 10 \rightarrow q_0, 0^2 11 \rightarrow r, \\ &\quad 0^3 00 \rightarrow s, 0^3 01 \rightarrow s, 0^3 10 \rightarrow q_0, 0^3 11 \rightarrow q_0\} \\ &\cup \{1^i \rightarrow q_i : 0 \leq i \leq m\} \\ &\cup \{1^i 011 \rightarrow r : 1 \leq i \leq m\} \\ &\cup \{t(l_1)t(l_2)t(l_3) \rightarrow s : (l_1 \vee l_2 \vee l_3) \text{ is a clause of } \phi\}.\end{aligned}$$

Let  $E_\phi$  be the intersection of all equivalence relations containing

$$\{(\sigma, \tau) : (\exists q \in Q_m)((\sigma \rightarrow q) \in A \text{ and } (\tau \rightarrow q) \in S)\}.$$

**Remark 21.** *The elements of  $Q_m$  in Definition 20 are thought of as states. The expression  $\sigma \rightarrow q$  is to be thought of as the statement that  $\delta(q_0, \sigma) = q$  where  $\delta$  is the transition function of a DFA  $M$  and  $q_0$  is the initial state. The equivalence relation  $E_\phi$  identifies two words as equivalent if they lead us to the same state. Thus such an  $M$  will cohere with  $\phi$ .*

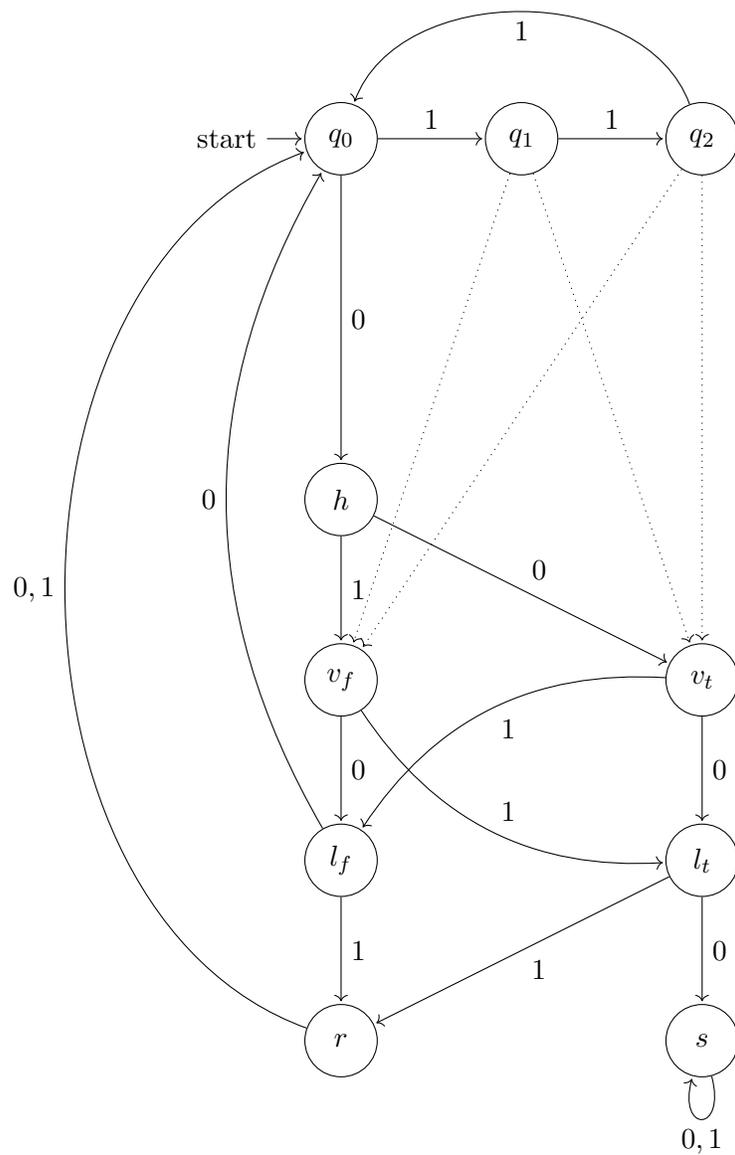


Figure 4: The automaton  $M'$  from the proof of Theorem 22 is given by the solid lines. Appropriate choice of two of the dotted lines gives the total DFA  $M$ . The case where the formula  $\phi$  has  $m = 2$  clauses is shown.

**Theorem 22.**  $\{E : A(E) = |E|\}$  is NP-complete.

*Proof.* It is immediate from the definitions that

$$\{E : A(E) = |E|\} = \{E : A(E) \leq |E|\}.$$

We reduce 3-SAT to  $\{E : A(E) \leq |E|\}$  using the mapping  $\phi \mapsto E_\phi$  from Definition 20. It induces a finite automaton  $M'$  which is deterministic but whose transition function  $\delta'$  is not total (Figure 4). We see that  $\phi$  is satisfiable iff there is a total DFA  $M$ , differing from  $M'$  only in that its transition function  $\delta \supseteq \delta'$  is total, such that  $M$  coheres with  $E_\phi$ . In particular  $M$  has no more states than  $M'$ . The possible extra transitions of  $M$  are shown in dotted lines in Figure 4. Thus

$$\begin{aligned} \phi \text{ is satisfiable} &\implies A(E_\phi) \leq |E_\phi|, \\ \phi \text{ is unsatisfiable} &\implies A(E_\phi) \not\leq |E_\phi|. \end{aligned}$$

□

**Theorem 23.**  $\{E : A(E) = |E| + 1\}$  is coNP-hard.

*Proof.* It suffices to use the same reduction as in Theorem 22 and demonstrate unconditionally, i.e., without any assumption on satisfiability of  $\phi$  or lack thereof, that

$$A(E_\phi) \leq |E_\phi| + 1.$$

The question is then how to add one more state to Figure 4 to make the resulting automaton  $M^+$  cohere with  $E_\phi$ . This is indicated in Figure 5. We ensure  $1^j 011 \rightarrow r$ , i.e.,  $\delta(q_0, 1^j 011) = r$ ,  $1 \leq j \leq m$  using a new state  $e$ . □

In the proof of Theorem 23, the state  $e$  is acting duplicitously, in a sense, copying some of the behavior of the “truth values”  $v_t$  and  $v_f$  without committing to a truth value.

**Definition 24** (Wechsung [13]). *The first two levels of the Boolean hierarchy for NP are given by*

$$\begin{aligned} \text{BH}_1 &= \text{NP}, \\ \text{BH}_2 &= \{L_1 \setminus L_2 : L_1, L_2 \in \text{NP}\}. \end{aligned}$$

**Definition 25.**

$$\text{SAT}(2) = \{(\phi_1, \phi_2) : \phi_1 \text{ is satisfiable and } \phi_2 \text{ is not}\}.$$

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<sup>1</sup>See Remark 21 for intuition.

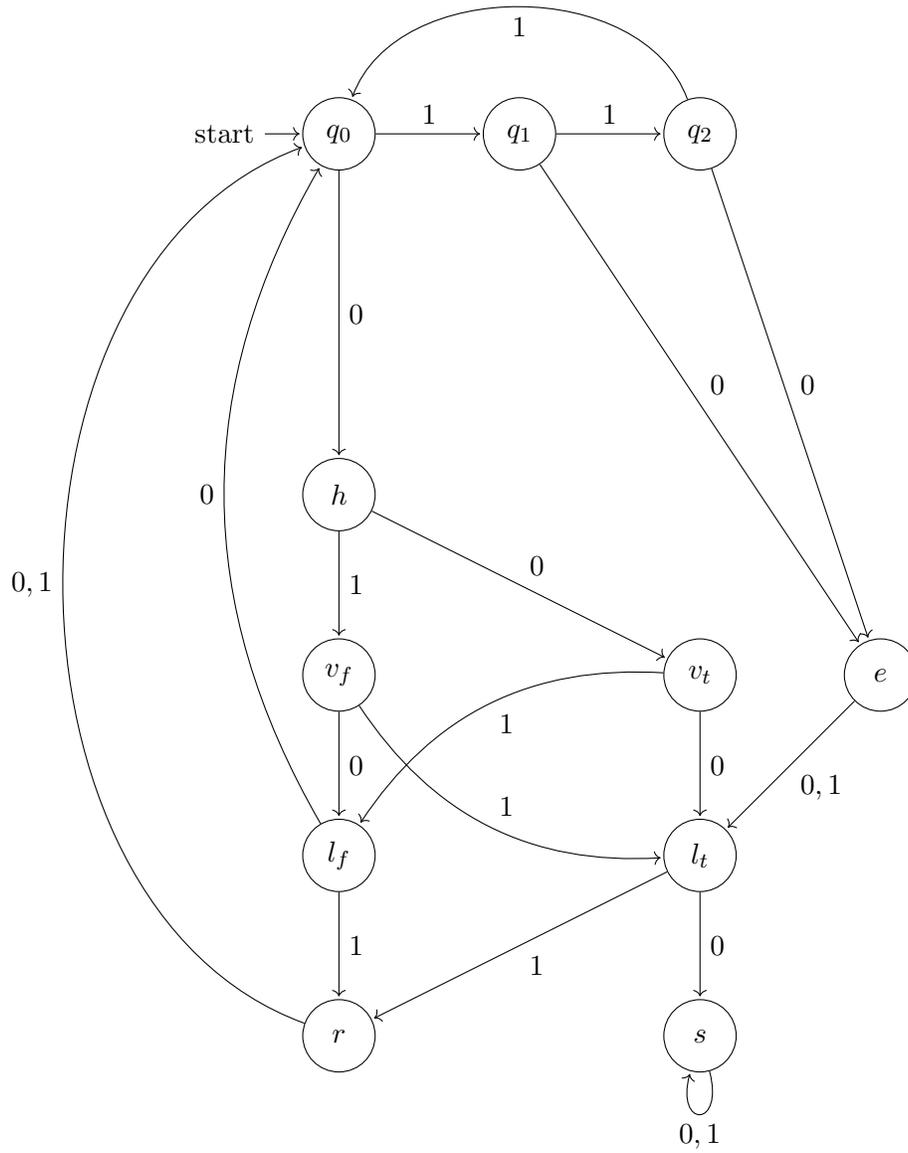


Figure 5: Automaton  $M^+$  used in Theorem 23. At the cost of adding a state  $e$ , we ensure that  $M^+$  coheres with  $E_\phi$ , whether or not  $\phi$  is satisfiable.

$\phi_1$ satisfiable?	$\phi_2$ satisfiable?	Number of extra states
no	no	$\ell + k$
no	yes	$\ell$
yes	no	$k$
yes	yes	0

Table 1: The number of extra states needed for Theorem 27.

**Theorem 26.** *SAT(2) is complete for  $\text{BH}_2$  with respect to polynomial-time many-one reductions.*

Theorem 26 can be found in Cai et al. [5, Theorem 5.2]. (In their notation,  $\text{BH}_2 = \text{NP}(2)$ .) We will use without proof the extension of Theorem 26 from SAT to 3-SAT.

**Theorem 27.** *For each  $k \geq 1$ ,  $\{E : A(E) = |E| + k\}$  is  $\text{BH}_2$ -complete.*

*Proof.* For each  $k \geq 1$ ,  $\{E : A(E) = |E| + k\}$  equals

$$\{E : A(E) \leq |E| + k\} \setminus \{E : A(E) \leq |E| + k - 1\} \in \text{BH}_2.$$

It remains to show  $\text{BH}_2$ -hardness. Let  $M_1$  and  $M_2$  be automata as indicated in Figure 5 for two 3-SAT instances  $\phi_1$  and  $\phi_2$ , respectively. Let  $k$  and  $\ell \neq k$  be positive integers and let  $\sigma_1, \dots, \sigma_{k+\ell}$  be incomparable words. We define  $M$  in a natural way so that

$$L(M) = \bigcup_{i=1}^{\ell} \{\sigma_i x : x \in L(M_1)\} \cup \bigcup_{i=\ell+1}^k \{\sigma_i x : x \in L(M_2)\}.$$

Note that the various “ $e$ ” states for distinct copies of  $M_1$  and  $M_2$  must be distinct, since they transition to distinct “ $l_t$ ” states. Thus, we have

$\phi_1$  is satisfiable, but  $\phi_2$  is not

if and only if

we need no extra state for  $\phi_1$ , but  $k$  extra states for  $\phi_2$ ,

if and only if (by Table 1)

we need  $k$  extra states overall,

if and only if  $A(E) = |E| + k$ . We make corresponding changes in the “axioms” (the elements of  $S$  in Definition 20) for  $E_{\phi_i}$ . Applying Theorem 26 completes the proof.  $\square$

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