Complexity and Inapproximability Results for Parallel Task Scheduling and Strip Packing^{*}

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Abstract

We study the Parallel Task Scheduling problem $Pm|size_j|C_{\max}$ with a constant number of machines. This problem is known to be strongly NP-complete for each $m \geq 5$, while it is solvable in pseudo-polynomial time for each $m \leq 3$. We give a positive answer to the long-standing open question whether this problem is strongly NP-complete for m = 4. As a second result, we improve the lower bound of $\frac{12}{11}$ for approximating pseudo-polynomial Strip Packing to $\frac{5}{4}$. Since the best known approximation algorithm for this problem has a ratio of $\frac{4}{3} + \varepsilon$, this result narrows the gap between approximation ratio and inapproximability result by a significant step. Both results are proven by a reduction from the strongly NP-complete problem 3-Partition.

1 Introduction

In the Parallel Task Scheduling problem, we have given m machines and a set of jobs J. Each job $j \in J$ has a processing time $p(j) \in \mathbb{N}$ and a number of required machines $q(j) \in \mathbb{N}$. A schedule σ is a combination of two functions $\sigma : J \to \mathbb{N}$ and $\rho : J \to \{M | M \subseteq \{1, \ldots, m\}\}$. The function σ maps each job to a start point in the schedule, while ρ maps each job to the set of machines it is processed on. A schedule is feasible if each machine processes at most one job at the time and each job is processed on the required number of machines. The objective is to find a feasible schedule σ minimising the makespan $T := \max_{i \in J} \sigma(i) + p(i)$. This problem is denoted with $P|size_j|C_{\max}$. If the number of machines is constant we write $Pm|size_j|C_{\max}$. For a given job $j \in J$ we define its work as $w(j) := p(j) \cdot q(j)$. For a subset $J' \subseteq J$ we define its total work as $w(J') := \sum_{i \in J'} w(j)$.

In the Strip Packing problem we have given a strip with a width $W \in \mathbb{N}$ and infinite height as well as a set of rectangular items I. Each item $i \in I$ has a width $w_i \in \mathbb{N}_{\leq W}$ and a height $h_i \in \mathbb{N}$. The objective is to find a feasible packing of the items I into the strip, which minimizes the packing height. A packing of the items I into the strip is a function $\rho : I \to \mathbb{Q}_0 \times \mathbb{Q}_0$, which assigns the left bottom corner of an item to a position in the strip, such that for each item $i \in I$ with $\rho(i) = (x_i, y_i)$ we have $x_i + w_i \leq W$. An *inner* point of i is a point from the set $inn(i) := \{(x, y) \in \mathbb{R} \times \mathbb{R} | x_i < x < x_i + w_i, y_i < y < y_i + h_i\}$. We say two items $i, j \in I$ overlap if they share an inner point (i.e if $inn(i) \cap inn(j) \neq \emptyset$). A packing is *feasible* if no two items overlap. The height of a packing is defined as $H := \max_{i \in I} y_i + h_i$.

A well known and interesting fact is that, in this setting, we can transform feasible packings to packings where all positions are integral, without enlarging the packing height [5]. This can be done by shifting all items downwards until they touch the upper border of an item or the bottom of the strip. Now all y-coordinates of the items are integral since each is given by the sum of some item heights, which are integral. The same can be done for the x-coordinate by shifting all items to the left as far as possible. Therefore we can assume that we have packings of the form $\rho: I \to \mathbb{N}_0 \times \mathbb{N}_0$.

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Strip packing is closely related to Parallel Task Scheduling. If we demand that jobs from the Parallel Task Scheduling are processed on contiguous machines, the resulting problem is equivalent to the Strip Packing problem. Although these problems are closely related, there are some instances that have a smaller optimal value on non-contiguous machines than on contiguous machines [30].

Related Work

Parallel Task Scheduling: In 1989 Du and Leung [9] proved the problem $Pm|size_j|C_{\max}$ to be strongly *NP*-complete for all $m \geq 5$, while $Pm|size_j|C_{\max}$ is solvable by a pseudo-polynomial algorithm for all $m \leq 3$. Amoura et al. [2], as well as Jansen and Porkolab [18], presented a polynomial time approximation scheme (in short PTAS) for the case that m is a constant. A PTAS is a family of algorithms that finds a solution with an approximation ratio of $(1 + \varepsilon)$ for any given value $\varepsilon > 0$. If m is polynomially bounded by the number of jobs, a PTAS still exists [21]. Nevertheless, if m is arbitrarily large, the problem gets harder. By a simple reduction from the Partition problem, one can see that there is no polynomial algorithm with approximation ratio $\alpha OPT + \beta$, where $\alpha < 3/2$ and β polynomial in n [22]. Parallel Task Scheduling with arbitrarily large m has been widely studied [12, 30, 24, 10]. The algorithm with the best known absolute approximation ratio of $\frac{3}{2} + \varepsilon$ was presented by Jansen [17].

Strip Packing: The Strip Packing problem was first studied in 1980 by Baker et al. [4]. They presented an algorithm with an absolute approximation ratio of 3. This ratio was improved by a series of papers [8, 27, 26, 28, 16]. The algorithm with the best known absolute approximation ratio by Harren, Jansen, Prädel and van Stee [15] achieves a ratio of $\frac{5}{3} + \varepsilon$. By a simple reduction from the Partition problem, one can see that it is impossible to find an algorithm with better approximation ratio than $\frac{3}{2}$, unless P = NP.

However, asymptotic algorithms can achieve asymptotic approximation ratios better than $\frac{3}{2}$ and have been studied in various papers [8, 14, 3]. Kenyon and Rémila [23] presented an asymptotic fully polynomial approximation scheme (in short AFPTAS) with additive term $\mathcal{O}(h_{\max}/\varepsilon^2)$, where h_{\max} is the largest occurring item height. An approximation scheme is fully polynomial if its running time is polynomial in $1/\varepsilon$ as well. This algorithm was simultaneously improved by Sviridenko [29] and Bougeret et al. [6] to an algorithm with an additive term of $\mathcal{O}(\log(1/\varepsilon)/\varepsilon)h_{\max}$. Furthermore, at the expense of the running time, Jansen and Soils-Oba [20] presented an asymptotic PTAS with an additive term of h_{\max} .

Recently the focus shifted to pseudo-polynomial algorithms. Jansen and Thöle [21] presented an pseudo-polynomial algorithm with approximation ratio of $\frac{3}{2} + \varepsilon$. Since the partition problem is solvable in pseudo-polynomial time, the lower bound of $\frac{3}{2}$ for polynomial time Strip Packing can be beaten by pseudo-polynomial algorithms. The first such algorithm with a better approximation ratio than $\frac{3}{2}$ was given by Nadiradze and Wiese [25]. It has an absolute approximation ratio of $\frac{7}{5} + \varepsilon$. Its approximation ratio was independently improved to $\frac{4}{3} + \varepsilon$ by Galvez, Grandoni, Ingala, and Khan [11] and by Jansen and Rau [19]. All these algorithms have a polynomial running time if the width of the strip W is bounded by a polynomial in the number of items.

In contrast to Parallel Task Scheduling, Strip Packing can not be approximated arbitrarily close to 1, if we allow pseudo-polynomial running time. Adamaszek, Kociumaka, Pilipczuk and Pilipczuk [1] proved this by presenting a lower bound of $\frac{12}{11}$. This result also implies that Strip Packing admits no quasi-polynomial time approximation scheme, unless $NP \subseteq DTIME(2^{\text{polylog}(n)})$. Christensen, Khan, Pokutta, and Tetali [7] list 10 major open problems related to multidimensional Bin Packing. As the 10th problem they name pseudo polynomial Strip Packing and underline the importance of finding tighter pseudo-polynomial time results for lower and upper bounds.

New Results In this paper, we present two hardness results. The first result answers the longstanding open question whether the problem $P4|size_j|C_{max}$ is strongly NP-complete.

Theorem 1. The Parallel Tasks Scheduling problem on 4 machines $P4|size_j|C_{max}$ is strongly NP-complete.



Figure 1: The upper and lower bounds for the best possible approximation for pseudo-polynomial Strip Packing achieved so far

The second result concerns pseudo-polynomial Strip Packing. We manage to adapt our reduction for $P4|size_j|C_{\text{max}}$ to Strip Packing, by transforming the optimal schedule into a packing of rectangles interpreting the makespan as the width of the strip. This adaptation leads to the following result:

Theorem 2. For each $\varepsilon > 0$ it is NP-Hard to approximate Strip Packing with a ratio of $\frac{5}{4} - \varepsilon$ in pseudo-polynomial time.

This improves the so far best lower bound of $\frac{12}{11}$ to $\frac{5}{4}$. In Figure 1 we display the results for pseudo-polynomial Strip Packing achieved so far.

Notation For a given schedule σ we define for $i \in J$ and any set of jobs $J' \subseteq J$ the value $\#_i J'$ as the number of jobs in J', which finish before $\sigma(i)$ (e.i. $\#_i J' = |\{j \in J' : \sigma(j) + p(j) \leq \sigma(i)\}|$). If the job is clear from the context we write #J' instead of $\#_i J'$. Furthermore, we will use a notation defined in [9] for swapping a part of the content of two machines. Let $i \in J$ be a job, that is processed by at least two machines \tilde{M} and $\tilde{M'}$ with start point $\sigma(i)$. We can swap the content of the machines \tilde{M} and $\tilde{M'}$ after time $\sigma(i)$ without violating any scheduling constraint. We define this swapping operation as $SWAP(\sigma(i), \tilde{M}, \tilde{M'})$.

Organization of this Paper In Section 2 we will prove that $P4|size_j|C_{\max}$ is strongly NPcomplete by a reduction from the strongly NP-complete Problem 3-Partition. First, we describe the jobs to construct for this reduction. Afterward, we prove: if the 3-Partition instance is a Yes-instance, then there is a schedule with a specific makespan, and if there is a schedule with this specific makespan then the 3-Partition instance has to be a Yes-instance. While the first can be seen directly, the proof of the second is more involved. Proving the second claim, we first show that it can be w.l.o.g. supposed that each machine contains a certain set of jobs. In the next step, we prove some implications on the order in which the jobs appear on the machines which finally leads to the conclusion that the 3-Partition instance has to be a Yes-instance. In Section 3 we discuss the implications for the inapproximability of pseudo-polynomial Strip Packing.

2 Hardness of Scheduling Parallel Tasks

In the following, we will prove Theorem 1 by a reduction from the 3-Partition problem. In the 3-Partition problem we have given a list $\mathcal{I} = (\iota_1, \ldots, \iota_{3z})$ of 3z positive integers, such that $\sum_{i=1}^{3z} \iota_i = zD$ and $D/4 < \iota_i < D/2$ for each $1 \leq i \leq 3z$. The problem is to decide whether there exists a partition of the set $I = \{1, \ldots, 3z\}$ into sets I_1, \ldots, I_z , such that $\sum_{i \in I_j} \iota_i = D$ for each $1 \leq j \leq z$. We define $SIZE(\mathcal{I}) = \sum_{i=1}^{3z} \log(\iota_i)$ as the input size of the problem. 3-Partition is strongly NP-complete [13]. Therefore, it can not be solved in pseudo-polynomial time, unless P = NP.

Construction First, we will describe how we generate an instance of $P4|size_j|C_{\max}$ from a given 3-Partition instance \mathcal{I} in polynomial time. Let $\mathcal{I} = (\iota_1, \ldots, \iota_{3z})$ be a 3-Partition instance with $\sum_{i=1}^{3z} \iota_i = zD$. If $D \leq 4z(7z+1)$, we scale each number with 4z(7z+1) such that we get a new instance $\mathcal{I}' := (4z(7z+1) \cdot \iota_1, \ldots, 4z(7z+1) \cdot \iota_{3z})$. For this instance, it holds that D' = 4z(7z+1)D > 4z(7z+1) and $SIZE(\mathcal{I}') \in \text{poly}(SIZE(\mathcal{I}))$. Furthermore, \mathcal{I} is a Yes-instance if and only if \mathcal{I}' is a Yes-instance. Therefore, we can w.l.o.g. assume that D > 4z(7z+1).

$$p(i) = \begin{cases} D^2 & i \in A \\ D^3 & i \in B \\ D^4 + D^6 + 3zD^7 & i \in a, \\ D^5 + D^6 + 3zD^7 & i \in b, \\ (z+j)D^7 + D^8 & i = c_j \in c, j \in \{0, \dots, z\} \\ D^3 + D^5 + 4zD^7 + D^8 & i \in \alpha \\ D^2 + D^4 + (4z-1)D^7 + D^8 & i \in \beta \\ D^5 + (3z-j)D^7 - D & i = \gamma_j \in \gamma, j \in \{1, \dots, z\} \\ D^4 + (3z-j)D^7 & i = \delta_j \in \delta, j \in \{1, \dots, z\} \\ D^3 + zD^7 + D^8 & i = \lambda_1 \\ D^2 + 2zD^7 + D^8 & i = \lambda_2 \end{cases}$$

Figure 2: Overview of the structure jobs

In the following, we describe the jobs constructed for the reduction; see Figure 2 for an overview. We generate two sets A and B of 3-processor jobs. A contains z + 1 jobs with processing time $p_A := D^2$ and B contains z + 1 jobs with processing time $p_B := D^3$. Furthermore, we generate three sets a, b and c of 2-processor jobs, such that a contains z jobs with processing time $p_a := D^4 + D^6 + 3zD^7$, b contains z jobs with processing time $p_b := D^5 + D^6 + 3zD^7$ while c contains one job c_j for each $0 \le j \le z$, having processing time $(z + j)D^7 + D^8$ resulting in z + 1 jobs total in c. Last we define five sets α , β , γ , δ , and λ of 1-processor jobs, such that α contains z jobs with processing time $p_{\beta} := D^2 + D^4 + (4z - 1)D^7 + D^8$, γ contains for each $1 \le j \le z$ one job γ_j with processing time $p_{\beta} := D^2 + D^4 + (4z - 1)D^7 + D^8$, γ contains for each $1 \le j \le z$ one job δ_j with processing time $D^5 + (3z - j)D^7 - D$ resulting in $|\gamma| = z$, δ contains for each $1 \le j \le z$ one job δ_j with processing time $p^4 + (3z - i)D^7$ resulting in $|\delta| = z$, and λ contains two jobs λ_1 and λ_2 with processing times $p(\lambda_1) := D^3 + zD^7 + D^8$ and $p(\lambda_2) := B + c_0 = D^2 + 2zD^7 + D^8$. We call these jobs structure jobs. Additionally, we generate for each $i \in \{1, \ldots, 3z\}$ one 1-processor job, called partition job, with processing time ι_i . We name the set of partition jobs P. Last, we define $W := (z+1)(D^2 + D^3 + D^8) + z(D^4 + D^5 + D^6) + z(7z+1)D^7$. Note that the work of the generated jobs adds up to 4W.

If we add the processing times of all generated jobs, the largest coefficient before a D^i is at most 4z(7z+1). Since 4z(7z+1) < D, it can never happen that in the total processing time of a set of jobs the value D^i , together with its coefficient, influences the coefficient of D^{i+1} . Furthermore, if the processing times of a set of jobs add up to a value where one of the coefficients is larger than the coefficients in W, it is not possible that in a schedule with no idle time one of the machines contains this set.

In the following sections, we will prove that there is a schedule with makespan W if and only if the 3-Partition instance is a Yes-instance.

Partition to Schedule Let \mathcal{I} be a Yes-instance with partition I_1, \ldots, I_z . One can easily verify that the *structure jobs* can be scheduled as shown in Figure 3. After each job γ_j , for each $1 \leq j \leq z$, we have a gap with processing time D. We schedule the *partition jobs* with indices out of I_j directly after γ_j . Their processing times add up to D, and therefore they fit into the gap. The resulting schedule has a makespan of W.

Schedule to Partition In this section, we will show that if there is a schedule with makespan W, then \mathcal{I} is a Yes-instance. Let a schedule S with makespan W be given. We will now step by step describe why \mathcal{I} has to be a Yes-instance. In the first step, we will show that we can transform the schedule, such that certain machines contain certain jobs.

Lemma 1. We can transform the schedule S into a schedule, where M_1 contains the jobs $A \cup a \cup \alpha \cup \lambda_1$, M_2 contains the jobs $A \cup B \cup c \cup \check{a} \cup \check{b} \cup \check{\gamma} \cup \check{\delta}$, M_3 contains the jobs $A \cup B \cup c \cup \hat{a} \cup \hat{b} \cup \hat{\gamma} \cup \hat{\delta}$



Figure 3: An optimal schedule, for a Yes-instance for $a_i \in a$, $b_i \in b$, $A_j \in A$, $B_j \in B$, $\alpha_i \in \alpha$ and $\beta_i \in \beta$.

and M_4 contains the jobs $B \cup b \cup \beta \cup \lambda_2$, with $\check{a} \subseteq a$, $\hat{a} = a \setminus \check{a}$, $\check{b} \subseteq b$, $\hat{b} = b \setminus \check{b}$, $\check{\gamma} \subseteq \gamma$, $\hat{\gamma} = \gamma \setminus \check{\gamma}$, and $\check{\delta} \subseteq \delta$, $\hat{\delta} = \delta \setminus \check{\delta}$. Furthermore, if the jobs are scheduled in this way, it holds that $|\check{a}| = |\check{\gamma}|$ and $|\check{b}| = |\check{\delta}|$.

Proof. First, we will show that the content of the machines can be swapped without enlarging the makespan, such that M_2 and M_3 each contain all the jobs in $A \cup B$. Let $x \in A \cup B$ be the job with the smallest starting point in this set. We can swap the complete content of the machines such that M_2 and M_3 contain x. Let us suppose that, after some swapping operations, M_2 and M_3 contain the first i jobs in $A \cup B$. Let $\tilde{M} \in \{M_1, M_4\}$ be the third machine containing the i-th job $x_i \in A \cup B$. Let \tilde{M}' be the machine not containing the (i + 1)-th job. If $\tilde{M}' \in \{M_2, M_3\}$, we transform the schedule such that M_2 and M_3 contain it, by performing one more swapping operation $SWAP(\sigma(x_i), \tilde{M}, \tilde{M}')$. Therefore, we can transform the given schedule without increasing its makespan such that M_2 and M_3 each contain all the jobs in $A \cup B$.

In the next step, we will determine the set of jobs contained by the machines M_1 and M_4 . The machines M_2 and M_3 contain besides the jobs in $A \cup B$ jobs with total processing time of $zD^4 + zD^5 + zD^6 + z(7z+1)D^7 + (z+1)D^8$. Hence, M_2 or M_3 can not contain jobs in $\alpha \cup \beta \cup \lambda$, since their processing times contain D^2 or D^3 . Therefore, each job in $A \cup B \cup \alpha \cup \beta \cup \lambda$ is either processed on M_1 or on M_4 . In addition to these jobs, M_1 and M_4 contain further jobs with a total processing time of $zD^4 + zD^5 + 2zD^6 + 6z^2D^7$ total. The only jobs with a processing time containing D^6 are the jobs in the set $a \cup b$. Therefore, each machine processes z jobs from the set $a \cup b$. Hence, a total processing time of $3z^2D^7$ is used by jobs in the set $a \cup b$ on each machine. This leaves a processing time of $(4z^2 + z)D^7$ for the jobs in $\alpha \cup \beta \cup \lambda$ on M_1 and M_4 corresponding to D^7 . All the 2(z+1) jobs in $\alpha \cup \beta \cup \lambda$ contain D^8 in their processing time. Therefore, each machine M_1 and M_4 processes z + 1 of them. We will swap the content of M_1 and M_4 such that λ_1 is scheduled on M_1 . As a consequence, M_1 processes z jobs from the set $\alpha \cup \beta \cup \{\lambda_2\}$, with processing times, which sum up to $4z^2D^7$ in the D^7 component. The jobs in α have with $4zD^7$ the largest amount of D^7 in their processing time. Therefore, M_1 processes all of them since $z \cdot 4zD^7 = 4z^2D^7$, while M_4 contains the jobs in $\beta \cup \{\lambda_2\}$. Since we have $p(\alpha \cup \{\lambda_1\}) = (z+1)D^3 + zD^5 + z(4z+1)D^7(z+1)D^8$, jobs from the set $A \cup B \cup a \cup b$ with total processing time of $(z+1)D^2 + zD^4 + zD^6 + 3z^2D^7$ have to be scheduled on M_1 . In this set, the jobs in A are the only jobs with processing times containing D^2 , while the jobs in a are the only jobs with a processing time containing D^4 . As a consequence, M_1 processes the jobs $A \cup a \cup \alpha \cup \{\lambda_1\}$. Analogously we can deduce that M_4 processes the jobs $B \cup b \cup \beta \cup \{\lambda_2\}.$

In the last step, we will determine which jobs are scheduled on M_2 and M_3 . As shown before, each of them contains the jobs $A \cup B$. Furthermore, since no job in c is scheduled on M_1 or M_4 , and they require two machines to be processed, machines M_2 and M_3 both contain the set c. Additionally, each job in $\gamma \cup \delta$ has to be scheduled on M_2 or M_3 since they are not scheduled on M_1 or M_4 . Each job in $a \cup b$ occupies one of the machines M_1 and M_4 . The second machine they occupy is either M_2 or M_3 . Let $\check{a} \subseteq a$ be the set of jobs, which is scheduled on M_2 and $\hat{a} \subseteq a$ be the set which is scheduled on M_3 . Clearly $\check{a} = a \setminus \hat{a}$. We define the sets $\hat{b}, \check{b}, \check{\delta}, \check{\delta}, \hat{\gamma}$, and $\check{\gamma}$ analogously. By this definition, M_2 contains the jobs $A \cup B \cup \check{a} \cup \check{b} \cup \check{\delta} \cup \check{\gamma} \cup c$ and M_3 contains

	M_1	M_2	M_3	M_4
x_2	$\#_i A$	$\#_i A$	#A	$\#_i\beta + \#_i\{\lambda_2\}$
x_3	$\#_i \alpha + \#_i \{\lambda_1\}$	$\#_i B$	$\#_i B$	$\#_i B$
x_4	$\#_i a$	$\#_i\check{a} + \#_i\check{\delta}$	$\#_i \hat{a} + \#_i \hat{\delta}$	$\#_i \beta$
x_5	$\#_i \alpha$	$\#_i\check{b} + \#_i\check{\gamma}$	$\#_i \hat{b} + \#_i \hat{\gamma}$	$\#_i b$
x_6	$\#_i a$	$\#_i\check{a} + \#_i\check{b}$	$\#_i \hat{a} + \#_i \hat{b}$	$\#_i b$
x_8	$\#_i \alpha + \#_i \{\lambda_1\}$	$\#_i c$	$\#_i c$	$\#_i\beta + \#_i\{\lambda_2\}$

Table 1: Overview of the values of the coefficients at the start point of a job i, if i is scheduled on machine M_j .

the jobs $A \cup B \cup \hat{a} \cup \hat{b} \cup \hat{\delta} \cup \hat{\gamma} \cup c$.

We still have to show that $|\check{a}| = |\check{\gamma}|$ and $|\check{b}| = |\check{\delta}|$. First, we notice that $|\check{a}| + |\check{b}| = z$ since these jobs are the only jobs with a processing time containing D^6 . So besides the jobs in $A \cup B \cup c \cup \check{a} \cup \check{b}$, M_2 contains jobs with total processing time of $(z - |\check{a}|)D^4 + (z - |\check{b}|)D^5 + \sum_{i=1}^{z}(3z - i)D^7 = |\check{b}|D^4 + |\check{a}|D^5 + \sum_{i=1}^{z}(3z - i)D^7$. Since the jobs in δ are the only jobs in $\delta \cup \gamma$ having a processing time containing D^4 , we have $|\check{\delta}| = |\check{b}|$ and analogously $|\check{\gamma}| = |\check{a}|$.

In the next steps, we will prove that it is possible to transform the order in which the jobs appear on the machines to the order in Figure 3. Notice that, since there is no idle time in the schedule, each start point of a job i is given by the sum of processing times of the jobs on the same machine scheduled before i. So the start position $\sigma(i)$ of a job i has the form

$$\sigma(i) = x_0 + x_2 D^2 + x_3 D^3 + x_4 D^4 + x_5 D^5 + x_6 D^6 + x_7 D^7 + x_8 D^9$$

for $-zD \leq x_0 \leq zD$ and $0 \leq x_j \leq 4z(7z+1) \leq D$ for each $2 \leq j \leq 8$. This allows us to make implications about the correlation between the number of jobs scheduled on different machines when a job from the set $A \cup B \cup a \cup b \cup c$ starts. For example, let us look at the coefficient x_2 . This value is just influenced by jobs with processing times containing D^2 . The only jobs with these processing times are the jobs in the set $A \cup \beta \cup \{\lambda_2\}$. The jobs in $\beta \cup \{\lambda_2\}$ are just processed on M_4 , while the jobs in A each are processed on the three machines M_1 , M_2 , and M_3 . Therefore, we know that at the starting point $\sigma(i)$ of a job i scheduled on machines M_1 , M_2 or M_3 we have that $x_2 = \#_i A$. Furthermore, if i is scheduled on M_4 we know that $x_2 = \#_i \beta + \#_i \{\lambda_2\}$. In Table 1 we present which sets influences which coefficients in which way when job i is started on the corresponding machine.

Let us consider the start point $\sigma(i)$ of a job *i*, which uses more than one machine. We know that $\sigma(i)$ is the same on all the used machines and therefore the coefficients are the same as well. In the following, we will study for each of the sets *A*, *B*, *a*, *b*, *c* what we can conclude for the starting times of these jobs. For each of the sets, we will present an equation, which holds at the start of each item in this set. These equations give us a strong set of tools for our further arguing.

First, we will consider the start points of the jobs in A. Each job $A' \in A$ is scheduled on machines M_1, M_2 and M_3 . Therefore, we know that at s(A') we have $\#_{A'}B =_{x_3} \#_{A'}\alpha + \#_{A'}\{\lambda_1\} =_{x_8} \#_{A'c}$. Furthermore, we know that $\#_{A'}a =_{x_6} \#_{A'}\check{a} + \#_{A'}\check{b} = \#_{A'}\hat{a} + \#_{A'}\hat{b}$. Since $\#_{A'}a = \#_{A'}\check{a} + \#_{A'}\hat{a}$ and $\#_{A'}b = \#_{A'}\check{b} + \#_{A'}\hat{b}$, we can deduce that $\#_{A'}a = \#_{A'}b$. Additionally, we know that $\#_{A'}\alpha =_{x_5} \#\check{b} + \#\check{\gamma} =_{x_5} \#\check{b} + \#\hat{\gamma}$. Thanks to this equality, we can show that $\#_{A'}\alpha = \#_{A'}b$. First, we show $\#_{A'}\alpha \geq \#_{A'}b$. Let $b' \in b$ be the last job in b scheduled before A' if there is any. Let us w.l.o.g assume that $b \in \hat{b}$. It holds that $\#_{A'}b = \#_{b'}b + 1 =_{x_5} \#_b \hat{b} + \#_b' \hat{\gamma} + 1 \leq \#_{A'}\hat{b} + \#_{A'}\hat{\gamma} =_{x_5} \#_{A'}\alpha$. If there is no such b' we have $\#_{A'}b = 0 \leq \#_{A'}\alpha$. Next, we show $\#_{A'}\alpha \leq \#_{A'}b$. Let $b'' \in A$ be the first job in b scheduled after A if there is any. Let us w.l.o.g assume that $b \in \check{b}$. It holds that $\#_{A'}b = \#_{b''}b + \#_{A'}\tilde{\gamma} =_{x_5} \#_{A'}\alpha$. If there is no such b'' we have $\#_{A'}b = \psi_{A'}b = \psi_{A'}b = \psi_{A'}b$. It holds that $\#_{A'}b = \#_{A'}b = \#_{A'}b$. It holds $\#_{A'}b = \#_{A'}b = \#_{A'}b$. It holds that $\#_{A'}b = \#_{A'}b = \#_{A'}b = \#_{A'}b$. It holds that $\#_{A'}b = \#_{A'}b = \#_$

$$\#_{A'}c - \#_{A'}\{\lambda_1\} = \#_{A'}B - \#_{A'}\{\lambda_1\} = \#_{A'}\alpha = \#_{A'}b = \#_{A'}a.$$
(1)

Analogously, we can deduce that at the start of each $B' \in B$ we have that

$$\#_{B'}c - \#_{B'}\{\lambda_2\} = \#_{B'}A - \#_{B'}\{\lambda_2\} = \#_{B'}\beta = \#_{B'}a = \#_{B'}b.$$
(2)

Each item $a' \in a$ is scheduled on machine M_1 and on one of the machines M_2 or M_3 . For each possibility, we can deduce the equation

$$\#_{a'}B =_{x_3} \#_{a'}\alpha + \#_{a'}\{\lambda_1\} =_{x_8} \#_{a'}c.$$
(3)

Analogously, we deduce for each $b' \in b$ that

$$\#_{b'}A =_{x_2} \#_{b'}\beta + \#_{b'}\{\lambda_2\} =_{x_8} \#_{b'}c.$$
(4)

Last, each item $c' \in c$ is scheduled on M_2 and M_3 . Let $a' \in a$ be the job with the smallest $\sigma(a') \geq \sigma(c')$. Let us w.l.o.g assume that $a' \in \hat{a}$. It holds that $\#_{c'}\check{a} + \#_{c'}\check{b} =_{x_6} \#_{c'}\hat{a} + \#_{c'}\hat{b} \leq \#_{a'}\hat{a} + \#_{a'}\hat{b} =_{x_6} \#_{a'}\hat{a} = \#_{a'}\hat{a} + \#_{a'}\check{a} = \#_{c'}\hat{a} + \#_{c'}\check{a}$. As a consequence, we have $\#_{c'}\check{b} \leq \#_{c'}\hat{a}$ and $\#_{c'}\hat{b} \leq \#_{c'}\check{a}$. Analogously, let $b' \in b$ be the job with the smallest $\sigma(b') \geq \sigma(c')$. Let us w.l.o.g assume that $b' \in \check{b}$. It holds that $\#_{c'}\hat{a} + \#_{c'}\hat{b} =_{x_6} \#_{c'}\check{a} + \#_{c'}\check{b} \leq \#_{b'}\check{a} + \#_{b'}\check{b} =_{x_6} \#_{b'}b = \#_{b'}\hat{b} + \#_{b'}\check{a} = \#_{c'}\hat{b} + \#_{c'}\check{b}$. Therefore, $\#_{c'}\check{a} \leq \#_{c'}\hat{b}$ and $\#_{c'}\hat{a} \leq \#_{c'}\check{b}$. As a consequence, we can deduce that

$$\#_{c'}b = \#_{c'}a \tag{5}$$

These equations give us the tools to analyze the given schedule with makespan W. First, we will show that in this schedule the first and last jobs have to be elements from the set $A \cup B$, (see Lemma 2). After that, we will prove that the jobs in A and jobs in B have to be scheduled alternating, (see Lemma 3). With the knowledge gathered in the proofs of Lemma 2 and Lemma 3, we can prove that the given schedule can be transformed such that all jobs are scheduled continuously, and that \mathcal{I} has to be a Yes-instance (see Lemma 3).

Lemma 2. The first and the last job on M_2 and M_3 are elements of $A \cup B$.

Proof. Let $i := \arg\min_{i \in A \cup B} s_i$ be the job with the smallest start point in $A \cup B$, (i.e. $\#_i A = 0 = \#_i B$). We have to consider each case $i \in A$ and $i \in B$ and to show that its starting time has the value $s_i = 0$.

If $i \in A$ it holds that $0 = \#_i B =_{(1)} \#_i \alpha + \#_i \{\lambda_1\} =_{(1)} \#_i a + \#_i \{\lambda_1\}$ and therefore $\#_i a = \#_i \alpha = 0 = \#_i \{\lambda_1\}$. The jobs $a \cup \alpha \cup \{\lambda_1\} \cup A$ are the only jobs, which are contained on machine M_1 . Since $\#_i A = 0$ as well, it has to be that $s_i = 0$ and therefore i is the first job on M_2 and M_3 .

If $i \in B$ it holds that $0 = \#_i A = (2) \#_i \beta + \#_i \{\lambda_2\} = (2) \#_i b + \#_i \{\lambda_2\}$ and therefore $\#_i b = \#_i \beta = 0 = \#_i \{\lambda_2\}$. The jobs $b \cup \beta \cup \{\lambda_2\} \cup B$ are the only jobs, which are contained on machine M_4 . Since $\#_i B = 0$ as well, it has to be that $s_i = 0$ and therefore i is the first job on M_2 and M_3 .

We have shown that the first job on M_2 and M_3 hast to be a job from the set $A \cup B$. Since the schedule stays valid, if we mirror the schedule such that the new start points are s'(i) = W - s(i) - p(i) for each job *i*, the last job has to be in the set $A \cup B$ as well.

Next, we will show that the items in the sets A and B have to be scheduled alternating. Let (A_0, \ldots, A_z) be the set A and (B_0, \ldots, B_z) be the set B each ordered by increasing size of the starting points.

Lemma 3. If the first item on M_2 is the job $B_0 \in B$ it holds for each item $i \in \{0, \ldots, z\}$ that

$$\#_{A_i}B - \#_{A_i}\{\lambda_1\} = \#_{A_i}A \tag{6}$$

with $\#_{A_i}\{\lambda_1\} = 1$.

Proof. We will prove this claim inductively and per contradiction.

Assume $\#_{A_0}B - \#_{A_0}\{\lambda_1\} > \#_{A_0}A = 0$. Therefore, we have $1 \leq \#_{A_0}B - \#_{A_0}\{\lambda_1\}$. Let $a' \in a, b' \in b$ and $c' \in c$ be the first started jobs in their sets. Since $\#_{A_0}b =_{(1)} \#_{A_0}a =_{(1)} \#_{A_0}c - \#_{A_0}\{\lambda_1\} =_{(1)} \#_{A_0}B - \#_{A_0}\{\lambda_1\} \geq 1$, the jobs a', b' and c' start before A_0 . It holds that $\#_{b'}c =_{(4)} \#_{b'}A = 0$. Therefore, c' has to start after b' resulting in $\#_{c'}b \geq 1$. Furthermore, we have $\#_{a'}c =_{(3)} \#_{a'}B \geq 1$. Hence, c' has to start before a' resulting in $\#_{c'}a = 0$. In total we have $1 \leq \#_{c'}b =_{(5)} \#_{c'}a = 0$ contradicting the assumption $\#_{A_0}B - \#_{A_0}\{\lambda_1\} > \#_{A_0}A = 0$. Therefore, we have $\#_{A_0}B - \#_{A_0}\{\lambda_1\} \leq \#_{A_0}A = 0$. As a consequence, it holds that $1 \leq \#_{A_0}B \leq \#_{A_0}\{\lambda_1\} \leq 1$ and we can conclude $\#_{A_0}B = 1 = \#_{A_0}\{\lambda_1\}$ as well as $\#_{A_0}B - \#_{A_0}\{\lambda_1\} = \#_{A_0}A$.

Choose $i \in \{0, \ldots, z\}$ such that $\#_{A_{i'}}B - \#_{A_{i'}}\{\lambda_1\} = \#_{A_{i'}}A$ for all $i' \in \{0, \ldots, i\}$. As a consequence, we have $\#_{B_i}B = i = \#_{A_i}A = \#_{A_i}B - 1$. Therefore B_i has to be scheduled before A_i . Additionally, we have $\#_{B_i}B - 1 = \#_{B_{i-1}}B = i - 1 = \#_{A_{i-1}}A = \#_{A_{i-1}}B - 1$, so B_i has to be scheduled after A_{i-1} . Therefore, we have $\#_{B_i}B = \#_{B_i}A$ and as a consequence

$$i = \#_{B_i}B = \#_{B_i}A = \#_{B'}c = \#_{B'}\beta + \#_{B'}\{\lambda_2\} = \#_{B'}a + \#_{B'}\{\lambda_2\} = \#_{B'}b + \#_{B'}\{\lambda_2\}.$$
 (7)

We will now prove our claim for A_{i+1} .

Claim. $\#_{A_{i+1}}B - \#_{A_{i+1}}\{\lambda_1\} \le \#_{A_{i+1}}A$

Assume for contradiction that $\#_{A_{i+1}}B - \#_{A_{i+1}}\{\lambda_1\} > \#_{A_{i+1}}A$. As a consequence, we have $\#_{A_{i+1}}B - \#_{A_{i+1}}\{\lambda_1\} - \#_{A_i}B + \#_{A_i}\{\lambda_1\} \ge 2$. Therefore, there are jobs $B_{i+1}, B_{i+2} \in B, a', a'' \in a, b', b'' \in b$ and $c', c'' \in c$, that are scheduled between A_i and A_{i+1} since equality (1) holds. Let us suppose that $\sigma(a') \le \sigma(a''), \sigma(b') \le \sigma(b'')$ and $\sigma(c') \le \sigma(c'')$.

Next, we will deduce in which order the jobs $a', a'', b', b'', c', c'', B_{i+1}$, and B_{i+2} appear in the schedule. It holds that $\#_{b''}c =_{(4)} \#_{b''}A = \#_{A_i}A + 1 = \#_{A_i}B =_{(1)} = \#_{A_i}c$. Therefore, b' and b'' have to start before c'. Furthermore we have $\#_{c'}a =_{(5)} \#_{c'}b \ge \#_{A_i}b + 2 =_{(1)} \#_{A_i}a + 2$. Hence, a'' hast to start before c' as well. Additionally, it holds that $\#_{B_{i+2}}c =_{(2)} \#_{B_{i+2}}A = \#_{A_i}A + 1 = \#_{A_i}B =_{(1)} \#_{A_i}c$. As a consequence, B_{i+2} has to start before c'. Additionally, a'' has to start before B_{i+1} , since $\#_{a''}B =_{(3)} \#_{a''}c = \#_{A_i}c =_{(1)} \#_{A_i}B$.

To this point, we have deduced that the jobs have to appear in the following order in the schedule: $A_i, a', a'', B_{i+1}, B_{i+2}, c', c'', A_{i+1}$. This schedule is not feasible, since we have $\#_{A_i}a + 2 \leq_S \#_{B_{i+1}}a \leq_{(2)} \#_{B_{i+1}}A =_S \#_{A_i}A + 1 =_{(1)} \#_{A_i}a + 1$, a contradiction to the assumption $\#_{A_{i+1}}B - \#_{A_{i+1}}\{\lambda_1\} > \#_{A_{i+1}}A$. Therefore, it holds that $\#_{A_{i+1}}B - \#_{A_{i+1}}\{\lambda_1\} \leq \#_{A_{i+1}}A$

Claim. $\#_{A_{i+1}}B - \#_{A_{i+1}}\{\lambda_1\} \ge \#_{A_{i+1}}A$

Assume for contradiction that $\#_{A_{i+1}}B - \#_{A_{i+1}}\{\lambda_1\} < \#_{A_{i+1}}A$. It follows that $\#_{A_{i+1}}B = \#_{A_i}B$ since $\#_{A_i}B - \#_{A_i}\{\lambda_1\} \leq \#_{A_{i+1}}B - \#_{A_{i+1}}\{\lambda_1\} \leq \#_{A_{i+1}}A - 1 = \#_{A_i}A = \#_{A_i}B - \#_{A_i}\{\lambda_1\}$. Furthermore, there has to be at least one job $B_{i+1} \in B$ that starts after A_{i+1} since |A| = |B|. Therefore, we have $\#_{B_{i+1}}c - \#_{B_i}c = \#_{B_{i+1}}A - \#_{B_i}A \geq 2$. As a consequence, there are jobs $c', c'' \in c$ which are scheduled between B_i and B_{i+1} . Let c' be the first job in c scheduled after B_i ans c'' be the next. Since we do not know the value of $\#_{B_i}\{\lambda_2\}$ or $\#_{B_{i+1}}\{\lambda_2\}$, we can just deduce from equation (2) that $\#_{B_{i+1}}a - \#_{B_i}a \geq 1$. Therefore, there has to be a job $a' \in a$ that is scheduled between B_i and B_{i+1} .

We will now look at the order in which the jobs A_i , A_{i+1} , c', c'' and a' have to be scheduled. First, we know that A_i and A_{i+1} have to be scheduled between c' and c'', since $\#_{A_i}c =_{(1)} \#_{A_i}B =_S \#_{B_i}B + 1 =_{(7)} \#_{B_i}A + 1 =_{(2)} \#_{B_i}c + 1$ and $\#_{A_{i+1}}c =_{(1)} \#_{A_{i+1}}B =_S \#_{B_i}B + 1 =_{(7)} \#_{B_i}A + 1 =_{(2)} \#_{B_i}c + 1$. Furthermore, we know that a' has to be scheduled between c' and c'' as well, since $\#_{a'}c =_{(3)} \#_{a'}B =_S \#_{B_i}B + 1 =_{(7)} \#_{B_i}A + 1 =_{(2)} \#_{B_i}c + 1$. As a consequence, we can deduce that there is a job $b' \in b$ which is scheduled between c' and c'', since $\#_{c''}b =_{(5)} \#_{c''}a \ge_S \#_{c'}a + 1 =_{(5)} \#_{c'}b + 1$. We know about this b' that $\#_{b'}A =_{(4)} \#_{b'}c =_S \#_{B_i}c + 1 =_{(2)} \#_{B_i}A + 1$, so b' has to be scheduled between A_i and A_{i+1} .

In summary, the jobs are scheduled as follows: $B_i, c', A_i, b', A_{i+1}, c'', B_{i+1}$. However, this schedule is infeasible since $\#_{A_i}b =_{(1)} \#_{A_i}B - \#_{A_i}\{\lambda_1\} =_S \#_{A_{i+1}}B - \#_{A_{i+1}}\{\lambda_1\} =_{(1)} \#_{A_{i+1}}b =_S \#_{A_i}b + 1$. This contradicts the assumption $\#_{A_{i+1}}B - \#_{A_{i+1}}\{\lambda_1\} < \#_{A_{i+1}}A$. Altogether, we have shown that $\#_{A_{i+1}}B - \#_{A_{i+1}}\{\lambda_1\} = \#_{A_{i+1}}A$.

A direct consequence of Lemma 3 is that the last job on M_2 is a job in A. Since the equations (1) and (2), as well as (3) and (4), are symmetric, we can deduce an analogue statement if the first job on M_2 is in A. More precisely in this case we can show that $\#_iA - \#_i\{\lambda_2\} = \#_iB$ and $\#_i\{\lambda_2\} = 1$ for each $i \in B$. This would imply that the last job on M_2 is a job in B. Since we can mirror the schedule such that the last job is the first job, we can suppose that the first job on M_2 is a job out of B. In this case a further direct consequence of Lemma 3 and equation (1) is the equation

$$i = \#_{A_i}A = \#_{A_i}B - 1 = \#_{A_i}c - 1 = \#_{A_i}\alpha = \#_{A_i}b = \#_{A_i}a$$
(8)

Lemma 4. \mathcal{I} is a Yes-instance and we can transform the schedule such that all jobs are scheduled on continuous machines.

Proof. First, we will show that λ_2 is scheduled after the last job in B. Assume there is an $i \in \{0, \ldots, z\}$ with $\#_{B_i}\{\lambda_2\} > 0$. Let i be the smallest of these indices. We know that

$$i - 1 =_{(7)} \#_{B_i} A - 1 = \#_{B_i} A - \#_{B_i} \{\lambda_2\} =_{(2)} \#_{B_i} a.$$

Since $\#_{A_i}b =_{(1)} \#_{A_i}a =_{(8)} i = \#_{B_i}a + 1 =_{(2)} \#_{B_i}b + 1$ there has to be an unique $a' \in a$ and an unique $b' \in b$ scheduled between B_i and A_i . Furthermore, since $\#_{A_i}c =_{(8)} i + 1$ and $\#_{B_i}c =_{(7)} i$, there has to be a $c' \in c$ scheduled between B_i and A_i as well. At the start of b' it holds that $\#_{b'}c =_{(4)} \#_{b'}A = \#_{A_{i-1}}A + 1 =_{(1)} \#_{A_{i-1}}c$, so b' has to start before c'. Additionally, at the start of a' we have $\#_{a'}c =_{(4)} \#_{a'}B = \#_{B_i}B + 1 =_{(7)} \#_{B_i}c + 1$ and therefore a' hast to start after c'. In total, the jobs appear in the following order: B_i, b', c', a', A_i . But this can not be the case, since we have $\#_{B_{i-1}}a =_S \#_{c'}a =_{(5)} \#_{c'}b =_S \#_{B_{i-1}}b + 1 = \#_{B_{i-1}}a + 1$. Hence, we have contradicted that assumption. As a consequence, we have $\#_{B_i}\{\lambda_2\} = 0$ for all $i \in \{0, \ldots, z\}$ and therefore

$$\#_{B_i}b = \#_{B_i}a = \#_{B_i}c = \#_{B_i}\beta = \#_{B_i}A = \#_{B_i}B = i.$$
(9)

In the next step, we will prove that M_1 processes the jobs $A \cup a \cup a \cup \{\lambda_1\}$ in the order $\lambda_1, A_0, a_1, \alpha_1, A_1, a_2, \alpha_2, A_2, \ldots, a_z, \alpha_z, A_z$, where $a_i \in a$ and $\alpha_i \in \alpha$ for each $i \in \{1, \ldots, z\}$. Equation (8) and Lemma 3 ensure that the first job on M_1 is the job λ_1 and the second job is A_0 . For each $i \in \{1, \ldots, z\}$ it holds that $\#_{A_i}\alpha =_{(8)} \#_{A_{i-1}}\alpha + 1$ and $\#_{A_i}a =_{(8)} \#_{A_{i-1}}a + 1$. Therefore, there is scheduled exactly one job $a_i \in a$ and one job $\alpha_i \in \alpha$ between the jobs A_{i-1} and A_i . It holds that $\#_{A_i-1}a + 1 =_{(8)} i =_{(9)} \#_{B_i}a$. Therefore, a_i has to be scheduled between A_{i-1} and B_i . As a consequence, we have $\#_{a_i}\alpha + 1 = \#_{a_i}\alpha + \#_{a_i}\{\lambda_1\} =_{(3)} \#_{a_i}B = \#_{B_i}B =_{(9)} \#_{B_i}a = \#_{a'}a + 1$. Therefore, a_i has to be scheduled before α_i and the jobs appear in machine M_1 in the described order. As a result, we know about the start point of A_i that

$$\begin{aligned} \sigma(A_i) &= p(\lambda_1) + i \cdot p_a + i \cdot p_a + i \cdot p_A \\ &= D^3 + zD^7 + D^8 + i(D^4 + D^6 + 3zD^7) + i(D^3 + D^5 + 4zD^7 + D^8) + iD^2 \\ &= iD^2 + (i+1)D^3 + iD^4 + iD^5 + iD^6 + (7zi+z)D^7 + (i+1)D^8. \end{aligned}$$

Now, we will show, that the machine M_4 processes the jobs $B \cup b \cup \beta \cup \{\lambda_2\}$ in the order $B_0, \beta_1, b_1, B_1, \beta_2, b_2, B_2, \ldots, \beta_z, b_z, B_z, \lambda_2$, where $b_i \in b$ and $\beta_i \in \beta$ for each $i \in \{1, \ldots, z\}$. The first job on M_4 is the job B_0 . Equation (9) ensures that between the jobs B_i and B_{i+1} there is scheduled exactly one job $b_{i+1} \in b$ and exactly one job $\beta_{i+1} \in \beta$. It holds that $\#_{A_i}b + 1 =_{(8)} i + 1 =_{(9)} \#_{B_{i+1}}b$. Therefore, b_{i+1} has to be scheduled between A_i and B_{i+1} . As a consequence, it holds that $\#_{b_{i+1}}\beta = \#_{b_{i+1}}\beta + \#_{b_{i+1}}\{\lambda_2\} = \#_{b_{i+1}}A = \#_{B_{i+1}}b = \#_{b_{i+1}}b + 1$. Hence, b_{i+1} has to be scheduled after β_{i+1} and the jobs on machine M_4 appear in the described order. As a result, we know about the start point of B_i that

$$\sigma(B_i) = ip_b + ip_\beta + ip_B$$

= $iD^2 + iD^3 + iD^4 + iD^5 + iD^6 + (i(7z - 1))D^7 + iD^8$.

Next, we can deduce, that the jobs in c are scheduled as shown in Figure 3. We have $\#_{B_i}c =_{(9)}$ $i =_{(8)} \#_{A_i}c - 1$. Therefore, there exists an $c' \in c$ for each $i \in \{0, \ldots, z\}$, which is scheduled between B_i and A_i . The processing time between B_i and A_i is exactly $\sigma(A_i) - \sigma(B_i) - p(B_i) = (z+i)D^7 + D^8$. As a consequence, one can see with an inductive argument that $c_i \in c$ with $p(c_i) = (z+i)D^7 + D^8$ has to be positioned between B_i and A_i , since the job in c with the largest processing time c_z only fits between B_z and A_z .

In this step, we will transform the schedule, such that all jobs are scheduled on continuous machines. To this point, this property is obviously fulfilled by the jobs in $A \cup B \cup c$. However, the jobs in $a \cup b$ might be scheduled on nonconsecutive machines. We know that the a_i and b_i are scheduled between A_{i-1} and B_i . One part of a_i is scheduled on M_1 and one part of b_i is scheduled on M_4 , while each second part is scheduled either on M_2 or on M_3 but both parts on different machines,

because $\sigma(B_i) - \sigma(A_{i-1}) - p(A_i) = D^4 + D^5 + D^6 + (6z-i)D^7 < D^4 + D^5 + 2D^6 + 6zD^7 = p(a_i) + p(b_i)$ for each $i \in \{0, \ldots, z\}$. Since A_i and B_{i+1} both are scheduled on machines M_2 and M_3 , we can swap the content of the machines between these jobs such that the second part of a_i is scheduled on M_2 and the second part of b_i is scheduled on M_3 . We do this swapping step for all $i \in \{0, \ldots, z-1\}$ such that all second parts of jobs in a are scheduled on M_2 and all second part of jobs in b are scheduled on M_3 respectively. After this swapping step, all jobs are scheduled on continuous machines.

Now, we will show that \mathcal{I} is a yes-instance. To this point we know that M_2 contains the jobs $A \cup B \cup a \cup c$. Since $\check{a} = a$, it has to hold by Lemma 1, using $|\check{a}| = |\check{\gamma}|$, that $\check{\gamma} = \gamma$ implying that M_2 contains all jobs in γ . Furthermore, since $\check{b} = \emptyset$ and $|\check{b}| = |\check{\delta}|$, we have $\check{\delta} = \emptyset$ and therefore M_2 does not contain any job in δ . Besides the jobs $A \cup B \cup a \cup c \cup \gamma$, M_2 processes further jobs with total processing time zD. Therefore, all the jobs in P are processed on M_2 . We will now analyse where the jobs in γ are scheduled. The only possibility where these jobs can be scheduled is the time between a_i and B_i for each $i \in \{1, \ldots, z\}$ since at each other time the machine is occupied by other jobs. The processing time between the end of a_i and the start of B_i is exactly $\sigma(B_i) - \sigma(A_{i-1}) - p(A_{i-1}) - p(a_i) = D^5 + (3z - i)D^7$. The job in γ with the largest processing time is the job γ_1 with $p(\gamma_1) = D^5 + (3z-1)D^7 - D$. This job only fits between a_i and B_1 . Inductively we can show that $\gamma_i \in \gamma$ with $p(\gamma_i) = D^5 + (3z - i)D^7 - D$ has to be scheduled between a_i and B_i on M_2 . Furthermore, since $p(\gamma_i) = D^5 + (3z - i)D^7 - D$ and the processing time between the end of a_i and the start of B_i is $D^5 + (3z - i)D^7$, there is exactly D processing time left. These processing time has to be occupied by the jobs in P since this schedule has no idle times. Therefore, we have for each $i \in \{1, \ldots, z\}$ a disjunct subset $P_i \subseteq P$ containing jobs with processing times adding up to D. As a consequence \mathcal{I} is a Yes-instance.

3 Hardness of Strip Packing

In the transformed schedule, all jobs are scheduled on contiguous machines. As a consequence, we have proven that this problem is strongly NP-complete even if we restrict the set of feasible solutions to those where all jobs are scheduled on continuous machines. We will now describe how this insight delivers a lower bound of $\frac{5}{4}$ for the best possible approximation ratio for pseudo-polynomial Strip Packing and in this way prove Theorem 2.

To show our hardness result for Strip Packing, let us consider the following instance. We define $W := (z+1)(D^2 + D^3 + D^8) + z(D^4 + D^5 + D^6) + z(7z+1)D^7$ as the width of the considered strip, so it is the same as the considered makespan in the scheduling problem. For each job j defined in the reduction above, we define an item i with w(i) = p(j) and height h(i) = q(j). Now, we can show analogously that if the 3-Partition instance is a Yes-instance there is a packing of height 4 (one example is the packing in Figure 3) and if there is a packing with height 4 then the 3-Partition instance has to be a Yes-instance. If the 3-Partition instance is a No-instance, the optimal packing has a height of at least 5 since the optimal height for this instance is integral. Therefore, we can not approximate Strip Packing in pseudo-polynomial time better than $\frac{5}{4}$.

4 Conclusion

In this paper, we positively answered the long standing open question whether $P4|size_j|C_{\max}$ is strongly NP-complete. Now, for each number of machines m it is known whether the problem $Pm|size_j|C_{\max}$ is strongly NP-complete. Furthermore, we have improved the lower bound for pseudo-polynomial Strip Packing to $\frac{5}{4}$. Since the best known algorithm has an approximation ratio of $\frac{4}{3}$, this still leaves a gap between the lower bound and the best known algorithm. With the techniques used in this paper, a lower bound of $\frac{4}{3}$ for pseudo-polynomial Strip Packing can not be proven, since $P3|size_j|C_{\max}$ is solvable in pseudo-polynomial time and in the generated solutions all jobs are scheduled contiguously. Moreover, we believe that it is possible to find an algorithm with approximation ratio $\frac{5}{4} + \varepsilon$.

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