

On the stab number of rectangle intersection graphs

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Abstract

We introduce the notion of *stab number* and *exact stab number* of rectangle intersection graphs, otherwise known as graphs of boxicity at most 2. A graph G is said to be a *k-stabtable rectangle intersection graph*, or *k-SRIG* for short, if it has a rectangle intersection representation in which k horizontal lines can be chosen such that each rectangle is intersected by at least one of them. If there exists such a representation with the additional property that each rectangle intersects exactly one of the k horizontal lines, then the graph G is said to be a *k-exactly stabtable rectangle intersection graph*, or *k-ESRIG* for short. The stab number of a graph G , denoted by $stab(G)$, is the minimum integer k such that G is a k -SRIG. Similarly, the exact stab number of a graph G , denoted by $estab(G)$, is the minimum integer k such that G is a k -ESRIG. In this work, we study the stab number and exact stab number of some subclasses of rectangle intersection graphs. A lower bound on the stab number of rectangle intersection graphs in terms of its pathwidth and clique number is shown. Tight upper bounds on the exact stab number of split graphs with boxicity at most 2 and block graphs are also given. We show that for $k \leq 3$, k -SRIG is equivalent to k -ESRIG and for any $k \geq 10$, there is a tree which is a k -SRIG but not a k -ESRIG. We also develop a forbidden structure characterization for block graphs that are 2-ESRIG and trees that are 3-ESRIG, which lead to polynomial-time recognition algorithms for these two classes of graphs. These forbidden structures are natural generalizations of asteroidal triples. Finally, we construct examples to show that these forbidden structures are not sufficient to characterize block graphs that are 3-SRIG or trees that are k -SRIG for any $k \geq 4$.

Keywords: Rectangle intersection graphs, interval graphs, stab number, k -SRIG, asteroidal triple, block graphs, forbidden structure characterization.

1 Introduction

A *rectangle intersection representation* of a graph is a collection of axis-parallel rectangles on the plane such that each rectangle in the collection represents a vertex of the graph and two rectangles intersect if and only if the vertices they represent are adjacent in the graph. The graphs that have rectangle intersection representation are called *rectangle intersection graphs*. The *boxicity* $box(G)$ of a graph G is the minimum d such that G is representable as a geometric intersection graph of d -dimensional (axis-parallel) hyper-rectangles. A graph G is an interval graph if $box(G) = 1$ and G is a rectangle intersection graph if $box(G) \leq 2$.

A *k-stabbed rectangle intersection representation* is a rectangle intersection representation, along with a collection of k horizontal lines called *stab lines*, such that every rectangle intersects at least one of the stab lines. A graph G is a *k-stabtable rectangle intersection graph* (k -SRIG), if there exists a k -stabbed rectangle intersection representation of G . The *stab number* of a rectangle intersection graph, denoted by $stab(G)$, is the minimum integer k such that there exists a k -stabbed rectangle intersection representation of G . In other words $stab(G)$ is the minimum integer k such that G is k -SRIG. Clearly, if a graph G has boxicity at most 2, then $stab(G)$ is finite. For graphs G with boxicity at least three, we define $stab(G) = \infty$.

A *k-exactly stabbed rectangle intersection representation* is a k -stabbed rectangle intersection representation in which every rectangle intersects exactly one of the stab lines. A graph G is a *k-exactly stabtable rectangle intersection graph*, or *k-ESRIG* for short, if there exists a k -exactly stabbed rectangle intersection representation of G . The *exact stab number* of a rectangle intersection graph, denoted by $estab(G)$, is the minimum integer k such that there exists a k -exactly stabbed rectangle intersection representation of G . In other words, $estab(G)$ is the minimum integer k such that G is k -ESRIG. When a graph G has no k -exactly stabbed rectangle intersection representation for any integer k , we define $estab(G) = \infty$. A graph G with $estab(G) < \infty$ is said to be an *exactly stabtable rectangle intersection graph*. Note that for a graph G ,

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$stab(G) \leq estab(G)$ and that a graph G is an interval graph if and only if $stab(G) = estab(G) = 1$, or in other words, the class of interval graphs, the class of 1-SRIGs, and the class of 1-ESRIGs are all the same.

For a subclass \mathcal{C} of rectangle intersection graphs, $stab(\mathcal{C}, n)$ is the minimum integer k such that any graph $G \in \mathcal{C}$ with n vertices satisfies $stab(G) \leq k$, and $estab(\mathcal{C}, n)$ is the minimum integer k such that for any graph $G \in \mathcal{C}$ with n vertices satisfies $estab(G) \leq k$. A *unit height rectangle intersection* graph G is a graph that has a rectangle intersection representation in which all rectangles have equal height. It is well-known that all unit height rectangle intersection graphs are exactly stabbable rectangle intersection graphs (for the sake of completion, we prove this in Theorem 5 in Section 3).

1.1 Motivation and related work

Boxicity of a graph has been an active field of research for many decades [2, 8–10, 17]. While recognizing graphs with boxicity at most d is NP-complete for all $d \geq 2$ [19, 23], there are efficient algorithms to recognize interval graphs, i.e. graphs with boxicity at most 1 [11, 21]. There seems to be a “jump in the difficulty level” of problems as the boxicity of the input graph increases from 1 to 2. For example, the MAXIMUM INDEPENDENT SET and CHROMATIC NUMBER problems, while being linear-time solvable for interval graphs, become NP-complete for rectangle intersection graphs (even with the rectangle intersection representation given as input) [18, 20]. Our goal is to understand the reason of this jump by studying graph classes that lie “in between” interval graphs and rectangle intersection graphs. For this purpose, we introduce a parameter called stab number for rectangle intersection graphs. The concept of stab number is a generalization of the idea behind a class of graphs known as “2SIG”, which was introduced in an earlier paper [6]. Even though our definitions of 2-SRIG and 2-ESRIG are both slightly different from that of “2SIG”, all three classes of graphs turn out to be equivalent (Theorem 2 shows that the classes k -SRIG and k -ESRIG are equivalent for any $k \leq 3$). A k -stabbed rectangle intersection representation of a graph involves rectangles and horizontal lines. Such combined arrangements of lines and rectangles have been popular topics of study in the geometric algorithms community. For example, such arrangements appear in the works of Agarwal et al. [3] and Chan [7], who gave approximation algorithms for the MAXIMUM INDEPENDENT SET problem in unit height rectangle intersection graphs, and also in a paper by Erlebach et al. [16], who proposed a PTAS for MINIMUM WEIGHT DOMINATING SET for unit square intersection graphs. Correa et al. [12] have studied the problems of computing independent and hitting sets for families of rectangles intersecting a diagonal line.

1.2 Contributions and organization of the paper

In this paper, we introduce the notion of “stab number” of a rectangle intersection graph and study this parameter for various subclasses of rectangle intersection graphs. In Section 2, we give some definitions and notation that will be used throughout the paper. We prove some basic results about k -SRIGs and k -ESRIGs in Section 3. We first show a simple necessary and sufficient condition for a graph to be a k -ESRIG and also show why the classes k -SRIG and k -ESRIG are equivalent when $k \leq 3$ (Theorem 2). Then we prove that the class of unit height rectangle intersection graphs is a proper subset of the class of rectangle intersection graphs with finite exact stab number (Theorem 3), which in turn is a proper subset of rectangle intersection graphs (Theorem 5). This leads us to the natural question of finding exactly stabbable graphs whose exact stab number is strictly greater than the stab number. We show that for each $k \geq 10$, there exist trees which are k -SRIG but not k -ESRIG (Theorem 36). Therefore, even for graphs that are exactly stabbable, like trees (Theorem 10), the stab number and the exact stab number may differ. We prove this result only in Section 6.3, after the machinery required for the proof is developed in Section 6. In Section 4, we show a lower bound on the stab number of rectangle intersection graphs in terms of the clique number and the pathwidth, and then study upper bounds on the stab number of rectangle intersection graphs that are also (a) split graphs, or (b) block graphs. In particular, we show (a) that all rectangle intersection graphs that are also split graphs have exact stab number at most 3 and that this bound is tight, and (b) an upper bound of $\lceil \log m \rceil$ on the exact stab number of block graphs with m blocks (this bound is shown to be asymptotically tight in Section 6.1). Then in Section 5, we describe a forbidden structure for k -SRIG and k -ESRIG, which we call “asteroidal-(non- $(k-1)$ -SRIG)” subgraphs and “asteroidal-(non- $(k-1)$ -ESRIG)” subgraphs respectively. These obstructions are a natural generalization of the well-known “asteroidal-triples” of Lekkerkerker and Boland [21], which are obstructions for interval graphs. In Section 5.2, we discuss some general properties possessed by the block-trees of graphs without these kinds of obstructions. In Section 6, we show that the absence of these forbidden structures is enough to characterize block graphs that are 2-ESRIG (Theorem 19)

and trees that are 3-ESRIG (Theorem 20). These results lead to polynomial-time algorithms to recognize block graphs that are 2-SRIG and trees that are 3-SRIG. In Section 6.2, we develop a geometric argument that allows us to show that this kind of forbidden structure is not sufficient to characterize block graphs that are 3-SRIG (Theorem 23) or trees that are k -SRIG, for any $k \geq 4$ (Theorem 24). We conclude by listing some open problems and suggesting some possible directions for further research on this topic.

2 Preliminaries

We present some definitions in this section. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $N(v) = \{u \in V(G) : uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$ denote the *open neighbourhood* and the *closed neighbourhood* of a vertex v , respectively. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph induced in G by the vertices in S , and by $G - S$ the graph obtained by removing the vertices in S from G . For an edge $e \in E(G)$, we denote by $G - e$ the graph on vertex set $V(G)$ having edge set $E(G) \setminus \{e\}$.

Let G be a rectangle intersection graph with rectangle intersection representation \mathcal{R} . A rectangle in \mathcal{R} corresponding to the vertex v is denoted as r_v . All rectangles considered in this article are closed rectangles. Denote by x_v^+ (x_v^-), the x -coordinate of the right (left) bottom corner of r_v . Also y_v^+ (y_v^-) is the y -coordinate of the left top (bottom) corner of r_v . In other words, $r_v = [x_v^-, x_v^+] \times [y_v^-, y_v^+]$. The *span* of a vertex u , denoted as $\text{span}(u)$, is the projection of r_u on the X -axis, i.e. $\text{span}(u) = [x_u^-, x_u^+]$. For two intervals $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$, we write $I_1 < I_2$ to indicate that $b_1 < a_2$. Clearly, $I_1 \cap I_2 = \emptyset$ if and only if $I_1 < I_2$ or $I_2 < I_1$. For an edge $uv \in E(G)$, we define $\text{span}(uv) = \text{span}(u) \cap \text{span}(v)$.

Let G be a k -SRIG with a k -stabbed rectangle intersection representation \mathcal{R} in which the stab lines are $y = a_1, y = a_2, \dots, y = a_k$, where $a_1 < a_2 < \dots < a_k$. The *top* (resp. *bottom*) stab line of \mathcal{R} is the stab line $y = a_k$ (resp. $y = a_1$). For $1 \leq i < k$, we say that $y = a_{i+1}$ is the stab line “just above” the stab line $y = a_i$ and that $y = a_i$ is the stab line “just below” the stab line $y = a_{i+1}$. We also say that the stab lines $y = a_i$ and $y = a_{i+1}$ are “consecutive”. A vertex $u \in V(G)$ is said to be “on” a stab line if r_u intersects that stab line. Two vertices u, v of G “have a common stab” if there is some stab line that intersects both r_u and r_v . Similarly, a set of vertices is said to have a common stab if there is one stab line that intersects the rectangles corresponding to each of them. It is easy to see that if $uv \in E(G)$, then there must be either a stab line such that u and v are on it or two consecutive stab lines such that u is on one of them and v is on the other. Whenever the k -stabbed rectangle intersection representation of a graph G under consideration is clear from the context, the terms $r_u, x_u^-, x_u^+, y_u^-, y_u^+$, for every vertex $u \in V(G)$ and usages such as “on a stab line”, “have a common stab”, “span” etc. are considered to be defined with respect to this representation. Clearly, both the classes k -SRIG and k -ESRIG are closed under taking induced subgraphs. We say that a graph is a non- k -SRIG (resp. non- k -ESRIG) if it is not a k -SRIG (resp. k -ESRIG). Similarly, we say that a graph is a non-interval graph if it is not an interval graph.

3 Basic Results

Given a collection \mathcal{I} of intervals, a *hitting set* X of \mathcal{I} is a subset of \mathbb{R} such that each interval in \mathcal{I} contains at least one element of X . The set X is an *exact hitting set* of \mathcal{I} if each interval in \mathcal{I} contains exactly one element of X . An interval graph G is said to have an exact hitting set of size k if there exists an interval representation \mathcal{I} of G that has an exact hitting set of cardinality k . Note that some collections of intervals may not have an exact hitting set of any cardinality. Also, there are interval graphs (for example, $K_{1,4}$) that have no exact hitting set.

Theorem 1. *A graph G is a k -ESRIG if and only if there exists two interval graphs I_1 and I_2 such that $V(G) = V(I_1) = V(I_2)$ and $E(G) = E(I_1) \cap E(I_2)$ and at least one of I_1, I_2 has an exact hitting set of size k .*

Proof. First we prove that if G has a k -ESRIG representation, then there exist two interval graphs I_1 and I_2 such that $V(G) = V(I_1) = V(I_2)$ and $E(G) = E(I_1) \cap E(I_2)$ and at least one of them has an exact hitting set of size k . Let \mathcal{R} be a k -exactly stabbed rectangle intersection representation of G and $\{y = a_1, y = a_2, \dots, y = a_k\}$ be the set of stab lines in \mathcal{R} . Let I_x, I_y be the interval graphs formed by taking the projections of the rectangles in \mathcal{R} on the X and Y axes, respectively. In other words, I_x is the interval graph given by the interval representation $\{[x_u^-, x_u^+]\}_{u \in V(G)}$ and I_y is the interval graph given by the interval representation $\{[y_u^-, y_u^+]\}_{u \in V(G)}$. It is clear that $V(G) = V(I_x) = V(I_y)$ and $E(G) = E(I_x) \cap E(I_y)$. Furthermore, the set

$S = \{a_1, a_2, \dots, a_k\}$ is an exact hitting set of the interval representation $\{[y_u^-, y_u^+]\}_{u \in V(G)}$ of I_y . Hence, I_y has an exact hitting set of size k .

Now assume that there exist two interval graphs I_1 and I_2 such that $V(G) = V(I_1) = V(I_2)$ and $E(G) = E(I_1) \cap E(I_2)$ and at least one of them, say I_1 , has an exact hitting set of size k . Let $S = \{a_1, a_2, \dots, a_k\}$ be an exact hitting set of an interval representation $\{[c_u, d_u]\}_{u \in V(G)}$ of I_1 . Also, let $\{[c'_u, d'_u]\}_{u \in V(G)}$ be an interval representation of I_2 . For each $u \in V(G)$, define $r_u = [c'_u, d'_u] \times [c_u, d_u]$. It is easy to see that $\mathcal{R} = \{r_u\}_{u \in V(G)}$ is a rectangle intersection representation of G . Further, the lines $y = a_1, y = a_2, \dots, y = a_k$ are horizontal lines such that each rectangle in \mathcal{R} intersects exactly one of them. Hence, \mathcal{R} , together with these lines, is a k -exactly stabbed rectangle intersection representation of G and therefore, G is a k -ESRIG. This completes the proof. \square

Theorem 2. *When $k \leq 3$, the classes k -SRIG and k -ESRIG are equivalent.*

Proof. If a graph G is k -ESRIG for some k , then G is also k -SRIG. Therefore it suffices to prove that if a graph G has a k -stabbed rectangle intersection representation for some $k \leq 3$, then G also has a k -exactly stabbed rectangle intersection representation. If $k = 1$, then there is nothing to prove. So we shall assume that $k \in \{2, 3\}$. Let \mathcal{R} be a k -stabbed rectangle intersection representation of a graph G with $k \leq 3$ with stab lines $y = 0, y = 1, \dots, y = k - 1$. We can assume without loss of generality that for any two distinct vertices $u, v \in V(G)$, we have $\{y_u^+, y_u^-\} \cap \{y_v^+, y_v^-\} = \emptyset$ and that for any vertex $v \in V(G)$, we have $\{y_v^+, y_v^-\} \cap \{0, 1, 2\} = \emptyset$ (note that if this is not the case, then the rectangles in \mathcal{R} can be perturbed slightly so that these conditions are satisfied). Let $S = \{y_v^+, y_v^-\}_{v \in V(G)} \cup \{0, 1, 2\}$ and ϵ be a positive real number such that $\epsilon < \min\{|a - b| : a, b \in S, a \neq b\}$. Let $M = \{u \in V(G) : r_u \text{ intersects the stab line } y = 1\}$. For each vertex $u \in M$, define $r'_u = [x_u^-, x_u^+] \times [y_u'^-, y_u'^+]$, where $y_u'^- = \max\{\epsilon, y_u^-\}$ and $y_u'^+ = \min\{2 - \epsilon, y_u^+\}$. Let \mathcal{R}' be the rectangle intersection representation given by the collection of rectangles $(\mathcal{R} \setminus \{r_u : u \in M\}) \cup \{r'_u : u \in M\}$. It is now easy to verify that \mathcal{R}' is a k -exactly stabbed rectangle intersection representation of G . Indeed, \mathcal{R}' is obtained from \mathcal{R} by the vertical shortening of some of the rectangles intersecting the stab line $y = 1$, and we only need to show that every rectangle that is so shortened still intersects with all the rectangles with which it originally has an intersection. The definition of ϵ guarantees that in \mathcal{R} , the bottom edge of any rectangle is no higher than $2 - \epsilon$ and the top edge of any rectangle is no lower than ϵ . So when a rectangle is shortened in the manner described above, it does not become disjoint from a rectangle with which it previously had a nonempty intersection. Therefore \mathcal{R}' is a valid rectangle intersection representation of G . It is clear that any rectangle that intersects the stab line $y = 1$ in \mathcal{R} intersects only the stab line $y = 1$ in \mathcal{R}' . This implies that \mathcal{R}' is a k -exactly stabbed rectangle intersection representation of G . \square

In the following theorem, we show that for $k = 4$, the classes k -SRIG and k -ESRIG differ.

Theorem 3. *There is a graph G such that $\text{stab}(G) \leq 4$ and $\text{estab}(G) = \infty$.*

Proof. We let $G = K_{4,4}$, i.e. the complete bipartite graph in which each partite set contains four vertices each. Clearly, G is a rectangle intersection graph with $\text{stab}(G) \leq 4$ (see Figure 1(a)). We shall prove that $\text{estab}(G) = \infty$, or in other words, G is not an exactly stabbable rectangle intersection graph. First we prove the following claim.

Claim. *Let C be a cycle of length four and $E(C) = \{ab, bc, cd, da\}$. There is no k -exactly stabbed rectangle intersection representation of C , for any integer k , in which a, c have a common stab and b, d have a common stab.*

Assume for the sake of contradiction that there is a k -exactly stabbed rectangle intersection representation \mathcal{R} of C , for some integer k , in which a, c have a common stab and b, d have a common stab. Clearly, a, b, c, d cannot all be on one stab line (as C is not an interval graph). Since every vertex is on exactly one stab line and because $ab \in E(C)$, we can assume without loss of generality that a, c are on the stab line just below the stab line on which b, d are. Since a, c and b, d are nonadjacent in C , again without loss of generality we can assume that $\text{span}(a) < \text{span}(c)$. Since $b \in N(a) \cap N(c)$, we can infer that $[x_a^+, x_c^-] \subset \text{span}(b)$. Similarly, we can show that $[x_a^+, x_c^-] \subset \text{span}(d)$. But this implies that $[x_a^+, x_c^-] \subset \text{span}(b) \cap \text{span}(d)$. Since b, d are on the same stab line, this means that $r_b \cap r_d \neq \emptyset$. As $bd \notin E(C)$, this contradicts the fact that \mathcal{R} is a rectangle intersection representation of C . This proves the claim.

Now suppose that G has a k -exactly stabbed rectangle intersection representation \mathcal{R} for some k . Let V_1, V_2 be the two partite sets of G (recall that G is isomorphic to $K_{4,4}$) and $v \in V_1$ be a vertex on some stab line ℓ . Since each vertex is on exactly one stab line, and all vertices of V_2 are adjacent to v , we know

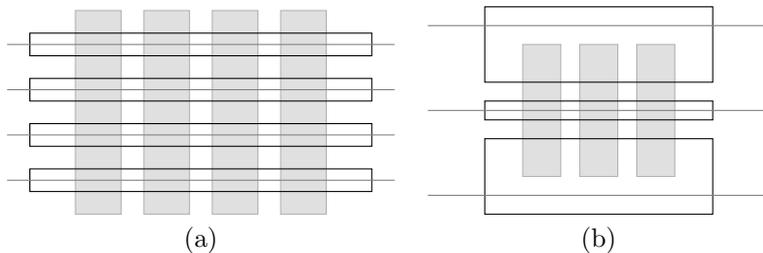


Figure 1: (a) A 4-stabbed rectangle intersection representation of $K_{4,4}$, (b) a 3-exactly stabbed rectangle intersection representation of $K_{3,3}$.

that each vertex of V_2 must be on the stab line ℓ , on the stab line just above ℓ , or on the stab line just below ℓ . By Pigeon Hole Principle, there exists $u, w \in V_2$ such that u and w are both on one of these stab lines, say ℓ_1 . Now, for the same reason as before, each vertex of V_1 must be on the stab line ℓ_1 , on the stab line just above ℓ_1 , or on the stab line just below ℓ_1 . Again by Pigeon Hole Principle, there are two vertices $u', w' \in V_1$ such that u' and w' are both on one of these stab lines. Now, consider the cycle C of length four with $E(C) = \{u'u, uw', w'w, wu'\}$, that is an induced subgraph of G . It can be seen that the rectangles in \mathcal{R} corresponding to the vertices of C form a k -exactly stabbed rectangle intersection representation of C in which u', w' have a common stab and u, w have a common stab. This contradicts the claim proved above. Therefore, G cannot have a k -exactly stabbed rectangle intersection representation for any k . \square

Corollary 4. *The class of exactly stabbable rectangle intersection graphs is a proper subset of the class of rectangle intersection graphs.*

The above theorem shows that there are graphs whose stab number is a constant but their exact stab number is infinite. Later on, in Theorem 36, we shall show that there are even trees whose stab number and exact number differ, even though both these parameters are finite for trees.

Theorem 5. *The class of unit height rectangle intersection graphs is a proper subset of the class of exactly stabbable rectangle intersection graphs.*

Proof. We shall first give a proof for the well-known fact that every unit height rectangle intersection graph is an exactly stabbable rectangle intersection graph. We shall prove the following stronger claim.

Claim. *Given a unit height rectangle intersection representation \mathcal{R} for a graph G , there exists a set of horizontal lines $y = a_1, y = a_2, \dots, y = a_k$ (for some integer k), where $a_1 < a_2 < \dots < a_k$, such that each rectangle in \mathcal{R} intersects exactly one of them and $a_1 = \min_{u \in V(G)} \{y_u^+\}$.*

Let $a = \min_{u \in V(G)} \{y_u^+\}$ and let $S = \{u: u \in V(G) \text{ and } a \in [y_u^-, y_u^+]\}$. Now consider the unit height rectangle intersection representation $\mathcal{R}' = \mathcal{R} \setminus \{r_u\}_{u \in S}$ of $G' = G - S$. By the induction hypothesis, there exists a set of horizontal lines $y = a'_1, y = a'_2, \dots, y = a'_{k'}$, for some integer k' , where $a'_1 < a'_2 < \dots < a'_{k'}$, such that each rectangle in \mathcal{R}' intersects exactly one of them and $a'_1 = \min_{u \in V(G')} \{y_u^+\}$. Since every rectangle in \mathcal{R}' lies completely above the horizontal line $y = a$, we have that $\min_{u \in V(G')} \{y_u^+\} > a + 1$. Therefore, we have $a'_1 - a > 1$. Since $a'_1 < a'_2 < \dots < a'_{k'}$, this means that for $1 \leq i \leq k'$, no rectangle of \mathcal{R} intersects both the horizontal lines $y = a'_i$ and $y = a$. Since every rectangle in $\{r_u\}_{u \in S}$ intersects the horizontal line $y = a$, and every rectangle in $\{r_u\}_{u \in V(G')}$ intersects exactly one of the horizontal lines $y = a'_1, y = a'_2, \dots, y = a'_{k'}$, it follows that each rectangle of \mathcal{R} intersects exactly one of the horizontal lines $y = a, y = a'_1, y = a'_2, \dots, y = a'_{k'}$. This proves the claim.

We shall now show the existence of an exactly stabbable rectangle intersection graph that is not a unit height rectangle intersection graph. Consider the graph $K_{3,3}$, i.e. the complete bipartite graph in which each partite set contains three vertices each. Clearly, $K_{3,3}$ is an exactly stabbable rectangle intersection graph (see Figure 1(b)). We shall prove that $K_{3,3}$ is not a unit height rectangle intersection graph.

A rectangle intersection representation \mathcal{R} of a graph G is *crossing-free* if for any two rectangles r_u and r_v in \mathcal{R} , the regions $r_u \setminus r_v$ and $r_v \setminus r_u$ are both arc-connected. Note that a unit height rectangle intersection representation of a graph is crossing-free. We shall show that if a triangle-free graph G has a crossing-free rectangle intersection representation, then G must be a planar graph. It then follows directly that $K_{3,3}$ is not a unit height rectangle intersection graph.

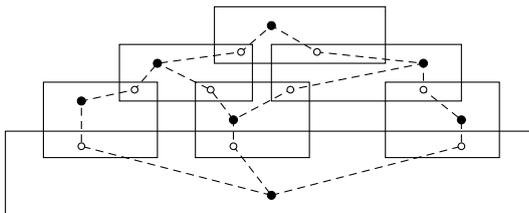


Figure 2: The dotted curves along with the solid points endpoints, give a planar embedding of the intersection graph of the rectangles in the figure. The hollow circle contained in the intersection region of two rectangles, say r_u and r_v , represents the point p_{uv} .

Let \mathcal{R} be a crossing-free rectangle intersection representation of a triangle-free graph G and let $S \subseteq V(G)$ be the set of vertices of G having degree one. Let $H = G - S$. Clearly, G is planar if and only if H is planar. Let \mathcal{R}' be obtained from \mathcal{R} by removing all the rectangles corresponding to the vertices in S . Note that H is a triangle-free graph and \mathcal{R}' is crossing-free.

Claim. *There is no rectangle in \mathcal{R}' which is contained in some other rectangle of \mathcal{R}' .*

Assume for the sake of contradiction that for vertices $u, v \in V(H)$ we have $r_u \subseteq r_v$ in \mathcal{R}' . Since u is a vertex of H , we know that u must have degree at least two in G . Let w be a neighbour of u other than v in G . Then in \mathcal{R} , we have $r_w \cap r_u \neq \emptyset$. Since $r_u \subseteq r_v$, this implies that $r_w \cap r_v \neq \emptyset$. But now u, v, w form a triangle in G , contradicting the fact that G is triangle-free. This proves the claim.

Since H is triangle-free, we have that in H , for any vertex $u \in V(H)$ and any two vertices in $v, w \in N(u)$, $r_v \cap r_w = \emptyset$. This, together with the fact that \mathcal{R}' is crossing free, implies that the region $r_u \setminus \bigcup_{w \in N(u)} r_w$ is arc-connected and non-empty. (To see this, observe that if $r_u \setminus \bigcup_{w \in N(u)} r_w$ is non-empty, but is not arc-connected, then there exists two points $x, y \in r_u$ and a simple curve $\mathbf{c} \subseteq \bigcup_{w \in N(u)} r_w$ such that x and y are in different arc-connected components of $r_u \setminus \mathbf{c}$. Since for any two vertices in $v, w \in N(u)$, we have $r_v \cap r_w = \emptyset$, we know that there exists some $z \in N(u)$ such that $\mathbf{c} \subseteq r_z$. But this means that x and y are in different arc-connected components of $r_u \setminus r_z$, contradicting the fact that \mathcal{R}' is crossing-free. If $r_u \setminus \bigcup_{w \in N(u)} r_w$ is empty, then $r_u \subseteq \bigcup_{w \in N(u)} r_w$. Again, since for any two vertices in $v, w \in N(u)$, we have $r_v \cap r_w = \emptyset$, it must be the case that there exists some $z \in N(u)$ such that $r_u \subseteq r_z$. But this contradicts the claim proved above.) Now choose for every vertex $u \in V(H)$, a point p_u in $r_u \setminus \bigcup_{w \in N(u)} r_w$. In other words, p_u is a point in r_u which is not contained in any rectangle other than r_u . For every edge $uv \in E(H)$, choose a point p_{uv} that is contained in the rectangular region $r_u \cap r_v$. Further, for each edge $uv \in E(H)$, choose a simple curve $\mathbf{s}_{\mathbf{u}, \mathbf{v}}$ between p_u and p_{uv} that is completely contained in r_u and a simple curve $\mathbf{s}_{\mathbf{v}, \mathbf{u}}$ between p_v and p_{uv} that is completely contained in r_v such that for any curve in the collection $\{\mathbf{s}_{\mathbf{u}, \mathbf{v}}, \mathbf{s}_{\mathbf{v}, \mathbf{u}}\}_{uv \in E(H)}$, none of its interior points are contained in any other curve in the collection. Now the set of simple curves $\{\mathbf{s}_{\mathbf{u}, \mathbf{v}} \cup \mathbf{s}_{\mathbf{v}, \mathbf{u}}\}_{uv \in E(H)}$ corresponds to the edges of H and gives a planar embedding of H (please see Figure 2 for an example). Hence, G is a planar graph. \square

4 Bounds on the stab number for some graph classes

In this section, we study the stab number of some subclasses of rectangle intersection graphs. We show a lower bound on $stab(G)$ for any rectangle intersection graph G , which is used to derive an asymptotically tight lower bound for the stab number of grids. We also derive upper bounds on $estab(G)$ when G is a split graph or a block graph.

4.1 Lower bounds

It is clear that given a k -stabbed rectangle intersection representation of a graph G , a set of $\omega(G)$ colours can be used to properly colour the vertices whose rectangles have a common stab (since the subgraph induced in G by these vertices is an interval graph). This means that if G is exactly stabbable, we can use two sets of $\omega(G)$ colours each to colour the vertices on alternate stab lines of a k -exactly stabbed representation of G (for some k) to obtain a proper colouring of G . Thus, if G is an exactly stabbable rectangle intersection graph, then $\chi(G) \leq 2\omega(G)$. For general rectangle intersection graphs, we can adapt the same colouring strategy to get the following observation.

Observation A. For a rectangle intersection graph G , we have $\chi(G) \leq \text{stab}(G) \cdot \omega(G)$, or in other words, $\text{stab}(G) \geq \frac{\chi(G)}{\omega(G)}$.

Remarks. Even though for a 3-SRIG G , the above observation gives only $\chi(G) \leq 3\omega(G)$, we can use Theorem 2 to infer that G is actually 3-ESRIG, and therefore $\chi(G) \leq 2\omega(G)$. Note that for any rectangle intersection graph G , $\chi(G) \leq 8\omega(G)^2$ [4]. The question of whether there exists an upper bound on $\chi(G)$ for rectangle intersection graphs that is linear in $\omega(G)$ is open.

We now strengthen the above observation and show that the $\chi(G)$ in the lower bound can be replaced by $\text{pw}(G) + 1$, where $\text{pw}(G)$ is the “pathwidth” of G . A *path decomposition* of a graph G is a collection X_1, X_2, \dots, X_t of subsets of $V(G)$, where t is some positive integer, such that for each edge $uv \in E(G)$, there exists $i \in \{1, 2, \dots, t\}$ such that $u, v \in X_i$ and for each vertex $u \in V(G)$, if $u \in X_i \cap X_j$, where $i < j$, then $u \in X_k$ for $i \leq k \leq j$. The *width* of a path decomposition X_1, X_2, \dots, X_t of G is defined to be $\max_{1 \leq i \leq t} \{|X_i|\} - 1$. The *pathwidth* of a graph G , denoted by $\text{pw}(G)$, is the width of a path decomposition of G of minimum width.

We adapt a proof by Suderman [22] to show that if a graph G is k -SRIG then G has pathwidth at most $k \cdot \omega(G) - 1$.

Theorem 6. Let G be a rectangle intersection graph. Then $\text{pw}(G) \leq \omega(G) \cdot \text{stab}(G) - 1$, or in other words, $\text{stab}(G) \geq \frac{\text{pw}(G)+1}{\omega(G)}$.

Proof. Let G be a rectangle intersection graph with $\text{stab}(G) = k$. We shall show that $\text{pw}(G) \leq k \cdot \omega(G) - 1$. Let \mathcal{R} be a k -stabbed rectangle intersection representation of G . Let $V(G) = \{u_1, u_2, \dots, u_n\}$ such that $x_{u_1}^+ \leq x_{u_2}^+ \leq \dots \leq x_{u_n}^+$. For $i \in \{1, 2, \dots, n\}$, let us define the subset $X_i = \{v \in V(G) : x_{u_i}^+ \in \text{span}(v)\}$. We claim that X_1, X_2, \dots, X_n is a path decomposition of G . To see this, note that for any edge $u_i u_j \in E(G)$, where $i < j$, $u_i, u_j \in X_i$. Also, if some vertex $v \in X_i \cap X_j$, where $i < j$, then $\text{span}(v)$ contains both $x_{u_i}^+$ and $x_{u_j}^+$, implying that it also contains $x_{u_k}^+$, for $i \leq k \leq j$. Therefore, $v \in X_k$, for $i \leq k \leq j$. To complete the proof, we only need to show that $\max_{1 \leq i \leq n} \{|X_i|\} \leq k \cdot \omega(G)$. Suppose that for some $i \in \{1, 2, \dots, n\}$, there exists $S \subseteq X_i$ such that $|S| \geq \omega(G) + 1$ and all the vertices of S have a common stab. Since $x_{u_i}^+ \in \bigcap_{u \in S} \text{span}(u)$ and the rectangles corresponding to the vertices of S all intersect a common stab line, we have that the vertices of S form a clique in G , which is a contradiction to the fact that $\omega(G)$ is the clique number of G . Therefore, for any $i \in \{1, 2, \dots, n\}$, there exists at most $\omega(G)$ vertices in X_i that have a common stab. Since there are only k stab lines in \mathcal{R} , we now have that $|X_i| \leq k \cdot \omega(G)$ for each $i \in \{1, 2, \dots, n\}$. \square

The (h, w) -grid is the undirected graph G with $V(G) = \{(x, y) : x, y \in \mathbb{Z}, 1 \leq x \leq h, 1 \leq y \leq w\}$ and $E(G) = \{(u, v)(x, y) : |u - x| + |v - y| = 1\}$.

Corollary 7. Let G be the (h, w) -grid. Then $\frac{1}{2}(\min\{h, w\} + 1) \leq \text{stab}(G) \leq \text{estab}(G) \leq \min\{h, w\}$.

Proof. It is clear that $\omega(G) \leq 2$ and from a result of [15] we know that the pathwidth of the (h, w) -grid is $\min\{h, w\}$. From these facts and Theorem 6, we can infer that, $\frac{1}{2}(\min\{h, w\} + 1) \leq \text{stab}(G)$. It is easy to see that the (h, w) -grid graph has a $\min\{h, w\}$ -exactly stabbed rectangle intersection representation as shown in Figure 3, and therefore $\text{estab}(G) \leq \min\{h, w\}$. The statement of the corollary now follows from the fact that $\text{stab}(G) \leq \text{estab}(G)$. \square

The above corollary shows that $\text{stab}(\text{GRIDS}, n) = \Theta(\sqrt{n})$. This also shows that there are triangle-free rectangle intersection graphs on n vertices whose stab number can be $\Omega(\sqrt{n})$. Moreover, these triangle-free rectangle intersection graphs are exactly stabbable.

4.2 Split graphs

A split graph is a graph whose vertex set can be partitioned into a clique and an independent set. It is known that split graphs can have arbitrarily high boxicity [13]. So it is natural to ask whether the split graphs within rectangle intersection graphs are all exactly stabbable rectangle intersection graphs. We show that any split graph with boxicity at most 2 is 3-ESRIG (Theorem 8) and that there exists a split graph with boxicity at most 2 which is not 2-ESRIG (Theorem 9). From Theorem 2, it then follows that the stab number and exact stab number are equal for any split graph that has boxicity at most 2. Adiga et al. showed that deciding whether a split graph has boxicity at most 3 is NP-complete [1]. But as far as we know, the problem of deciding whether the boxicity of a split graph is at most 2 is not known to be polynomial-time

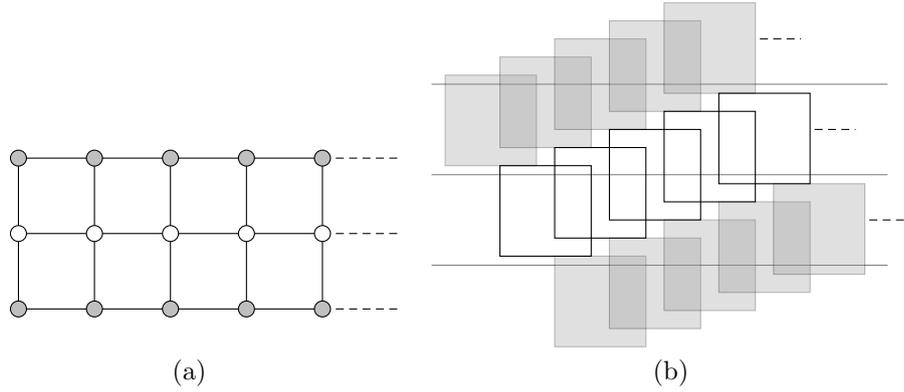


Figure 3: Illustration of $\min\{h, w\}$ -exactly stabbed rectangle intersection representation of the (h, w) -grid: (a) The $(3, n)$ -grid with $n \geq 3$; (b) a 3-exactly stabbed rectangle intersection representation of the $(3, n)$ -grid.

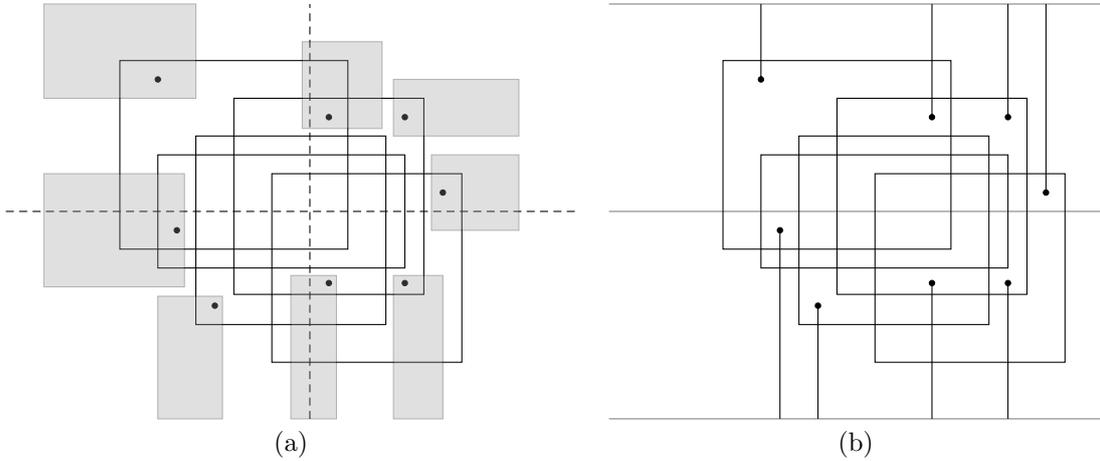


Figure 4: Representation of split graphs with boxicity at most 2. (a) The shaded rectangles represent vertices of the independent set of the split graph and the dots indicate the points p_u , for each vertex u in the independent set. (b) The 3-ESRIG representation derived from the rectangle intersection representation given in (a).

solvable or NP-complete. By our observations below, it follows that this problem is equivalent to deciding whether a given split graph is 3-ESRIG (or equivalently, 3-SRIG).

Theorem 8. *A split graph G is a rectangle intersection graph if and only if G is a 3-ESRIG.*

Proof. As G is a split graph, there exists a partition of $V(G)$ into sets C and I such that C is a clique and I is an independent set. If G is a 3-ESRIG then G is a rectangle intersection graph. Now let G be a split graph having a rectangle intersection representation \mathcal{R} such that for any two vertices $u, v \in V(G)$, $\{x_u^-, x_u^+, y_u^-, y_u^+\} \cap \{x_v^-, x_v^+, y_v^-, y_v^+\} = \emptyset$ (note that such a rectangle intersection representation exists for any rectangle intersection graph). We shall assume without loss of generality that in this representation, the origin is contained in $\bigcap_{v \in C} r_v$. For every vertex $u \in I$, define the region $A_u = \bigcap_{v \in N[u]} r_v$. It is easy to see that $A_u \subseteq r_u$. It follows that for vertices $u, v \in V(G)$ such that $u \in I$ and $v \notin N[u]$, $A_u \cap r_v = \emptyset$. Also, A_u is a rectangle (by the Helly property of rectangles) with non-zero height and width. This means that we can choose a point p_u in A_u that is not on the X -axis for each vertex $u \in I$, while satisfying the additional property that no two points in $\{p_u\}_{u \in I}$ have the same x -coordinate. Consider $u \in I$. Since the degenerate rectangle given by the point p_u intersects all the rectangles in $\{r_v\}_{v \in N(u)}$, we can replace the rectangle r_u with the degenerate rectangle given by the point p_u to obtain a new rectangle intersection representation of G . Let \mathcal{R}' be the rectangle intersection representation of G obtained in this fashion, i.e. $\mathcal{R}' = (\mathcal{R} \setminus \{r_u\}_{u \in I}) \cup \{p_u\}_{u \in I}$ (see Figure 4(a)).

Let I^+ (respectively I^-) be the set of vertices $\{u \in I : p_u \text{ is above (respectively, below) the } X\text{-axis}\}$. Let $y_{max} = \max\{y_v^+ : v \in C\}$ and $y_{min} = \min\{y_v^- : v \in C\}$. For each vertex $u \in I^+$, we define s_u to be the

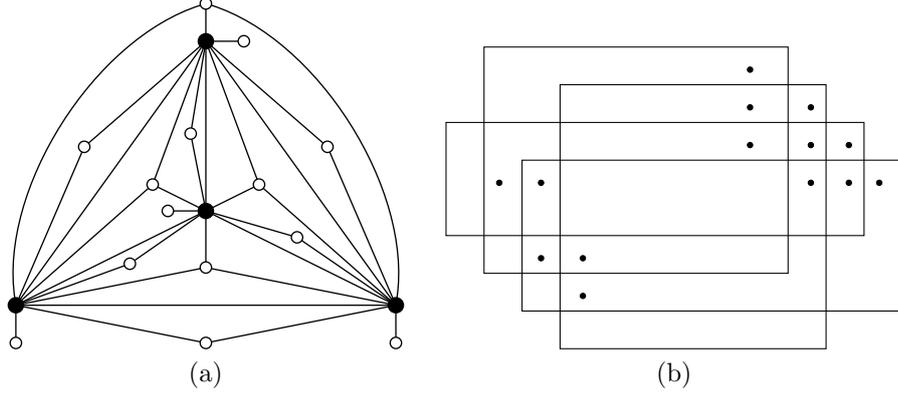


Figure 5: (a) A planar split graph which is 3-ESRIG but not 2-ESRIG. The clique vertices are coloured black and the remaining vertices are independent vertices. (b) A rectangle intersection representation of the graph shown in (a). The vertices corresponding to the independent set are represented as points.

degenerate rectangle given by the vertical line segment whose bottom end point is p_u and top end point has y -coordinate $y_{max} + 1$. Similarly, for each vertex $u \in I^-$, we define s_u to be the degenerate rectangle given by the vertical line segment whose top end point is p_u and bottom end point has y -coordinate $y_{min} - 1$. As each rectangle in \mathcal{R}' corresponding to a vertex in C contains the origin, we have that for any $u, v \in V(G)$ such that $u \in I$ and $v \in C$, the rectangle r_v intersects s_u if and only if r_v contains p_u . Therefore, the collection of rectangles given by $(\mathcal{R}' \setminus \{p_u\}_{u \in I}) \cup \{s_u\}_{u \in I}$ is a rectangle intersection representation of G . It is easy to see that this rectangle intersection representation, together with the horizontal lines $y = y_{min} - 1$, $y = 0$, and $y = y_{max} + 1$, forms a 3-ESRIG representation of G (see Figure 4(b)). \square

Theorem 9. *There is a split graph G which is a rectangle intersection graph but not a 2-ESRIG.*

Proof. Let G be the split graph whose vertex set is partitioned into a clique C on four vertices and an independent set I of 14 vertices, and whose edges are defined as follows. Let \mathcal{X} be the set of all subsets X of C with $1 \leq |X| \leq 3$. For every $X \in \mathcal{X}$, there is exactly one vertex $u_X \in I$ such that $N(u_X) = X$. See Figure 5(a) for a drawing of the graph G . Clearly, G has a rectangle intersection representation as shown in Figure 5(b).

Now assume for the sake of contradiction that G has a 2-ESRIG representation \mathcal{R} . We can assume that the stab lines are $y = 0$ and $y = 1$. We shall further assume that all the rectangles are contained in the strip of the plane between the two stab lines, i.e. for each $v \in V(G)$, we have $y_v^- \geq 0$ and $y_v^+ \leq 1$ (it is easy to see that every 2-ESRIG representation can be converted to such a 2-ESRIG representation by “trimming” the parts of the rectangles that lie above the top stab line and below the bottom stab line).

Observe that for each $X \in \mathcal{X}$, the rectangle r_{u_X} intersects all the rectangles in $\{r_v\}_{v \in X}$ and is disjoint from each rectangle in $\{r_v\}_{v \in C \setminus X}$. Now choose a point $p_X \in r_{u_X} \cap \bigcap_{v \in X} r_v$. Clearly, $p_X \in \bigcap_{v \in X} r_v$ and $p_X \notin \bigcup_{v \in C \setminus X} r_v$.

Let $a, b \in C$ (not necessarily distinct) such that $x_a^- = \max\{x_v^-\}_{v \in C}$ and $x_b^+ = \min\{x_v^+\}_{v \in C}$. Let c, d be two distinct vertices in $C \setminus \{a, b\}$. By our choice of a and b , we have $[x_a^-, x_b^+] \subseteq span(c)$ and $[x_a^-, x_b^+] \subseteq span(d)$, or in other words $[x_a^-, x_b^+] \subseteq span(c) \cap span(d)$.

Claim. *The vertices c and d have a common stab.*

Suppose for the sake of contradiction that c and d do not have a common stab. Then, since $[x_a^-, x_b^+] \subseteq span(c) \cap span(d)$ and $r_c \cap r_d \neq \emptyset$, it follows that the rectangle $[x_a^-, x_b^+] \times [0, 1] \subseteq r_c \cup r_d$. We thus have $r_a \cap r_b \subseteq [x_a^-, x_b^+] \times [0, 1] \subseteq r_c \cup r_d$. But this contradicts the fact that there exists a point $p_{\{a,b\}}$ such that $p_{\{a,b\}} \in r_a \cap r_b$ and $p_{\{a,b\}} \notin r_c \cup r_d$. This proves the claim.

By the above claim, we shall assume without loss of generality that c and d are on the stab line $y = 0$ and that $y_c^+ \leq y_d^+$. This implies that $[x_a^-, x_b^+] \times [0, y_c^+] \subseteq [x_a^-, x_b^+] \times [0, y_d^+] \subseteq r_d$ (recall that $[x_a^-, x_b^+] \subseteq span(d)$). Note that $r_a \cap r_b \cap r_c \subseteq [x_a^-, x_b^+] \times [0, y_c^+]$, implying that $r_a \cap r_b \cap r_c \subseteq r_d$. But this contradicts the fact that there exists a point $p_{\{a,b,c\}}$ such that $p_{\{a,b,c\}} \in r_a \cap r_b \cap r_c$ and $p_{\{a,b,c\}} \notin r_d$. \square

4.3 Block graphs

A graph G is a block graph if every block (i.e 2-connected component) of G is a clique. Note that all trees are block graphs. It is not hard to see that all trees, and indeed all block graphs, are rectangle intersection graphs. We show that all block graphs are exactly stabble rectangle intersection graphs and give an upper bound of $\lceil \log m \rceil$ for the exact stab number of block graphs with m blocks, where $m \geq 2$. Note that this implies an upper bound of $\lceil \log n \rceil$ for the exact stab number of trees on n vertices. We shall show in Section 6.1 that this bound is asymptotically tight, by constructing trees whose stab number is $\Omega(\log n)$.

Let G be a block graph. Given a k -exactly stabbed rectangle intersection representation \mathcal{R} of G , we say that a set of vertices $S \subseteq B$, where B is a block in G , is *accessible* if all vertices in S are on the bottom stab line of \mathcal{R} and for any vertex $v \notin S$ either v is not on the bottom stab line or $x_u^- < x_v^-$ for every vertex $u \in S$.

Theorem 10. *For any block graph G with m blocks, $\text{estab}(G) \leq \max\{1, \lceil \log m \rceil\}$.*

Proof. Note that we only need the statement of the theorem to be proved for connected graphs. In fact, we shall prove the following stronger claim for connected graphs.

Claim. *Let G be any connected block graph with m blocks and let $k = \max\{1, \lceil \log m \rceil\}$. Then for any block B of G , any subset S of B , any $a, b \in \mathbb{R}$ such that $a < b$, and any $h \in \mathbb{R}$ such that $0 \leq h < 1$, there is a k -exactly stabbed rectangle intersection representation $\mathcal{R}(S, a, b, h)$ of G with stab lines $y = 0, y = 1, y = 2, \dots, y = k - 1$ such that:*

- S is accessible,
- for every vertex $u \in V(G)$, $\text{span}(u) \subseteq (a, b)$,
- for every vertex $u \in V(G)$ that is on the bottom stab line, we have $y_u^+ > h$, and
- for every vertex $u \in V(G)$ that is not on the bottom stab line, we have $y_u^- > h$.

Proof. We prove the claim by induction on m . When $m \leq 2$, G is an interval graph. It is not hard to see that the statement of the claim is true in this case. From here onwards, we shall assume that $m \geq 3$, and that the statement of the claim is true when the number of blocks in the graph is lesser than m .

Let \mathcal{H} be the set of components of $G - B$. It is easy to see that each graph $H \in \mathcal{H}$ is a block graph and at most one of them can have greater than $\frac{m}{2}$ blocks. We shall denote the graph in \mathcal{H} that has greater than $\frac{m}{2}$ blocks, if it exists, as H^* . For a vertex $u \in B$, let $\mathcal{H}_u = \{H \in \mathcal{H} : N(u) \cap V(H) \neq \emptyset\}$. Note that for $u, v \in B$ such that $u \neq v$, $\mathcal{H}_u \cap \mathcal{H}_v = \emptyset$. Also, since G is connected, $\{\mathcal{H}_u\}_{u \in B}$ is a partition of \mathcal{H} . If H^* exists, let $u^* \in B$ be the vertex such that $H^* \in \mathcal{H}_{u^*}$.

Let $\mathcal{I}_B = \{[c_u, d_u]\}_{u \in B}$ be an interval representation of $G[B]$ (which is a complete graph) such that all endpoints of intervals are distinct, $[c_u, d_u] \subseteq (a, \frac{a+b}{2})$ for any $u \in V(G)$, and for any $u \in S$ and $v \in B \setminus S$, we have $c_u < c_v$. Let $B = \{u_1, u_2, \dots, u_{|B|}\}$, where $c_{u_1} < c_{u_2} < \dots < c_{u_{|B|}}$. We shall define $c_{u_{|B|+1}} = d_{u_{|B|}}$ (this shall be used later on). Choose $|B|$ real numbers $h_1, h_2, \dots, h_{|B|}$ such that $h < h_1 < h_2 < \dots < h_{|B|} < 1$. We define r_u for every vertex $u \in B$ other than u^* as follows: $r_u = [c_u, d_u] \times [0, h_i]$, where $i \in \{1, 2, \dots, |B|\}$ is such that $u = u_i$. We shall show how to define r_{u^*} , in case u^* exists, later.

For each $i \in \{1, 2, \dots, |B|\}$, let $\mathcal{H}'_i = \mathcal{H}_{u_i} \setminus \{H^*\}$, if H^* exists, and $\mathcal{H}'_i = \mathcal{H}_{u_i}$ otherwise. Let $t_i = |\mathcal{H}'_i|$. For each $i \in \{1, 2, \dots, |B|\}$, let $\mathcal{H}'_i = \{H_{i,1}, H_{i,2}, \dots, H_{i,t_i}\}$ and for each $j \in \{1, 2, \dots, t_i\}$, let $S_{i,j} = N(u_i) \cap V(H_{i,j})$ (which is nonempty by the definition of \mathcal{H}_{u_i}). For each $i \in \{1, 2, \dots, |B|\}$, choose $t_i + 1$ real numbers $q_{i,1} < q_{i,2} < \dots < q_{i,t_i+1} < c_{u_{i+1}}$ (recall that $c_{u_{|B|+1}} = d_{u_{|B|}}$). Now consider any $i \in \{1, 2, \dots, |B|\}$ and any $j \in \{1, 2, \dots, t_i\}$. As the number of blocks in $H_{i,j}$ is at most $\frac{m}{2}$, we can apply the induction hypothesis on $H_{i,j}$ to conclude that there is a $\max\{1, \lceil \log m \rceil - 1\}$ -exactly stabbed rectangle intersection representation $\mathcal{R}'_{i,j} = \mathcal{R}(S_{i,j}, q_{i,j}, q_{i,j+1}, 0)$ of $H_{i,j}$. Since $m \geq 3$, we know that $k \geq 2$ and that $\max\{1, \lceil \log m \rceil - 1\} = k - 1$. Thus, $\mathcal{R}'_{i,j}$ uses the stab lines $y = 0, y = 1, \dots, y = k - 2$. For each vertex $v \in V(H_{i,j})$, let $r'_v = [x_v^-, x_v^+] \times [y_v^-, y_v^+]$ be the rectangle corresponding to v in $\mathcal{R}'_{i,j}$. We now define r_v for each vertex $v \in V(H_{i,j})$ for all $i \in \{1, 2, \dots, |B|\}$ and $j \in \{1, 2, \dots, t_i\}$ as follows. If $v \in S_{i,j}$, then $r_v = [x_v^-, x_v^+] \times [h_i, y_v^+ + 1]$. If $v \in V(H_{i,j}) \setminus S_{i,j}$ and v is on the bottom stab line of $\mathcal{R}'_{i,j}$, then we define $r_v = [x_v^-, x_v^+] \times [1, y_v^+ + 1]$. Lastly, if $v \in V(H_{i,j}) \setminus S_{i,j}$, but v is not on the bottom stab line of $\mathcal{R}'_{i,j}$, then we define $r_v = [x_v^-, x_v^+] \times [y_v^- + 1, y_v^+ + 1]$.

(*) For an integer $i \in \{1, 2, \dots, |B|\}$ and a vertex v of some $H \in \mathcal{H}'_i$, we have $[x_v^-, x_v^+] \subset [c_{u_i}, c_{u_{i+1}}]$.

(+) For an integer $i \in \{1, 2, \dots, |B|\}$ and for any two distinct integers $j, k \in \{1, 2, \dots, t_i\}$ let u be a vertex in $V(H_{i,j})$ and v be a vertex in $V(H_{i,k})$. Then $r_u \cap r_v = \emptyset$ (since $[x_u^-, x_u^+], [x_v^-, x_v^+]$ belong respectively to the intervals $(q_{i,j}, q_{i,j+1}), (q_{i,k}, q_{i,k+1})$ which are disjoint).

(++) Let i, j be two distinct integers in $\{1, 2, \dots, |B|\}$. Let u be a vertex in some graph in \mathcal{H}'_i and v be a vertex in some graph in \mathcal{H}'_j . Then $r_u \cap r_v = \emptyset$ (since $[x_u^-, x_u^+]$, $[x_v^-, x_v^+]$ belong respectively to the intervals $(c_{u_i}, c_{u_{i+1}})$, $(c_{u_j}, c_{u_{j+1}})$ which are disjoint).

We now define a rectangle r_v for each vertex $v \in V(H^*)$ and the rectangle r_{u^*} for u^* , in case H^* exists. Let $S^* = N[u^*] \cap V(H^*)$. Since H^* contains less than m blocks, and recalling that $k = \max\{1, \lceil \log m \rceil\}$, we have by the induction hypothesis that H^* has a k -exactly stabbed rectangle intersection representation $\mathcal{R}^* = \mathcal{R}(S^*, \frac{a+b}{2}, b, h_{|B|})$ that uses the stab lines $y = 0, y = 1, \dots, y = k - 1$. Let the rectangle in \mathcal{R}^* corresponding to a vertex $v \in V(H^*)$ be denoted by $r_v^* = [x_v^{*-}, x_v^{*+}] \times [y_v^{*-}, y_v^{*+}]$. We define $r_v = r_v^*$ for every vertex $v \in V(H^*)$. We now let $r_{u^*} = [c_{u^*}, \max\{x_v^{*-} : v \in S^*\}] \times [0, h_i]$, where $i \in \{1, 2, \dots, |B|\}$ is such that $u^* = u_i$.

(+++) Let i be any integer in $\{1, 2, \dots, |B|\}$. Let u be a vertex of some graph in \mathcal{H}'_i and v be a vertex of H^* . Then $r_u \cap r_v = \emptyset$ (since $[x_u^-, x_u^+]$, $[x_v^-, x_v^+]$ belong respectively to the intervals $(a, \frac{a+b}{2})$, $(\frac{a+b}{2}, b)$ which are disjoint).

We now verify that $\mathcal{R} = \{r_u\}_{u \in V(G)}$ forms a $\lceil \log m \rceil$ -exactly stabbed rectangle intersection representation of G that satisfies all the requirements to be $\mathcal{R}(S, a, b, h)$. For a vertex $u \in V(G)$, let $x_u^-, x_u^+, y_u^-, y_u^+$ be such that $r_u = [x_u^-, x_u^+] \times [y_u^-, y_u^+]$.

From the construction of \mathcal{R} , it is clear that all the vertices in B , and therefore all the vertices in S , are on the bottom stab line. It is also easy to see that the only vertices on the bottom stab line other than the vertices in B are some vertices in $V(H^*)$. For any vertex $u \in B$ and $v \in V(H^*)$, we have $x_u^- < \frac{a+b}{2} < x_v^-$. Note that for any vertex $u \in B$, we have $x_u^- = c_u$. Therefore, for vertices $u, v \in B$ such that $u \in S$ and $v \in B \setminus S$, we have $x_u^- < x_v^-$ (recall that $c_u < c_v$ in this case). From this, we can infer that S is accessible in \mathcal{R} .

It is clear that for each $u \in V(G)$, $r_u \subset (a, b)$. Now consider any vertex v that is on the bottom stab line in \mathcal{R} . As explained before, v is either in B or in $V(H^*)$. If $v \in B$, then $v = u_i$ for some $i \in \{1, 2, \dots, |B|\}$, and $y_v^+ = h_i > h$. On the other hand, if $v \in V(H^*)$, then $r_v = r_v^*$, the rectangle corresponding to v in \mathcal{R}^* . Since $\mathcal{R}^* = \mathcal{R}(S^*, \frac{a+b}{2}, b, h_{|B|})$, we know that $y_v^{*+} > h_{|B|} > h$, and therefore we have $y_v^+ > h$. Therefore, for every vertex $v \in V(G)$ that is on the bottom stab line, we have $y_v^+ > h$. Now consider a vertex $v \in V(G)$ that is not on the bottom stab line in \mathcal{R} . It is clear that $v \notin B$. If $v \in V(H)$, where $H \neq H^*$ and $H \in \mathcal{H}_{u_i}$, for some $i \in \{1, 2, \dots, |B|\}$, then by our construction, $y_v^- \geq h_i > h$. If $v \in V(H^*)$, then we know that since v is not on the bottom stab line of \mathcal{R} , it is also not on the bottom stab line of \mathcal{R}^* . Since $\mathcal{R}^* = (S^*, \frac{a+b}{2}, b, h_{|B|})$, this means that $y_v^{*-} > h_{|B|} > h$. As $y_v^- = y_v^{*-}$, we now have $y_v^- > h$. This shows that \mathcal{R} satisfies the four conditions to be chosen as $\mathcal{R}(S, a, b, h)$.

As it can be easily verified that each rectangle in \mathcal{R} is intersected by exactly one of the stab lines $y = 0, y = 1, \dots, y = k - 1$, it only remains to be shown that \mathcal{R} is a rectangle intersection representation of G . Even though this is more or less clear from the construction, we give a proof for the sake of completeness. Consider $u, v \in V(G)$. We shall show that $uv \in E(G)$ if and only if $r_u \cap r_v \neq \emptyset$.

- (i) First, let us consider the case when $u, v \in V(H^*)$. Since we have $r_u = r_u^*$ and $r_v = r_v^*$, $r_u \cap r_v \neq \emptyset \Leftrightarrow r_u^* \cap r_v^* \neq \emptyset$. Since \mathcal{R}^* is a valid representation of H^* , we have $r_u \cap r_v \neq \emptyset \Leftrightarrow uv \in E(H^*) \Leftrightarrow uv \in E(G)$.
- (ii) Next, let us consider the case when $u \in B$ and $v \in V(H^*)$. If $u \neq u^*$ then $H^* \notin \mathcal{H}_u$ and thus $uv \notin E(G)$. Also, we have $[x_u^-, x_u^+] \subseteq (a, \frac{a+b}{2})$ (since $u \neq u^*$) and $[x_v^-, x_v^+] \subseteq (\frac{a+b}{2}, b)$. Hence $r_u \cap r_v = \emptyset$. Now assume that $u = u^* = u_i$ (for some $i \in \{1, 2, \dots, |B|\}$). Recall that $S^* = N(u) \cap V(H^*)$. Suppose first that $v \in S^*$. Then $uv \in E(G)$. Now from the definition of \mathcal{R}^* and $r_{u^*} = r_u$, we have that both r_v and r_u intersect the stab line $y = 0$, $x_v^- = x_v^{*-}$ and that $x_u^+ = \max\{x_w^{*-} : w \in S^*\}$. Combining these, we have $x_v^- \leq x_u^+$. This gives us $x_u^- < \frac{a+b}{2} < x_v^- \leq x_u^+$, implying that $r_u \cap r_v \neq \emptyset$. Now assume that $v \notin S^*$, from which it follows that $uv \notin E(G)$. If $r_v = r_v^*$ intersects the stab line $y = 0$, then since $\mathcal{R}^* = \mathcal{R}(S^*, \frac{a+b}{2}, b, h_{|B|})$, we have that $\max\{x_w^{*-} : w \in S^*\} < x_v^-$, implying that $x_u^+ < x_v^-$ (recall that $u = u^*$). Therefore, $r_u \cap r_v = \emptyset$. The only remaining case is if r_v does not intersect the bottom stab line. Then, since $r_v = r_v^*$ and $\mathcal{R}^* = \mathcal{R}(S^*, \frac{a+b}{2}, b, h_{|B|})$, we have $y_v^{*-} > h_{|B|} \geq h_i = y_u^+$, where $i \in \{1, 2, \dots, |B|\}$ is such that $u = u^* = u_i$. Therefore $r_u \cap r_v = \emptyset$.
- (iii) Next, let u be a vertex of some graph in \mathcal{H}'_i for some $i \in \{1, 2, \dots, |B|\}$ and v be a vertex in H^* . Then clearly $uv \notin E(G)$ and by (+++) we have that $r_u \cap r_v = \emptyset$.
- (iv) Next, suppose that $u, v \in B$. Note that for every vertex $u \in B \setminus \{u^*\}$, we have $x_u^- = c_u$ and $x_u^+ = d_u$. Since we have $x_{u^*}^- = c_{u^*}$ and $x_{u^*}^+ = \max\{x_v^{*-} : v \in S^*\} > \frac{a+b}{2} > d_{u^*}$, we can conclude that for every vertex $u \in B$, $[c_u, d_u] \subseteq [x_u^-, x_u^+]$. As $G[B]$ is a clique, we have $uv \in E(G)$. By our construction, both

u and v are on the bottom stab line, and since $[c_u, d_u] \cap [c_v, d_v] \neq \emptyset$, we have $[x_u^-, x_u^+] \cap [x_v^-, x_v^+] \neq \emptyset$. We thus have $r_u \cap r_v \neq \emptyset$.

- (v) Next, let us consider the case when $u \in B$ and v is a vertex of some graph in \mathcal{H}'_i . First, let us consider the case when $u = u_i$. Let v be a vertex in $H_{i,j}$ for some $j \in \{1, 2, \dots, t_i\}$. If $uv \in E(G)$, then $v \in S_{i,j}$ (recall that $S_{i,j} = N(u_i) \cap V(H_{i,j})$). In this case, we have by (*) that $[x_v^-, x_v^+] \subset [c_{u_i}, c_{u_{i+1}}]$ and thus $[x_v^-, x_v^+] \subset [x_u^-, x_u^+]$. Furthermore, we have by construction that $y_v^- = h_i = y_u^+$, allowing us to conclude that $r_u \cap r_v \neq \emptyset$. If $uv \notin E(G)$, then $v \notin S_{i,j}$, and therefore by construction, we know that $y_v^- \geq 1$ whereas $y_u^+ = h_i < 1$. Therefore the two rectangles r_u and r_v do not intersect. Now let us consider the case when $u \neq u_i$. In this case, we have $uv \notin E(G)$. Let $u = u_j$ and assume $j < i$. Then from our construction, we have that $y_v^- \geq h_i > h_j = y_u^+$ and therefore $r_u \cap r_v = \emptyset$. Now assume $j > i$. Then from (*), we know that $x_v^+ < c_{u_{i+1}} \leq x_u^-$ and therefore conclude that $r_u \cap r_v = \emptyset$.
- (vi) Next, let i, j be two distinct integers in $\{1, 2, \dots, |B|\}$. Let u be a vertex of some graph in \mathcal{H}'_i and v be a vertex of some graph in \mathcal{H}'_j . Then clearly $uv \notin E(G)$ and by (++) we have that $r_u \cap r_v = \emptyset$.
- (vii) Next, let i be an integer in $\{1, 2, \dots, |B|\}$ and j, k be two distinct integers in $\{1, 2, \dots, t_i\}$. Let u be a vertex in $H_{i,j}$ and v be a vertex of $H_{i,k}$. Then clearly $uv \notin E(G)$ and by (+) we have that $r_u \cap r_v = \emptyset$.
- (viii) Finally, let i be an integer in $\{1, 2, \dots, |B|\}$ and j be an integer in $\{1, 2, \dots, t_i\}$. Let $u, v \in V(H_{i,j})$. Let $\{r'_w\}_{w \in V(H_{i,j})} = \mathcal{R}'_{i,j}$. Also, let $r'_w = [x'_w, x'_w] \times [y'_w, y'_w]$. Then we have $[x_u^-, x_u^+] = [x'_u, x'_u]$, $[x_v^-, x_v^+] = [x'_v, x'_v]$, $y_u^+ = y'_u + 1$, $y_v^+ = y'_v + 1$, $y_u^- \in \{1, h_i, y'_u + 1\}$, and $y_v^- \in \{1, h_i, y'_v + 1\}$. Let us assume without loss of generality that $y_u^- \leq y_v^-$. We now have $[y_u^-, y_u^+] \cap [y_v^-, y_v^+] = \emptyset \Leftrightarrow y_u^+ < y_v^- \Leftrightarrow y'_u + 1 < y_v^-$. Recall that $y_v^- \in \{1, h_i, y'_v + 1\}$. If $y'_u + 1 < y_v^-$ and $y_v^- \in \{1, h_i\}$, then we have $y'_u + 1 < 0$, which is not possible (as no stab line of $\mathcal{R}'_{i,j}$ could have intersected r'_u). We can thus continue the derivation as $y'_u + 1 < y_v^- \Leftrightarrow y'_u + 1 < y'_v + 1 \Leftrightarrow y'_u < y'_v \Leftrightarrow [y'_u, y'_u] \cap [y'_v, y'_v] = \emptyset$. Since we have $[x_u^-, x_u^+] = [x'_u, x'_u]$ and $[x_v^-, x_v^+] = [x'_v, x'_v]$, it is clear that $[x_u^-, x_u^+] \cap [x_v^-, x_v^+] = \emptyset \Leftrightarrow [x'_u, x'_u] \cap [x'_v, x'_v] = \emptyset$. We can thus conclude that $r_u \cap r_v = \emptyset \Leftrightarrow r'_u \cap r'_v = \emptyset$. Since $\mathcal{R}'_{i,j}$ is a valid representation of $H_{i,j}$, we have $r_u \cap r_v = \emptyset \Leftrightarrow uv \notin E(H_{i,j}) \Leftrightarrow uv \notin E(G)$.

This completes the proof. \square

5 Asteroidal subgraphs in a graph

In this section, we present a forbidden structure for k -SRIGs and k -ESRIGs that generalizes the ‘‘asteroidal triples’’ of Lekkerkerker and Boland [21]. We then study the block-trees of graphs in the context of these forbidden structures, to derive some preliminary observations which shall be used in the proofs in Section 6. First, we give some basic definitions.

We say that two subgraphs G_1, G_2 of a graph G are *neighbour-disjoint* if for any vertex $v \in V(G_1)$, $N[v] \cap V(G_2) = \emptyset$. In other words, $V(G_1)$ and $V(G_2)$ are disjoint and there is no edge between a vertex in $V(G_1)$ and a vertex in $V(G_2)$.

Let $G = (V, E)$ be any graph. Given a vertex $v \in V(G)$, we say that a path P *misses* v , if no vertex in P is a neighbour of v . Similarly, given a subgraph H of G we say that P *misses* H if P misses each vertex in $V(H)$; in other words, P misses H exactly when P and H are neighbour-disjoint.

Definition 1. *Given a graph G , three vertices $a, b, c \in V(G)$ are said to form an asteroidal triple, or AT for short, in G if there exists a path between any two vertices in $\{a, b, c\}$ that misses the third.*

A graph is said to be *AT-free* if it contains no asteroidal triple. A graph is *chordal* if it contains no induced subgraph isomorphic to a cycle on 4 or more vertices.

Theorem 11 ([21]). *A graph G is an interval graph if and only if G is chordal and AT-free.*

Definition 2. *Three connected induced subgraphs G_1, G_2, G_3 of G that are pairwise neighbour-disjoint are said to be asteroidal in G if for each $i \in \{1, 2, 3\}$, for any i, j, k such that $\{i, j, k\} = \{1, 2, 3\}$, there is a path from some vertex of G_i to some vertex of G_j that misses G_k .*

Suppose G_1, G_2, G_3 are asteroidal in a graph G . Then from the above definition, they are pairwise neighbour-disjoint and each of them is connected. This implies that for any i, j, k such that $\{i, j, k\} = \{1, 2, 3\}$, and for any $u \in V(G_i)$ and any $v \in V(G_j)$, there is some path between u and v that misses G_k .

Definition 3. Let \mathcal{C} be a class of graphs and let G be any graph. Let G_1, G_2, G_3 be asteroidal in G and let $G_i \in \mathcal{C}$ for $i \in \{1, 2, 3\}$. Then we say that G_1, G_2, G_3 are asteroidal- \mathcal{C} in G .

Definition 4. We say that a graph G is asteroidal- \mathcal{C} -free if there does not exist three subgraphs that are asteroidal- \mathcal{C} in G .

5.1 A forbidden structure for k -SRIGs and k -ESRIGs

We now show that no k -SRIG can contain three subgraphs that are asteroidal-(non- $(k-1)$ -SRIG) in it. The same technique can be used to show that a k -ESRIG cannot contain three subgraphs that are asteroidal-(non- $(k-1)$ -ESRIG) in it. The intuition is that if a k -SRIG G contains subgraphs G_1, G_2, G_3 which are asteroidal-(non- $(k-1)$ -SRIG) in G , then in any k -stabbed rectangle intersection representation of G , the rectangles corresponding to vertices in G_i , for each $i \in \{1, 2, 3\}$, together occupy all the stab lines (as each G_i is a non- $(k-1)$ -SRIG). Coupled with the fact that the three subgraphs are pairwise neighbour-disjoint, this enforces a kind of “left-to-right” order on the subgraphs: that is, in the k -SRIG representation, for distinct $i, j \in \{1, 2, 3\}$, the collection of rectangles corresponding to vertices of G_i can be thought of as being “to the left of” or “to the right of” the collection of rectangles corresponding to the vertices of G_j . If we take this left-to-right order of subgraphs to be G_1, G_2, G_3 , then it can be shown that any path from a vertex of G_1 to a vertex of G_3 must contain a vertex whose rectangle intersects a rectangle belonging to a vertex of G_2 , thus contradicting the fact that G_1, G_2, G_3 are asteroidal in G . We give the formal proof below.

Theorem 12. k -SRIGs are asteroidal-(non- $(k-1)$ -SRIG)-free.

Proof. Assume for the sake of contradiction that G is a k -SRIG with a k -stabbed rectangle intersection representation \mathcal{R} and has three connected induced non- $(k-1)$ -SRIG subgraphs G_1, G_2, G_3 that are asteroidal in G . As each of G_1, G_2, G_3 are non- $(k-1)$ -SRIGs, but are k -SRIGs (as they are induced subgraphs of G), for each $i \in \{1, 2, 3\}$, there exists a walk W_i in G_i such that W_i contains at least one vertex on each stab line of \mathcal{R} (for example, W_i can be chosen to be any path in G_i between a vertex on the top stab line and a vertex on the bottom stab line). This further implies that for each $i \in \{1, 2, 3\}$, there exists a vertex v_i in W_i that is on the bottom stab line. As G_1, G_2, G_3 are pairwise neighbour-disjoint, we know that $\text{span}(v_1), \text{span}(v_2), \text{span}(v_3)$ are pairwise disjoint. Therefore we can assume without loss of generality that $\text{span}(v_1) < \text{span}(v_2) < \text{span}(v_3)$. Now consider the set of vertices $S = \{w : w \in N[w'] \text{ for some } w' \in W_2\}$.

Consider the region X of the plane defined by $X = \bigcup_{u \in W_2} r_u$. Since W_2 is connected and has a vertex on each stab line, X is an arc-connected region that intersects all the stab lines. Clearly, for any vertex x such that $r_x \cap X \neq \emptyset$ we can conclude that $x \in S$. Now let B be the rectangle with diagonally opposite corners (x_1, y_1) and (x_2, y_2) where $x_1 = \min\{x_v^- : v \in V(G)\}$, $x_2 = \max\{x_v^+ : v \in V(G)\}$, $y = y_1$ is the bottom stab line and $y = y_2$ is the top stab line of \mathcal{R} .

Claim. The rectangles $B \cap r_{v_1}$ and $B \cap r_{v_3}$ are completely contained in different arc-connected regions of $B \setminus X$.

Since v_1 and v_3 have no neighbours in W_2 , and therefore are not in S , we can infer from our earlier observation that the rectangles $B \cap r_{v_1}$ and $B \cap r_{v_3}$ are disjoint from X . This means that each of these rectangles are completely contained in some arc-connected region of $B \setminus X$. Assume for the sake of contradiction that the rectangles $B \cap r_{v_1}$ and $B \cap r_{v_3}$ are completely contained in the same connected region of $B \setminus X$. This implies that there exists a curve \mathbf{s} in $B \setminus X$ that connects some point in $B \cap r_{v_1}$ that is on the bottom stab line to some point in $B \cap r_{v_3}$ that is also on the bottom stab line. Now consider the points $p, q \in X$ such that p is on the top stab line and q is a point in r_{v_2} that is on the bottom stab line. Since X is connected, there is a curve \mathbf{s}' in X that connects p, q . Since $\text{span}(v_1) < \text{span}(v_2) < \text{span}(v_3)$ and \mathbf{s}, \mathbf{s}' are curves that are completely contained in B , we can conclude that the curves \mathbf{s} and \mathbf{s}' intersect. But this is a contradiction, as \mathbf{s} is a curve in $B \setminus X$ and hence cannot contain any point in $\mathbf{s}' \subseteq X$. This completes the proof of the claim.

As G_1, G_2, G_3 are asteroidal in G , there is a path P between v_1 and v_3 that misses G_2 . This means that the path P does not contain any vertex from S , and therefore the rectangle corresponding to no vertex in P intersects X . Since every rectangle in the representation intersects B , this means that $\bigcup_{w \in V(P)} B \cap r_w$ is an arc-connected set in $B \setminus X$ that contains both $B \cap r_{v_1}$ and $B \cap r_{v_3}$. This is a contradiction to the above claim. \square

The following theorem can be proved using the similar arguments, and hence we omit the proof.

Theorem 13. k -ESRIGs are asteroidal-(non- $(k-1)$ -ESRIG)-free.

5.2 The coloured block-tree of a graph

A *hereditary* class of graphs is a class of graphs that is closed under taking induced subgraphs. A class of graphs is said to be *closed under vertex addition* if adding a vertex (and an arbitrary set of edges incident on it) to any graph in the class results in another graph that is in the class. It can be seen that a class of graphs is closed under vertex addition if and only if its complement class (the set of graphs that are not in the class) is hereditary. Therefore, the class of non- k -SRIGs and the class of non- k -ESRIGs, for any positive integer k , are both closed under vertex addition. In this section, we study the block-tree (defined below) of an asteroidal- \mathcal{C} -free graph, where \mathcal{C} is some graph class that is closed under vertex addition. The lemmas derived in this section will be useful in the next section.

For any graph G , let $\mathcal{B}(G)$ be the set of blocks in it and $C(G)$ the set of cut-vertices in it. The *block-tree* of G (denoted as T_G) is the graph with $V(T_G) = \mathcal{B}(G) \cup C(G)$ and $E(T_G) = \{Bc : B \in \mathcal{B}(G), c \in C(G), \text{ and } c \in B\}$. For any graph G , the graph T_G turns out to be a tree, justifying the name ‘‘block-tree of G ’’ [14].

For $e = Bc \in E(T_G)$, where $c \in C(G)$ and $B \in \mathcal{B}(G)$, we denote by $T_G(e)$ the connected component of $T_G - e$ containing B . Also, let us define

$$G_e = G \left[\bigcup_{B \in T_G(e)} B \setminus \{c\} \right]$$

In other words, G_e is the component of $G - \{c\}$ that contains the vertices of B other than c . Note that G_e is a connected induced subgraph of G . The following observation is a direct consequence of the structure of the block-tree.

Observation B. *The vertices of G other than c that belong to blocks not in $T_G(e)$ are neither in G_e nor are adjacent to any vertex in G_e .*

Let \mathcal{C} be a class of graphs. Let us now colour red those edges e of T_G such that $G_e \in \mathcal{C}$. Further, let us colour red those cut-vertices in T_G that have at least two red edges incident on them. Note that if two red edges e_1 and e_2 are incident on a cut-vertex u in T_G , then G_{e_1} and G_{e_2} are two components of $G - \{u\}$. As the final step of colouring, we colour red those block-vertices of T_G that are adjacent to at least two cut-vertices that are red. We now say that the tree T_G is *coloured with respect to \mathcal{C}* .

Lemma 14. *Let \mathcal{C} be a class of graphs that is closed under vertex addition. Let G be any graph and let T_G be coloured with respect to \mathcal{C} . Then the subgraph of T_G induced by the set of red vertices is connected.*

Proof. We only need to prove that for any $u, v \in V(T_G)$ that are coloured red, any vertex $w \in V(T_G)$ that lies on the path in T_G between u and v is also red. Let P be the path between u and v in T_G . If u is a cut-vertex, then let $u' = u$ and if u is a block-vertex, then let u' be a red cut-vertex that is adjacent to u but is not on P . Similarly, if v is a cut-vertex, then we let $v' = v$ and if v is a block-vertex, we let v' be a red cut-vertex that is adjacent to v but is not on P . Clearly, the path P' in T_G between u' and v' also contains w . It can be seen that there is a red edge e_u that is incident on u' but does not belong to P' and a red edge e_v that is incident on v' but does not belong to P' . As e_u and e_v are red edges, we know that $G_{e_u}, G_{e_v} \in \mathcal{C}$. Now consider any edge e that is in P' . From the structure of the block-tree, it follows that either $V(G_{e_u}) \subseteq V(G_e)$ or $V(G_{e_v}) \subseteq V(G_e)$. (To see this, let z be the cut-vertex in e and assume that u' is closer to z than v' in T_G . Then, $T_G(e_v)$ is a subtree of $T_G(e)$. Note that z is not adjacent to any block-vertex of $T_G(e_v)$, implying that z is not contained in any block that appears as a block-vertex in $T_G(e_v)$. We now have that $V(G_{e_v}) \subseteq V(G_e)$.) Since \mathcal{C} is closed under vertex addition, we now have that $G_e \in \mathcal{C}$, which implies that e is red. Therefore, every edge in P' is red. It now follows that every cut-vertex in P' other than u' and v' are incident with at least two red edges. Therefore every cut-vertex in P' is red (recall that u' and v' are red by definition). This tells us that every block-vertex in P' is adjacent to two red cut-vertices, and is therefore red. This proves that w is red. \square

Lemma 15. *Let G be a graph and \mathcal{C} a class of graphs closed under vertex addition. Let T_G be coloured with respect to \mathcal{C} and let \mathcal{B} be the set of block-vertices of T_G that have at least one red neighbour (or equivalently, the blocks of G that contain at least one cut-vertex that is red in T_G). Furthermore, assume that T_G has at least one red vertex. Let H be any component of $G - \bigcup_{B \in \mathcal{B}} B$. Then:*

- (a) *there exists exactly one vertex $u \in V(G) \setminus V(H)$ such that $N(u) \cap H \neq \emptyset$, and*
- (b) *$H \notin \mathcal{C}$.*

Proof. Let us mark the block-vertices in T_G corresponding to blocks of G that contain at least one vertex of H and also mark the cut-vertices in T_G corresponding to cut-vertices of G that are in H . Clearly, the block-vertices that are marked are not in \mathcal{B} . Since H is connected, it follows from the structure of the block-tree that the marked vertices of T_G form a subtree of T_G whose leaves are all marked block-vertices. Further, it is clear that any unmarked cut-vertex that is adjacent to a marked block-vertex belongs to some block in \mathcal{B} (otherwise, that cut-vertex would have been in H and therefore marked). Now suppose there exist two distinct edges $e = uX$ and $e' = u'X'$ of T_G where X, X' are marked block-vertices and u, u' are unmarked cut-vertices. Let B, B' be the blocks in \mathcal{B} that contain u, u' respectively. As $B, B' \in \mathcal{B}$, there exist red cut-vertices v, v' adjacent to B, B' respectively where $u \neq v$ and $u' \neq v'$. From Lemma 14, we know that the red vertices in T_G induce a connected subtree of T_G . Therefore, every vertex in the path in T_G between v and v' has to be red. This implies that u is red, which further implies that $X \in \mathcal{B}$. But this contradicts the fact that X is a marked block-vertex. We can therefore conclude that there exist at most one marked block-vertex X that has an unmarked neighbour in T_G . Since T_G contains at least one marked vertex and at least one unmarked vertex (as $V(H) \neq \emptyset$ and $\mathcal{B} \neq \emptyset$), we have that there is exactly one marked block-vertex X such that it has an unmarked neighbour u in T_G . It now follows from the structure of the block-tree that $H = G_{uX}$. This implies that no vertex in H can have a neighbour in $V(G) \setminus V(H)$ other than u . This proves (a).

We shall now prove (b). Suppose for the sake of contradiction that $H \in \mathcal{C}$, or in other words, $G_{uX} \in \mathcal{C}$. So, the edge uX is red in T_G .

Claim. The cut-vertex u of T_G is red.

As observed earlier, u is in some block that is in \mathcal{B} . Let $B \in \mathcal{B}$ be a block containing u . So uB is an edge of T_G . Since $B \in \mathcal{B}$, there must be some red cut-vertex u' in T_G that is adjacent to B . Clearly, $u' \neq u$, as otherwise, X would have been adjacent to a red cut-vertex, and hence it would have been in \mathcal{B} . But this cannot happen as X contains vertices from H . Since u' is a red cut-vertex, it has at least two red edges incident on it and therefore there is a red edge e incident on u' that is different from $u'B$. From the definition of red edges, we have that $G_e \in \mathcal{C}$. It follows from the structure of the block-tree that G_e is an induced subgraph of $G_{u'B}$. As \mathcal{C} is closed under vertex addition, we have that $G_{u'B} \in \mathcal{C}$, implying that the edge uB is red in T_G . We now have two red edges, uX and uB , incident on u , which means that u is a red cut-vertex of T_G .

From the above claim, it follows that X is a block-vertex of T_G that is incident to a red cut-vertex u , and hence it is in \mathcal{B} . But this is a contradiction as B contains vertices of H . This proves (b). \square

Lemma 16. *Let \mathcal{C} be a class of graphs that is closed under vertex addition. Let G be an asteroidal- \mathcal{C} -free graph and let T_G be coloured with respect to \mathcal{C} . Then the subgraph T_r of T_G induced by the set of red vertices is either empty or is a path.*

Proof. If there are no red vertices in T_G , then there is nothing to prove. So let us suppose that T_r is not empty. From Lemma 14, it follows that T_r is connected. It only remains to be shown that every vertex has degree at most two in T_r . Suppose for the sake of contradiction that u is a red vertex that has three red neighbours u_1, u_2, u_3 .

Let us first consider the case when u is a block-vertex. Then, clearly u_1, u_2, u_3 are all cut-vertices. Since each u_i , for $i \in \{1, 2, 3\}$ is red, we know that there are two red edges incident on each of them. This means that for each $i \in \{1, 2, 3\}$ there is a red edge e_i different from uu_i that is incident on u_i . It is clear from Observation B that $G_{e_1}, G_{e_2}, G_{e_3}$ are pairwise neighbour-disjoint connected induced subgraphs of G . Because e_1, e_2, e_3 are red, we know that $G_{e_1}, G_{e_2}, G_{e_3} \in \mathcal{C}$. For each $i \in \{1, 2, 3\}$, let v_i be a neighbour of u_i in G_{e_i} . Let the block-vertex u in T_G correspond to a block B in G . From the definition of the block-tree, we know that $u_1, u_2, u_3 \in B$. Since B is a 2-connected subgraph of G , for any i, j, k such that $\{i, j, k\} = \{1, 2, 3\}$, there exists a path P_{ij} in B between u_i and u_j that does not contain u_k . Let $P'_{ij} = P_{ij} \cup \{u_i v_i, u_j v_j\}$. From Observation B, it follows that P'_{ij} misses G_{e_k} . This means that $G_{e_1}, G_{e_2}, G_{e_3}$ are asteroidal- \mathcal{C} in G , contradicting the fact that G is asteroidal- \mathcal{C} -free.

Next, let us consider the case when u is a cut-vertex. Then, u_1, u_2, u_3 are block-vertices that are coloured red. Since each of them have to be adjacent to at least two red cut-vertices, we know that for each $i \in \{1, 2, 3\}$, there is a red cut-vertex u'_i different from u that is adjacent to u_i . Then again, as for each $i \in \{1, 2, 3\}$, u'_i is red, we can infer that there is a red edge e_i different from $u'_i u_i$ that is incident on u'_i . As before, $G_{e_1}, G_{e_2}, G_{e_3}$ form neighbour-disjoint connected induced subgraphs of G that all belong to \mathcal{C} . For each $i \in \{1, 2, 3\}$, let v_i be a neighbour of u'_i in G_{e_i} . It is now clear from the structure of the block-tree that for any i, j, k such that

$\{i, j, k\} = \{1, 2, 3\}$, there is a path P_{ij} in G between u'_i and u'_j that does not contain u'_k . We can now infer using Observation B that the path $P'_{ij} = P_{ij} \cup \{v_i u'_i, v_j u'_j\}$ misses G_{e_k} . So we again have that $G_{e_1}, G_{e_2}, G_{e_3}$ are asteroidal- \mathcal{C} in G , contradicting the fact that G is asteroidal- \mathcal{C} -free. \square

Lemma 17. *Let \mathcal{C} be a class of graphs that is closed under vertex addition. Let G be a graph and let T_G be coloured with respect to \mathcal{C} . If there are no red vertices in T_G , then there exists a block B in G such that no component of $G - B$ is in \mathcal{C} .*

Proof. Note that if there exists a cut-vertex u in G such that each component of $G - \{u\}$ is not in \mathcal{C} , then clearly, removal of any block that contains u from G will result in a graph whose components are not in \mathcal{C} (recall that \mathcal{C} is closed under vertex addition). Therefore, we shall assume that for any cut-vertex u of G , there is some component of $G - \{u\}$ that is in \mathcal{C} . Since $\{G_e : e \text{ incident on } u\}$ are the components of $G - \{u\}$, this implies that in T_G , every cut-vertex has at least one edge e incident on it such that $G_e \in \mathcal{C}$. In other words, every cut-vertex of T_G has at least one red edge incident on it. Since T_G contains no red vertices, we can now conclude that every cut-vertex in T_G has exactly one red edge incident on it.

For a cut-vertex u in G , let us define $f(u)$ to be the only red edge incident on u in T_G . Let v be the cut-vertex in G that minimizes $|V(G_{f(v)})|$. Let $f(v) = vB$, where B is a block-vertex of T_G . Recall that in T_G , every edge incident on v other than vB is a non-red edge. In other words, none of the components of $G - v$ other than $G_{f(v)}$ belong to \mathcal{C} . We now claim that every edge in T_G incident on B is red. Suppose that there is a non-red edge wB in T_G . Since w is a cut-vertex, there is a red edge $f(w)$ incident on w . Since wB is non-red, $f(w)$ is different from wB . From the structure of the block-tree, it is evident that $V(G_{f(w)}) \subset V(G_{f(v)})$ ($w \in V(G_{f(v)}) \setminus V(G_{f(w)})$). But this contradicts our choice of v as we now have $(|V(G_{f(w)})| < |V(G_{f(v)})|)$. Therefore, every edge that is incident on B in T_G is red.

For any block-vertex X in T_G , we shall define $F_X = \{wY : wX \in E(T_G) \text{ and } X \neq Y\}$. In other words, F_X consists of exactly those edges of T_G that are not incident on X but are incident on some cut-vertex adjacent to X . Note that $\{G_e : e \in F_X\}$ are exactly the components of $G - X$. Since for the block-vertex B under consideration, we know that every edge incident on it is red, we can infer that every edge in F_B is non-red (as every cut-vertex has exactly one red edge incident on it). This means that each of $\{G_e : e \in F_B\}$ is a graph that is not in \mathcal{C} ; in other words, no component of $G - B$ belongs to \mathcal{C} . We have thus found the required block. \square

6 Trees and block graphs

A question asked in Babu et al. [5] is whether it can be determined in polynomial-time if an input tree has a rectangle intersection representation in which each rectangle is a square of unit height and width. Instead of restricting the rectangles to be unit squares, we study a different restriction. In particular, we ask if, given a tree and an integer k , it can be determined in polynomial-time whether the tree has a k -SRIG or k -ESRIG representation. We show that the problem is polynomial-time solvable if $k \leq 3$. In fact, we show that we can determine in polynomial-time if the input graph G is 2-ESRIG (equivalently 2-SRIG, by Theorem 2) if G is guaranteed to be a block graph. We also show that it can be determined in polynomial-time if an input tree is 3-ESRIG (equivalently 3-SRIG, by Theorem 2). Our algorithms depend on a forbidden structure characterization for block graphs that are 2-ESRIG and trees that are 3-ESRIG. In fact, in both cases, the algorithm is a search for the presence of these forbidden structures in the input graph, and therefore it is a ‘‘certifying algorithm’’, in the sense that the algorithm outputs a representation whenever the answer is ‘‘Yes’’ and a forbidden structure in the graph whenever the answer is ‘‘No’’.

The forbidden structure characterizations of block graphs that are 2-ESRIG and trees that are 3-ESRIG are obtained as follows. In the previous section, we showed that a necessary condition for a graph to be a 2-ESRIG is that it has to be asteroidal-(non-interval)-free. We show in this section that for block graphs, this necessary condition is also sufficient. We later on show that for trees that are 3-ESRIG, the necessary condition of being asteroidal-(non-2-ESRIG)-free is again a sufficient condition. First, we need the following lemma.

Lemma 18. *Let \mathcal{C} be a class of graphs that is closed under vertex addition. Let G be a block graph that is asteroidal- \mathcal{C} -free and let T_G be coloured with respect to \mathcal{C} . Then there exists a set $S \subseteq V(G)$ such that $G[S]$ is an interval graph and no component of $G - S$ is in \mathcal{C} .*

Proof. When T_G contains at least one red vertex, let \mathcal{B} be the set of block-vertices of T_G that have at least one red neighbour. If T_G contains no red vertices, then by Lemma 17, there is a block B in G whose removal

gives us components, none of which are in \mathcal{C} . In this case, let $\mathcal{B} = \{B\}$. We shall let S be the set of vertices which are contained in some block in \mathcal{B} , or in other words, $S = \bigcup_{B \in \mathcal{B}} B$. By the above observation and Lemma 15, we can assume from here onwards that no component of $G - S$ is in \mathcal{C} . If there are no red vertices in T_G , then $G[S]$ is a complete graph, and therefore an interval graph. To complete the proof, we only need to show that if T_G contains at least one red vertex, then $G[S]$ is an interval graph.

Suppose that T_G contains at least one red vertex. Then from Lemma 16, we know that the red vertices in T_G form a path. Since block graphs are chordal, by Theorem 11, we need to only show that $G[S]$ is AT-free in order to prove that $G[S]$ is an interval graph. Suppose for the sake of contradiction that there exists an asteroidal triple $\{a, b, c\} \subseteq S$ in $G[S]$. Since $\{a, b, c\}$ has to be an independent set in G , we know that there is no block that contains any two of them. We shall say that a cut-vertex in G is red if that cut-vertex is coloured red in T_G . Note that from the definition of S , every vertex in S is adjacent to at least one red cut-vertex (since each vertex of S is in some block that also contains a cut-vertex that is coloured red in T_G , and each block is a complete graph). Let a', b', c' denote red cut-vertices that are adjacent to a, b, c respectively. Suppose that $a' = b' = x$. Then, it is clear from the structure of the block-tree that either every path between a and c contains x or every path between b and c contains x . But this contradicts the fact that a, b, c form an AT in $G[S]$, since x is a neighbour of both a and b . We can therefore assume that a', b', c' are distinct red cut-vertices. Since the red vertices form a path in T_G , the vertices a', b', c' must lie on a path in T_G . Let us assume without loss of generality that b' lies on the path in T_G between a' and c' . This means that every path between a' and c' in $G[S]$ contains b' . We now claim that every path in $G[S]$ between a and c goes through b' . Suppose for the sake of contradiction that there exists a path P between a and c in $G[S]$ that does not contain b' . Then the path $a'a \cup P \cup cc'$ is a path between a' and c' in $G[S]$ that does not contain b' , contradicting the fact that every path in $G[S]$ between a' and c' contains b' . So, we have that every path between a and c in $G[S]$ contains b' , which is a neighbour of b . This contradicts the fact that a, b, c forms an AT in $G[S]$. \square

Theorem 19. *A block graph G is 2-ESRIG if and only if G is asteroidal-(non-interval)-free.*

Proof. Let G be a block graph. We know by Theorem 13 that if G is a 2-ESRIG then G is asteroidal-(non-interval)-free. Now we prove that if G is asteroidal-(non-interval)-free then G is a 2-ESRIG.

By letting \mathcal{C} be the class of non-interval graphs, we have by Lemma 18 that there exists a set $S \subseteq V(G)$ such that $G[S]$ is an interval graph and each component of $G - S$ is also an interval graph.

Let $\mathcal{R} = \{[c_u, d_u]\}_{u \in S}$ be an interval representation of $G[S]$ such that all endpoints of intervals are distinct. Let $\epsilon \in \mathbb{R}^+$ be such that $\epsilon < \min\{|c_u - c_v| : u, v \in S, u \neq v\}$. Also, let $L, R \in \mathbb{R}$ such that $L < \min_{u \in S} c_u$ and $R > \max_{u \in S} d_u$. For each vertex $u \in S$, define $t_u = \frac{c_u - L}{R - L}$. Let \mathcal{H} be the set of components of $G - S$. For a vertex $u \in S$, let $\mathcal{H}_u = \{H \in \mathcal{H} : N(u) \cap H \neq \emptyset\}$. From Lemma 15(a), it is clear that for each component $H \in \mathcal{H}$, there is a exactly one vertex in S that has neighbours in H . Therefore, it follows that $\{\mathcal{H}_u\}_{u \in S}$ is a partition of \mathcal{H} (recall that G is connected). Since each component of \mathcal{H} is an interval graph, and because disjoint unions of interval graphs are again interval graphs, we know that for $u \in S$, the graph I_u formed by the disjoint union of the components in \mathcal{H}_u is an interval graph. It is easy to see that $\{I_u\}_{u \in S}$ is a collection of neighbour-disjoint interval graphs. For each $u \in S$, let \mathcal{R}_u be an interval representation $\{[c'_v, d'_v]\}_{v \in V(I_u)}$ for the interval graph I_u such that every interval in it is contained in the interval $[c_u, c_u + \epsilon]$. Note that for distinct $a, b \in S$, no interval of \mathcal{R}_a intersects with any interval of \mathcal{R}_b . Also let $b'_v = 1$ if $v \notin N(u)$ and $b'_v = t_u$ if $v \in N(u)$. From here onwards, we shall assume that for every vertex $v \in V(G) \setminus S$, the interval $[c'_v, d'_v]$ and the value b'_v are defined.

We shall now define a rectangle $r_u = [x_u^-, x_u^+] \times [y_u^-, y_u^+]$ for each vertex $u \in V(G)$. For a vertex $u \in S$, we let $x_u^- = c_u$, $x_u^+ = d_u$, $y_u^- = 0$ and $y_u^+ = t_u$. For a vertex $u \in V(G) \setminus S$, we let $x_u^- = c'_u$, $x_u^+ = d'_u$, $y_u^- = b'_u$ and $y_u^+ = 1$. We leave it to the reader to verify that the rectangles $\{r_u\}_{u \in V(G)}$ form a 2-exactly stabbed rectangle intersection representation of G . \square

Remarks. Let \mathcal{C} be the class of non-interval graphs and G be a block graph with n vertices and m edges. Since checking whether G is in \mathcal{C} or not is possible in $O(n + m)$ time [11], we can infer that coloring the edges of T_G with respect to \mathcal{C} is possible in $O(n^2 + nm)$ time. The construction procedure described in the above proof can also be performed in $O(n^2 + nm)$ time, thus giving a polynomial time algorithm to recognize block graphs that are 2-ESRIG.

Theorem 20. *A tree G is 3-ESRIG if and only if G is asteroidal-(non-2-ESRIG)-free.*

Proof. Let G be a tree. We know by Theorem 13 that if G is a 3-ESRIG then G is asteroidal-(non-2-ESRIG)-free. Now we prove that if G is asteroidal-(non-2-ESRIG)-free then G is a 3-ESRIG.

By letting \mathcal{C} be the class of non-2-ESRIGs, we have by Lemma 18 that there exists a set $S \subseteq V(G)$ such that $G[S]$ is an interval graph and each component of $G - S$ is a 2-ESRIG.

Let $\mathcal{R} = \{[c_u, d_u]\}_{u \in S}$ be an interval representation of $G[S]$ such that all endpoints of intervals are distinct. Let $\epsilon \in \mathbb{R}^+$ be such that $\epsilon < \min\{|c_u - c_v| : u, v \in S, u \neq v\}$. Also, let $L, R \in \mathbb{R}$ such that $L < \min_{u \in S} c_u$ and $R > \max_{u \in S} d_u$. For each vertex $u \in S$, define $t_u = \frac{c_u - L}{R - L}$. Let \mathcal{H} be the set of components of $G - S$. For a vertex $u \in S$, let $\mathcal{H}_u = \{H \in \mathcal{H} : N(u) \cap H \neq \emptyset\}$. From Lemma 15(a), it is clear that for each component $H \in \mathcal{H}$, there is exactly one vertex in S that has neighbours in H . Therefore, it follows that $\{\mathcal{H}_u\}_{u \in S}$ is a partition of \mathcal{H} (recall that G is connected). Now let H be a component of \mathcal{H}_u . Since G is a tree, there is exactly one vertex w of H which is adjacent to u in G . It is easy to see that there is a 2-exactly stabbed rectangle intersection representation of H such that w is on the bottom stab line (take any 2-exactly stabbed rectangle intersection representation of H , and if the rectangle corresponding to w does not intersect the bottom stab line, then reflect the whole representation about the X -axis).

Since each component of \mathcal{H} is a 2-ESRIG, and because disjoint unions of 2-ESRIGs are again 2-ESRIG, we know that for $u \in S$, the graph I_u formed by the disjoint union of the components in \mathcal{H}_u is a 2-ESRIG. Let $\mathcal{R}_u = \{r'_v\}_{v \in I_u}$ be a 2-exactly stabbed rectangle intersection representation of I_u with the stab lines $y = 1$ and $y = 2$ such that for any vertex v of I_u , $\text{span}(v) \subset [c_u, c_u + \epsilon]$, and for each vertex $w \in N(u) \cap V(I_u)$ the rectangle r'_w intersects the stab line $y = 1$. Let I_u^1 be the subgraph induced in I_u by the vertices that are on the stab line $y = 1$ in \mathcal{R}_u . Similarly, I_u^2 be the subgraph induced in I_u by the vertices that are on the stab line $y = 2$ in \mathcal{R}_u . For any vertex $v \in I_u$, let c'_v, d'_v, t'_v, b'_v be such that $r'_v = [c'_v, d'_v] \times [b'_v, t'_v]$.

We shall now define a rectangle r_u for each vertex $u \in V(G)$ as follows. For a vertex $u \in S$, we let $r_u = [c_u, d_u] \times [0, t_u]$. Consider a vertex $v \in V(G) \setminus S$. Let u be the vertex in S such that $v \in V(I_u)$. If $v \in V(I_u^2)$, then we let $r_v = r'_v$. If $v \in V(I_u^1)$ and $v \notin N(u)$, then we let $r_v = [c'_v, d'_v] \times [1, t'_v]$. If $v \in V(I_u^1)$ and $v \in N(u)$, then we let $r_v = [c'_v, d'_v] \times [t_u, t'_v]$. We leave it to the reader to verify that the rectangles $\{r_u\}_{u \in V(G)}$ form a 3-exactly stabbed rectangle intersection representation of G . \square

Remarks. Let \mathcal{C} be the class of non-2-ESRIG graphs and T be a tree with n vertices. Since checking whether T is in \mathcal{C} or not is possible in $O(n^2)$ time, we can infer that coloring the edges of block-tree of T with respect to \mathcal{C} is possible in $O(n^3)$ time. The construction procedure described in the above proof can also be performed in $O(n^3)$ time, thus giving a polynomial time algorithm to recognize trees that are 3-ESRIG.

We show in Section 6.2 that the forbidden structure characterizations of Theorems 19 and 20 do not extend to block graphs that are 3-ESRIG (equivalently 3-SRIG, by Theorem 2) or trees that are k -SRIG for any $k \geq 4$. First, we explore the natural question of whether there exists a constant c such that every tree is a c -SRIG. We give a negative answer to this question in the following section. The construction used will come in handy in Sections 6.2 and 6.3.

6.1 Constructing trees with high stab number

For a rooted tree T , let $\text{root}(T)$ be the root vertex of T . The following observation is easy to see.

Observation C. *Let T be a tree and T' be a subtree of T such that $T - V(T')$ has only one component.*

- (i) *For any edge $e \in E(T')$, at least one component of $T - e$ is a proper subtree of T' .*
- (ii) *For any vertex $v \in V(T')$, all but one component of $T - \{v\}$ are proper subtrees of T' .*

First we describe a recursive procedure to construct a rooted tree G_l for all $l \geq 1$. For $l = 1$, let G_1 be the rooted tree containing only one vertex. For any integer l greater than 1, we construct G_l as follows. Let T_1, T_2 and T_3 be three rooted trees each isomorphic to G_{l-1} . Take a $K_{1,3}$ with vertex set $\{u, u_1, u_2, u_3\}$, where u_1, u_2, u_3 are the pendant vertices, and construct G_l by adding edges between u_i and $\text{root}(T_i)$ for each $i \in \{1, 2, 3\}$. Also let $\text{root}(G_l) = u$. For any rooted tree T with root r , we can define the ‘‘ancestor’’ relation on $V(T)$ in the usual way: i.e., for $u, v \in V(T)$, u is an *ancestor* of v if and only if the path in T between r and v contains u . We prove the following lemma.

Lemma 21.

- (i) *For $l > 1$, G_l is not $(l - 1)$ -SRIG.*
- (ii) *For $l \geq 1$, there is an l -exactly stabbed rectangle intersection representation \mathcal{R} of G_l such that for $v, w \in V(G_l)$, $\text{span}(v) \subseteq \text{span}(w)$ if w is an ancestor of v and the vertices on the top stab line of \mathcal{R} are exactly the vertices in $N[\text{root}(G_l)]$.*
- (iii) *Let T and T' be two trees each isomorphic to G_l , for some $l \geq 1$. Let F_l be the tree obtained by taking a new vertex u and joining it to the root vertices of T, T' using paths of length two.*

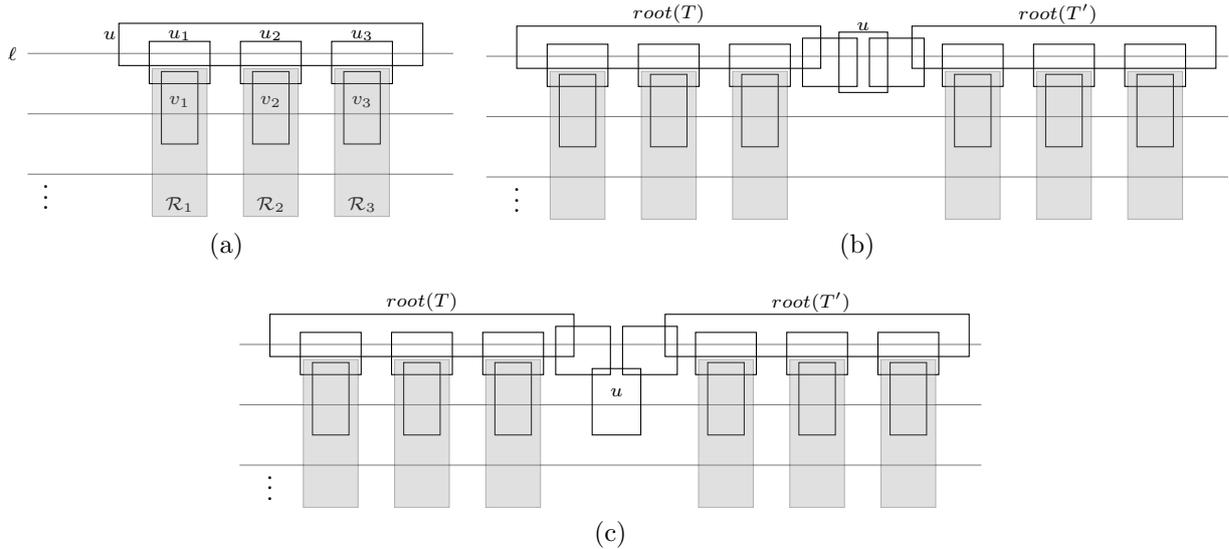


Figure 6: Construction of G_l and F_l . The shaded region denotes a collection of rectangles. In (a), for $i \in \{1, 2, 3\}$, v_i is the vertex $\text{root}(T_i)$. Figures (b) and (c) show different l -exactly stabbed rectangle intersection representations of F_l as described in Lemma 21(iii)(a) and Lemma 21(iii)(b).

- (a) For $l \geq 1$, there is an l -exactly stabbed rectangle intersection representation \mathcal{R}' of F_l such that for $v, w \in V(F_l)$, $\text{span}(v) \subseteq \text{span}(w)$ if w is an ancestor of v in T or T' , and all vertices in the path between $\text{root}(T)$ and $\text{root}(T')$ are on the top stab line of \mathcal{R}' .
- (b) For $l \geq 2$, there is an l -exactly stabbed rectangle intersection representation \mathcal{R}'' of F_l such that for $v, w \in V(F_l)$, $\text{span}(v) \subseteq \text{span}(w)$ if w is an ancestor of v in T or T' , and only the vertices in $N[\text{root}(T)] \cup N[\text{root}(T')]$ are on the top stab line of \mathcal{R}'' .
- (iv) For $l \geq 2$, there does not exist two vertex-disjoint subtrees in G_l such that they are both non- $(l-1)$ -ESRIG.
- (v) For $l \geq 1$, $\text{estab}(G_l) = \text{stab}(G_l) = \log_3(n+2)$, where $n = |V(G_l)|$.

Proof. We prove each statement separately by induction on l . When $l = 1$, G_l consists of a single vertex and therefore all the statements are true. Now we assume that the above statements are true for all integers less than l .

Recall that G_l is obtained by taking three rooted trees T_1, T_2, T_3 , each isomorphic to G_{l-1} , and then making each root adjacent to a unique pendant vertex of a $K_{1,3}$. Let u be the vertex of degree 3 and u_1, u_2, u_3 be the pendant vertices of the $K_{1,3}$. Also recall that $\text{root}(G_l) = u$.

To prove (i), note that as T_i is isomorphic to G_{l-1} for each $i \in \{1, 2, 3\}$, we have by our induction hypothesis that T_i is not $(l-2)$ -SRIG. Therefore, T_1, T_2, T_3 are asteroidal-(non- $(l-2)$ -SRIG) in G_l . Using Theorem 12, we can conclude that G_l is not $(l-1)$ -SRIG.

To prove (ii), note that by our induction hypothesis, for each $i \in \{1, 2, 3\}$, T_i has an $(l-1)$ -exactly stabbed rectangle intersection representation \mathcal{R}_i such that for $v, w \in V(T_i)$, $\text{span}(v) \subseteq \text{span}(w)$ if w is an ancestor of v and only the vertices in $N[\text{root}(T_i)]$ are on the top stab line of \mathcal{R}_i . Since T_1, T_2, T_3 are vertex disjoint, it is easy to see that there is an $(l-1)$ -exactly stabbed rectangle intersection representation \mathcal{R} of the subgraph induced in G_l by $\cup_{i=1}^3 V(T_i)$ such that only the vertices in $\cup_{i=1}^3 N[\text{root}(T_i)]$ are on the top stab line of \mathcal{R} : we can just place $\mathcal{R}_1, \mathcal{R}_2$ and \mathcal{R}_3 side by side as shown in Figure 6(a). Now by introducing a new stab line ℓ above the top stab line of \mathcal{R} and new rectangles corresponding to the vertices in $N[\text{root}(G_l)] = \{u, u_1, u_2, u_3\}$ into the representation such that they all intersect ℓ , and for each $i \in \{1, 2, 3\}$, the rectangle corresponding to u_i intersects the rectangle corresponding to $\text{root}(T_i)$ as shown in Figure 6(a), we can get the desired l -exactly stabbed rectangle intersection representation of G_l .

Now we prove (iii)(a) and (iii)(b). Since T and T' are both isomorphic to G_l and vertex disjoint, we can infer using (ii) that there is an l -exactly stabbed rectangle intersection representation \mathcal{R} of $F_l[V(T) \cup V(T')]$ such that for $v, w \in V(F_l)$, $\text{span}(v) \subseteq \text{span}(w)$ if w is an ancestor of v in T or T' , and only the vertices in $N[\text{root}(T)] \cup N[\text{root}(T')]$ are on the top stab line of \mathcal{R} (we can obtain \mathcal{R} by placing the representations of

T and T' as given by (ii) side by side as shown in Figure 6(b)). Let P be the path that joins $\text{root}(T)$ and $\text{root}(T')$ in F_l . As shown in Figure 6(b), we can represent P such that all the vertices in P are on the top stab line of \mathcal{R} . This proves (iii)(a). Similarly, if $l \geq 2$, then as shown in Figure 6(c), we can represent P such that only the vertices in $N[\text{root}(T)] \cup N[\text{root}(T')]$ are on the top stab line of \mathcal{R} . This proves (iii)(b).

Now we prove (iv). Assume for the sake of contradiction that X_1, X_2 are two vertex-disjoint subtrees in G_l such that they are both non- $(l-1)$ -ESRIG. Since G_l is connected, there exists an edge e in G_l such that if X'_1 and X'_2 are the two components in $G_l - e$, then for each $i \in \{1, 2\}$, X_i is a subtree of X'_i . This implies that both X'_1 and X'_2 are non- $(l-1)$ -ESRIG. Suppose that $e \in E(T_i)$ for some $i \in \{1, 2, 3\}$. Note that $G_l - V(T_i)$ has only one component. Therefore, using Observation C(i) we can infer that there exists $X \in \{X'_1, X'_2\}$ such that X is a proper subtree of T_i . But as T_i , being isomorphic to G_{l-1} , is $(l-1)$ -ESRIG by (ii), this implies that X is $(l-1)$ -ESRIG. This contradicts the fact that both X'_1 and X'_2 are non- $(l-1)$ -ESRIG. Therefore, we can assume without loss of generality that e is either uu_1 or the edge between u_1 and $\text{root}(T_1)$. If e is the edge between u_1 and $\text{root}(T_1)$, then one of the components of $T - e$ is T_1 , which is $(l-1)$ -ESRIG by (ii), contradicting the fact that both components of $T - e$ are non- $(l-1)$ -ESRIG. If e is the edge uu_1 , then one of the components of $T - e$ is isomorphic to F_{l-1} , and therefore by (iii), is $(l-1)$ -ESRIG. This again contradicts the fact that both components of $T - e$ are non- $(l-1)$ -ESRIG.

To prove (v), we can solve the recurrence $|V(G_l)| = 3|V(G_{l-1})| + 4$ to obtain $n = |V(G_l)| = 3^l - 2$. Now, using (i) and (ii), we can conclude that $\text{stab}(G_l) = \text{stab}(G_l) = \log_3(n + 2)$. \square

From Theorem 10, we have that for any tree T on n vertices with $n \geq 3$, $\text{stab}(T) \leq \lceil \log(n-1) \rceil$. Also, using Theorem 10 and Lemma 21(v), we have the following corollary.

Corollary 22. $\text{stab}(\text{TREES}, n) = \Theta(\log n)$, $\text{stab}(\text{TREES}, n) = \Theta(\log n)$, $\text{stab}(\text{BLOCK GRAPHS}, n) = \Theta(\log n)$, and $\text{stab}(\text{BLOCK GRAPHS}, n) = \Theta(\log n)$.

Although the stab number and exact stab number were equal for the trees that we constructed in this section, we shall show in Theorem 36 there are trees for which these parameters differ. The graph G_l and the observations in Lemma 21 will be used frequently in the remainder of the paper.

6.2 Absence of asteroidal subgraphs is not sufficient

We showed in Theorem 12 that being asteroidal-(non- $(k-1)$ -SRIG)-free is a necessary condition for a graph to be k -SRIG. Theorem 19 showed that this necessary condition is also sufficient for block graphs when $k \leq 2$ and Theorem 20 demonstrated that this necessary condition is sufficient for trees when $k \leq 3$. In this section, we shall show that this necessary condition is not sufficient for block graphs for any $k \geq 3$ and it is not sufficient for trees for any $k \geq 4$. In particular, we shall prove the following two theorems.

Theorem 23. *There exists a block graph that is asteroidal-(non-2-SRIG)-free, but is not 3-SRIG.*

Note that by Theorem 2, the above theorem also means that there exists a block graph that is asteroidal-(non-2-ESRIG)-free, but is not 3-ESRIG.

Theorem 24. *For each integer $k \geq 4$, there exists a tree T that is asteroidal-(non- $(k-1)$ -ESRIG)-free, but is not k -SRIG.*

It is easy to see that Theorem 24 directly gives the following two corollaries, which tell us that the necessary conditions derived in Theorem 12 and Theorem 13 for a tree to be a k -SRIG and k -ESRIG respectively, are not sufficient for any $k \geq 4$.

Corollary 25. *For each integer $k \geq 4$, there exists a tree T that is asteroidal-(non- $(k-1)$ -SRIG)-free, but is not k -SRIG.*

Corollary 26. *For each integer $k \geq 4$, there exists a tree T that is asteroidal-(non- $(k-1)$ -ESRIG)-free, but is not k -ESRIG.*

In order to prove these theorems, we develop some tools to study k -stabbed rectangle intersection representations using special kinds of curves in the representation that are derived from induced paths in the graph.

Consider a k -stabbed rectangle intersection representation \mathcal{R} of a graph G . In this representation, we say that a curve is *rectilinear* if it consists of vertical and horizontal line segments and each horizontal line segment in it lies on a stab line. Given an induced path $P = u_1 u_2 \dots u_s$ in G and two distinct points $p \in r_{u_1}$

and $p' \in r_{u_s}$ such that p, p' lie on stab lines, a rectilinear curve *through P from p to p'* is a simple rectilinear curve \mathbf{p} that starts at p and ends at p' , where $\mathbf{p} \subseteq \bigcup_{i=1}^s r_{u_i}$ and $\mathbf{p} \cap r_{u_i}$ is arc-connected (and nonempty) for each $i \in \{1, 2, \dots, s\}$. Note that such a curve always exists and that for each $i \in \{1, 2, \dots, s\}$, the curve contains some point in r_{u_i} that is on a stab line.

Given a set X of consecutive stab lines $y = a_1, y = a_2, \dots, y = a_t$, such that $a_1 < a_2 < \dots < a_t$, we say that $y = a_1$ is the bottom stab line in X and $y = a_t$ is the top stab line in X . Further, we say that a connected induced subgraph H of G is X -spanning if there is some vertex in H on each stab line in X . An induced path in G is said to be an X -spanning path if its starting and ending vertices are on the top and bottom stab lines of X respectively. Note that if a subgraph H of G is X -spanning, then there is an X -spanning path in H (to see this, consider the shortest path between two vertices u and v in H such that u is on the top stab line in X and v is on the bottom stab line in X).

In the following, we use the term “region” to denote an arc-connected region of the plane that is bounded by a closed rectilinear curve which is the union of four simple rectilinear curves that satisfy some special properties (we assume that a region does not contain the points on its boundary). Suppose $\mathbf{t}, \mathbf{l}, \mathbf{b}$, and \mathbf{r} are four simple rectilinear curves such that $\mathbf{l} \cap \mathbf{r} = \emptyset$, $\mathbf{t} \cap \mathbf{b} = \emptyset$, and for each $(\mathbf{x}, \mathbf{y}) \in \{(\mathbf{t}, \mathbf{l}), (\mathbf{l}, \mathbf{b}), (\mathbf{b}, \mathbf{r}), (\mathbf{r}, \mathbf{t})\}$, the curves \mathbf{x} and \mathbf{y} have exactly one point in common which is also an end point of both of them. Then, the region $R = (\mathbf{t}, \mathbf{l}, \mathbf{b}, \mathbf{r})$ is the bounded arc-connected component of $\mathbb{R}^2 \setminus (\mathbf{t} \cup \mathbf{l} \cup \mathbf{b} \cup \mathbf{r})$. The closed rectilinear curve $\mathbf{t} \cup \mathbf{l} \cup \mathbf{b} \cup \mathbf{r}$ is called the “boundary” of R . For a region R , we let $\mathcal{L}_{\mathcal{R}}(R)$ denote the set of stab lines of \mathcal{R} that intersect R . Also, let G_R denote the subgraph induced in G by the vertices whose rectangles lie completely inside R .

Observation D. *Let ℓ_t, ℓ_b be the stab lines just above and just below the top and bottom stab lines in $\mathcal{L}_{\mathcal{R}}(R)$ respectively. Then, no point on the boundary of R lies above ℓ_t or below ℓ_b .*

Proof. Suppose that the boundary of R contains a point p that is above ℓ_t . Let p' be an arbitrary point in R that is on the top stab line in $\mathcal{L}_{\mathcal{R}}(R)$. It is easy to see that there exists a simple curve from p' to p all of whose points except p belong to R . Since p' is below ℓ_t and p above it, there must be a point on this curve that lies on ℓ_t . But this would mean that R intersects ℓ_t , contradicting the fact that $\ell_t \notin \mathcal{L}_{\mathcal{R}}(R)$. Using similar arguments, we can prove that no point on the boundary of R lies below ℓ_b . \square

Definition 5. *A region $R = (\mathbf{t}, \mathbf{l}, \mathbf{b}, \mathbf{r})$ is said to be “good” if it has the following properties:*

- (i) *the parts of \mathbf{l} and \mathbf{r} that are above the top stab line in $\mathcal{L}_{\mathcal{R}}(R)$ and below the bottom stab line in $\mathcal{L}_{\mathcal{R}}(R)$ consist of just a vertical segment each, or in other words, every horizontal segment of \mathbf{l} and \mathbf{r} lies on a stab line in $\mathcal{L}_{\mathcal{R}}(R)$,*
- (ii) *no point of \mathbf{t} lies below the bottom stab line of $\mathcal{L}_{\mathcal{R}}(R)$, and*
- (iii) *no point of \mathbf{b} lies above the top stab line of $\mathcal{L}_{\mathcal{R}}(R)$.*

For a good region $R = (\mathbf{t}, \mathbf{l}, \mathbf{b}, \mathbf{r})$, we let $\mathbf{top}(R) = \mathbf{t}$ and $\mathbf{bottom}(R) = \mathbf{b}$.

Let $R = (\mathbf{t}, \mathbf{l}, \mathbf{b}, \mathbf{r})$ be a good region with $|\mathcal{L}_{\mathcal{R}}(R)| \geq 1$. Let P_1 and P_2 be two neighbour-disjoint $\mathcal{L}_{\mathcal{R}}(R)$ -spanning paths in G_R . For $i \in \{1, 2\}$, let u_i, v_i be the endvertices of P_i that are on the top and bottom stab lines in $\mathcal{L}_{\mathcal{R}}(R)$ respectively. For $i \in \{1, 2\}$, let \mathbf{p}_i be a rectilinear curve that starts at a point $(x_i, y_i) \in r_{u_i}$ on the top stab line in $\mathcal{L}_{\mathcal{R}}(R)$ and ends at a point $(x'_i, y'_i) \in r_{v_i}$ on the bottom stab line in $\mathcal{L}_{\mathcal{R}}(R)$, with the following additional properties:

- (i) The only point in \mathbf{p}_i that is in r_{u_i} and is also on the top stab line in $\mathcal{L}_{\mathcal{R}}(R)$ is (x_i, y_i) , and
- (ii) The only point in \mathbf{p}_i that is in r_{v_i} and is also on the bottom stab line in $\mathcal{L}_{\mathcal{R}}(R)$ is (x'_i, y'_i) .

It is not difficult to see that the curves $\mathbf{p}_1, \mathbf{p}_2$ always exist. (Take any rectilinear curve \mathbf{q} through P_i between some point on the top stab line in r_{u_i} and some point on the bottom stab line in r_{v_i} . Let (x_i, y_i) be the last point in \mathbf{q} that is both in r_{u_i} and is on the top stab line and let (x'_i, y'_i) be the first point in \mathbf{q} that is both in r_{v_i} and is on the bottom stab line. Then the subcurve of \mathbf{q} between (x_i, y_i) and (x'_i, y'_i) can be taken as \mathbf{p}_i .)

Suppose that there is a path in G_R between a vertex of P_1 and a vertex of P_2 . Then, let P be the induced path in G_R between a vertex w_1 in P_1 and a vertex w_2 in P_2 such that all other vertices of P belong to neither P_1 nor P_2 . Let p_1, p_2 be points on stab lines where for $i \in \{1, 2\}$, $p_i \in \mathbf{p}_i \cap r_{w_i}$, such that there exists a rectilinear curve \mathbf{p} through P from p_1 to p_2 , whose interior points belong to neither \mathbf{p}_1 nor \mathbf{p}_2 (note that p_1, p_2 and \mathbf{p} always exist — take arbitrary points p, p' on stab lines such that $p \in \mathbf{p}_1 \cap r_{w_1}, p' \in \mathbf{p}_2 \cap r_{w_2}$ and consider the rectilinear curve \mathbf{p}' through P between p and p' ; p_1, p_2 can be chosen to be the closest pair of points on \mathbf{p}' such that $p_1 \in \mathbf{p}_1, p_2 \in \mathbf{p}_2$, and the part of \mathbf{p}' between p_1 and p_2 can be chosen as \mathbf{p}).

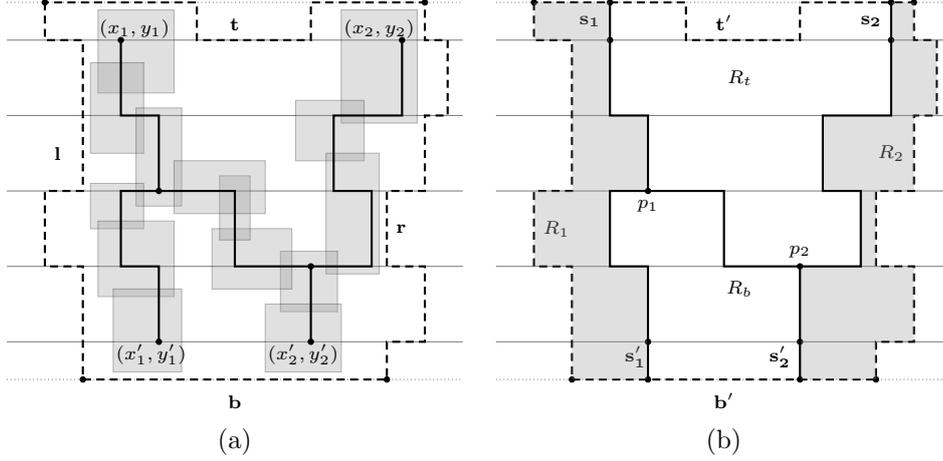


Figure 7: An example of a good region $R = (\mathbf{t}, \mathbf{l}, \mathbf{b}, \mathbf{r})$ (whose boundary is shown using thick dashed lines) containing the rectangles corresponding to minimal spanning paths P_1 and P_2 and a path P connecting them. (a) shows the rectilinear curves \mathbf{p}_1 , \mathbf{p}_2 and \mathbf{p} through these paths using thick solid lines. (b) shows the partition of R into the four regions R_1 , R_2 , R_t and R_b .

Please refer to Figure 7(a) for an example showing the different curves in R . For $i \in \{1, 2\}$, let \mathbf{s}_i be the shortest vertical line segment with its bottom endpoint being (x_i, y_i) and top endpoint being a point on the boundary of R . Similarly, for $i \in \{1, 2\}$, let \mathbf{s}'_i be the shortest vertical line segment with its top endpoint being (x'_i, y'_i) and bottom endpoint being a point on the boundary of R (refer Figure 7(b)).

Observation E. For each $i \in \{1, 2\}$, the top endpoint of \mathbf{s}_i lies on the stab line just above the top stab line in $\mathcal{L}_{\mathcal{R}}(R)$ and on a horizontal segment of \mathbf{t} and the bottom endpoint of \mathbf{s}'_i lies on the stab line just below the bottom stab line in $\mathcal{L}_{\mathcal{R}}(R)$ and on a horizontal segment of \mathbf{b} .

Proof. For $i \in \{1, 2\}$, we know that the top endpoint of \mathbf{s}_i lies on the boundary of R , and hence on a horizontal segment of the boundary of R . This implies that the top endpoint of \mathbf{s}_i lies on a stab line. Also, note that the bottom endpoint of \mathbf{s}_i is a point in R that is on the top stab line in $\mathcal{L}_{\mathcal{R}}(R)$. This means that the top endpoint of \mathbf{s}_i lies above the top stab line in $\mathcal{L}_{\mathcal{R}}(R)$. Since the top endpoint of \mathbf{s}_i lies on the boundary of R , we immediately have from Observation D that it lies on the stab line just above the top stab line in $\mathcal{L}_{\mathcal{R}}(R)$. Also, since it lies on a horizontal segment of the boundary of R , it lies on some horizontal segment that belongs to one of the curves $\mathbf{t}, \mathbf{l}, \mathbf{b}, \mathbf{r}$. Since R is good, we know that no horizontal segment of \mathbf{l}, \mathbf{r} or \mathbf{b} lies above the top stab line in $\mathcal{L}_{\mathcal{R}}(R)$. This means that the top endpoint of \mathbf{s}_i lies on a horizontal segment of \mathbf{t} . Using similar reasoning, it can be seen that for $i \in \{1, 2\}$, the bottom endpoint of \mathbf{s}'_i lies on the stab line just below the bottom stab line in $\mathcal{L}_{\mathcal{R}}(R)$ and on a horizontal segment of \mathbf{b} . \square

Let $\mathbf{t}' \subseteq \mathbf{t}$ be the portion of the curve \mathbf{t} that starts at the top endpoint of \mathbf{s}_1 and ends at the top endpoint of \mathbf{s}_2 . Similarly, let $\mathbf{b}' \subseteq \mathbf{b}$ be the portion of the curve \mathbf{b} that starts at the bottom endpoint of \mathbf{s}'_1 and ends at the bottom endpoint of \mathbf{s}'_2 .

For $i \in \{1, 2\}$, let the curve $\mathbf{p}_i^{\mathbf{t}}$ be the connected portion of \mathbf{p}_i that starts at (x_i, y_i) and ends at the common point of \mathbf{p}_i and \mathbf{p} (denoted as p_i previously) and let the curve $\mathbf{p}_i^{\mathbf{b}}$ be the connected portion of \mathbf{p}_i that starts at the common point of \mathbf{p}_i and \mathbf{p} and ends at (x'_i, y'_i) .

Let R_1, R_2, R_t, R_b be the regions into which the region R gets split by the union of the curves $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}, \mathbf{s}_1, \mathbf{s}'_1, \mathbf{s}_2, \mathbf{s}'_2$, where R_i , for $i \in \{1, 2\}$, is the region whose boundary contains \mathbf{p}_i , $R_t = (\mathbf{t}', \mathbf{s}_1 \cup \mathbf{p}_1^{\mathbf{t}}, \mathbf{p}, \mathbf{p}_2^{\mathbf{t}} \cup \mathbf{s}_2)$, and $R_b = (\mathbf{p}, \mathbf{p}_1^{\mathbf{b}} \cup \mathbf{s}'_1, \mathbf{b}', \mathbf{s}'_2 \cup \mathbf{p}_2^{\mathbf{b}})$ (please refer to Figure 7(b)).

Observation F. From the definition of R_t and R_b , we have:

- (i) $\mathbf{top}(R_t) \subseteq \mathbf{top}(R)$ and $\mathbf{bottom}(R_b) \subseteq \mathbf{bottom}(R)$.
- (ii) $\mathbf{bottom}(R_t) = \mathbf{top}(R_b)$.
- (iii) If x is a vertex in P , then r_x intersects $\mathbf{bottom}(R_t)$ ($= \mathbf{top}(R_b)$).
- (iv) Let $x \in V(G)$ such that r_x intersects $\mathbf{bottom}(R_t)$ ($= \mathbf{top}(R_b)$). Then x has a neighbour in P .

For the rest of this section, for a good region R and paths P_1, P_2, P such that:

- P_1 and P_2 are two neighbour-disjoint $\mathcal{L}_{\mathcal{R}}(R)$ -spanning paths in G_R , and
- P is an induced path in G_R between a vertex in P_1 and a vertex in P_2 such that all vertices of P other than its end vertices belong to neither P_1 nor P_2 (note that such a path will exist if there is some path in G_R between a vertex of P_1 and a vertex of P_2),

we shall denote by $\Delta(\mathcal{R}, R, P_1, P_2, P)$ the ordered pair (R_t, R_b) , where the regions R_t and R_b are obtained using the procedure described above. We shall now prove some observations about the regions R_t and R_b .

Lemma 27.

- The curve \mathbf{t}' (resp. \mathbf{b}') does not intersect the bottom (resp. top) stab line in $\mathcal{L}_{\mathcal{R}}(R)$.
- R_t does not intersect the bottom stab line in $\mathcal{L}_{\mathcal{R}}(R)$ and R_b does not intersect the top stab line in $\mathcal{L}_{\mathcal{R}}(R)$.

Proof. Let us first prove (a). We shall only show that the curve \mathbf{t}' does not intersect the bottom stab line in $\mathcal{L}_{\mathcal{R}}(R)$ as the other case is similar. Let the rectilinear curve \mathbf{q} be $\mathbf{p}_1^{\mathbf{t}} \cup \mathbf{p} \cup \mathbf{p}_2^{\mathbf{t}}$. Note that \mathbf{q} is a simple rectilinear curve. Let ℓ be the stab line just above the top stab line of $\mathcal{L}_{\mathcal{R}}(R)$. From Observation E, we have that the top endpoints of \mathbf{s}_1 and \mathbf{s}_2 lie on ℓ . Let the horizontal line segment (that lies entirely on ℓ) between these two points be denoted by \mathbf{s} . Let R' be the region bounded by $\mathbf{s}_1 \cup \mathbf{q} \cup \mathbf{s}_2 \cup \mathbf{s}$. From Observation D, it is then clear that \mathbf{t}' lies entirely in $R' \cup \mathbf{s}$ (recall that R' consists only of the points in the interior of the region bounded by $\mathbf{s}_1 \cup \mathbf{q} \cup \mathbf{s}_2 \cup \mathbf{s}$). Since the points in \mathbf{q} all belong to rectilinear curves through paths in G_R , every horizontal segment of \mathbf{q} is on a stab line in $\mathcal{L}_{\mathcal{R}}(R)$. Since the endpoints of \mathbf{q} lie on the top stab line in $\mathcal{L}_{\mathcal{R}}(R)$, and \mathbf{q} is a simple rectilinear curve, it follows that every point in \mathbf{q} is on or above the bottom stab line in $\mathcal{L}_{\mathcal{R}}(R)$. As the points in $\mathbf{s}_1 \cup \mathbf{s}_2 \cup \mathbf{s}$ lie on or above the top stab line in $\mathcal{L}_{\mathcal{R}}(R)$, this means that all the points on the boundary of R' lie on or above the bottom stab line in $\mathcal{L}_{\mathcal{R}}(R)$, implying that R' does not intersect the bottom stab line in $\mathcal{L}_{\mathcal{R}}(R)$. As \mathbf{s} lies on the stab line just above the top stab line in $\mathcal{L}_{\mathcal{R}}(R)$, we now have that $R' \cup \mathbf{s}$ does not intersect the bottom stab line in $\mathcal{L}_{\mathcal{R}}(R)$. From our earlier observation that \mathbf{t}' lies entirely in $R' \cup \mathbf{s}$, we now have that \mathbf{t}' does not intersect the bottom stab line in $\mathcal{L}_{\mathcal{R}}(R)$.

To prove (b), we shall only prove that R_t does not intersect the bottom stab line in $\mathcal{L}_{\mathcal{R}}(R)$ as the case for R_b involves similar arguments. Note that the boundary of R_t is $\mathbf{t}' \cup \mathbf{s}_1 \cup \mathbf{q} \cup \mathbf{s}_2$. From the arguments in the previous paragraph, it is easy to see that all the points in $\mathbf{s}_1 \cup \mathbf{q} \cup \mathbf{s}_2$ lie on or above the bottom stab line in $\mathcal{L}_{\mathcal{R}}(R)$. Combining this with (a), we now have that all the points on the boundary of R_t lie on or above the bottom stab line in $\mathcal{L}_{\mathcal{R}}(R)$. Hence we can conclude that the bottom stab line in $\mathcal{L}_{\mathcal{R}}(R)$ does not intersect R_t . \square

An $\mathcal{L}_{\mathcal{R}}(R)$ -spanning path P is said to be a *minimal $\mathcal{L}_{\mathcal{R}}(R)$ -spanning path* if there is no $\mathcal{L}_{\mathcal{R}}(R)$ -spanning path P' such that $V(P') \subset V(P)$. Note that the existence of an $\mathcal{L}_{\mathcal{R}}(R)$ -spanning path in a graph implies the existence of a minimal $\mathcal{L}_{\mathcal{R}}(R)$ -spanning path in the graph.

Lemma 28. *Suppose that P_1 and P_2 are minimal $\mathcal{L}_{\mathcal{R}}(R)$ -spanning paths. Let $R' \in \{R_t, R_b\}$ such that $|\mathcal{L}_{\mathcal{R}}(R')| \geq |\mathcal{L}_{\mathcal{R}}(R)| - 1$. Then R' is good.*

Proof. We shall prove this only for the case when $R' = R_t$ as the other case is similar. As $|\mathcal{L}_{\mathcal{R}}(R_t)| \geq |\mathcal{L}_{\mathcal{R}}(R)| - 1$, and by Lemma 27(b), R_t does not intersect the bottom stab line, we know that $\mathcal{L}_{\mathcal{R}}(R_t)$ consists of all the stab lines in $\mathcal{L}_{\mathcal{R}}(R)$ other than the bottom stab line in $\mathcal{L}_{\mathcal{R}}(R)$.

Recall that $R_t = (\mathbf{t}', \mathbf{s}_1 \cup \mathbf{p}_1^{\mathbf{t}}, \mathbf{p}, \mathbf{p}_2^{\mathbf{t}} \cup \mathbf{s}_2)$. Since the paths P_1 and P_2 are minimal, we know that for $i \in \{1, 2\}$, u_i is the only vertex on P_i that is on the top stab line in $\mathcal{L}_{\mathcal{R}}(R)$ and v_i is the only vertex on P_i that is on the bottom stab line in $\mathcal{L}_{\mathcal{R}}(R)$. Therefore, from the definition of curves \mathbf{p}_1 and \mathbf{p}_2 , we have that for $i \in \{1, 2\}$, the only points of \mathbf{p}_i that lie on the top and bottom stab lines in $\mathcal{L}_{\mathcal{R}}(R)$ are the endpoints of \mathbf{p}_i , which further implies that \mathbf{p}_i does not contain any horizontal segment on the top or bottom stab lines in $\mathcal{L}_{\mathcal{R}}(R)$. It follows that for $i \in \{1, 2\}$, $\mathbf{p}_i^{\mathbf{t}}$, and therefore $\mathbf{s}_i \cup \mathbf{p}_i^{\mathbf{t}}$, also does not contain any horizontal segment on the top or bottom stab lines in $\mathcal{L}_{\mathcal{R}}(R)$. As $\mathcal{L}_{\mathcal{R}}(R_t)$ consists of all the stab lines in $\mathcal{L}_{\mathcal{R}}(R)$ other than the bottom stab line in $\mathcal{L}_{\mathcal{R}}(R)$, we have that $\mathbf{s}_1 \cup \mathbf{p}_1^{\mathbf{t}}$ and $\mathbf{s}_2 \cup \mathbf{p}_2^{\mathbf{t}}$ do not contain any horizontal segment that lies above the top stab line in $\mathcal{L}_{\mathcal{R}}(R_t)$ or below the bottom stab line in $\mathcal{L}_{\mathcal{R}}(R_t)$. Therefore, R_t satisfies property (i) of Definition 5. From Lemma 27(a), we have that \mathbf{t}' does not intersect the bottom stab line in $\mathcal{L}_{\mathcal{R}}(R)$. Since the endpoints of \mathbf{t}' lie above the top stab line in $\mathcal{L}_{\mathcal{R}}(R)$, we can then conclude using the definition of rectilinear curves that no point of \mathbf{t}' lies below the bottom stab line of $\mathcal{L}_{\mathcal{R}}(R_t)$. Thus, R_t satisfies property (ii) of Definition 5. Since the points in \mathbf{p} all belong to rectangles contained in R and \mathbf{p} is a simple rectilinear curve, we know that all of them are on or below the top stab line in $\mathcal{L}_{\mathcal{R}}(R)$ and hence on or below the top stab line in $\mathcal{L}_{\mathcal{R}}(R_t)$. Therefore, R_t satisfies property (iii) of Definition 5 as well. This completes the proof. \square

Observation G. Let $v \in V(G)$. For $i \in \{1, 2\}$, if $r_v \cap \mathbf{t}' = \emptyset$ (resp. $r_v \cap \mathbf{b}' = \emptyset$) and r_v intersects \mathbf{s}_i (resp. \mathbf{s}'_i), then r_v contains the point (x_i, y_i) (resp. (x'_i, y'_i)).

Proof. Suppose that $r_v \cap \mathbf{t}' = \emptyset$, but r_v intersects \mathbf{s}_i . As the top endpoint of \mathbf{s}_i is contained in \mathbf{t}' , we can infer that r_v does not contain the top endpoint of \mathbf{s}_i . If r_v also does not contain the bottom endpoint of \mathbf{s}_i , then there is no stab line that intersects r_v , as the top and bottom endpoints of \mathbf{s}_i are on consecutive stab lines. We can therefore conclude that the bottom endpoint of \mathbf{s}_i , which is (x_i, y_i) , is contained in r_v . The arguments for the other case are similar and are therefore omitted. \square

Lemma 29. Let $v \in V(G)$ such that r_v intersects the boundary of R_t (resp. R_b). Then either r_v intersects $\mathbf{t}' = \mathbf{top}(R_t)$ (resp. $\mathbf{b}' = \mathbf{bottom}(R_b)$) or v has a neighbour on at least one of the paths P_1, P_2 , or P .

Proof. We shall prove this lemma only for R_t as the arguments for R_b are similar. Suppose there exists a vertex $v \in V(G)$ such that r_v intersects the boundary of R_t , but r_v does not intersect \mathbf{t}' and v does not have a neighbour on any of the paths P_1, P_2 , or P . Then r_v does not intersect any of the curves $\mathbf{p}_1^t, \mathbf{p}$, or \mathbf{p}_2^t . From this, it follows that r_v does not contain the points (x_1, y_1) or (x_2, y_2) . By Observation G, we now have that r_v does not intersect \mathbf{s}_1 or \mathbf{s}_2 . Since the boundary of R_t is $\mathbf{t}' \cup \mathbf{s}_1 \cup \mathbf{p}_1^t \cup \mathbf{p} \cup \mathbf{p}_2^t \cup \mathbf{s}_2$, this means that r_v does not intersect the boundary of R_t , which is a contradiction. \square

Lemma 30. Let $v \in V(G_R)$ such that P misses v and there is a path in G_R from v to a vertex in P that misses both P_1 and P_2 . Then r_v is contained in R_t or R_b .

Proof. As v is not adjacent to any vertex in P, P_1 or P_2 , the rectangle r_v does not intersect \mathbf{p}, \mathbf{p}_1 or \mathbf{p}_2 . Also, as $v \in V(G_R)$, r_v does not intersect \mathbf{t}' or \mathbf{b}' . Then, by Observation G, we can further infer that r_v does not intersect $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}'_1$ or \mathbf{s}'_2 . This means that r_v is contained in one of the regions R_1, R_2, R_t or R_b . Now suppose for the sake of contradiction that r_v is contained in R_1 . We know from the statement of the lemma that there is at least one path in G_R from v to some vertex in P that misses both P_1 and P_2 . Let Q be such a path of minimum length and let u be the endpoint of Q other than v . It is clear that $V(P) \cap V(Q) = \{u\}$. Let p' be a point in $r_u \cap \mathbf{p}$ that is on a stab line (recall from the definition of rectilinear curves through paths that such a point exists). As u has no neighbour on P_1 or P_2 , it can be seen that p' is not an endpoint of \mathbf{p} , i.e., p' is an interior point of \mathbf{p} . Now consider the rectilinear path through Q from some point in r_v (that is on a stab line) to p' . As the point p' is not inside or on the boundary of R_1 , this rectilinear curve must cross the boundary of R_1 at some point p'' . It is clear that there is a vertex x in Q such that $p'' \in r_x$. Since r_x is contained in R , we can infer that p'' is on $\mathbf{s}_1 \cup \mathbf{p}_1 \cup \mathbf{s}'_1$ and also that r_x does not intersect \mathbf{t}' or \mathbf{b}' . If p'' is on \mathbf{s}_1 or \mathbf{s}'_1 , we have by Observation G that r_x intersects \mathbf{p}_1 . So we can conclude that in any case, r_x intersects \mathbf{p}_1 . Since from the definition of \mathbf{p}_1 , every point of \mathbf{p}_1 belongs to the rectangle corresponding to some vertex of P_1 , this implies that x is adjacent to some vertex of P_1 . This contradicts the fact that Q misses P_1 . We can thus conclude that r_v is not contained in R_1 . Using similar arguments, we can also infer that r_v is not contained in R_2 . This completes the proof. \square

Lemma 31. Let $v, w \in V(G_R)$ such that r_v is contained in $R' \in \{R_t, R_b\}$ and there is a path in G_R between v and w that misses P_1, P_2 and P . Then r_w is contained in R' .

Proof. We shall prove the statement of the lemma only for the case $R' = R_t$ as the proof for the case $R' = R_b$ is similar. Let Q be the path between v and w in G_R that misses P_1, P_2 and P . Let x be any vertex on Q . Clearly, x has no neighbour on P_1, P_2 or P . As $x \in V(G_R)$, the rectangle r_x is contained in R , implying that r_x does not intersect the boundary of R . As we have $\mathbf{top}(R_t) \subseteq \mathbf{top}(R)$ by Observation F(i), this means that r_x does not intersect $\mathbf{top}(R_t)$. By Lemma 29, we now have that r_x does not intersect the boundary of R_t . Therefore, no rectangle corresponding to a vertex in Q can intersect the boundary of R_t . Since r_v is contained in R_t , this means that the rectangle corresponding to each vertex of Q , and hence r_w , is contained in R_t . \square

We shall use the technical details about good regions and rectilinear curves only for the proof of Theorem 24. We now give a lemma that shall be sufficient for most of the other proofs. Given a graph G and a representation \mathcal{R} of G , we shall define $\mathcal{L}_{\mathcal{R}}(H)$, for any connected induced subgraph H of G , to be the set of stab lines of \mathcal{R} that intersect the rectangle corresponding to some vertex in $V(H)$. Note that $\mathcal{L}_{\mathcal{R}}(H)$ will contain a consecutive set of stab lines of \mathcal{R} .

Lemma 32. Let G be a connected k -SRIG and \mathcal{R} a k -stabbed rectangle intersection representation of it. Let H_1 and H_2 be two neighbour-disjoint connected induced subgraphs of G such that $\mathcal{L}_{\mathcal{R}}(H_1) = \mathcal{L}_{\mathcal{R}}(H_2) =$

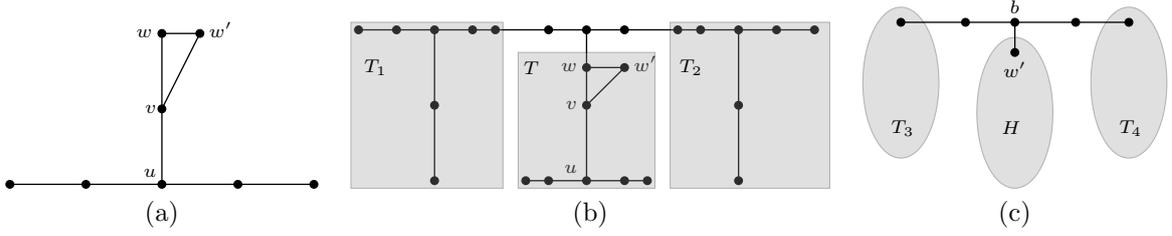


Figure 8: Construction of block graph for Proof of Theorem 23. (a) Construction of T . (b) Construction of H . T_1 and T_2 are isomorphic to G_2 . (c) Construction of G . T_3 and T_4 are isomorphic to G_3 and H is the block graph shown in (b).

$\mathcal{L}_{\mathcal{R}}(G) = k$. Let P be an induced path in G between some vertex in $V(H_1)$ and some vertex in $V(H_2)$ such that no internal vertex of P is in $V(H_1)$ or $V(H_2)$. Let H be a connected induced subgraph of G that is neighbour-disjoint from H_1 , H_2 and P such that there is a vertex in H from which there is a path to a vertex of P that misses both H_1 and H_2 . Then, $\mathcal{L}_{\mathcal{R}}(H) \subset \mathcal{L}_{\mathcal{R}}(G)$.

Proof. We shall augment \mathcal{R} to a new representation \mathcal{R}' by adding two new stab lines, one above the top stab line and the other below the bottom stab line of \mathcal{R} . Notice that for any connected induced subgraph G' of G , we have $\mathcal{L}_{\mathcal{R}'}(G') = \mathcal{L}_{\mathcal{R}}(G')$. Let A be a good region that contains all the rectangles of \mathcal{R}' , i.e., $G_A = G$ (note that such a region exists; we can consider a rectangle with top and bottom edges on the top and bottom stab lines such that it contains all the rectangles of \mathcal{R}'). As the only two stab lines that are not intersected by any rectangle in \mathcal{R}' are the top and bottom stab lines (recall that $\mathcal{L}_{\mathcal{R}}(G)$ contains all the stab lines of \mathcal{R}), it follows that $\mathcal{L}_{\mathcal{R}'}(A) = \mathcal{L}_{\mathcal{R}'}(G)$. It is clear that for any induced subgraph G' of G , $\mathcal{L}_{\mathcal{R}'}(G') = \mathcal{L}_{\mathcal{R}}(G')$. Therefore, we have $\mathcal{L}_{\mathcal{R}'}(H_1) = \mathcal{L}_{\mathcal{R}'}(H_2) = \mathcal{L}_{\mathcal{R}'}(G)$, which implies that there are $\mathcal{L}_{\mathcal{R}'}(A)$ -spanning paths in each of them. Let P_1 and P_2 be minimal $\mathcal{L}_{\mathcal{R}'}(A)$ -spanning paths in H_1 and H_2 respectively. As H_1 and H_2 are neighbour-disjoint, P_1 and P_2 are neighbour-disjoint. It is not hard to see that there exists an induced path P' in $G[V(H_1) \cup V(P) \cup V(H_2)]$ that contains P as a subpath, such that P' connects some vertex of P_1 to some vertex of P_2 and no internal vertex of P' belongs to either P_1 or P_2 . Let $(A_t, A_b) = \Delta(\mathcal{R}', A, P_1, P_2, P')$.

We know that there exists a vertex, say v , in H such that there is a path from v to a vertex of P that misses both H_1 and H_2 . Clearly, this is also a path from v to a vertex in P' that misses both P_1 and P_2 . As H is neighbour-disjoint from P , we know that P misses v . By Lemma 30, we know that r_v is contained in A_t or A_b . Let us assume without loss of generality that r_v is contained in A_t . Since H is a connected induced subgraph of G that is neighbour-disjoint from H_1 , H_2 and P , we know that there is a path from v to each vertex of H that misses H_1 , H_2 and P . This means that there is a path from v to each vertex of H that misses P_1 , P_2 and P' . Now, we can use Lemma 31 to conclude that the rectangles corresponding to the vertices of H are all contained in A_t . Since by Lemma 27(b), we know that $\mathcal{L}_{\mathcal{R}'}(A_t) \subset \mathcal{L}_{\mathcal{R}'}(A)$, we can now conclude that $\mathcal{L}_{\mathcal{R}'}(H) \subset \mathcal{L}_{\mathcal{R}'}(G)$, and therefore $\mathcal{L}_{\mathcal{R}}(H) \subset \mathcal{L}_{\mathcal{R}}(G)$. \square

Proof of Theorem 23.

Let T be the block graph obtained by taking a copy of the tree G_2 (defined in Section 6.1) and then introducing a true twin for one of the leaves. Let w, w' be the two true twins in T , v be their common neighbour and u the degree 3 vertex adjacent to v . See Figure 8(a) for a drawing of T . Notice that the graph G_2 is non-interval (folklore, or by Lemma 21(i)).

Let T_1 and T_2 be trees each isomorphic to G_2 . Let H be the graph obtained by taking the disjoint union of T_1 , T_2 and T and then doing the following: introduce a new vertex a , connect a to a leaf T_1 and to a leaf of T_2 using paths of length 2 and then make a adjacent to w (see Figure 8(b)).

Claim 1. H is non-(2-SRIG).

Proof. Note that $T - \{w\}$ is isomorphic to G_2 , and hence is non-interval. As $T_1, T_2, T - \{w\}$ are asteroidal-(non-interval) in H , by Theorem 12, we have that H is non-(2-SRIG).

It is easy to see that $H - \{w'\}$ is asteroidal-(non-interval)-free. Hence, by Theorem 19, we have that $H - \{w'\}$ is 2-SRIG.

Claim 2. The vertices w and v do not have a common stab in any 2-stabbed rectangle intersection representation of $H - \{w'\}$.

Proof. Let $H' = H - \{w'\}$. Let \mathcal{R} be any 2-stabbed rectangle intersection representation of H' . Since T_1 and T_2 are neighbour-disjoint connected induced subgraphs of H' that are non-interval, we have that $|\mathcal{L}_{\mathcal{R}}(T_1)| = |\mathcal{L}_{\mathcal{R}}(T_2)| = 2$. Let P be the (induced) path between T_1 and T_2 in H' . Notice that $T - \{w, w'\}$ is a connected induced subgraph of H' that is neighbour-disjoint from T_1, T_2 and P . Moreover, there is a path from the vertex v of $T - \{w, w'\}$ to the vertex a of P that misses T_1 and T_2 . We can now use Lemma 32 to conclude that $|\mathcal{L}_{\mathcal{R}}(T - \{w, w'\})| = 1$. Let $\mathcal{L}_{\mathcal{R}}(T - \{w, w'\}) = \{\ell\}$. It is clear that for each vertex of $T - \{w, w'\}$, and hence also for v , the only stab line that intersects the rectangle corresponding to it is ℓ . If r_w also intersects ℓ , then the collection $\{\ell \cap r_x\}_{x \in V(T - \{w, w'\})}$ would form an interval representation of G_2 , which contradicts the fact that G_2 is non-interval. This completes the proof of the claim.

We shall now construct the desired block graph G that satisfies the requirements in the statement of Theorem 23. Let T_3 and T_4 be trees that are isomorphic to G_3 (defined in Section 6.1). Let G' be the graph formed by taking the disjoint union of T_3 and T_4 and then doing the following: add a new vertex b and connect it to a vertex of T_3 using a path of length 2 and a vertex of T_4 using a path of length 2. The graph G is constructed by taking the disjoint union of H and G' and then adding an edge between b and w' (see Figure 8(c) for a schematic diagram of G).

Claim 3. G is not 3-SRIG.

Proof. Suppose for the sake of contradiction that G is 3-SRIG. Let \mathcal{R} be a 3-stabbed rectangle intersection representation of G . Since T_3 and T_4 are neighbour-disjoint connected induced subgraphs of G that are non-(2-SRIG) (recall that T_3 and T_4 are isomorphic to G_3 and that G_3 is non-(2-SRIG) by Lemma 21(i)), we have that $|\mathcal{L}_{\mathcal{R}}(T_3)| = |\mathcal{L}_{\mathcal{R}}(T_4)| = 3$. Let P be the path between T_3 and T_4 in G . Notice that $H - \{w'\}$ is a connected induced subgraph of G that is neighbour-disjoint from T_3, T_4 and P . Moreover, there is a path from the vertex w of $H - \{w'\}$ to the vertex b of P that misses T_3 and T_4 . We can now use Lemma 32 to conclude that $|\mathcal{L}_{\mathcal{R}}(H - \{w'\})| = 2$. This means that in \mathcal{R} , the rectangles corresponding to $H - \{w'\}$ form a 2-stabbed rectangle intersection representation of $H - \{w'\}$. Then, by Claim 2, we know that neither of the two stab lines in $\mathcal{L}_{\mathcal{R}}(H - \{w'\})$ intersects both r_w and r_v . Since w' is adjacent to both w and v , this implies that $r_{w'}$ intersects at least one of the two stab lines in $\mathcal{L}_{\mathcal{R}}(H - \{w'\})$. But then, the rectangles corresponding to the vertices of H , together with the stab lines in $\mathcal{L}_{\mathcal{R}}(H - \{w'\})$, form a 2-stabbed rectangle intersection representation of H . This contradicts Claim 1.

To complete the proof of the theorem, we only need to show that G is asteroidal-(non-2-SRIG)-free. Suppose for the sake of contradiction that there exist induced subgraphs X_1, X_2, X_3 that are asteroidal-(non-2-SRIG) in G . First we need the following claim, whose proof is left to the reader.

Claim. In any block graph that contains three induced subgraphs that are asteroidal- \mathcal{C} in it, for some graph class \mathcal{C} , there exists either a cutvertex that has no neighbour in each of the three subgraphs, or a triangle, whose removal results in a graph in which each of the three subgraphs is in a different component.

From the above claim, we have that either there exists a vertex $x \in V(G)$ such that $G - \{x\}$ has three components X'_1, X'_2, X'_3 such that for each $i \in \{1, 2, 3\}$, $V(X_i) \subseteq V(X'_i) \setminus N[x]$, or X_1, X_2, X_3 are each contained in a different component of $G - \{w, w', v\}$ (since the only triangle in G is formed by w, w' and v). Let us first suppose that X_1, X_2, X_3 are each contained in a different component of the three components in $G - \{w, w', v\}$. It is easy to see that the component of $G - \{w, w', v\}$ that contains a neighbour of v is a path and is therefore 1-SRIG, contradicting the fact that it contains one of the non-(2-SRIG) graphs X_1, X_2, X_3 . So we can assume that there exists a vertex $x \in V(G)$ such that $G - x$ has three components X'_1, X'_2, X'_3 such that for each $i \in \{1, 2, 3\}$, $V(X_i) \subseteq V(X'_i) \setminus N[x]$. Note that since $G - \{x\}$ contains at least three components, degree of x is at least 3 and $x \notin \{w, w', v\}$.

Let us first suppose that $x \in V(G')$. If $x = b$, one of the three components of $G - \{x\}$, say X'_1 , is H . But now, $V(X'_1) \setminus N[x] = H - \{w'\}$, which is 2-SRIG by our earlier observation. This contradicts the fact that $V(X_1) \subseteq V(X'_1) \setminus N[x]$ as X_1 is non-(2-SRIG). If $x \neq b$, then $x \in V(T_3)$ or $x \in V(T_4)$. Suppose that $x \in V(T_3)$. As G' is a tree, we know that $G' - \{x\}$ contains at least three components. Also, as $G' - V(T_3)$ has only one component, we can use Observation C(ii) to conclude that all components of $G' - \{x\}$ except the component Y that contains b are proper subtrees of T_3 . Since the only edge between $V(G) \setminus V(G')$ and $V(G')$ is $w'b$, we can see that every component of $G' - \{x\}$ other than Y is also a component of $G - \{x\}$. This means that at least two components, say X'_1, X'_2 , of $G - \{x\}$ are also components of $G' - \{x\}$. Since $V(X_1) \subseteq V(X'_1)$ and $V(X_2) \subseteq V(X'_2)$, we have that X'_1 and X'_2 are non-(2-SRIG) neighbour-disjoint induced

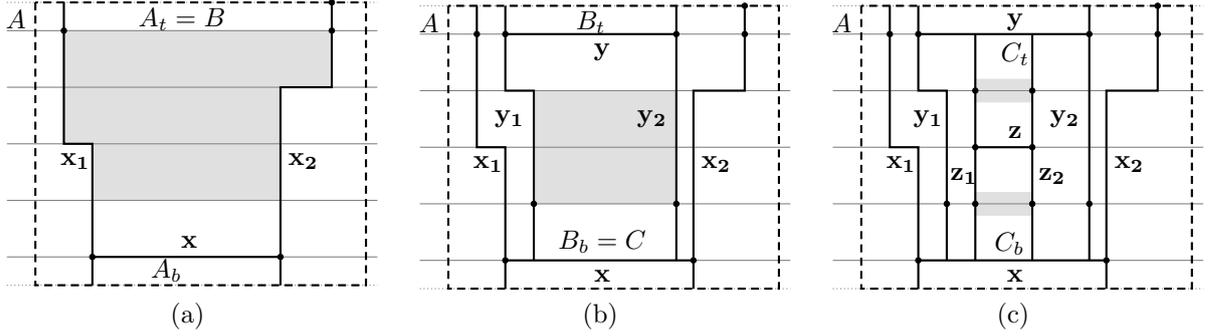


Figure 9: An illustration of various stages of the proof of Theorem 33. The region bounded by the dashed curve is A . The solid curves represent the rectilinear curves through paths chosen in the proof to split the regions. For example, the solid curve labelled \mathbf{x}_1 is the rectilinear curve through the path X_1 , the solid curve labelled \mathbf{y} is the rectilinear curve through the path Y and so on. The shaded region indicates the possible locations of the rectangle r_c as the proof proceeds.

subgraphs of T_3 . As T_3 is isomorphic to G_3 , this is a contradiction to Lemma 21(iv). For the same reason, we can also conclude that $x \notin V(T_4)$. This means that $x \in V(H)$.

But if $x \in V(H)$, then since $x \notin \{w, w', v\}$, it is clear from the construction of G that at least one of the components, say X'_1 , of $G - \{x\}$ is an induced subgraph of $H - \{w'\}$. As $H - \{w'\}$ is 2-SRIG by our earlier observation, this means that X'_1 is 2-SRIG, which contradicts the fact that it contains the non-(2-SRIG) graph X_1 as an induced subgraph. This shows that G is asteroidal non-(2-SRIG)-free and hence completes the proof. \square

We shall now prove a general theorem that will later be used to prove Theorem 24.

Theorem 33. *Let $k \geq 4$. For each $i \in \{k, k-1, k-2\}$, let T_i, T'_i be two graphs that are i -SRIG but not $(i-1)$ -SRIG and let $a_i \in V(T_i)$ and $a'_i \in V(T'_i)$. For $i \in \{k, k-1, k-2\}$, let H_i be the graph obtained by adding a new vertex b_i to the disjoint union of T_i and T'_i and connecting it to a_i and a'_i using paths of length at least two. Let T be the graph obtained by adding a new vertex c to the disjoint union of H_k, H_{k-1} and H_{k-2} and then connecting c to each of b_k, b_{k-1} and b_{k-2} using paths of length at least two. Then T is not k -SRIG.*

Proof. Suppose for the sake of contradiction that T is k -SRIG. Let \mathcal{R} be a $(k+2)$ -stabbed rectangle intersection representation of T in which the top and bottom stab lines do not intersect any rectangle. Let A be a good region that contains all the rectangles of \mathcal{R} , i.e., $T_A = T$ (note that such a region exists; we can consider a rectangle with top and bottom edges on the top and bottom stab lines such that it contains all the rectangles of \mathcal{R}). As the only two stab lines that are not intersected by any rectangle in \mathcal{R} are the top and bottom stab lines (recall that T_A is not $(k-1)$ -SRIG as it contains T_k and T'_k), it follows that $|\mathcal{L}_{\mathcal{R}}(A)| = k$. As T_k and T'_k are k -SRIG but not $(k-1)$ -SRIG, we know that there are $\mathcal{L}_{\mathcal{R}}(A)$ -spanning paths in each of them. Let X_1 and X_2 be minimal $\mathcal{L}_{\mathcal{R}}(A)$ -spanning paths in T_k and T'_k respectively. It is easy to see that X_1 and X_2 are neighbour-disjoint. Let X be an induced path in T_A that connects some vertex of X_1 and some vertex of X_2 such that no internal vertex of X belongs to either X_1 or X_2 . Note that X is a subgraph of H_k that contains b_k . Let $(A_t, A_b) = \Delta(\mathcal{R}, A, X_1, X_2, X)$.

(+) By Observation F(iv), if for $x \in V(T)$, the rectangle r_x intersects $\mathbf{bottom}(A_t)$, then x has a neighbour on X .

Since there is a path in T_A from $c \in V(T_A)$ to a vertex in X (in this case, b_k) that misses both X_1 and X_2 , we know by Lemma 30 that r_c is contained in A_t or A_b . We shall assume without loss of generality that r_c is contained in A_t (see Figure 9(a)). Let us define $B = A_t$. Let T^* be the graph obtained by removing the vertices in $V(H_k)$ and their neighbours from T , or in other words, $T^* = T - (V(H_k) \cup N[b_k])$. Note that there is a path in T_A from c to each vertex of T^* that misses X_1, X_2 and X . We can now infer using Lemma 31 that the rectangles corresponding to the vertices in T^* are all contained in $A_t = B$. In other words, T^* is a connected induced subgraph of T_B .

Since T^* contains T_{k-1} and T'_{k-1} as induced subgraphs, and is therefore not $(k-2)$ -SRIG, we have $|\mathcal{L}_{\mathcal{R}}(B)| \geq k-1$. By Lemma 28, this means that $B = A_t$ is a good region. Since B does not contain the bottom stab line in $\mathcal{L}_{\mathcal{R}}(A)$ by Lemma 27(b), we can conclude that $|\mathcal{L}_{\mathcal{R}}(B)| = k-1$. Now, T_{k-1} and T'_{k-1} are two neighbour-disjoint subgraphs of T^* that are $(k-1)$ -SRIG but not $(k-2)$ -SRIG. Since the rectangles

corresponding to the vertices in them are all contained in B (recall that T^* is an induced subgraph of T_B), there is at least one vertex of T_{k-1} and at least one vertex of T'_{k-1} on every stab line in $\mathcal{L}_{\mathcal{R}}(B)$. This means that there exist minimal $\mathcal{L}_{\mathcal{R}}(B)$ -spanning paths Y_1 in T_{k-1} and Y_2 in T'_{k-1} , and it is clear that Y_1 and Y_2 are neighbour-disjoint. Let Y be an induced path in T^* that connects some vertex of Y_1 and some vertex of Y_2 such that no internal vertex of Y belongs to either Y_1 or Y_2 . Note that Y is a subgraph of H_{k-1} that contains b_{k-1} . Let $(B_t, B_b) = \Delta(\mathcal{R}, B, Y_1, Y_2, Y)$.

(++) By Observation F(iv), if for $x \in V(T)$, the rectangle r_x intersects $\mathbf{top}(B_b)$, then x has a neighbour on Y .

Since there is a path in T^* from c to a vertex in Y (in this case, b_{k-1}) that misses both Y_1 and Y_2 , we know by Lemma 30 that r_c is contained in B_t or B_b . Suppose that r_c is contained in B_t . Note that the path Q in T between c and b_k misses Y_1 , Y_2 and Y . As b_k lies on the path X , we know by Observation F(iii) that r_{b_k} intersects $\mathbf{bottom}(B)$. This means that r_{b_k} contains some points from outside B and hence some points from outside B_t . Since r_c is contained in B_t , this can only mean that there exists some vertex x in Q such that the rectangle r_x intersects the boundary of B_t . Since x has no neighbour on Y_1 , Y_2 or Y , we know by Lemma 29 that r_x intersects $\mathbf{top}(B_t)$. Since $B = A_t$ and A are good regions, we have by Observation F(i) that $\mathbf{top}(B_t) \subseteq \mathbf{top}(A_t) \subseteq \mathbf{top}(A)$. This implies that r_x intersects the boundary of A , which is a contradiction to the fact that $T = T_A$ (or in other words, all rectangles corresponding to vertices of T are contained in A). Thus, we can conclude that r_c is not contained in B_t , and hence is contained in B_b (See Figure 9(b)). Let us define $C = B_b$.

Let T^{**} be the graph obtained by removing the vertices in $V(H_{k-1})$ and their neighbours from T^* , or in other words, $T^{**} = T^* - (V(H_{k-1}) \cup N[b_{k-1}])$. Note that $c \in V(T^{**})$ and that there is a path in T^* from c to each vertex of T^{**} that misses Y_1 , Y_2 and Y . We can now infer using Lemma 31 that the rectangles corresponding to the vertices in T^{**} are all contained in C . In other words, T^{**} is a connected induced subgraph of T_C .

Since T^{**} contains T_{k-2} and T'_{k-2} as induced subgraphs, and is therefore not $(k-3)$ -SRIG, we have $|\mathcal{L}_{\mathcal{R}}(C)| \geq k-2$. By Lemma 28, this means that C is a good region. Since C does not contain the top stab line in $\mathcal{L}_{\mathcal{R}}(B)$ by Lemma 27(b), we can conclude that $|\mathcal{L}_{\mathcal{R}}(C)| = k-2$. Now, T_{k-2} and T'_{k-2} are two neighbour-disjoint subgraphs of T^{**} that are $(k-2)$ -SRIG but not $(k-3)$ -SRIG. Since T^{**} is an induced subgraph of T_C , at least one vertex of T_{k-2} and at least one vertex of T'_{k-2} are on every stab line in $\mathcal{L}_{\mathcal{R}}(C)$. This means that there exist minimal $\mathcal{L}_{\mathcal{R}}(C)$ -spanning paths Z_1 in T_{k-2} and Z_2 in T'_{k-2} , which are neighbour-disjoint. Let Z be an induced path in T^{**} that connects some vertex of Z_1 and some vertex of Z_2 such that no internal vertex of Z belongs to either Z_1 or Z_2 . Note that Z is a subgraph of H_{k-2} that contains b_{k-2} . Let $(C_t, C_b) = \Delta(\mathcal{R}, C, Z_1, Z_2, Z)$.

Since there is a path in T^{**} from c to a vertex in Z (in this case, b_{k-2}) that misses both Z_1 and Z_2 , we know by Lemma 30 that r_c is contained in C_t or C_b (See Figure 9(c)). Suppose that r_c is contained in C_t . Note that the path Q in T between c and b_k misses Z_1 , Z_2 , Z and Y . As b_k lies on the path X , we know by Observation F(iii) that r_{b_k} intersects $\mathbf{bottom}(B)$. This means that r_{b_k} contains some points from outside B , and hence some points from outside C_t . Since r_c is contained in C_t , this can only mean that there exists some vertex x in Q such that the rectangle r_x intersects the boundary of C_t . Since x has no neighbour on Z_1 , Z_2 or Z , we know by Lemma 29 that r_x intersects $\mathbf{top}(C_t)$. Since $C = B_b$ is a good region, we have by Observation F(i) that $\mathbf{top}(C_t) \subseteq \mathbf{top}(B_b)$, implying that r_x intersects $\mathbf{top}(B_b)$. By (++), we now have that x has a neighbour on Y , which is a contradiction to the fact that Q misses Y . This means that r_c is contained in C_b .

Now consider the path Q in T between c and b_{k-1} . It is clear that Q misses Z_1 , Z_2 , Z and X . As b_{k-1} lies on the path Y , we know by Observation F(iii) that $r_{b_{k-1}}$ intersects $\mathbf{top}(C)$. This means that $r_{b_{k-1}}$ contains some points from outside C , and hence some points from outside C_b . Since r_c is contained in C_b , this can only mean that there exists some vertex x in Q such that the rectangle r_x intersects the boundary of C_b . Since x has no neighbour on Z_1 , Z_2 or Z , we know by Lemma 29 that r_x intersects $\mathbf{bottom}(C_b)$. Since $C = B_b$ and $B = A_t$ are good regions, we have by Observation F(i) that $\mathbf{bottom}(C_b) \subseteq \mathbf{bottom}(B_b) \subseteq \mathbf{bottom}(A_t)$, implying that r_x intersects $\mathbf{bottom}(A_t)$. By (+), we now have that x has a neighbour on X , which is a contradiction to the fact that Q misses X . This completes the proof. \square

Proof of Theorem 24.

Let k be any integer greater than or equal to 4. For each $i \in \{k, k-1, k-2\}$, let T_i, T'_i be two rooted trees that are each isomorphic to G_i (defined in Section 6.1). From Lemma 21(i) and Lemma 21(ii) we know that T_i and T'_i are i -SRIG but not $(i-1)$ -SRIG. Let $a_i = \mathit{root}(T_i)$ and $a'_i = \mathit{root}(T'_i)$. For $i \in \{k, k-1, k-2\}$,

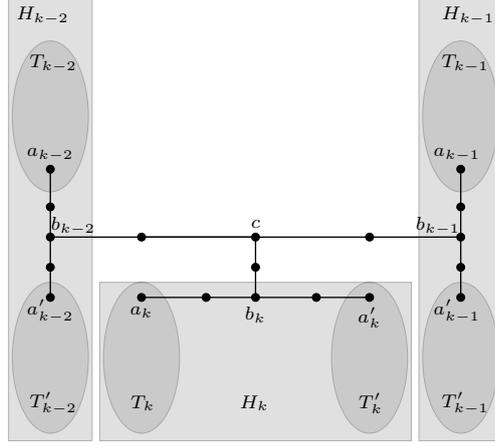


Figure 10: A schematic diagram of T . For each $i \in \{k, k-1, k-2\}$, let T_i, T'_i be two rooted trees that are each isomorphic to G_i (defined in Section 6.1) and rooted at a_i and a'_i respectively.

let H_i be the tree obtained by adding a new vertex b_i to the disjoint union of T_i and T'_i and connecting it to a_i and a'_i using paths of length two. Note that H_i is isomorphic to F_i (also defined in Section 6.1). Let T be the tree obtained by adding a new vertex c to the disjoint union of H_k, H_{k-1} and H_{k-2} and then connecting c to each of b_k, b_{k-1} and b_{k-2} using paths of length at least two. See Figure 10 for a schematic diagram of T . From Theorem 33, we know that T is not k -SRIG.

We now show that T is asteroidal-(non- $(k-1)$ -ESRIG)-free. For the sake of contradiction, assume that there are three subtrees X_1, X_2, X_3 that are asteroidal-(non- $(k-1)$ -ESRIG) in T . The following claim is easy to see.

Claim. *There is a vertex v in T of degree at least 3 such that $T - \{v\}$ contains three components X'_1, X'_2, X'_3 where for each $i \in \{1, 2, 3\}$, X'_i is an induced subtree of $X'_i - N[v]$.*

Let v be the vertex in T of degree at least 3 such that $T - \{v\}$ contains three components X'_1, X'_2, X'_3 where for each $i \in \{1, 2, 3\}$, X'_i is an induced subtree of $X'_i - N[v]$. For each $i \in \{1, 2, 3\}$, since X'_i is non- $(k-1)$ -ESRIG, we also have that X'_i is non- $(k-1)$ -ESRIG. Let us assume that v is a vertex of T_k . Note that $T - V(T_k)$ has only one component. Then by Observation C(ii), all but one component of $T - \{v\}$ are proper subtrees of T_k . This implies that there exist distinct $X, Y \in \{X'_1, X'_2, X'_3\}$ such that X, Y are proper subtrees of T_k . Therefore, X and Y are vertex-disjoint (in fact, neighbour-disjoint) subtrees of T_k that are both non- $(k-1)$ -ESRIG. But since T_k is isomorphic to G_k , this is a contradiction to Lemma 21(iv). Hence, v is not a vertex of T_k and for similar reasons, v is not a vertex of T'_k . Let $T^* = T - (V(H_k) \cup N[b_k])$.

Claim. *The tree T^* is $(k-1)$ -ESRIG.*

Proof. From the definition of T , we know that T^* is the union of H_{k-1}, H_{k-2} and the path in T between b_{k-1} and b_{k-2} (which contains the vertex c). Recall that H_{k-1} is obtained by adding a new vertex b_{k-1} to the disjoint union of T_{k-1} and T'_{k-1} and connecting their roots (i.e. a_{k-1} and a'_{k-1} respectively) to b_{k-1} using paths of length two. Therefore, H_{k-1} is isomorphic to F_{k-1} . Let \mathcal{R}_1 be the $(k-1)$ -exactly stabbed rectangle intersection representation of H_{k-1} that is given by Lemma 21(iii)(a). Similarly, H_{k-2} is isomorphic to F_{k-2} . Let \mathcal{R}_2 be the $(k-2)$ -exactly stabbed rectangle intersection representation of H_{k-2} that is given by Lemma 21(iii)(b), in which the only vertices on the top stab line are those in $N[a_{k-2}] \cup N[a'_{k-2}]$. It can now be seen that the two representations \mathcal{R}_1 and \mathcal{R}_2 can be combined as shown in Figure 11 to obtain a $(k-1)$ -exactly stabbed rectangle intersection representation \mathcal{R} of $T^*[V(H_{k-1}) \cup V(H_{k-2})]$ that satisfies the following properties: (i) all vertices of the path between a_{k-1} and a'_{k-1} are on the top stab line of \mathcal{R} , (ii) a vertex $u \in V(H_{k-2})$ is on the stab line just below the top stab line of \mathcal{R} if and only if $u \in N[a_{k-2}] \cup N[a'_{k-2}]$, and (iii) for any vertex $u \in V(H_{k-2})$, we have that $\text{span}(u) \subset \text{span}(b_{k-1})$. We leave it to the reader to verify that \mathcal{R} can be extended to a $(k-1)$ -exactly stabbed rectangle intersection representation of T^* by adding the rectangles corresponding to the three vertices in the path between b_{k-1} and b_{k-2} (refer to Figure 11). Therefore we conclude that T^* is $(k-1)$ -ESRIG.

Now suppose v is a vertex of H_k . Since we have already concluded that $v \notin V(T_k) \cup V(T'_k)$, we can infer that v must be the vertex b_k . Recalling the definition of T , we can infer that $T - \{b_k\}$ has exactly three

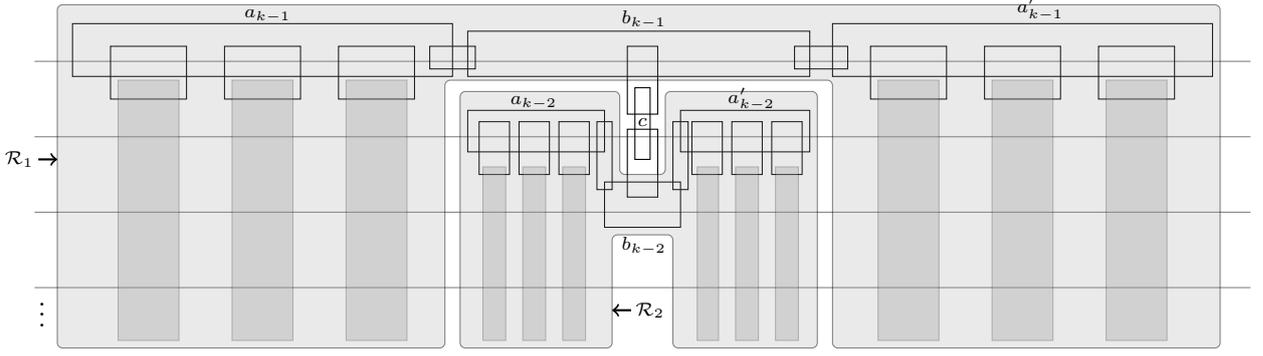


Figure 11: A schematic diagram of the $(k-1)$ -stabbed rectangle intersection representation \mathcal{R} of T^* .

components and since $b_k = v$ we know that they are X'_1, X'_2, X'_3 . Also from the definition of T , it follows that there exists $i \in \{1, 2, 3\}$ such that $X'_i = T - V(H_k)$. We know from the definition of v that X_i is a subtree of $X'_i - N[v] = T^*$. But then by the above claim, we have that X_i is $(k-1)$ -ESRIG, which contradicts the fact that X_i is non- $(k-1)$ -ESRIG.

From the above arguments, we infer that v must lie in the tree $T - V(H_k)$. Since v has degree at least 3, we can infer from the construction of T that $v \in V(T^*)$. Notice that $T - V(T^*)$ has only one component. Then by Observation C(ii), all but one component of $T - \{v\}$ are proper subtrees of T^* . This implies that there is a component $X \in \{X'_1, X'_2, X'_3\}$ such that X is a proper subtree of T^* . But by the above claim, we now have that X is $(k-1)$ -ESRIG, contradicting our earlier observation that X'_1, X'_2, X'_3 are all non- $(k-1)$ -ESRIG. This completes the proof. \square

6.3 Trees that are k -SRIG but not k -ESRIG

We define the tree D_l , for $l > 1$, as follows. Let T_1, T_2, \dots, T_7 be seven rooted trees, each isomorphic to G_{l-1} . Take a $K_{1,7}$ with vertex set $\{u, u_1, u_2, \dots, u_7\}$, where u_1, u_2, \dots, u_7 are the leaves, and add edges between u_i and $root(T_i)$ for each $i \in \{1, 2, \dots, 7\}$. The resulting graph is D_l and we let $root(D_l) = u$.

Lemma 34. *Let $l > 1$.*

- (i) D_l is not $(l-1)$ -SRIG.
- (ii) *There is an l -exactly stabbed rectangle intersection representation \mathcal{R} of D_l such that for $v, w \in V(D_l)$, $span(v) \subseteq span(w)$ if w is an ancestor of v and the rectangles intersecting the top stab line of \mathcal{R} are exactly the vertices in $N[root(D_l)]$.*
- (iii) *Let T and T' be two trees each isomorphic to D_l . Let J_l be the tree obtained by taking a new vertex x and joining it to the root vertices of T, T' using paths of length two.*
 - (a) *There is an l -exactly stabbed rectangle intersection representation \mathcal{R}' of J_l such that for $v, w \in V(J_l)$, $span(v) \subseteq span(w)$ if w is an ancestor of v in T or T' , and all vertices in the path between $root(T)$ and $root(T')$ are on the top stab line of \mathcal{R}' .*
 - (b) *If $l \geq 6$, then in any l -exactly stabbed rectangle intersection representation of J_l , $root(T)$ and $root(T')$ are either both on the top stab line or both on the bottom stab line.*
- (iv) *In any l -exactly stabbed rectangle intersection representation \mathcal{R} of D_l , $root(D_l)$ is on the top or bottom stab line of \mathcal{R} .*

Proof. For (i), it is easy to see that G_l is an induced subgraph of D_l , and therefore by Lemma 21(i), D_l is not $(l-1)$ -SRIG. It is also easy to see that the constructions in the proofs of Lemma 21(ii) and Lemma 21(iii)(a) can be easily extended to prove (ii) and (iii)(a) respectively.

We shall now prove (iv). Suppose for the sake of contradiction that there exists an l -exactly stabbed rectangle intersection representation \mathcal{R} of D_l in which $root(D_l)$ is not on the top or bottom stab lines. Recall that D_l is constructed by taking a $K_{1,7}$ with vertex set $\{u, u_1, u_2, \dots, u_7\}$ with leaves u_1, u_2, \dots, u_7 and making each u_i adjacent to the root of a tree T_i that is isomorphic to G_{l-1} . For each $i \in \{1, 2, \dots, 7\}$, let $T'_i = D_l[\{u_i\} \cup V(T_i)]$. Suppose that there exists $I \subseteq \{1, 2, \dots, 7\}$ with $|I| = 3$ such that for each $i \in I$, there is no vertex in T'_i that is on the top stab line. Then, since $u = root(D_l)$ is not on the top stab line,

the rectangles corresponding to the vertices of $\{u\} \cup \bigcup_{i \in I} V(T'_i)$ form an $(l-1)$ -(exactly) stabbed rectangle intersection representation of a tree isomorphic to G_l . This contradicts Lemma 21(i). Therefore, there are at most two trees in $\{T'_1, T'_2, \dots, T'_7\}$ such that none of their vertices are on the top stab line. In similar fashion, we can conclude that there are at most two trees in $\{T'_1, T'_2, \dots, T'_7\}$ such that none of their vertices are on the bottom stab line. This means that there are at least three trees in $\{T'_1, T'_2, \dots, T'_7\}$, say T'_1, T'_2, T'_3 , such that $|\mathcal{L}_{\mathcal{R}}(T'_1)| = |\mathcal{L}_{\mathcal{R}}(T'_2)| = |\mathcal{L}_{\mathcal{R}}(T'_3)| = l$. For $i \in \{1, 2, 3\}$, let P_i be an $\mathcal{L}_{\mathcal{R}}(T'_i)$ -spanning induced path in T'_i starting at a vertex x_i that is on the top stab line and ending at a vertex y_i that is on the bottom stab line. Let \mathbf{p}_i be a rectilinear curve through P_i starting at some point on the top stab line in r_{x_i} and ending at some point on the bottom stab line in r_{y_i} . As T'_1, T'_2, T'_3 are pairwise neighbour-disjoint, we know that P_1, P_2, P_3 are also pairwise neighbour-disjoint, implying that the curves $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are pairwise disjoint. Therefore one of the curves, say \mathbf{p}_2 , is between the other two. Then, it is easy to see that any path between a vertex of T'_1 and a vertex of T'_3 contains a vertex whose rectangle intersects \mathbf{p}_2 , which means that this vertex has a neighbour on P_2 . Now consider the path $u_1 u u_3$. As the only vertex on this path that has a neighbour in $V(T_2)$ is $u = \text{root}(D_l)$, we can infer that r_u intersects \mathbf{p}_2 . It follows from the definition of rectilinear curves that there is a point $q \in r_u \cap \mathbf{p}_2$ that is also on a stab line, say ℓ . As $u = \text{root}(D_l)$ is on ℓ , we can conclude that ℓ is neither the top nor the bottom stab line of \mathcal{R} . Since the point $q \in \mathbf{p}_2$, it belongs to the rectangle corresponding to a vertex on P_2 that intersects r_u . Note that if u has a neighbour on P_2 , then it has to be u_2 . This lets us conclude that u_2 is on P and also that $q \in r_{u_2}$, which implies that u_2 is on ℓ . As \mathcal{R} is an l -exactly stabbed rectangle intersection representation, we infer that u_2 is neither on the top nor the bottom stab line. Then, $u_2 \notin \{x_i, y_i\}$. But this means that $x_i, y_i \in V(T_i)$, implying that the path P_2 does not contain u_2 . This contradicts our earlier observation that u_2 is on P_2 .

It only remains to prove (iii)(b). Let $l \geq 6$ and let \mathcal{R} be any l -exactly stabbed rectangle intersection representation of J_l . Let ℓ, ℓ' be the stab lines that intersect $r_{\text{root}(T)}$ and $r_{\text{root}(T')}$ respectively. By (iv), we know that each of ℓ, ℓ' is either the top stab line or the bottom stab line. Since there is a path of length 4 between $\text{root}(T)$ and $\text{root}(T')$ in J_l , we can infer that ℓ and ℓ' have no more than 3 stab lines between them. Since $l \geq 6$, this means that it is not possible that one of ℓ, ℓ' is the top stab line and the other the bottom stab line. So ℓ, ℓ' are either both the top stab line or both the bottom stab line. \square

Lemma 35. *Let \mathcal{R} be a k -exactly stabbed rectangle intersection representation of a graph G and let R be a good region in this representation. Let P_1 and P_2 be minimal $\mathcal{L}_{\mathcal{R}}(R)$ -spanning paths in G_R that are neighbour-disjoint and let P be an induced path in G_R between some vertex in $V(P_1)$ and some vertex in $V(P_2)$ such that no internal vertex of P is on P_1 or P_2 . Let $(R_t, R_b) = \Delta(\mathcal{R}, R, P_1, P_2, P)$. Suppose that there are two nonadjacent vertices $x_1, x_2 \in V(P)$ that are on the top (bottom) stab line in $\mathcal{L}_{\mathcal{R}}(R)$ such that the subpath P' of P between x_1 and x_2 has length at most d , for some $d \geq 2$. Then there does not exist a connected induced subgraph H of G_{R_t} (G_{R_b}) which is neighbour-disjoint from P_1, P_2, P and satisfies the following properties:*

- (i) $|\mathcal{L}_{\mathcal{R}}(H)| > \lceil \frac{d-1}{2} \rceil$, and
- (ii) H contains a vertex c such that there exists a path in G_R from c to some vertex in P' that misses x_1, x_2, P_1, P_2 and $P - V(P')$.

Proof. We shall prove the lemma only for the case when x_1 and x_2 are on the top stab line in $\mathcal{L}_{\mathcal{R}}(R)$, as the other case can be proved in similar fashion. Suppose there exists a connected component H of G_{R_t} that is neighbour-disjoint from P_1, P_2 and P such that $|\mathcal{L}_{\mathcal{R}}(H)| > \lceil \frac{d-1}{2} \rceil$, and there exists $c \in V(H)$ from which there is a path Q in G_R to some vertex in P' that misses x_1, x_2, P_1, P_2 and $P - V(P')$. For $i \in \{1, 2\}$, let u_i, v_i be the endvertices of P_i on the top and bottom stab lines in $\mathcal{L}_{\mathcal{R}}(R)$ respectively, and let $V(P_i) \cap V(P) = \{w_i\}$. Let us assume without loss of generality that x_1 appears before x_2 when traversing the path P from w_1 to w_2 . For $i \in \{1, 2\}$, define P'_i to be the path obtained by the union of the subpath of P_i between v_i and w_i and the subpath of P between w_i and x_i . It is clear that P'_1 and P'_2 are neighbour-disjoint $\mathcal{L}_{\mathcal{R}}(R)$ -spanning paths in G_R and that P' is an induced path in G_R between a vertex in $V(P'_1)$ and a vertex in $V(P'_2)$ none of whose internal vertices are on either P'_1 or P'_2 . Let $(R'_t, R'_b) = \Delta(\mathcal{R}, R, P'_1, P'_2, P')$. As \mathcal{R} is a k -exactly stabbed rectangle intersection representation and P' has length d , it follows that $|\mathcal{L}_{\mathcal{R}}(R'_i)| \leq \lceil \frac{d-1}{2} \rceil$.

Since P' misses c and there is the path Q in G_R between c and a vertex of P' that misses both P'_1 and P'_2 , we can apply Lemma 30 to conclude that r_c is contained in R'_t or R'_b . It is easy to see that for any vertex z that misses P_1, P_2 and P , the rectangle r_z is contained in R'_b if and only if it is contained in R_b . As we know that $c \in V(G_{R_t})$, which implies that r_c is contained in R_t and therefore not in R_b , we can now conclude that r_c is contained in R'_t . Since H is neighbour-disjoint from P_1, P_2 and P , it is also neighbour-disjoint from P'_1, P'_2 and P' . As H is connected, this means that there is a path in G_R from c to each vertex of H that

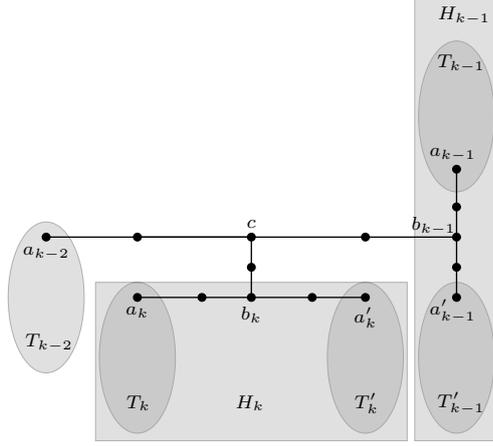


Figure 12: A schematic diagram of T . For each $i \in \{k, k-1\}$, let T_i, T'_i two rooted trees that are each isomorphic to D_i and rooted at a_i and a'_i respectively. T_{k-2} is isomorphic to D_{k-2} and is rooted at a_{k-2} .

misses P'_1, P'_2 and P' . By Lemma 31, we now have that H is an induced subgraph of $G_{R'_t}$. This means that $|\mathcal{L}_{\mathcal{R}}(R'_t)| > \lceil \frac{d-1}{2} \rceil$, contradicting our earlier observation. \square

Theorem 36. *For every $k \geq 10$, there is a tree which is k -SRIG but not k -ESRIG.*

Proof. Let k be any integer greater than or equal to 10. For each $i \in \{k, k-1\}$, let T_i, T'_i be two rooted trees that are each isomorphic to D_i and let T_{k-2} be a tree isomorphic to D_{k-2} . From Lemma 34(i) and Lemma 34(ii), we know that for $i \in \{k, k-1\}$, T_i and T'_i are i -SRIG but not $(i-1)$ -SRIG. For $i \in \{k, k-1\}$, let $a_i = \text{root}(T_i)$ and $a'_i = \text{root}(T'_i)$. Further, let H_i be the tree obtained by adding a new vertex b_i to the disjoint union of T_i and T'_i and connecting it to a_i and a'_i using paths of length two. Let $a_{k-2} = \text{root}(T_{k-2})$. Let T be the tree obtained by adding a new vertex c to the disjoint union of H_k, H_{k-1} and T_{k-2} and then connecting c to each of b_k, b_{k-1} and a_{k-2} using paths of length two. See Figure 12 for a schematic diagram of T . We claim that T is k -SRIG but not k -ESRIG.

We will first show that T is k -SRIG. Let $\ell_1, \ell_2, \dots, \ell_k$ be k horizontal lines, ordered from bottom to top. Since H_k is isomorphic to J_k , we know from Lemma 34(iii)(a) that there is a k -(exactly) stabbed rectangle intersection representation \mathcal{R}_1 of H_k using stab lines $\ell_1, \ell_2, \dots, \ell_k$ such that for $v, w \in V(H_k)$, $\text{span}(v) \subseteq \text{span}(w)$ if w is an ancestor of v in T_k or T'_k , and all vertices in the path in T between a_k and a'_k are on the bottom stab line ℓ_1 . Similarly, there is a $(k-1)$ -(exactly) stabbed rectangle intersection representation \mathcal{R}_2 of H_{k-1} using stab lines $\ell_2, \ell_3, \dots, \ell_k$ such that for $v, w \in V(H_{k-1})$, $\text{span}(v) \subseteq \text{span}(w)$ if w is an ancestor of v in T_{k-1} or T'_{k-1} , and all vertices in the path in T between a_{k-1} and a'_{k-1} are on the top stab line ℓ_k . By Lemma 34(ii), there exists a $(k-2)$ -(exactly) stabbed rectangle intersection representation \mathcal{R}_3 of T_{k-2} using stab lines $\ell_2, \ell_3, \dots, \ell_{k-1}$ such that for $v, w \in V(T_{k-2})$, $\text{span}(v) \subseteq \text{span}(w)$ if w is an ancestor of v in T_{k-2} , and the only vertices in T_{k-2} that are on the stab line ℓ_{k-2} are the ones in $N[a_{k-2}]$. It can be seen as shown in Figure 13 that $\mathcal{R}_1, \mathcal{R}_2$ and \mathcal{R}_3 can be combined and rectangles for the vertices in $N[c]$ can be added to obtain a k -stabbed rectangle intersection representation of T in which for any $x \in V(H_{k-1})$, $\text{span}(x) \subseteq \text{span}(b_k)$ and for any $x \in V(T_{k-2})$, $\text{span}(x) \subseteq \text{span}(b_{k-1})$.

Suppose for the sake of contradiction that T is k -ESRIG. This part of the proof proceeds very similarly to the proof of Theorem 33. As in that proof, we let \mathcal{R} be a $(k+2)$ -exactly stabbed rectangle intersection representation of T in which the top and bottom stab lines do not intersect any rectangle and let A be a good region that contains all the rectangles of \mathcal{R} . As T_k and T'_k are k -SRIG but not $(k-1)$ -SRIG, we have $|\mathcal{L}_{\mathcal{R}}(A)| = k$ and there are $\mathcal{L}_{\mathcal{R}}(A)$ -spanning paths in both T_k and T'_k . Let X_1 and X_2 be minimal $\mathcal{L}_{\mathcal{R}}(A)$ -spanning paths in T_k and T'_k respectively. Let X be an induced path in T that connects some vertex of X_1 and some vertex of X_2 such that no internal vertex of X belongs to either X_1 or X_2 . Note that X is a subgraph of H_k that contains b_k . Let $(A_t, A_b) = \Delta(\mathcal{R}, A, X_1, X_2, X)$.

Since there is a path in $T_A = T$ from $c \in V(T_A)$ to a vertex in X (in this case, b_k) that misses both X_1 and X_2 , we know by Lemma 30 that r_c is contained in A_t or A_b . We shall assume without loss of generality that r_c is contained in A_t . Let $T^* = T - (V(H_k) \cup N[b_k])$. Since there is a path in T_A from c to each vertex of T^* that misses X_1, X_2 and X , we can use Lemma 31 to infer that T^* is a connected induced subgraph of T_{A_t} .

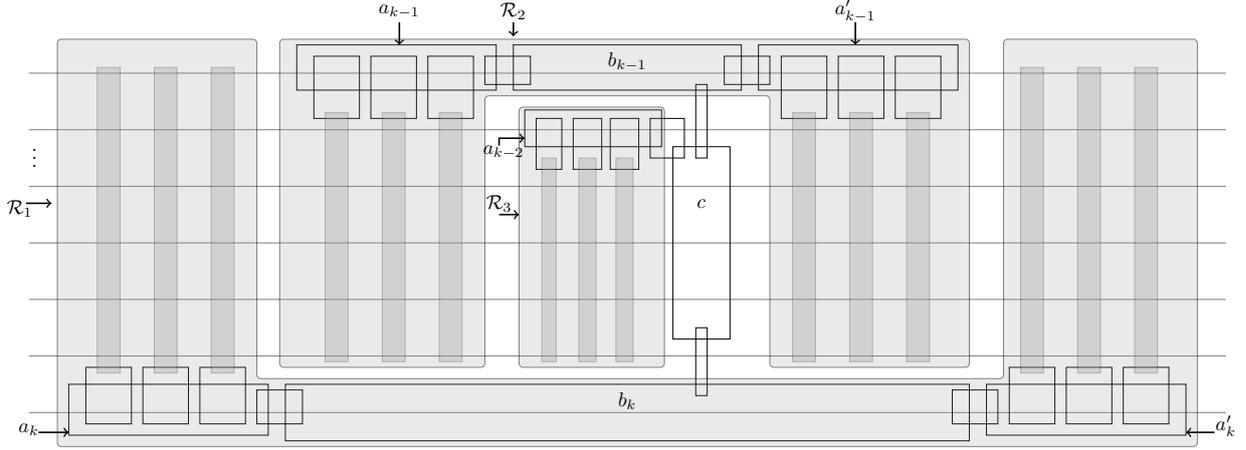


Figure 13: A schematic diagram of a k -stabbed rectangle intersection representation of T .

Claim. Both the vertices a_k and a'_k are on the bottom stab line in $\mathcal{L}_{\mathcal{R}}(A)$.

Proof. Let X' be the path in T_A between a_k and a'_k . Clearly, X' has length 4 and is a subpath of X . The tree T^* contains T_{k-1} as an induced subgraph, and is therefore not $(k-2)$ -SRIG by Lemma 34(i). Hence, $|\mathcal{L}_{\mathcal{R}}(T^*)| \geq k-1$. Since T^* contains the vertex c that has a path to a vertex in X' which misses a_k, a'_k, X_1, X_2 and $X - V(X')$, we can use Lemma 35 to infer that at least one of a_k and a'_k is not on the top stab line in $\mathcal{L}_{\mathcal{R}}(A)$. Notice that the graph induced by $V(T_k) \cup V(T'_k) \cup V(X')$ in $T = T_A$ is isomorphic to J_k . This means that there is a k -exactly stabbed rectangle intersection representation of J_k contained in the region A . Using Lemma 34(iii)(b), we can now conclude that both a_k and a'_k are on the bottom stab line in $\mathcal{L}_{\mathcal{R}}(A)$. This completes the proof of the claim.

From here onwards, we shall let $B = A_t$, for ease of notation. From the above arguments, we know that $|\mathcal{L}_{\mathcal{R}}(T^*)| \geq k-1$ and T^* is a connected induced subgraph of T_B . Therefore, $|\mathcal{L}_{\mathcal{R}}(B)| \geq k-1$. By Lemma 28, this means that B is a good region and by Lemma 27(b), we can conclude that $|\mathcal{L}_{\mathcal{R}}(B)| = k-1$. Now, T_{k-1} and T'_{k-1} are two neighbour-disjoint subtrees of T^* that are $(k-1)$ -SRIG but not $(k-2)$ -SRIG. This means that there exist minimal $\mathcal{L}_{\mathcal{R}}(B)$ -spanning induced paths Y_1 in T_{k-1} and Y_2 in T'_{k-1} . Let Y be an induced path in T^* that connects some vertex of Y_1 and some vertex of Y_2 such that no internal vertex of Y belongs to either Y_1 or Y_2 . Note that Y is a subgraph of H_{k-1} that contains b_{k-1} . Let $(B_t, B_b) = \Delta(\mathcal{R}, B, Y_1, Y_2, Y)$.

Since there is a path in T^* from c to a vertex in Y (in this case, b_{k-1}) that misses both Y_1 and Y_2 , we know by Lemma 30 that r_c is contained in B_t or B_b . As explained in the proof of Theorem 33, it can be shown that r_c is contained in B_b (if r_c is contained in B_t , then there could not have been a path in T between c and the vertex b_k in X that misses Y_1, Y_2 and Y). Let $T^{**} = T^* - (V(H_{k-1}) \cup N[b_{k-1}])$. Since there is a path in T_B from c to each vertex of T^{**} that misses Y_1, Y_2 and Y , we can use Lemma 31 to infer that T^{**} is a connected induced subgraph of T_{B_b} .

Claim. Both the vertices a_{k-1} and a'_{k-1} are on the top stab line in $\mathcal{L}_{\mathcal{R}}(A)$.

Proof. Let Y' be the path in T between a_{k-1} and a'_{k-1} . Clearly, Y' has length 4 and is a subpath of Y . The tree T^{**} contains T_{k-2} as an induced subgraph, and is therefore not $(k-3)$ -SRIG, implying that $|\mathcal{L}_{\mathcal{R}}(T^{**})| \geq k-2$. Since T^{**} contains the vertex c that has a path to a vertex in Y' which misses $a_{k-1}, a'_{k-1}, Y_1, Y_2$ and $Y - V(Y')$, we can use Lemma 35 to infer that at least one of a_{k-1} and a'_{k-1} is not on the bottom stab line in $\mathcal{L}_{\mathcal{R}}(B)$. Notice that the graph induced by $V(T_{k-1}) \cup V(T'_{k-1}) \cup V(Y')$ in T^* is isomorphic to J_{k-1} . This means that there is a $(k-1)$ -exactly stabbed rectangle intersection representation contained in the region B . Using Lemma 34(iii)(b), we can now conclude that both a_{k-1} and a'_{k-1} are on the top stab line in $\mathcal{L}_{\mathcal{R}}(B)$. Now since $B = A_t$ and $|\mathcal{L}_{\mathcal{R}}(B)| = |\mathcal{L}_{\mathcal{R}}(A)| - 1$, we know by Lemma 27(b) that the top stab line in $\mathcal{L}_{\mathcal{R}}(B)$ is also the top stab line in $\mathcal{L}_{\mathcal{R}}(A)$. This completes the proof of the claim.

Let $\ell_1, \ell_2, \dots, \ell_k$ be the stab lines in $\mathcal{L}_{\mathcal{R}}(A)$ in order from bottom to top. Now, the fact that each rectangle in \mathcal{R} intersects exactly one stab line gives us several observations. Since there is a path of length 2 between b_k and a_k in T , and because our first claim tells us that a_k is on ℓ_1 , we can conclude that b_k is not on any of the stab lines in $\{\ell_4, \ell_5, \dots, \ell_k\}$. Similarly, our second claim tells us that a_{k-1} is on ℓ_k , and then the fact that there is a path of length 2 between a_{k-1} and b_{k-1} implies that b_{k-1} cannot be on any stab line

in $\{\ell_{k-3}, \ell_{k-4}, \dots, \ell_2, \ell_1\}$. Now, since there is a path of length 4 between b_k and b_{k-1} , there can be at most 3 stab lines between ℓ_3 and ℓ_{k-2} . But this contradicts the fact that $k \geq 10$. \square

7 Conclusions

A direction of further research could be to investigate the class of 2-SRIGs and try to characterize this class of graphs.

Question 1. *Develop a forbidden structure characterization and/or a polynomial-time recognition algorithm for 2-SRIGs.*

Note that Theorem 19 gives such a characterization of the 2-SRIGs within the class of block graphs. This theorem shows that within the class of block graphs, those graphs that do not contain asteroidal-(non-interval) subgraphs are exactly the 2-SRIGs. From the characterization of interval graphs by Lekkerkerker and Boland (Theorem 11), we know that the absence of asteroidal triples characterizes the 1-SRIGs within chordal graphs. Therefore, a natural question is whether the absence of asteroidal-(non-interval) subgraphs is enough to characterize the 2-SRIGs within chordal graphs (note that block graphs are a subclass of chordal graphs). The answer to this question is negative, as we have shown in Theorem 9 that there are split graphs that are not 2-SRIG. Split graphs are chordal and clearly, no split graph can contain asteroidal-(non-interval) subgraphs, as for any three connected induced subgraphs that are pairwise neighbour-disjoint in a split graph, at least two of them will contain just one vertex each. This gives rise to the following question.

Question 2. *Find a forbidden structure characterization for chordal graphs (resp. split graphs) that are 2-SRIG. Can chordal graphs (resp. split graphs) that are 2-SRIG be recognized in polynomial-time?*

We have shown that any split graph with boxicity at most 2 is 3-SRIG and that there exists a split graph which is 3-SRIG but not 2-SRIG. Therefore, following question is interesting.

Question 3. *What is the complexity of recognizing split graphs that are 3-SRIG?*

Note that by Theorem 8, the above problem is equivalent to the problem of recognizing split graphs that have boxicity at most 2. This problem assumes significance in light of the fact that recognizing split graphs that have boxicity at most 3 is NP-complete [1].

We constructed polynomial-time algorithms that check if $stab(G) \leq 2$ for any block graph G , and if $stab(T) \leq 3$ for any tree T . Therefore, the following are natural questions in this direction.

Question 4. *For a given block graph G , is it possible to determine $stab(G)$ in polynomial-time?*

Question 5. *For a given tree T , is it possible to determine $stab(T)$ in polynomial-time?*

We showed that $K_{4,4}$ is not k -ESRIG for any finite k , but is 4-SRIG. Here, the question arises as to how high the exact stab number of an exactly stabbable graph can be with respect to its stab number. Theorem 10 shows that trees are exactly stabbable and Theorem 36 shows a tree T such that $estab(T) > stab(T)$ (in fact, it is an easy exercise to show that $estab(T) = stab(T) + 1$). The following questions are therefore of interest.

Question 6. *Is there a constant c such that for any tree T we have, $estab(T) - stab(T) \leq c$ or $\frac{estab(T)}{stab(T)} \leq c$?*

Question 7. *For a given tree T , is it possible to determine $estab(T)$ in polynomial-time?*

We constructed graphs on n vertices ((\sqrt{n}, \sqrt{n}) -grids) which have stab number $\Omega(\sqrt{n})$. It can be asked if there are families of graphs which have asymptotically larger stab number.

Question 8. *Is there a class \mathcal{C} of rectangle intersection graphs such that $stab(\mathcal{C}, n) = \omega(\sqrt{n})$?*

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