

On the Computational Complexity of Decision Problems about Multi-Player Nash Equilibria*

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Abstract

We study the computational complexity of decision problems about Nash equilibria in m -player games. Several such problems have recently been shown to be computationally equivalent to the decision problem for the existential theory of the reals, or stated in terms of complexity classes, $\exists\mathbb{R}$ -complete, when $m \geq 3$. We show that, unless they turn into trivial problems, they are $\exists\mathbb{R}$ -hard even for 3-player *zero-sum* games.

We also obtain new results about several other decision problems. We show that when $m \geq 3$ the problems of deciding if a game has a Pareto optimal Nash equilibrium or deciding if a game has a strong Nash equilibrium are $\exists\mathbb{R}$ -complete. The latter result rectifies a previous claim of NP-completeness in the literature. We show that deciding if a game has an irrational valued Nash equilibrium is $\exists\mathbb{R}$ -hard, answering a question of Bilò and Mavronicolas, and address also the computational complexity of deciding if a game has a rational valued Nash equilibrium. These results also hold for 3-player zero-sum games.

Our proof methodology applies to corresponding decision problems about symmetric Nash equilibria in symmetric games as well, and in particular our new results carry over to the symmetric setting. Finally we show that deciding whether a symmetric m -player game has a *non-symmetric* Nash equilibrium is $\exists\mathbb{R}$ -complete when $m \geq 3$, answering a question of Garg, Mehta, Vazirani, and Yazdanbod.

1 Introduction

Given a finite strategic form m -player game the most basic algorithmic problem is to compute a Nash equilibrium, shown always to exist by Nash [22]. The computational complexity of this problem was characterized in seminal work by Daskalakis, Goldberg, and Papadimitriou [13] and Chen and Deng [11] as PPAD-complete for 2-player games and by Etessami and Yannakakis [14] as FIXP-complete for m -player games, when $m \geq 3$. Any 2-player game may be viewed as a 3-player *zero-sum* game by adding a dummy player, thereby making the class of 3-player zero-sum games a natural class of games intermediate between 2-player and 3-player games. The problem of

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computing a Nash equilibrium for a 3-player zero-sum game is clearly PPAD-hard and belongs to FIXP, but its precise complexity appears to be unknown.

Rather than settling for *any* Nash equilibrium, one might be interested in a Nash equilibrium that satisfies a given property, e.g. giving each player at least a certain payoff. Such a Nash equilibrium might of course not exist and therefore results in the basic computational problem of deciding existence. In the setting of 2-player games, the computational complexity of several such problems was proved to be NP-complete by Gilboa and Zemel [17]. Conitzer and Sandholm [12] revisited these problems and showed them, together with additional problems, to be NP-complete even for symmetric games.

Only recently was the computational complexity of analogous problems in m -player games determined, for $m \geq 3$. Schaefer and Štefankovič [25] obtained the first such result by proving $\exists\mathbb{R}$ -completeness of deciding existence of a Nash equilibrium in which no action is played with probability larger than $\frac{1}{2}$ by any player. Garg, Mehta, Vazirani, and Yazdanbod [15] used this to also show $\exists\mathbb{R}$ -completeness for deciding if a game has more than one Nash equilibrium, whether each player can ensure a given payoff in a Nash equilibrium, and for the two problems of deciding whether the support sets of the mixed strategies of a Nash equilibrium can belong to given sets or contain given sets. In addition, by a symmetrization construction, they show that the analogue to the latter two problems for symmetric Nash equilibria are $\exists\mathbb{R}$ -complete as well. Bilò and Mavronicolas [5, 6] subsequently extended the results of Garg et al. to further problems both about Nash equilibria and about symmetric Nash equilibria. They show $\exists\mathbb{R}$ -completeness of deciding existence of a Nash equilibrium where all players receive at most a given payoff, where the total payoff of the players is at least or at most a given amount, whether the size of the supports of the mixed strategies all have a certain minimum or maximum size, and finally whether a Nash equilibrium exists that is *not* Pareto optimal or that is *not* a strong Nash equilibrium. All the analogous problems about symmetric Nash equilibria are shown to be $\exists\mathbb{R}$ -complete as well.

1.1 Our Results

We revisit the problems about existence of Nash equilibria in m -player games, with $m \geq 3$, considered by Garg et al. and Bilò and Mavronicolas. In a zero-sum game the total payoff of the players in any Nash equilibrium is of course 0, and any Nash equilibrium is Pareto optimal. This renders the corresponding decision problems trivial in the case of zero-sum games. We show except for these, all the problems considered by Garg et al. and Bilò and Mavronicolas remain $\exists\mathbb{R}$ -hard for 3-player zero-sum games. We obtain our results building on a recent more direct and simple proof of $\exists\mathbb{R}$ -hardness of the initial $\exists\mathbb{R}$ -complete problem of Schaefer and Štefankovič due to Hansen [18]. For completeness we give also comparably simpler proofs of $\exists\mathbb{R}$ -hardness for the problems about total payoff and existence of a non Pareto optimal Nash equilibrium.

We next show that deciding existence of a strong Nash equilibrium in an m -player game with $m \geq 3$ is $\exists\mathbb{R}$ -complete, and likewise for the similar problem of deciding existence of a Pareto optimal Nash equilibrium. Gatti, Rocco, and Sandholm [16] proved earlier that deciding if a given (rational valued) strategy profile x is a strong Nash equilibrium can be done in polynomial time. They then erroneously concluded that the problem of deciding existence of a strong Nash equilibrium is, as a consequence NP-complete. A problem with this reasoning is that if a strong Nash equilibrium exists, there is no guarantee that a rational valued strong Nash equilibrium exists. Even if one disregards a concern about irrational valued strong Nash equilibria, it is possible that

even when a rational valued strong Nash equilibrium exists, any rational valued strong Nash equilibrium could require *exponentially* many bits to describe in standard binary notation the numerators and denominators of the probabilities of the equilibrium strategy profile. Nevertheless, our proof of $\exists\mathbb{R}$ -membership build on the idea behind the polynomial time algorithm of Gatti et al. Our reduction for proving $\exists\mathbb{R}$ -hardness produces non-zero-sum games. The case of deciding existence of a Pareto optimal Nash equilibrium is, as already noted, trivial for the case of 3-player zero-sum games. We leave the complexity of the deciding existence of a strong Nash equilibrium in 3-player zero-sum games an open problem.

In another work, Bilò and Mavronicolas [4] considered the problems of deciding whether an irrational valued Nash equilibrium exists and whether a rational valued Nash equilibrium exists, proving both problems to be NP-hard. Bilò and Mavronicolas asked if the problem about existence of an irrational valued Nash equilibria is hard for the so-called square-root-sum problem. We confirm this, showing the problem to be $\exists\mathbb{R}$ -hard. We relate the problem about existence of rational valued Nash equilibria to the existential theory of the rationals.

We next use a symmetrization construction similar to Garg et al. to translate all problems considered to the analogous setting of decision problems about symmetric Nash equilibria. Here we do not obtain qualitative improvements on existing results, but give for completeness the simple proofs of these results in addition to our new results.

A final problem we consider is of deciding existence of a *nonsymmetric* Nash equilibrium of a given symmetric game. Mehta, Vazirani, and Yazdanbod [21] proved that this problem is NP-complete for 2-player games, and Garg et al. [15] raised the question of the complexity for m -player games with $m \geq 3$. We show this problem to be $\exists\mathbb{R}$ -complete.

Our hardness proofs are presented for the special case of 3-player games, but extend to m -player games for any fixed $m > 3$, in a similar way to previous works [15, 5, 6]. For the case of nonsymmetric games this is achieved by adding $m - 3$ dummy players with suitably chosen actions sets and payoff functions (cf. [5]). Zero-sum games are, of course, mainly interesting for 3-player games. For the case of symmetric games, the $m - 3$ dummy players can be introduced prior to the symmetrization construction and this together with the reductions that follow are easily generalized to m players.

2 Preliminaries

2.1 Existential Theory of the Reals and Rationals

The existential theory of the reals $\text{Th}_{\exists}(\mathbb{R})$ is the set of all true sentences over \mathbb{R} of the form $\exists x_1, \dots, x_n \in \mathbb{R} : \phi(x_1, \dots, x_n)$, where ϕ is a quantifier free Boolean formula of equalities and inequalities of polynomials with integer coefficients. The complexity class $\exists\mathbb{R}$ is defined [25] as the closure of $\text{Th}_{\exists}(\mathbb{R})$ under polynomial time many-one reductions. Equivalently, $\exists\mathbb{R}$ is the constant-free Boolean part of the class $\text{NP}_{\mathbb{R}}$ [8], which is the analogue class to NP in the Blum-Shub-Smale model of computation [7]. It is straightforward to see that $\text{Th}_{\exists}(\mathbb{R})$ is NP-hard (cf. [9]) and the decision procedure by Canny [10] shows that $\text{Th}_{\exists}(\mathbb{R})$ belongs to PSPACE. Thus it follows that $\text{NP} \subseteq \exists\mathbb{R} \subseteq \text{PSPACE}$.

We may similarly consider the existential theory over the rationals $\text{Th}_{\exists}(\mathbb{Q})$ and likewise form the complexity class $\exists\mathbb{Q}$ as the closure of $\text{Th}_{\exists}(\mathbb{Q})$ under polynomial time

many-one reductions. While it is a long-standing open problem whether $\text{Th}_{\exists}(\mathbb{Q})$ is decidable, Koenigsmann [20] recently showed that already $\text{Th}_{\forall\exists}(\mathbb{Q})$, consisting of true sentences in prenex form with a single block of universal quantifiers followed by a single block of existential quantifiers, is undecidable. In contrast, the entire first order theory $\text{Th}(\mathbb{R})$ of the reals is decidable in EXPSPACE [23]. Schaefer and Štefankovič [25] show that the problem of deciding feasibility of a system of *strict* inequalities is complete for $\exists\mathbb{R}$. Since a system of strict inequalities that is feasible over \mathbb{R} is also feasible over \mathbb{Q} , it follows that $\exists\mathbb{R} \subseteq \exists\mathbb{Q}$.

The basic complete problem for $\exists\mathbb{R}$ and for $\exists\mathbb{Q}$, is the problem of deciding whether a system of quadratic equations with integer coefficients has a solution over \mathbb{R} and over \mathbb{Q} , respectively [7]. We denote this problem over \mathbb{R} as QUAD and the problem over \mathbb{Q} as QUAD $_{\mathbb{Q}}$.

2.2 Strategic Form Games and Nash Equilibrium

A finite strategic form game \mathcal{G} with m players is given by sets S_1, \dots, S_m of actions (*pure strategies*) together with *utility functions* $u_1, \dots, u_m : S_1 \times \dots \times S_m \rightarrow \mathbb{R}$. A choice of an action $a_i \in S_i$ for each player together form a pure strategy profile $a = (a_1, \dots, a_m)$.

The game \mathcal{G} is *symmetric* if $S_1 = \dots = S_m$ and for every permutation π on $[m]$, every $i \in [m]$ and every $(a_1, \dots, a_m) \in S_1 \times \dots \times S_m$ it holds that $u_i(a_1, \dots, a_m) = u_{\pi(i)}(a_{\pi(1)}, \dots, a_{\pi(m)})$. In other words, a game is symmetric if the players share the same set of actions and the utility function of a player depends only on the action of the player together with the *multiset* of actions of the other players.

Let $\Delta(S_i)$ denote the set of probability distributions on S_i . A (*mixed*) *strategy* for Player i is an element $x_i \in \Delta(S_i)$. The *support* $\text{Supp}(x_i)$ is the set of actions given strictly positive probability by x_i . We say that x_i is *fully mixed* if $\text{Supp}(x_i) = S_i$. A strategy x_i for each player i together form a strategy profile $x = (x_1, \dots, x_m)$. The utility functions extend to strategy profiles by letting $u_i(x) = E_{a \sim x} u_i(a_1, \dots, a_m)$. We shall also refer to $u_i(x)$ as the *payoff* of Player i .

Given a strategy profile x we let $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m)$ denote the strategies of all players except Player i . Given a strategy $y \in S_i$ for Player i , we let $(x_{-i}; y)$ denote the strategy profile $(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_m)$ formed by x_{-i} and y . We may also denote $(x_{-i}; y)$ by $x \setminus y$. We say that y is a *best reply* for Player i to x (or to x_{-i}) if $u_i(x \setminus y) \geq u_i(x \setminus y')$ for all $y' \in \Delta(S_i)$.

A *Nash equilibrium* (NE) is a strategy profile x where each individual strategy x_i is a best reply to x . As shown by Nash [22], every finite strategic form game \mathcal{G} has a Nash equilibrium. In a symmetric game \mathcal{G} , a *symmetric Nash equilibrium* (SNE) is a Nash equilibrium where the strategies of all players are identical. Nash also proved that every symmetric game has a symmetric Nash equilibrium.

A strategy profile x is *Pareto optimal* if there is no strategy profile x' such that $u_i(x) \leq u_i(x')$ for all i , and $u_j(x) < u_j(x')$ for some j . A Nash equilibrium strategy profile need not be Pareto optimal and a Pareto optimal strategy profile need not be a Nash equilibrium. A strategy profile that is both a Nash equilibrium and is Pareto optimal is called a Pareto optimal Nash equilibrium. The existence of a Pareto optimal Nash equilibrium is not guaranteed.

A *strong Nash equilibrium* [1] (strong NE) is a strategy profile x for which there is no non-empty set $B \subseteq [m]$ for which *all* players $i \in B$ can increase their payoff by different strategies assuming players $j \in [m] \setminus B$ play according to x . Equivalently, x is a strong Nash equilibrium if for every strategy profile $x' \neq x$ there exist i such that $x_i \neq x'_i$ and $u_i(x') \leq u_i(x)$. The existence of a strong Nash equilibrium is not guaranteed.

3 Decision Problems about Nash Equilibria

Below we define the decision problems under consideration with names generally following Bilò and Mavronicolas [5]. The given input is a finite strategic form game \mathcal{G} , together with auxiliary input depending on the particular problem. We let u denote a rational number, k an integer, and $T_i \subseteq S_i$ a set of actions of Player i , for every i . We describe the decision problem by stating the property a Nash equilibrium x whose existence is to be determined should satisfy. The problems are grouped together in four groups each of which are covered in a separate subsection below.

Except for the last four problems, it is straightforward to prove membership in $\exists\mathbb{R}$ by an explicit existentially quantified first-order formula. We prove $\exists\mathbb{R}$ membership of $\exists\text{PARETOOPTIMALNE}$ and $\exists\text{STRONGNE}$ in subsection 3.3 and discuss decidability of $\exists\text{IRRATIONALNE}$ and $\exists\text{RATIONALNE}$ in subsection 3.4.

Problem	Condition
$\exists\text{NEWITHLARGEPAFFOFFS}$	$u_i(x) \geq u$ for all i .
$\exists\text{NEWITHSMALLPAFFOFFS}$	$u_i(x) \leq u$ for all i .
$\exists\text{NEWITHLARGETOTALPAFFOFF}$	$\sum_i u_i(x) \geq u$.
$\exists\text{NEWITHSMALLTOTALPAFFOFF}$	$\sum_i u_i(x) \leq u$.
$\exists\text{NEINABALL}$	$x_i(a_i) \leq u$ for all i and $a_i \in S_i$.
$\exists\text{SECONDNE}$	x is not the only NE.
$\exists\text{NEWITHLARGESUPPORTS}$	$ \text{Supp}(x_i) \geq k$ for all i .
$\exists\text{NEWITHSMALLSUPPORTS}$	$ \text{Supp}(x_i) \leq k$ for all i .
$\exists\text{NEWITHRESTRICTINGSUPPORTS}$	$T_i \subseteq \text{Supp}(x_i)$ for all i .
$\exists\text{NEWITHRESTRICTEDSUPPORTS}$	$\text{Supp}(x_i) \subseteq T_i$ for all i .
$\exists\text{NONPARETOOPTIMALNE}$	x is not Pareto optimal.
$\exists\text{NONSTRONGNE}$	x is not a strong NE.
$\exists\text{PARETOOPTIMALNE}$	x is Pareto optimal.
$\exists\text{STRONGNE}$	x is a strong NE.
$\exists\text{IRRATIONALNE}$	$x_i(a_i) \notin \mathbb{Q}$ for some i and $a_i \in S_i$.
$\exists\text{RATIONALNE}$	$x_i(a_i) \in \mathbb{Q}$ for all i and $a_i \in S_i$.

A key step (implicitly present) in the proof of the first $\exists\mathbb{R}$ -hardness result about Nash equilibrium in 3-player games by Schaefer and Štefankovič is a result due to Schaefer [24] that QUAD remains $\exists\mathbb{R}$ -hard under the *promise* that either the given quadratic system has no solutions or a solution exists in the unit ball $\mathbf{B}(\mathbf{0}, 1)$. For our purposes the following variation [18, Proposition 2] will be more directly applicable (which may easily be proved from the latter, cf. Section 3.4). Here we denote by Δ_c^n the standard corner n -simplex $\{x \in \mathbb{R}^n \mid x \geq 0 \wedge \sum_{i=1}^n x_i \leq 1\}$.

Proposition 1. *It is $\exists\mathbb{R}$ -hard to decide if a given system of quadratic equations in n variables and with integer coefficients has a solution under the promise that either the system has no solutions or a solution z exists that is in the interior of Δ_c^n and also satisfies $z_i \leq \frac{1}{2}$ for all i and that $\sum_{i=1}^n z_i \geq \frac{1}{2}$.*

Schaefer and Štefankovič showed that $\exists\text{NEINABALL}$ is $\exists\mathbb{R}$ -hard for 3-player games by first proving that the following problem is $\exists\mathbb{R}$ -hard: Given a continuous function $f : \mathbf{B}(\mathbf{0}, 1) \rightarrow \mathbf{B}(\mathbf{0}, 1)$ mapping the unit ball to itself, where each coordinate function f_i is given as a polynomial, and given a rational number r , is there a fixed point of

f in the ball $B(\mathbf{0}, r)$? The proof was then concluded by a transformation of Brouwer functions into 3-player games by Etesammi and Yannakakis [14]. This latter reduction is rather involved and goes through an intermediate construction of 10-player games. More recently, Hansen [18] gave a simple and direct reduction from the above promise version of QUAD to $\exists\text{NEINABALL}$.

The first step of this as well as our reductions is to transform the given quadratic system over the corner simplex Δ_c^n into a homogeneous bilinear system over the standard n -simplex $\{x \in \mathbb{R}^{n+1} \mid x \geq 0 \wedge \sum_{i=1}^{n+1} x_i = 1\}$ which we denote by Δ^n . In short, this is done by introducing a set of new variables y_i and new equations $x_i - y_i = 0$, replacing quadratic terms $x_i x_j$ by bilinear quadratic terms $x_i y_j$, and finally homogenizing the entire system using the two equations $\sum_{i=1}^{n+1} x_i = 1$ and $\sum_{i=1}^{n+1} y_i = 1$ where x_{n+1} and y_{n+1} are new *slack variables*. Doing this we arrive at the following statement (cf. [18, Proposition 3]).

Proposition 2. *It is $\exists\mathbb{R}$ -complete to decide if a system of homogeneous bilinear equations $q_k(x, y) = 0$, $k = 1, \dots, \ell$ with integer coefficients has a solution $x, y \in \Delta^n$. It remains $\exists\mathbb{R}$ -hard under the promise that either the system has no such solution or a solution (x, x) exists where x belong to the relative interior of Δ^n and further satisfies $x_i \leq \frac{1}{2}$ for all i .*

3.1 Payoff Restricted Nash Equilibria

For proving the $\exists\mathbb{R}$ -hardness results we start by showing that it is $\exists\mathbb{R}$ -hard to decide if a given zero-sum game has a Nash equilibrium in which each player receives payoff 0. This is in contrast to the earlier work of Garg et al. [15] and Bilò and Mavronicolas [5, 6] that reduce from the $\exists\text{NEINABALL}$ problem. On the other hand we do show $\exists\mathbb{R}$ -hardness even under the promise that the Nash equilibrium also satisfies the condition of $\exists\text{NEINABALL}$. The construction and proof below are modifications of proofs by Hansen [18, Theorem 1 and Theorem 2].

Definition 1 (The 3-player zero-sum game \mathcal{G}_0). *Let \mathcal{S} be a system of homogeneous bilinear polynomials $q_1(x, y), \dots, q_\ell(x, y)$ with integer coefficients in variables $x = (x_1, \dots, x_{n+1})$ and $y = (y_1, \dots, y_{n+1})$,*

$$q_k(x, y) = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} a_{ij}^{(k)} x_i y_j .$$

We define the 3-player game $\mathcal{G}_0(\mathcal{S})$ as follows. The strategy set of Player 1 is the set $S_1 = \{1, -1\} \times \{1, 2, \dots, \ell\}$. The strategy sets of Player 2 and Player 3 are $S_2 = S_3 = \{1, 2, \dots, n+1\}$. The (integer) utility functions of the players are defined by

$$\frac{1}{2} u_1((s, k), i, j) = -u_2((s, k), i, j) = -u_3((s, k), i, j) = s a_{ij}^{(k)} .$$

When the system \mathcal{S} is understood by the context, we simply write $\mathcal{G}_0 = \mathcal{G}_0(\mathcal{S})$. We think of the strategy (s, k) of Player 1 as corresponding to the polynomial q_k together with a sign s , the strategy i of Player 2 as corresponding to x_i and the strategy j of Player 3 as corresponding to y_j . We may thus identify mixed strategies of Player 2 and Player 3 with assignments to variables $x, y \in \Delta^n \subseteq \mathbb{R}^{n+1}$.

The following observation is immediate from the definition of \mathcal{G}_0 .

Lemma 1. Any strategy profile (x, y) of Player 2 and Player 3 satisfies for every $(s, k) \in S_1$ the equation

$$\frac{1}{2}u_1((s, k), x, y) = -u_2((s, k), x, y) = -u_3((s, k), x, y) = sq_k(x, y) . \quad (1)$$

Hence $u_1(z, x, y) = u_2(z, x, y) = u_3(z, x, y) = 0$ when z is the uniform distribution on S_1 . Consequentially, any Nash equilibrium payoff profile is of the form $(2u, -u, -u)$, where $u \geq 0$.

Next we relate solutions to the system \mathcal{S} to Nash equilibria in \mathcal{G}_0 .

Proposition 3. Let \mathcal{S} be a system of homogeneous bilinear polynomials $q_k(x, y)$, $k = 1, \dots, \ell$. If \mathcal{S} has a solution $(x, y) \in \Delta^n \times \Delta^n$, then letting z be the uniform distribution on S_1 , the strategy profile $\sigma = (z, x, y)$ is a Nash equilibrium of \mathcal{G}_0 in which every player receives payoff 0. If in addition (x, y) satisfies the promise of Proposition 2, then σ is fully mixed, Player 2 and Player 3 use identical strategies, and no action is chosen with probability more than $\frac{1}{2}$ by any player. Conversely, if (z, x, y) is a Nash equilibrium of \mathcal{G}_0 in which every player receives payoff 0, then (x, y) is a solution to \mathcal{S} .

Proof. Suppose first that $(x, y) \in \Delta^n \times \Delta^n$ is a solution to \mathcal{S} and let z be the uniform distribution on S_1 . By Equation (1) the strategy profile (x, y) of Player 2 and Player 3 ensures that all players receive payoff 0 regardless of which strategy is played by Player 1, and likewise the strategy z of Player 1 ensures that all players receive payoff 0 regardless of the strategies of Player 2 and Player 3. This shows that σ is a Nash equilibrium of \mathcal{G}_0 , in which by Lemma 1 every player receives payoff 0. If (x, y) in addition satisfies that the promise of Proposition 2 we have $0 < x_i = y_i \leq \frac{1}{2}$. From this and our choice of z , we have that σ is a fully mixed and that no action is chosen by a strategy of σ with probability more than $\frac{1}{2}$.

Suppose on the other hand that $\sigma = (z, x, y)$ is a Nash equilibrium of \mathcal{G}_0 in which every player receives payoff 0. Suppose that $q_k(x, y) \neq 0$ for some k . Then by Equation (1) we get that $u_1((\text{sgn}(q_k(x, y)), k), x, y) = |2q_k(x, y)| > 0$, contradicting that σ is a Nash equilibrium. Thus (x, y) is a solution to \mathcal{S} . \square

Theorem 1. $\exists\text{NEWITHLARGEPAyOFFS}$ and $\exists\text{NEWITHSMALLPAyOFFS}$ are $\exists\mathbb{R}$ -complete, even for 3-player zero-sum games.

Proof. For a strategy profile x in a zero-sum game \mathcal{G} we have that $u_i(x) = 0$, for all i , if and only if $u_i(x) \geq 0$, for all i , if and only if $u_i(x) \leq 0$, for all i .

Thus Proposition 3 gives a reduction from the promise problem of Proposition 2, thereby establishing $\exists\mathbb{R}$ -hardness of the problems $\exists\text{NEWITHLARGEPAyOFFS}$ and $\exists\text{NEWITHSMALLPAyOFFS}$. \square

A simple change to the game \mathcal{G}_0 give $\exists\mathbb{R}$ -hardness for the two problems $\exists\text{NEWITHLARGETOTALPAyOFF}$ and $\exists\text{NEWITHSMALLTOTALPAyOFF}$. Naturally we must give up the zero-sum property of the game.

Theorem 2 (Bilò and Mavronicolas [5]). $\exists\text{NEWITHLARGETOTALPAyOFF}$ and $\exists\text{NEWITHSMALLTOTALPAyOFF}$ are $\exists\mathbb{R}$ -complete, even for 3-player games.

Proof. Define the game \mathcal{G}'_0 from \mathcal{G}_0 with new utility functions $u'_1(x) = u_1(x)$ and $u'_2(x) = u'_3(x) = -u_1(x)$, and thus also $u'_2(x) = u'_3(x) = 2u_2(x) = 2u_3(x)$, where u_1, u_2 , and u_3 are the utility functions of \mathcal{G}_0 . Clearly \mathcal{G}'_0 has the same set of Nash equilibria as \mathcal{G}_0 . Now $u'_1(x) + u'_2(x) + u'_3(x) = -u_1(x)$ and it follows that $u'_1(x) + u'_2(x) + u'_3(x) \geq 0$ if

and only if $u_1(x) \leq 0$. By Lemma 1, any Nash equilibrium x must satisfy the inequality $u_1(x) \geq 0$. Thus, a Nash equilibrium x satisfies the inequality $u'_1(x) + u'_2(x) + u'_3(x) \geq 0$ if and only if $u_1(x) = u_2(x) = u_3(x) = 0$. We conclude that Proposition 3 gives a reduction from the promise problem of Proposition 2 to $\exists\text{NEWITHLARGETOTALPAYOFF}$ thereby showing $\exists\mathbb{R}$ -hardness.

Similarly, define the game \mathcal{G}_0'' from \mathcal{G}_0 with new utility functions $u''_1(x) = 3u_1(x)$ and $u''_2(x) = u''_3(x) = -u_1(x)$. Again, \mathcal{G}_0'' clearly has the same set of Nash equilibria as \mathcal{G}_0 . Now $u''_1(x) + u''_2(x) + u''_3(x) = u_1(x)$ and it follows that $u''_1(x) + u''_2(x) + u''_3(x) \leq 0$ if and only if $u_1(x) \leq 0$. Analogously to above we then obtain $\exists\mathbb{R}$ -hardness for $\exists\text{NEWITHSMALLTOTALPAYOFF}$. \square

3.2 Probability Restricted Nash Equilibria

A key property of the game \mathcal{G}_0 is that Player 1 may ensure all players receive payoff 0. We now give *all* players this choice by playing a new additional action \perp . We then design the utility functions involving \perp in such a way that the pure strategy profile (\perp, \perp, \perp) is always a Nash equilibrium, and every other Nash equilibrium is a Nash equilibrium in \mathcal{G}_0 in which all players receive payoff 0.

Definition 2. For $u \geq 0$, let $\mathcal{H}_1 = \mathcal{H}_1(u)$ be the 3-player zero-sum game where each player has the action set $\{G, \perp\}$ and the payoff vectors are given by the entries of the following two matrices, where Player 1 selects the matrix, Player 2 selects the row, Player 3 selects the column.

$$\begin{array}{cc}
 & \begin{array}{cc} G & \perp \end{array} \\
 \begin{array}{c} G \\ \perp \end{array} & \begin{array}{|cc|} \hline (2u, -u, -u) & (1, -1, 0) \\ \hline (1, 0, -1) & (-4, 2, 2) \\ \hline \end{array}
 \end{array}
 \qquad
 \begin{array}{cc}
 & \begin{array}{cc} G & \perp \end{array} \\
 \begin{array}{c} G \\ \perp \end{array} & \begin{array}{|cc|} \hline (0, 0, 0) & (2, -3, 1) \\ \hline (2, 1, -3) & (-2, 1, 1) \\ \hline \end{array}
 \end{array}$$

It is straightforward to determine the Nash equilibria of \mathcal{H}_1 .

Lemma 2. When $u > 0$, the only Nash equilibrium of $\mathcal{H}_1(u)$ is the pure strategy profile (\perp, \perp, \perp) . When $u = 0$ the only Nash equilibria of $\mathcal{H}_1(u)$ are the pure strategy profiles (G, G, G) and (\perp, \perp, \perp) .

Proof. Let p_i be the probability of Player i choosing the action G . Consider first the case of $u > 0$. Then the action \perp is strictly dominating the action G for both Player 2 and Player 3. Hence any Nash equilibrium would require $p_2 = p_3 = 0$. The only best reply for Player 1 is then $p_1 = 0$ as well. Consider next the case of $u = 0$. In case $p_1 < 1$, again the action \perp is strictly dominating the action G for both Player 2 and Player 3, and we conclude that $p_1 = p_2 = p_3 = 0$ as before. Suppose now that $p_1 = 1$. In a Nash equilibrium we would have either $p_2 = p_3 = 1$ or $p_2 = p_3 = 0$. The former clearly gives a Nash equilibrium whereas for $p_2 = p_3 = 0$ the only best reply for Player 1 is $p_1 = 0$. \square

We use the game $\mathcal{H}_1(u)$ to extend the game \mathcal{G}_0 . The action G of \mathcal{H}_1 represents selecting an action from \mathcal{G}_0 , and the payoff vector $(2u, -u, -u)$ that is the result of all players playing the action G is precisely of the form of the Nash equilibrium payoff profile of \mathcal{G}_0 .

Definition 3 (The 3-player zero-sum game \mathcal{G}_1). Let $\mathcal{G}_1 = \mathcal{G}_1(\mathcal{S})$ be the game obtained from $\mathcal{G}_0(\mathcal{S})$ as follows. Each player is given an additional action \perp . When no player plays the action \perp , the payoffs are the same as in \mathcal{G}_0 . When at least one player is playing the action \perp the payoff are the same as in \mathcal{H}_1 , where each action different from \perp is translated to action G .

We next characterize the Nash equilibria in \mathcal{G}_1 .

Proposition 4. The pure strategy profile (\perp, \perp, \perp) is a Nash equilibrium of \mathcal{G}_1 . Any other Nash equilibrium x in \mathcal{G}_1 is also a Nash equilibrium of \mathcal{G}_0 and is such that every player receives payoff 0.

Proof. By Lemma 1 a Nash equilibrium of \mathcal{G}_1 induces a Nash equilibrium of $\mathcal{H}_1(u)$, where $(2u, -u, -u)$ is a Nash equilibrium payoff profile of \mathcal{G}_0 , by letting each player play the action G with the total probability of which the actions of \mathcal{G}_0 are played. By Lemma 2, any Nash equilibrium in \mathcal{G}_1 different from (\perp, \perp, \perp) must then be a Nash equilibrium of \mathcal{G}_0 with Nash equilibrium payoff profile $(0, 0, 0)$ as claimed. \square

Theorem 3. The following problems are $\exists\mathbb{R}$ -complete, even for 3-player zero-sum games: $\exists\text{NEINABALL}$, $\exists\text{SECONDNE}$, $\exists\text{NEWITHLARGESUPPORTS}$, $\exists\text{NEWITHRESTRICTINGSUPPORTS}$, and $\exists\text{NEWITHRESTRICTEDSUPPORTS}$.

Proof. Proposition 3 and Proposition 4 together gives a reduction from the promise problem of Proposition 2 to all of the problems under consideration when setting the additional parameters as follows. For $\exists\text{NEINABALL}$ we let $u = \frac{1}{2}$, we let $k = 2$ for $\exists\text{NEWITHLARGESUPPORTS}$, and lastly we let T_i be the set of all actions of Player i except \perp for both of the problems $\exists\text{NEWITHRESTRICTINGSUPPORTS}$ and $\exists\text{NEWITHRESTRICTEDSUPPORTS}$. \square

Remark 1. Except for the case of $\exists\text{SECONDNE}$, the results of Theorem 3 can also be proved with the slightly simpler construction of adding an additional action \perp to the players in \mathcal{G}_0 which when played by at least one player results in all players receiving payoff 0.

To adapt the reduction of Theorem 3 to $\exists\text{NEWITHSMALLSUPPORTS}$ we need to replace the trivial Nash equilibrium (\perp, \perp, \perp) by a Nash equilibrium with large support.

Definition 4. Define the 2-player zero-sum game $\mathcal{H}_2(k)$ as follows. The two players, which we denote Player 2 and Player 3, have the same set of pure strategies $S_2 = S_3 = \{0, 1, \dots, k-1\}$. The utility functions are defined by

$$u_2(a_2, a_3) = -u_3(a_2, a_3) = \begin{cases} 1 & \text{if } a_2 = a_3 \\ -1 & \text{if } a_2 \equiv a_3 + 1 \pmod{k} \\ 0 & \text{otherwise} \end{cases}$$

We omit the easy analysis of the game $\mathcal{H}_2(k)$.

Lemma 3. For any $k \geq 2$, in the game $\mathcal{H}_2(k)$ the strategy profile in which each action is played with probability $\frac{1}{k}$ is the unique Nash equilibrium and yields payoff 0 to both players.

Definition 5 (The 3-player zero-sum game \mathcal{G}_2). Let $\mathcal{G}_2 = \mathcal{G}_2(\mathcal{S})$ be the game obtained from \mathcal{G}_1 as follows. The action \perp of Player 2 and Player 3 are replaced by the set of actions (\perp, i) , $i \in \{0, 1, \dots, k-1\}$, where k is the maximum number of actions of a player in \mathcal{G}_1 . The payoff vector of the pure strategy profile $(\perp, (\perp, a_2), (\perp, a_3))$ is $(-2, 1 + u_2(a_2, a_3), 1 + u_3(a_2, a_3))$, where u_2 and u_3 are the utility functions of the game $\mathcal{H}_2(k)$. Otherwise, when at least one player plays the action G , the payoff is as in \mathcal{H}_1 , where actions of the form (\perp, i) are translated to the action \perp .

Theorem 4. $\exists\text{NEWITHSMALLSUPPORTS}$ is $\exists\mathbb{R}$ -complete, even for 3-player zero-sum games.

Proof. In \mathcal{G}_2 , the strategy profile where Player 1 plays \perp and Player 2 and Player 3 play (\perp, i) , with i chosen uniformly at random, is a Nash equilibrium that takes the role of the Nash equilibrium (\perp, \perp, \perp) in \mathcal{G}_1 . Consider now an arbitrary Nash equilibrium in \mathcal{G}_2 . In case all players play the action G with probability less than 1, Player 2 and Player 3 must choose each action of the form (\perp, i) with the same probability, since \mathcal{H}_2 has a unique Nash equilibrium. The Nash equilibrium induces a strategy profile in \mathcal{G}_1 , letting Player 2 and Player 3 play the action \perp with the total probability each player placed on the actions (\perp, i) . By definition of $\mathcal{H}_2(k)$ the payoff vector of (\perp, \perp, \perp) in \mathcal{G}_1 differs by at most 1 in each entry from the payoff vectors of $(\perp, (\perp, a_2), (\perp, a_3))$. The proof of Lemma 2 and Proposition 4 still holds when changing the payoff vector of (\perp, \perp, \perp) by at most 1 in each coordinate. The strategy profile induced in \mathcal{G}_1 must therefore be a Nash equilibrium in \mathcal{G}_1 . We conclude that in a Nash equilibrium x of \mathcal{G}_2 , either Player 2 and Player 3 use strategies with support of size k or x is a Nash equilibrium of \mathcal{G}_0 , where every player uses a strategy of support size strictly less than k and where every player receives payoff 0. Proposition 3 thus gives a reduction showing $\exists\mathbb{R}$ -hardness. \square

3.3 Pareto Optimal and Strong Nash Equilibria

For showing $\exists\mathbb{R}$ -hardness for $\exists\text{NONSTRONGNE}$ we first analyze the Strong Nash equilibria in the game \mathcal{H}_1 .

Lemma 4. For $u \geq 0$, the Nash equilibrium (\perp, \perp, \perp) of $\mathcal{H}_1(u)$ is a strong Nash equilibrium. For $u = 0$, the Nash equilibrium (G, G, G) of $\mathcal{H}_1(u)$ is not a strong Nash equilibrium.

Proof. Consider first $u = 0$ and the Nash equilibrium (G, G, G) . This is not a strong Nash equilibrium, since for instance Player 1 and Player 2 could both increase their payoff by playing the strategy profile (\perp, \perp, G) . Consider next $u \geq 0$ and the Nash equilibrium (\perp, \perp, \perp) . Since \mathcal{H}_1 is a zero-sum game it is sufficient to consider possible coalitions of two players. Player 2 and Player 3 are already receiving the largest possible payoff given that Player 1 is playing the strategy \perp , and hence they do not have a profitable deviation. Consider then, by symmetry, the coalition formed by Player 1 and Player 2, and let them play G with probabilities p_1 and p_2 . A simple calculation shows that to increase the payoff of Player 1 requires $p_1 p_2 + 4p_2 - 2p_1 > 0$ and to increase the payoff of Player 2 requires $p_1 p_2 - 4p_2 + p_1 > 0$. Adding these gives $p_1(2p_2 - 1) > 0$ which implies $p_2 > \frac{1}{2}$. But then $p_1 p_2 - 4p_2 + p_1 < 0$. Thus (\perp, \perp, \perp) is a strong Nash equilibrium. \square

Theorem 5. $\exists\text{NONSTRONGNE}$ is $\exists\mathbb{R}$ -complete, even for 3-player zero-sum games.

Proof. Proposition 3 and Proposition 4 together give a reduction establishing $\exists\mathbb{R}$ -hardness, since by Lemma 4 the Nash equilibrium (\perp, \perp, \perp) is a strong Nash equilibrium, and a Nash equilibrium of \mathcal{G}_0 where every player receives payoff 0 is not a strong Nash equilibrium. \square

In a zero-sum game, every strategy profile is Pareto optimal. Thus for showing $\exists\mathbb{R}$ -hardness of $\exists\text{NONPARETOOPTIMALNE}$ we consider non-zero-sum games.

Definition 6. For $u \geq 0$, let $\mathcal{H}_3 = \mathcal{H}_3(u)$ be the 3-player game given by the following matrices, where Player 1 selects the matrix, Player 2 selects the row, Player 3 selects the column.

$$\begin{array}{c} \begin{array}{cc} & \begin{array}{c} G \\ \perp \end{array} \\ \begin{array}{c} G \\ \perp \end{array} & \begin{array}{|c|c|} \hline (2u, -u, -u) & (0, 0, 0) \\ \hline (0, 0, 0) & (1, 1, 1) \\ \hline \end{array} \end{array} & \begin{array}{cc} & \begin{array}{c} G \\ \perp \end{array} \\ \begin{array}{c} G \\ \perp \end{array} & \begin{array}{|c|c|} \hline (0, 0, 0) & (1, 1, 1) \\ \hline (1, 1, 1) & (2, 2, 2) \\ \hline \end{array} \end{array} \\ G & \perp \end{array}$$

Lemma 5. When $u > 0$, the only Nash equilibrium of $\mathcal{H}_3(u)$ is the pure strategy profile (\perp, \perp, \perp) . When $u = 0$ the only Nash equilibria of $\mathcal{H}_3(u)$ are the pure strategy profiles (G, G, G) and (\perp, \perp, \perp) . For $u \geq 0$, (\perp, \perp, \perp) is Pareto optimal. For $u = 0$, (G, G, G) is not Pareto optimal.

Proof. When $u = 0$, clearly (G, G, G) is a Nash equilibrium, which is Pareto dominated by (\perp, \perp, \perp) . Likewise, clearly (\perp, \perp, \perp) is always a Pareto optimal Nash equilibrium. When $u > 0$, the action G is strictly dominated by the action \perp for Player 2 and Player 3, and hence they play \perp with probability 1 in a Nash equilibrium. The only best reply of Player 1 is to play \perp with probability 1 as well. \square

Analogously to Definition 3 we define the game $\mathcal{G}_3 = \mathcal{G}_3(\mathcal{S})$ to be the game extending \mathcal{G}_0 with \mathcal{H}_3 replacing the role of \mathcal{H}_1 and analogously to Proposition 4 any Nash equilibrium in \mathcal{G}_3 different from (\perp, \perp, \perp) , which is Pareto optimal, must by Lemma 5 be a Nash equilibrium of \mathcal{G}_0 with payoff profile $(0, 0, 0)$, which is not Pareto optimal. This gives the $\exists\mathbb{R}$ -hardness part of the following theorem.

Theorem 6 (Bild and Mavronicolas [5]). $\exists\text{NONPARETOOPTIMALNE}$ is $\exists\mathbb{R}$ -complete, even for 3-player games.

We next consider the problems $\exists\text{STRONGNE}$ and $\exists\text{PARETOOPTIMALNE}$. We first outline a proof of membership in $\exists\mathbb{R}$, building on ideas of Gatti et al [16] and Hansen, Hansen, Miltersen, and Sørensen [19]. Gatti et al. proved that deciding whether a given strategy profile x of an m -player game \mathcal{G} is a strong Nash equilibrium can be done in polynomial time. The crucial insight behind this result that the question of whether a coalition of $k \leq m$ players may all improve their payoff by together changing their strategies can be recast into a question in a derived game about the minmax value of an additional *fictional* player that has only k strategies. Hansen et al. proved that in such a game, the minmax value may be achieved by strategies of the other players that are of support at most k .

Lemma 6 (Hansen et al. [19]). Let \mathcal{G} be a $m + 1$ player game and let $k = |S_{m+1}|$. If there exists a strategy profile x of the first m players such that $u_{m+1}(x; a) \leq 0$ for all $a \in S_{m+1}$ then there also exists a strategy profile x' of the first m players in which each strategy has support size at most k and $u_{m+1}(x'; a) \leq 0$ for all $a \in S_{m+1}$.

We next give a generalization of the auxiliary game construction of Gatti et al. that also allows us to treat Pareto optimal Nash equilibria at the same time.

Definition 7 (cf. Gatti et al [16]). *Let \mathcal{G} be an m -player game with strategy sets S_i and utility functions u_i . Let x be a strategy profile of \mathcal{G} and let $B_1 \dot{\cup} B_2 \dot{\cup} B_3 = [m]$ be a partition of the players, let $k_i = |B_i|$ and $k = k_1 + k_2$. For $\varepsilon > 0$ consider the $(m+1)$ -player auxiliary game $\mathcal{G}' = \mathcal{G}'_{x,\varepsilon,(B_1,B_2,B_3)}$ defined as follows. For $i \in B_1 \cup B_2$ the strategy set of Player i is $S'_i = S_i$. For $i \in B_3$ the strategy set of Player i is $S_i = \{\perp\}$. Finally, the strategy set of Player $m+1$ is $B_1 \cup B_2$. The utility function of Player $m+1$ is defined as follows. Let $a = (a'_1, \dots, a'_m, j)$ be a pure strategy profile of \mathcal{G}' . Define the strategy profile x^a of \mathcal{G} letting $x^a_i = a_i$ for $i \in B_1 \cup B_2$ and $x^a_i = x_i$ for $i \in B_3$. We then let $u'_{m+1}(a) = u_j(x) - u_j(x^a) + \varepsilon$ for $j \in B_1$ and $u'_{m+1}(a) = u_j(x) - u_j(x^a)$ for $j \in B_2$.*

The following is immediate from the definition of \mathcal{G}' .

Lemma 7. *There exist a strategy profile x' in \mathcal{G} that satisfies $u_i(x') > u_i(x)$ when $i \in B_1$, $u_i(x') \geq u_i(x)$ when $i \in B_2$, and $x'_i = x_i$ when $i \in B_3$ if and only if there exist $\varepsilon > 0$ and a strategy x' in $\mathcal{G}'_{x,\varepsilon,(B_1,B_2,B_3)}$ of the first m players such that $u'_{m+1}(x', j) \leq 0$ for all $j \in B_1 \cup B_2$.*

The task of deciding if a strategy x is Pareto optimal amounts to checking the condition of Lemma 7 for $B_1 = \{i\}$ and $B_2 = [m] \setminus \{i\}$ for all i and to decide whether x is a strong Nash equilibrium amounts to checking the condition for all nonempty $B_1 \subseteq [m]$ while letting $B_2 = \emptyset$.

According to Lemma 6 we may restrict our attention to strategies x' in \mathcal{G}' of supports of size at most m . Fixing such a set of supports $T_i \subseteq S_i$ for $i \in B_1 \cup B_2$, we may formulate the question of existence of a strategy x' , with $\text{Supp}(x'_i) \subseteq T_i$ for $i \in B_1 \cup B_2$ that satisfies the conditions of Lemma 7 as an existentially quantified first-order formula over the reals. For a fixed x we need only $1 + m^2$ existentially quantified variables to describe ε and the strategy x' . Since this is a constant number of variables, when as in our case m is a constant, the general decision procedure of Basu, Pollack, and Roy [2] runs in polynomial time in the bitsize of coefficients, number of polynomials, and their degrees, resulting in an overall polynomial time algorithm. Now, adding a step of simply enumerating over all nonempty $B_1 \subseteq [m]$ and all support sets of size m we obtain the result of Gatti et al. that deciding whether a given strategy profile x is a strong Nash equilibrium can be done in polynomial time. The same holds in a similar way for checking that a strategy profile is a Pareto optimal Nash equilibrium.

In our case, when proving $\exists\mathbb{R}$ membership the only input is the game \mathcal{G} , whereas the strategy profile x will be given by a block of existentially quantified variables. We then need to show how to express that x is a Pareto optimal or a strong Nash equilibrium by a quantifier free formula over the reals with free variables x . This will be possible by the fact that quantifier elimination, rather than just decision, is possible for the first order theory of the reals. The quantifier elimination procedure of Basu et al. [2] runs in time exponential in the number of free variables, so we cannot apply it directly.

Instead we express the condition of Lemma 7 for a strategy profile x' that is constrained by $\text{Supp}(x'_i) \subseteq T_i$ for $i \in B_1 \cup B_2$ in terms of additional free variables \tilde{u}' that take the place of the values of the utility function u' of \mathcal{G}' . Since the supports of x' are restricted to size m , just m^{m+1} variables are needed to represent the utility to Player $m+1$ on every such pure strategy profile. For constant m , this is a constant number of variables, and thus the quantifier elimination procedure of Basu et al. runs in polynomial time and outputs a quantifier free formula over the reals with free variables \tilde{u}' that expresses the condition of Lemma 7 when the utilities u' are given by \tilde{u}' . After this we

substitute expressions for the utilities u' in terms of the variables x for the variables \tilde{u}' . The final formula is obtained, in an analogous way to the decision question, by enumerating over the appropriate sets B_1 and B_2 as well as all possible supports T_i , obtaining a formula for each such choice and combining them to a single formula with free variables x expressing either that x is Pareto optimal or that x is a strong Nash equilibrium. To the former we add the simple conditions of x being a Nash equilibrium. Finally we existentially quantify over x and obtain a formula expressing either that \mathcal{G} has a Pareto optimal Nash equilibrium or that \mathcal{G} has a strong Nash equilibrium. Since this formula was computed in polynomial time given \mathcal{G} we obtain the following result.

Proposition 5. \exists STRONGNE and \exists PARETOOPTIMALNE both belong to $\exists\mathbb{R}$.

For showing $\exists\mathbb{R}$ -hardness we construct a new extension of \mathcal{G}_0 .

Definition 8. For $u \geq 0$, let $\mathcal{H}_4 = \mathcal{H}_4(u)$ be the 3-player game given by the following matrices, where Player 1 selects the matrix, Player 2 selects the row, Player 3 selects the column.

$$\begin{array}{c}
 \begin{array}{cc}
 & \begin{array}{cc} G & \perp \end{array} \\
 \begin{array}{cc} G & \perp \end{array} & \begin{array}{|cc|} \hline (2u, -u, -u) & (-3, -3, 0) \\ \hline (-3, 0, -3) & (-2, -2, -2) \\ \hline \end{array} \\
 & \begin{array}{c} G \\ \perp \end{array}
 \end{array}
 \qquad
 \begin{array}{cc}
 & \begin{array}{cc} G & \perp \end{array} \\
 \begin{array}{cc} G & \perp \end{array} & \begin{array}{|cc|} \hline (0, -3, -3) & (-2, -2, -2) \\ \hline (-2, -2, -2) & (-1, -1, -1) \\ \hline \end{array} \\
 & \begin{array}{c} G \\ \perp \end{array}
 \end{array}
 \end{array}$$

Lemma 8. When $u > 0$, the only Nash equilibrium of $\mathcal{H}_4(u)$ is the pure strategy profile (\perp, \perp, \perp) . When $u = 0$, the only Nash equilibria of $\mathcal{H}_4(u)$ are the pure strategy profiles (G, G, G) and (\perp, \perp, \perp) . Furthermore, when $u = 0$, the Nash equilibrium (G, G, G) is both a Pareto optimal and a strong Nash equilibrium.

Proof. When $u = 0$, clearly (G, G, G) is a Nash equilibrium, which is both Pareto optimal and a strong Nash equilibrium. Likewise, clearly (\perp, \perp, \perp) is always a Nash equilibrium. When $u > 0$, the action G is strictly dominated by the action \perp for Player 2 and Player 3, and hence they play \perp with probability 1 in a Nash equilibrium. The only best reply of Player 1 is to play \perp with probability 1 as well. \square

Analogously to Definition 3 we define the game $\mathcal{G}_4 = \mathcal{G}_4(\mathcal{S})$ to be the game extending \mathcal{G}_0 with \mathcal{H}_4 replacing the role of \mathcal{H}_1 . We next establish $\exists\mathbb{R}$ -hardness

Theorem 7. \exists PARETOOPTIMALNE and \exists STRONGNE are $\exists\mathbb{R}$ -complete, even for 3-player games.

Proof. In \mathcal{G}_4 , the strategy profile (\perp, \perp, \perp) , with payoff profile $(-1, -1, -1)$, is a Nash equilibrium that is neither Pareto optimal or a strong Nash equilibrium, since by Lemma 1 a strategy profile in \mathcal{G}_0 in which Player 1 plays an action according to the uniform distribution has payoff profile $(0, 0, 0)$.

Similarly to the proof of Theorem 3, any Nash equilibrium x in \mathcal{G}_4 different from (\perp, \perp, \perp) must by Lemma 8 be a Nash equilibrium of \mathcal{G}_0 with payoff profile $(0, 0, 0)$. Since \mathcal{G}_0 is a zero-sum game, any strategy that is Pareto dominating x must involve the strategy \perp and is thus ruled out by Lemma 8. Therefore x is Pareto-optimal. Now, x is not necessarily a strong Nash equilibrium, but by Lemma 1, letting Player 1 instead play an action of \mathcal{G}_0 according to the uniform distribution is also a Nash equilibrium of \mathcal{G}_0 with payoff profile $(0, 0, 0)$, that furthermore ensures that any strategy profile of Player 2 and Player 3 in \mathcal{G}_0 does not improve their payoffs. Also, by Lemma 1, no

coalition involving Player 1 can improve their payoff without playing the action \perp . No coalition can however improve their payoff by a strategy profile involving the action \perp , since all such payoff profiles result in a player receiving negative payoff. Thus x' is a strong Nash equilibrium.

We conclude that Proposition 3 gives a reduction showing $\exists\mathbb{R}$ -hardness of both $\exists\text{PARETOOPTIMALNE}$ and $\exists\text{STRONGNE}$, thereby together with Proposition 5 completing the proof. \square

3.4 Irrational and Rational Nash Equilibria

Starting with a quadratic system in which every solution must involve an irrational valued variable allows us to obtain $\exists\mathbb{R}$ -hardness for $\exists\text{IRRATIONALNE}$.

Theorem 8. $\exists\text{IRRATIONALNE}$ is $\exists\mathbb{R}$ -hard, even for 3-player zero-sum games.

Proof. The proof of Proposition 1 constructs a polynomial time computable function that takes a system \mathcal{S} of quadratic equations and produces a new system \mathcal{S}' of quadratic equations. From this construction it follows that there is an affine function F given by a matrix and a vector with rational entries such that the set of solutions of \mathcal{S}' is the inverse image under F of the set of solutions of \mathcal{S} . Adding to \mathcal{S} the equation $x^2 - 2 = 0$, where x is a new variable, ensures that every solution of \mathcal{S} and hence \mathcal{S}' is not rational valued. This also holds for the homogeneous bilinear system of equations \mathcal{S}'' obtained from \mathcal{S}' by Proposition 2. By Proposition 3 any Nash equilibrium of $\mathcal{G}_0(\mathcal{S}'')$ with payoff profile $(0, 0, 0)$ is thereby not rational valued. We conclude that Proposition 4 gives a reduction showing $\exists\mathbb{R}$ -hardness of $\exists\text{IRRATIONALNE}$, since the Nash equilibrium (\perp, \perp, \perp) of $\mathcal{G}_1(\mathcal{S}'')$ is a rational valued strategy profile. \square

While Theorem 8 shows that deciding whether a Nash equilibrium that is not rational valued exists is $\exists\mathbb{R}$ -hard, we do not know whether the problem $\exists\text{IRRATIONALNE}$ is even decidable.

We next consider the question of deciding whether a given game has a rational valued Nash equilibrium. This problem is naturally expressible in the existential theory of the rationals $\text{Th}_{\exists}(\mathbb{Q})$, which is however not known to be decidable. It is natural to ask whether the problem $\exists\text{RATIONALNE}$ is also $\exists\mathbb{Q}$ -hard. An obstacle for such a result however, is that we do not know a bound on the magnitude of coordinates of rational solutions to quadratic equations similar to the case of real numbers. We can however start from a promise version of $\text{QUAD}_{\mathbb{Q}}$ and construct a reduction to $\exists\text{RATIONALNE}$. We sketch the construction below.

Definition 9. Let $\text{QUAD}_{\mathbb{Q}}(\mathbf{B}(\mathbf{0}, 1))$ denote the promise problem given by $\text{QUAD}_{\mathbb{Q}}$ together with the promise that if the given quadratic system has a solution over \mathbb{Q} , then a solution over \mathbb{Q} exists in the unit ball $\mathbf{B}(\mathbf{0}, 1)$.

A simple scaling and translation give a reduction from the promise problem of Definition 9 to the analogue over \mathbb{Q} of the promise problem of Proposition 1 and then further to the analogue over \mathbb{Q} of the promise problem of Proposition 2. We shall then construct a modification of \mathcal{G}_1 in which the Nash equilibrium (\perp, \perp, \perp) is replaced by an irrational valued Nash equilibrium. Several examples of 3-player games are known that are without rational valued Nash equilibria. We give below a simple 3-player zero-sum game with a unique Nash equilibrium that is irrational valued.

Definition 10. Let \mathcal{H}_5 be the 3-player zero-sum game where each player has the action set $\{1, 2\}$, and the payoff vectors are given by the following two matrices, where Player 1 selects the matrix, Player 2 selects the row, Player 3 selects the column.

$$\begin{array}{c}
 \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{array}{|cc|} \hline (-4, 2, 2) & (-2, 1, 1) \\ \hline (-2, 1, 1) & (0, 0, 0) \\ \hline \end{array} \\ & \begin{array}{cc} 1 & 2 \end{array} \end{array}
 \qquad
 \begin{array}{c}
 \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{array}{|cc|} \hline (0, 0, 0) & (-2, 1, 1) \\ \hline (-2, 1, 1) & (-6, 3, 3) \\ \hline \end{array} \\ & \begin{array}{cc} 1 & 2 \end{array} \end{array}
 \end{array}$$

We omit the straightforward but tedious analysis of the game \mathcal{H}_5 .

Lemma 9. The unique Nash equilibrium of \mathcal{H}_5 has Player 1 playing action 1 with probability $1 - 1/\sqrt{6}$, and both Player 2 and Player 3 playing the action 1 with probability $3 - \sqrt{6}$. The Nash equilibrium payoff profile is $(-4(3 - \sqrt{6}), 2(3 - \sqrt{6}), 2(3 - \sqrt{6}))$.

We can now provide our hardness statement for \exists RATIONALNE.

Theorem 9. There is a polynomial time reduction from the promise problem $\text{QUAD}_{\mathbb{Q}}(\mathbf{B}(\mathbf{0}, 1))$ to \exists RATIONALNE, and the output of the reduction is a 3-player zero-sum game.

Proof. Let \mathcal{S} be a system of quadratic equations in n variables such that either \mathcal{S} has no solutions in \mathbb{Q}^n or has a solution in $\mathbb{Q}^n \cap \mathbf{B}(\mathbf{0}, 1)$. As explained above we may in polynomial time transform \mathcal{S} into a system \mathcal{S}' of homogeneous bilinear polynomials in $2(n+1)$ variables such that \mathcal{S} has a solution in \mathbb{Q}^n if and only if \mathcal{S}' has a solution in $(\mathbb{Q}^{n+1} \times \mathbb{Q}^{n+1}) \cap (\Delta^n \times \Delta^n)$. Define the 3-player zero-sum game $\mathcal{G}_5 = \mathcal{G}_5(\mathcal{S}')$ to be the game obtained from $\mathcal{G}_1(\mathcal{S}')$ as follows, similarly to the definition of \mathcal{G}_2 .

The action \perp is for all players replaced by actions $(\perp, 1)$ and $(\perp, 2)$. When the players choose the pure strategy profile $((\perp, a_1), (\perp, a_2), (\perp, a_3))$ Player 1 receive utility $-2 + \frac{1}{6}u_1(a_1, a_2, a_3)$ and Player 2 and Player 3 both receive utility $1 + \frac{1}{6}u_2(a_1, a_2, a_3) = 1 + \frac{1}{6}u_3(a_1, a_2, a_3)$, where u_1, u_2 , and u_3 are the utility functions of the game \mathcal{H}_5 . Thus the payoff profile $(-2, 1, 1)$ of the strategy profile (\perp, \perp, \perp) is perturbed by the payoffs of the game \mathcal{H}_6 , scaled by $\frac{1}{6}$ in order to ensure that each entry is perturbed by at most 1. As in the proof of Theorem 4, a Nash equilibrium x is either a Nash equilibrium of \mathcal{G}_0 in which every player receives payoff 0, or is such that the players choose the actions (\perp, a) according to the unique Nash equilibrium of \mathcal{H}_5 . Since the latter is irrational valued we conclude that if x is a rational valued Nash equilibrium then x is a rational valued Nash equilibrium of $\mathcal{G}_0(\mathcal{S}')$ in which every player receives payoff 0, which by Proposition 3 implies a rational valued solution to \mathcal{S}' . Likewise a rational valued solution of \mathcal{S}' in $\Delta^n \times \Delta^n$ gives a rational valued Nash equilibrium of $\mathcal{G}_1(\mathcal{S}')$, thereby completing the proof. \square

4 Decision Problems about Nash Equilibria in Symmetric Games

In this section we consider variations of all the decision problems considered in Section 3, where the given input is now a finite strategic form *symmetric* game \mathcal{D} , where every player share the same set S of pure strategies, together with auxiliary input. As

before, u denotes a rational number, k an integer, whereas we now consider a single subset $T \subseteq S$ of actions. The decision problems are described by stating the property that a *symmetric* Nash equilibrium x whose existence is to be determined should satisfy. We use the same grouping as the problems of Section 3, but now we cover all problems in the same section.

Problem	Condition
\exists SNEWITHLARGEPAFFS	$u_i(x) \geq u$ for all i .
\exists SNEWITHSMALLPAFFS	$u_i(x) \leq u$ for all i .
\exists SNEWITHLARGETOTALPAFF	$\sum_i u_i(x) \geq u$.
\exists SNEWITHSMALLTOTALPAFF	$\sum_i u_i(x) \leq u$.
\exists SNEINABALL	$x_i(a_i) \leq u$ for all i and $a_i \in S_i$.
\exists SECONDSNE	x is not the only SNE.
\exists SNEWITHLARGESUPPORTS	$ \text{Supp}(x_i) \geq k$ for all i .
\exists SNEWITHSMALLSUPPORTS	$ \text{Supp}(x_i) \leq k$ for all i .
\exists SNEWITHRESTRICTINGSUPPORTS	$T \subseteq \text{Supp}(x_i)$ for all i .
\exists SNEWITHRESTRICTEDSUPPORTS	$\text{Supp}(x_i) \subseteq T$ for all i .
\exists NONPARETOOPTIMALSNE	x is not Pareto optimal.
\exists NONSTRONGSNE	x is not a strong NE.
\exists PARETOOPTIMALSNE	x is Pareto optimal.
\exists STRONGSNE	x is a strong NE.
\exists IRRATIONALSNE	$x_i(a_i) \notin \mathbb{Q}$ for some i and $a_i \in S_i$.
\exists RATIONALSNE	$x_i(a_i) \in \mathbb{Q}$ for all i and $a_i \in S_i$.

In addition to the above problems about symmetric Nash equilibria, we also shall consider the problem \exists NONSYMMETRICNE, that given a finite strategic form symmetric game \mathcal{D} , asks whether \mathcal{D} has a Nash equilibrium x that is *nonsymmetric*.

$\exists\mathbb{R}$ membership of all these problems, except for those of the last group above, follows analogously to the case of their non-symmetric counterparts and will not be discussed further.

4.1 Symmetrization

Garg et al. [15] constructed a *symmetrization* transformation of 3-player games to symmetric 3-player games. This was used to give reductions from the two problems \exists NEWITHRESTRICTINGSUPPORTS and \exists NEWITHRESTRICTEDSUPPORTS to their symmetric counterparts, and these were the first problems about symmetric Nash equilibria shown to be $\exists\mathbb{R}$ -complete. Bilò and Mavronicolas [6], then constructed further reductions starting from \exists SNEWITHRESTRICTEDSUPPORTS.

We can apply a different, but similar symmetrization transformation to the game $\mathcal{G}_0(\mathcal{S})$ of Section 3 obtaining a symmetric game $\mathcal{D}_0(\mathcal{S})$ that will form the base of further reduction as well as giving a direct proof of $\exists\mathbb{R}$ -completeness for the problem \exists SNEWITHLARGEPAFFS. In addition to our new results, we give for completeness also proofs of the previous $\exists\mathbb{R}$ -completeness results.

The idea of symmetrization is to take a game \mathcal{G} , with strictly positive payoffs, and construct a new symmetric game \mathcal{D} in which the players can take the role of any player of \mathcal{G} . The game \mathcal{G} is then played when the players choose distinct roles. The players are in the construction of Garg et al. [15, Lemma 5.1] incentivized to have this behavior

by the choice of payoffs (0 or 1) in case the roles of the players overlap. In our case we can simply let the players be incentivized by the given payoff requirement alone.

Definition 11 (The symmetric 3-player game \mathcal{D}_0). Let $\mathcal{G}_+ = \mathcal{G}_+(\mathcal{S})$ be the game obtained from $\mathcal{G}_0(\mathcal{S})$ as follows. Let $u_1, u_2,$ and u_3 be the utility functions of \mathcal{G}_0 . Let M be the smallest (positive) integer such that $-M < u_1(x) < M$ for all pure strategy profiles x . Define the utility functions u'_1 and $u'_2 = u'_3$ of \mathcal{G}_+ by $u'_1(x) = u_1(x) + M$ and $u'_2(x) = u'_3(x) = -u_1(x) + M$. Thus also, $u'_2(x) = u'_3(x) = 2u_2(x) + M = 2u_3(x) + M$.

For a permutation π of $\{1, 2, 3\}$ we denote by $\mathcal{G}_+^\pi = \mathcal{G}_+^{\pi(1), \pi(2), \pi(3)}$ the game where Player i has the set of actions $S_{\pi(i)}$ and the utility function given by $u'_{\pi(i)}(a_{\pi^{-1}(1)}, a_{\pi^{-1}(2)}, a_{\pi^{-1}(3)})$, where $a_i \in S_{\pi(i)}$ is the action chosen by Player i . Thus \mathcal{G}_+^π is just a reordering of the players of \mathcal{G}_+ such that Player i in \mathcal{G}_+^π assumes the role of Player $\pi(i)$ in \mathcal{G}_+ .

Define the game $\mathcal{D}_0 = \mathcal{D}_0(\mathcal{S})$ to be the 3-player symmetric form game in which the players have the set of actions $S = S_1 \cup S_2 \cup S_3$, which is the disjoint union of the set of actions S_1, S_2 and S_3 of the players in \mathcal{G}_0 . We also view $S_1, S_2,$ and S_3 as disjoint sets below. When the players play actions $a_1, a_2,$ and a_3 , such that there exists a permutation π of $\{1, 2, 3\}$ satisfying that $a_i \in S_{\pi(i)}$, for all i , then Player i receives utility $u'_{\pi(i)}(a_{\pi^{-1}(1)}, a_{\pi^{-1}(2)}, a_{\pi^{-1}(3)})$. Otherwise, Player i simply receives utility 0. The payoffs vectors of \mathcal{D}_0 are illustrated below as a block tensor of payoff vectors, where Player 1 selects the matrix slice, Player 2 selects the row, and Player 3 selects the column. We let $\mathbf{0}$ denote a payoff tensor of any appropriate dimensions in which every payoff is 0.

	S_1	S_2	S_3
S_1	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
S_2	$\mathbf{0}$	$\mathbf{0}$	$\mathcal{G}_+^{(1,2,3)}$
S_3	$\mathbf{0}$	$\mathcal{G}_+^{(1,3,2)}$	$\mathbf{0}$

S_1

	S_1	S_2	S_3
S_1	$\mathbf{0}$	$\mathbf{0}$	$\mathcal{G}_+^{(2,1,3)}$
S_2	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
S_3	$\mathcal{G}_+^{(2,3,1)}$	$\mathbf{0}$	$\mathbf{0}$

S_2

	S_1	S_2	S_3
S_1	$\mathbf{0}$	$\mathcal{G}_+^{(3,1,2)}$	$\mathbf{0}$
S_2	$\mathcal{G}_+^{(3,2,1)}$	$\mathbf{0}$	$\mathbf{0}$
S_3	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$

S_3

We next relate symmetric Nash equilibria in \mathcal{D}_0 to Nash equilibria in \mathcal{G}_0 .

Lemma 10. *The games $\mathcal{G}_0(\mathcal{S})$ and $\mathcal{G}_+(\mathcal{S})$ have the same set of Nash equilibria. All players receive payoff 0 in \mathcal{G}_0 if and only if the total payoff of the players in \mathcal{G}_+ is $3M$, which is also maximum possible total payoff of the players in \mathcal{G}_+ in any Nash equilibrium.*

Proof. Since the utility functions of \mathcal{G}_+ are obtained from those of \mathcal{G}_0 by scaling with a positive constant and adding a constant, the games have the same set of Nash equilibria. Note now that $u'_1(x) + u'_2(x) + u'_3(x) = 3M - u_1(x)$. Since $u_1(x) \geq 0$ in any

Nash equilibrium by Lemma 1, the maximum total equilibrium payoff in \mathcal{G}_+ is $3M$. In \mathcal{G}_0 all players receive payoff 0 if and only if $u_1(x) = 0$, from which the conclusion follows. \square

Proposition 6. *Define $K = \frac{2M}{9}$, where M is given in Definition 11. Let (x_1, x_2, x_3) be a Nash equilibrium of \mathcal{G}_0 in which every player receive payoff 0. Then the strategy profile (y, y, y) in which every player chooses $i \in \{1, 2, 3\}$, each with probability $\frac{1}{3}$, and plays an action according to x_i is a symmetric Nash equilibrium in $\mathcal{D}_0(\mathcal{S})$ in which all players receive payoff K . Conversely let (y, y, y) be a symmetric Nash equilibrium of \mathcal{D}_0 in which every player receives payoff K , which is also the maximum possible payoff of a symmetric Nash equilibrium of \mathcal{D}_0 . Then the total probability given to actions of each set S_i is exactly $\frac{1}{3}$. Define x_i to be the conditional probability distribution on S_i obtained from y given that an action of S_i is played. Then (x_1, x_2, x_3) is a Nash equilibrium of \mathcal{G}_0 in which every player receives payoff 0.*

Proof. First, let $x = (x_1, x_2, x_3)$ be a Nash equilibrium of \mathcal{G}_0 in which every player receive payoff 0. By Lemma 10 x is also a Nash equilibrium of \mathcal{G}_+ given total payoff $3M$. Let y be the strategy that selects each $i \in \{1, 2, 3\}$ with probability $\frac{1}{3}$ and then chooses an action according to a x_i . Then (y, y, y) must be a symmetric Nash equilibrium of \mathcal{D}_0 , since if a player could improve payoff by a change to a different strategy y' , there would also be a way for one of the players to improve the payoff in \mathcal{G}_+ . Each player takes part in playing \mathcal{G}_+ a total of 6 times, each chosen with probability $\frac{1}{27}$, and taking the role of each player 2 times. The payoff to each player is therefore equal to $\frac{2}{27}3M = K$ by Lemma 10.

Assume now that (y, y, y) is a symmetric Nash equilibrium of \mathcal{D}_0 . Let p_i be the total probability given to actions of S_i , for $i \in \{1, 2, 3\}$. Clearly, if $p_i = 0$ for some i the players receive payoff 0 due to the symmetrization construction of \mathcal{D}_0 . Assume now that $p_i > 0$ for all i . The conditional probability distributions x_i , obtained from y given that an action of S_i is played, are therefore well defined. The strategy profile $x = (x_1, x_2, x_3)$ is a Nash equilibrium of \mathcal{G}_+ as otherwise a player could improve the payoff in \mathcal{D}_0 as well. By Lemma 10 the total payoff U to the players in \mathcal{G}_+ is at most $3M$, and equals $3M$ exactly when x gives payoff 0 to all players in \mathcal{G}_0 . The total payoff to the players in \mathcal{D}_0 is therefore equal to $6p_1p_2p_3U \leq p_1p_2p_318M$. By the AM-GM inequality $p_1p_2p_3 \leq (\frac{1}{3}(p_1 + p_2 + p_3))^3 = \frac{1}{27}$ with equality if and only if $p_1 = p_2 = p_3 = \frac{1}{3}$. The maximum total payoff of the players is thus $\frac{2}{3}M = 3K$, and obtaining this requires both that $p_1 = p_2 = p_3 = \frac{1}{3}$ and $U = M$. Thus by Lemma 10, if (y, y, y) give all players payoff K in \mathcal{D}_0 then (x_1, x_2, x_3) give all players payoff 0 in \mathcal{G}_0 . \square

4.2 Decision Problems for Symmetric Nash Equilibria

From Proposition 6 together with Theorem 1 we immediately obtain the first $\exists\mathbb{R}$ -hardness result about symmetric Nash equilibria.

Theorem 10 (Bild and Mavronicolas [6]). $\exists\text{SNEWITHLARGEPAYOFFS}$ and $\exists\text{SNEWITHLARGETOTALPAYOFF}$ are $\exists\mathbb{R}$ -complete, even for 3-player games.

As done for the game \mathcal{G}_0 we now construct simple extensions of the game \mathcal{D}_0 . We describe these constructions below. For some of the results we give only a proof sketch.

Definition 12 (The symmetric 3-player game \mathcal{D}_1). Let $\mathcal{D}_1 = \mathcal{D}_1(\mathcal{S})$ be the game obtained from $\mathcal{D}_0(\mathcal{S})$ as follows. Each player is given an additional action \perp . When no

player plays the action \perp , the payoffs are the same as in \mathcal{D}_0 . When exactly one player is playing \perp , every player receives payoff K . When more than one player is playing \perp , every player receives payoff $K + 1$.

Proposition 7. *The pure strategy profile (\perp, \perp, \perp) is a symmetric Nash equilibrium of \mathcal{D}_1 in which every player receives payoff $K + 1$. Any other symmetric Nash equilibrium is also a symmetric Nash equilibrium of \mathcal{D}_0 and is such that every player receives payoff K .*

Proof. Let (y, y, y) be a symmetric Nash equilibrium of \mathcal{D}_1 that is different from (\perp, \perp, \perp) . Let y' be the probability distribution obtained from y given that \perp is not played. Then (y', y', y') must be a symmetric Nash equilibrium of \mathcal{D}_0 in which every player receive payoff K , since otherwise a player could improve the payoff in \mathcal{D}_1 by always playing \perp . Also it follows that \perp is actually played with probability 0 by y , since otherwise a player could improve the payoff in \mathcal{D}_1 by always playing \perp . Thus $y = y'$ and the result follows. \square

The game \mathcal{D}_1 gives, together with Proposition 3, reductions from the promise problem of Proposition 2 to most of the problems under consideration, showing $\exists\mathbb{R}$ -completeness. Except for $\exists\text{SNEINABALL}$, this was shown earlier by Garg et al [15] and Bilò and Mavronicolas [6].

Theorem 11 (Garg et al [15]; Bilò and Mavronicolas [6]). *The following problems are $\exists\mathbb{R}$ -complete, even for 3-player games:*

$\exists\text{SNEWITHSMALLPAYOFFS}$,	$\exists\text{SNEWITHSMALLTOTALPAYOFF}$,
$\exists\text{SNEINABALL}$,	$\exists\text{SECONDSNE}$,
$\exists\text{SNEWITHLARGESUPPORTS}$,	$\exists\text{SNEWITHRESTRICTINGSUPPORTS}$,
$\exists\text{NONPARETOOPTIMALSNE}$,	$\exists\text{SNEWITHRESTRICTEDSUPPORTS}$,
$\exists\text{NONSTRONGSNE}$.	

Proof. Proposition 3, Proposition 6, and Proposition 7 together give a reduction from the promise problem of Proposition 2 to all the problems under consideration thereby showing $\exists\mathbb{R}$ -hardness, when setting the additional parameters as follows. We let $u = K$ for $\exists\text{SNEWITHSMALLPAYOFFS}$ and we let $u = 3K$ for the similar problem $\exists\text{SNEWITHSMALLTOTALPAYOFF}$. For $\exists\text{SNEINABALL}$ we let $u = \frac{1}{2}$ and for $\exists\text{SNEWITHLARGESUPPORTS}$ we let $k = 2$. We let T be the set of all actions except i for $\exists\text{SNEWITHRESTRICTINGSUPPORTS}$ and $\exists\text{SNEWITHRESTRICTEDSUPPORTS}$. \square

We can proceed in a similar way as Section 3 for the remaining problems concerning symmetric Nash equilibria. In order to adapt the proof of Theorem 4, we need to replace the Nash equilibrium (\perp, \perp, \perp) in \mathcal{D}_1 by a symmetric Nash equilibrium with large supports. Bilò and Mavronicolas [6, Lemma 4] construct for any k a symmetric m -player zero-sum game with a unique symmetric Nash equilibrium that is fully mixed on a set of k strategies. We may use this to perturb the payoff profile of (\perp, \perp, \perp) in \mathcal{D}_1 analogously to the proof of Theorem 4 thereby obtaining an alternative proof of $\exists\mathbb{R}$ -hardness of $\exists\text{SNEWITHSMALLSUPPORTS}$.

Theorem 12 (Bilò and Mavronicolas [6]). $\exists\text{SNEWITHSMALLSUPPORTS}$ is $\exists\mathbb{R}$ -complete, even for 3-player games.

For the problems $\exists\text{PARETOOPTIMALSNE}$ and $\exists\text{STRONGSNE}$ we define the game $\mathcal{D}_4 = \mathcal{D}_4(\mathcal{S})$ extending \mathcal{D}_0 in an analogous way to the game \mathcal{G}_4 . Namely, each player is given an additional action \perp . When no player plays the action \perp , the payoffs are the same as in \mathcal{D}_0 . When exactly one player is playing \perp , that player receives payoff K , whereas the other two players receive payoff $K - 3$. When exactly two players are playing \perp , every player receives payoff $K - 2$. Finally, when all players are playing \perp , every player receives payoff $K - 1$. Thus the utilities of the players when a player is playing the action \perp are those of \mathcal{H}_4 added to K . In an analogous way to the proof of Theorem 7 we may then obtain the following result.

Theorem 13. $\exists\text{PARETOOPTIMALSNE}$ and $\exists\text{STRONGSNE}$ are $\exists\mathbb{R}$ -complete, even for 3-player games.

We now turn to irrational and rational valued symmetric Nash equilibria. Analogously to the proof of Theorem 8, starting with a quadratic system \mathcal{S} in which every solution must involve an irrational valued variable gives via the game \mathcal{D}_1 a reduction showing $\exists\mathbb{R}$ -hardness for $\exists\text{IRRATIONALSNE}$.

Theorem 14. $\exists\text{IRRATIONALSNE}$ is $\exists\mathbb{R}$ -hard, even for 3-player games.

To make a symmetric analogue of Theorem 9 we need a 3-player symmetric game with unique Nash equilibrium that is irrational valued. Rather than giving an explicit example, we note that the symmetrization transformation of Garg et al. [15] applied to, say, the game \mathcal{H}_3 gives precisely such a symmetric game. Using that to extend \mathcal{D}_1 and perturb the payoff profile of (\perp, \perp, \perp) we may obtain the following hardness result.

Theorem 15. *There is a polynomial time reduction from the promise problem $\text{QUAD}_{\mathbb{Q}}(\mathbf{B}(\mathbf{0}, 1))$ to $\exists\text{RATIONALNE}$.*

4.3 A Decision Problem about Nonsymmetric Equilibria

Our final result is concerned with the existence of a non-symmetric Nash equilibrium in a symmetric game. Our hardness proof is based by a modification of the games \mathcal{D}_0 and \mathcal{D}_1 . We note that the game \mathcal{D}_0 was defined to be a symmetrization of the game \mathcal{G}'_0 , used in the $\exists\mathbb{R}$ -hardness proof of Theorem 2 of for the problem $\exists\text{NEWITHLARGETOTALPAYOFF}$, with M added to every payoff in order to make all payoffs strictly positive. This is the appropriate choice for studying symmetric Nash equilibria, since in a symmetric Nash equilibria of \mathcal{D}_0 each player takes the role of every player of \mathcal{G}'_0 , thereby accumulating the payoffs of each player (scaled appropriately). For studying nonsymmetric Nash equilibria the idea is force the players to take on the role of just one player of \mathcal{G}'_0 .

Define $\mathcal{G}'_+ = \mathcal{G}'_+(\mathcal{S})$ to be the game obtained from \mathcal{G}'_0 by adding M to all payoffs, where M is the smallest positive integer such that $-M < u_1(x) < M$. Define \mathcal{D}'_0 analogously to \mathcal{D}_0 with the game \mathcal{G}'_+ taking the role of \mathcal{G}'_+ . Next, define the game $\mathcal{D}'_1 = \mathcal{D}'_1(\mathcal{S})$ obtained from $\mathcal{D}'_0(\mathcal{S})$ by giving each player an additional action \perp , and defining the utility function as follows. When no player plays the action \perp , the payoffs are the same as in \mathcal{D}'_0 . When exactly one player is playing \perp , every player receives payoff M . When exactly two players are playing \perp , every player receives payoff $M + 1$. Finally, when all players are playing \perp , every player receives payoff $M + 2$.

Theorem 16. $\exists\mathbb{R}$ -NONSYMMETRICNE is $\exists\mathbb{R}$ -complete, even for 3-player games.

Proof. We show $\exists\mathbb{R}$ -hardness by reduction from the promise problem of Proposition 2 by the game $\mathcal{D}'_1(\mathcal{S})$. Consider a strategy profile $x' = (x_1, x_2, x_3)$ in the game \mathcal{D}'_0 . Since \mathcal{G}_0 is a zero-sum game, the total payoff received by the players is at most $3M$. Furthermore, this is by the construction of \mathcal{D}'_0 achievable only when there is a permutation π of $\{1, 2, 3\}$ such that $\text{Supp}(x'_i) \subseteq S_{\pi(i)}$, where S_1, S_2 , and S_3 are the strategy sets of the players in \mathcal{G}_0 . Thus when the total payoff of the players is $3M$ we may view the strategy profile x' as a strategy profile of $\mathcal{G}_0(\mathcal{S})$.

If there exists a strategy profile x' in \mathcal{G}_0 in which every player receives payoff 0, we may conversely view this as a (nonsymmetric) strategy profile of $\mathcal{D}_0(\mathcal{S})$ in which every player receives payoff M . This is also a Nash equilibrium in \mathcal{D}_1 which is nonsymmetric.

Conversely, consider a Nash equilibrium x of \mathcal{D}'_1 that is nonsymmetric, and therefore different from (\perp, \perp, \perp) . No player can play \perp with probability 1, since then \perp would be the unique best reply of the other players. Thus we may consider the strategy profile x' of \mathcal{D}'_0 obtained from x conditioned on that no player is playing \perp . This must be a Nash equilibrium of \mathcal{D}'_0 in which every player receives payoff M , since otherwise x would not be a Nash equilibrium of \mathcal{D}'_1 . As argued above this means that x' gives a Nash equilibrium of \mathcal{G}_0 in which every player receives payoff 0, thereby completing the proof using Proposition 3. \square

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