

Sharp estimates for the global attractor of scalar reaction-diffusion equations with a Wentzell boundary condition

Ciprian G. Gal
 Department of Mathematics
 University of Missouri,
 Columbia, MO 65211, USA
 ciprian@math.missouri.edu

October 24, 2018

Abstract

In this paper, we derive optimal upper and lower bounds on the dimension of the attractor \mathcal{A}_W for scalar reaction-diffusion equations with a Wentzell (dynamic) boundary condition. We are also interested in obtaining explicit bounds on the constants involved in our asymptotic estimates, and to compare these bounds to previously known estimates for the dimension of the global attractor \mathcal{A}_K , $K \in \{D, N, P\}$, of reaction-diffusion equations subject to Dirichlet, Neumann and periodic boundary conditions. The explicit estimates we obtain show that the dimension of the global attractor \mathcal{A}_W is of different order than the dimension of \mathcal{A}_K , for each $K \in \{D, N, P\}$, in all space dimensions that are greater or equal than three.

1 Introduction

It is well-known that the long-time behaviour of solutions of partial differential equations arising in mathematical physics can, in many cases, be described in terms of global attractors of the associated semigroups (see [3, 6, 32, 46] and references therein). For a large class of equations of mathematical physics, including parabolic partial differential equations modelling reaction, diffusion and drift, hyperbolic type equations, and so on, the corresponding

¹2000 Mathematics Subject Classification: 35K57, 35K55, 35B40, 35B41, 37L30, 35Q80, 35Q86

attractor has finite Hausdorff and fractal dimensions. Thus, the dynamics on the attractor happens to be finite-dimensional, even though the system is governed by a set of partial differential equations. As the dimension of the attractor is indicative of the number of degrees of freedom needed to simulate a given dynamical system, it is then crucial to obtain more realistic estimates for its dimension in terms of observable physical quantities.

Aside from some applied motivation, much of the mathematical interest nowadays is centered on the dynamics of boundary value problems with *static* boundary conditions of Dirichlet and Neumann-Robin type, or even periodic boundary conditions. The influence of these dissipative boundary conditions on a given model has only been recently investigated in connection with a class of reaction-diffusion systems. In [33], a first contribution is made to the understanding of this problem with a Robin boundary condition. In particular, it is shown, for a fixed nonlinearity, how the flow defined by the reaction-diffusion system depends on the interaction between diffusion ν and another parameter θ involved in the boundary condition (cf. also [34]). A classification of points in (ν, θ) -space, as structurally stable, or bifurcation points, for a one-dimensional scalar reaction-diffusion equation with a cubic nonlinearity is discussed in detail in [33]. Other studies on the influence of boundary conditions upon the solution structures of partial differential equations have also been done by other scientists. These studies have analyzed the detailed effect of boundary conditions on the structure of global attractors (see, e.g., [8, 30, 36, 44]). If the equilibrium is nonhyperbolic and a bifurcation occurs, the structure of attractors may vary with respect to boundary conditions. This has been observed in the analysis of pattern formation in a 1D reaction-diffusion system [8], in lattice systems [43], in the study of steady state bifurcations [30, 36], and finally in [44], on mode-jumping of the von Karman equations.

Although the global attractors of these systems will depend, for a given nonlinearity, on the choice of the boundary conditions, their finite dimension does generally *not*. This result can be easily formulated for a scalar reaction-diffusion equation, as follows. Consider the

parabolic partial differential equation

$$\partial_t u = \nu \Delta u - f(u) + \lambda u + g, \quad (x, t) \in \Omega \times (0, +\infty), \quad (1.1)$$

where $u = u(x, t) \in \mathbb{R}$, $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded domain with sufficiently smooth boundary Γ , $g = g(x)$, and ν, λ are positive constants. The function $f : \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be $C^{1,1}$, that is, continuous and with a Lipschitz continuous first derivative, which satisfies, among other natural growth conditions (see, e.g., [6, Chapter II]),

$$f'(y) \geq -c_f, \quad \text{for all } y \in \mathbb{R}, \text{ for some } c_f > 0.$$

We may ask that u satisfy either a Dirichlet ($K = D$) boundary condition or a Neumann ($K = N$) boundary condition, and even a periodic ($K = P$) boundary condition. It is well-known that equation (1.1), supplemented with an appropriate initial condition, generates a semigroup $\{S_t\}$ acting on a suitable phase space H . This semigroup possesses the global attractor \mathcal{A}_K , which may depend on the choice of the boundary conditions, and \mathcal{A}_K has finite fractal dimension for each $K \in \{D, N, P\}$. In particular, the Hausdorff and fractal dimensions of \mathcal{A}_K , for any $K \in \{D, N, P\}$, satisfy the following upper and lower bounds:

$$c_0 \left(\frac{\lambda}{\nu} \right)^{n/2} |\Omega| \leq \dim_H \mathcal{A}_K \leq \dim_F \mathcal{A}_K \leq c_1 \left(1 + \frac{c_f + \lambda}{\nu} |\Omega|^{2/n} \right)^{n/2}, \quad (1.2)$$

for some positive constants c_0, c_1 that depend only on n, f and the shape of Ω (see, e.g., [3, Chapter III]; cf. also [6], [46, Chapter VI]). Here, $|\Omega|$ stands for the Lebesgue measure of Ω . For a fixed domain Ω , we observe that these estimates are sharp with respect to $\nu \rightarrow 0^+$ (for each fixed $\lambda > 0$), or sufficiently large λ (for each fixed $\nu > 0$). Hence, these bounds for the dimension of \mathcal{A}_K are of the same order for each $K \in \{D, N, P\}$. These remarkable estimates also depend linearly on the "volume" of the spatial domain Ω , which is consistent

with physical intuition. This property of the dimension of the attractor has not been proved for all equations, such as, the Kuramoto-Sivashinski equation.

Our main goal in this paper is to investigate the dependance of the dimension of the global attractor for equation (1.1) subject to a completely new class of boundary conditions, which are sometimes dubbed as *Wentzell* boundary conditions, and which have some applications in probability theory, specifically, Markov processes. But what are they really? To put them into a context, let L be an elliptic differential operator of the second order (e.g., $L = \nu\Delta$) with coefficients that are well-defined over $\overline{\Omega}$. It is known that there is a one-to-one correspondence between (C_0) -semigroups and Markov processes in $\overline{\Omega}$ which are homogeneous in time and satisfy the condition of Feller [9] (that is, the range of the resolvent operator coincides with a prescribed set). Thus, to each such Markov process there is a corresponding semigroup of operators

$$T_t v(x) = \int_{\overline{\Omega}} v(y) P(t, x, dy),$$

where the Markov transition function $P(t, x, B)$ satisfies $P(t, x, B) \geq 0$, for $t \geq 0$, $x \in \overline{\Omega}$ and any Borel set $B \subseteq \overline{\Omega}$. As a function of B , $P(t, x, \cdot)$ is a probability measure. What are the most general boundary conditions which restrict the given operator L (more correctly, its closure) to the infinitesimal operator of a semigroup of positive contraction operators acting on $C(\overline{\Omega})$? Wentzell [47] gave a partial answer to this question in higher space dimensions by finding a sufficiently large class of boundary conditions which involve differential operators on the boundary that are of the same order as the operator acting in Ω . He discovered the following form of boundary conditions:

$$Lu + \nu b \partial_{\mathbf{n}}^L u = 0, \text{ on } \Gamma \times (0, +\infty), \tag{1.3}$$

where \mathbf{n} denotes the outward normal at Γ , b is a positive constant and $\partial_{\mathbf{n}}^L u$ is the outward

co-normal derivative of u with respect to L . We refer also to the pioneering work of [10], for generation theorems for L with Wentzell boundary conditions in one space dimension. Until the work of [11], the study of the operator L with Wentzell boundary conditions was usually confined to generation properties of this operator in the space $C(\overline{\Omega})$. In 2002, the authors in [11] have found a way to introduce the Wentzell boundary condition (1.3) in an L^p -context, which led to the discovery of the natural space for these type of problems (see Section 2). The reader is referred to [4, 27] for an extensive survey of these results and some history.

For the homogeneous linear heat equation (1.1) (that is, $f = g = \lambda = 0$), the Wentzell boundary condition (1.3) is equivalent to a purely differential equation of the form

$$\partial_t u + \nu b \partial_{\mathbf{n}} u = 0, \quad \text{on } \Gamma \times (0, \infty). \quad (1.4)$$

Thus, the main attraction here is that there is a dynamic element introduced into the boundary condition. The heat equation, supplemented by either (1.3) or (1.4), corresponds to the situation where there is a heat source (if $b > 0$) or sink (if $b < 0$) acting on the boundary Γ . Mathematically speaking, this kind of conditions (1.4) arises due to the presence of additional boundary terms in the free energy, which must also account for the action of a source on Γ (see [22]). We refer the reader to [26] (cf. also [16]), for an extensive derivation and physical interpretation of (1.4) for (1.1). For the nonlinear parabolic equation (1.1), the boundary condition (1.3) can be formally be transformed into a condition of the form

$$\partial_t u + \nu b \partial_{\mathbf{n}} u + f(u) - \lambda u = g, \quad \text{on } \Gamma \times (0, \infty). \quad (1.5)$$

Generally, one may replace $f - \lambda$ in (1.5) by another arbitrary function h , satisfying suitable assumptions. With more sophisticated arguments, using techniques from semigroup theory, and a variation of parameter formula, it is possible to prove that the regularity of the solution

for (1.1),(1.5) increases as f , Ω and g become more regular (see Section 2). In particular, for $g = 0$, if Ω is a bounded C^∞ domain and f is a C^∞ function, regularity theory implies that the solution $u(t)$ to (1.1),(1.5) belongs $H^k(\Omega)$, for all $k \geq 0$ and all positive times. At least in this case, the boundary condition (1.3), for equation (1.1), is equivalent to the boundary condition (1.5). However, in general, this may not be so if the solution, for the semilinear problem (1.1) and the Wentzell condition (1.5), is not smooth enough. Since we wish to treat the most general case, by imposing the least regularity assumptions on f , g and Ω , we will devote our attention only to the study of (1.1), subject to linear boundary conditions of the form (1.4). Our results below can be immediately extended to other classes of nonlinear Wentzell boundary conditions (see, e.g., [22] and references therein). Boundary conditions of the form (1.5) arise for many known equations of mathematical physics. They are motivated by heat control problems formulated in the book of Duvaut and Lions [7], problems in phase-transition phenomena [5, 15, 17, 20, 21, 24, 25, 37, 38] (and their references), special flows in hydrodynamics [12, 22, 42, 39], Stefan problems [1, 35, 41], models in climatology [40], and many others. The reader is referred to [18] for a more complete list of references involving the application of such boundary conditions to real-world phenomena.

By keeping our treatment of the boundary condition simple, we wish to prove that the problem (1.1), (1.4) generates a dynamical system on a suitable phase-space, possessing a finite dimensional global attractor \mathcal{A}_W . Then, we establish that the Hausdorff and fractal dimensions of \mathcal{A}_W satisfy the following upper and lower bounds:

$$c_1 \left(\frac{\lambda}{C_W(\Omega, \Gamma) \nu} \right)^{n-1} \leq \dim_H \mathcal{A}_W \leq \dim_F \mathcal{A}_W \leq c_2 \left(1 + \frac{c_f + \lambda}{C_W(\Omega, \Gamma) \nu} \right)^{n-1}, \quad (1.6)$$

for $n \geq 2$, and

$$c_3 \left(\frac{\lambda}{C_D(\Omega) \nu} \right)^{1/2} \leq \dim_H \mathcal{A}_W \leq \dim_F \mathcal{A}_W \leq c_4 \left(1 + \frac{c_f + \lambda}{C_D(\Omega) \nu} \right)^{1/2}, \quad (1.7)$$

in one space dimension. The positive constants c_i , $i = 1, \dots, 4$, depend only on n , f and the shape of Ω , while explicit estimates and formulas for $C_W(\Omega, \Gamma)$ and $C_D(\Omega)$, respectively, are provided in the Appendix. We note again that, for a fixed domain Ω , these estimates are sharp with respect to $\nu \rightarrow 0^+$ (for each fixed $\lambda > 0$), and for sufficiently large λ (if $\nu > 0$ is fixed). We remark that the bounds we obtain in (1.6)-(1.7) are quite simple and their explicit dependence on the physical parameters is transparent. Moreover, a careful analysis of the constants involved in (1.6) yields the following more explicit two-sided estimate,

$$c'_1 \left(\frac{\lambda}{\nu b} \right)^{n-1} |\Gamma| \leq \dim_H \mathcal{A}_W \leq \dim_F \mathcal{A}_W \leq c'_2 \left(1 + \frac{c_f + \lambda}{\nu b} |\Gamma|^{1/(n-1)} \right)^{n-1}, \quad (1.8)$$

in all space dimensions $n \geq 3$. It is worth pointing out that the bounds in (1.8) are proportional to the "surface area" $|\Gamma|$ of Γ , and *not* the "volume" $|\Omega|$ of Ω . This is remarkable; most nonlinear equations arising in mathematical physics, involving the Laplacian on bounded domains, have the dimension of the attractor of the order of $|\Omega|^\alpha$, for some $\alpha > 0$ and for sufficiently large domains. This property may have profound implications in the prediction of weather and climate. The reader is referred to Section 4 where this interesting physical observation is further discussed for the balance equations governing the large-scale oceanic motion.

Our paper is organized as follows. In Section 2, we obtain upper bounds (cf. Theorem 2.7) for the fractal dimension of the global attractor for equation (1.1) with dynamic boundary conditions of the form (1.4). In Section 3, we employ the same technique of [3] to derive a lower bound for the dimension of the unstable manifold of a constant stationary solution u^* of (1.1), (1.4). As a consequence, we find a lower bound on the dimension of \mathcal{A}_W (see Theorem 3.1). Finally, in the Appendix, we recall some useful results on the so-called Wentzell Laplacian, and prove an auxiliary inequality, namely, we derive some kind of Sobolev-Lieb-Thirring inequality that is required to prove the upper bound in (1.6).

2 Upper bounds on the dimension

We use the standard notation and facts from the dynamic theory of parabolic equations (see, for instance, [4], [11], [17], [22]). We denote by $\|\cdot\|_p$ and $\|\cdot\|_{p,\Gamma}$, the norms on $L^p(\Omega)$ and $L^p(\Gamma)$, respectively. In the case $p = 2$, $\langle \cdot, \cdot \rangle_2$ stands for the usual scalar product. The norms on $H^r(\Omega)$ and $H^r(\Gamma)$ are indicated by $\|\cdot\|_{H^r(\Omega)}$ and $\|\cdot\|_{H^r(\Gamma)}$, respectively, for any $r > 0$.

The natural phase-space for problem (1.1), (1.4) is

$$\mathbb{X}^p := L^p(\Omega) \oplus L^p(\Gamma) = \left\{ F = \begin{pmatrix} f \\ g \end{pmatrix} : f \in L^p(\Omega), g \in L^p(\Gamma) \right\},$$

for all $p \in [1, \infty]$, endowed with the norm

$$\|F\|_{\mathbb{X}^p}^p = \int_{\Omega} |f(x)|^p dx + \int_{\Gamma} |g(x)|^p \frac{dS}{b}, \quad b > 0, \quad (2.1)$$

if $p \in [1, \infty)$, and

$$\|F\|_{\mathbb{X}^\infty} := \|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\Gamma)}.$$

In the definition of \mathbb{X}^p , dx denotes the Lebesgue measure on Ω , and dS denotes the natural surface measure on Γ . Moreover, we have [11],

$$\mathbb{X}^p = L^p(\overline{\Omega}, d\mu), \quad p \in [1, \infty],$$

where the measure $d\mu = dx|_{\Omega} \oplus \frac{dS}{b}|_{\Gamma}$, on $\overline{\Omega}$, is defined for any measurable set $B \subset \overline{\Omega}$ by $\mu(B) = |B \cap \Omega| + |B \cap \Gamma|$. The Dirichlet trace map $Tr_D : C^\infty(\overline{\Omega}) \rightarrow C^\infty(\Gamma)$, defined by $Tr_D(u) = u|_{\Gamma}$ extends to a linear continuous operator $Tr_D : H^r(\Omega) \rightarrow H^{r-1/2}(\Gamma)$, for all $r > 1/2$, which is onto for $1/2 < r < 3/2$. This map also possesses a bounded right inverse $Tr_D^{-1} : H^{r-1/2}(\Gamma) \rightarrow H^r(\Omega)$ such that $Tr_D(Tr_D^{-1}\psi) = \psi$, for any $\psi \in H^{r-1/2}(\Gamma)$. Identifying each function $v \in C(\overline{\Omega})$ with the vector $V = \begin{pmatrix} v \\ Tr_D(v) \end{pmatrix} \in C(\overline{\Omega}) \times C(\Gamma)$, it follows that $C(\overline{\Omega})$

is a dense subspace of \mathbb{X}^p , for every $p \in [1, \infty)$, and a closed subspace of \mathbb{X}^∞ . Finally, we can also introduce the subspaces of $H^r(\Omega) \times H^{r-1/2}(\Gamma)$,

$$\mathbb{V}_r := \left\{ \begin{pmatrix} u \\ \psi \end{pmatrix} \in H^r(\Omega) \times H^{r-1/2}(\Gamma) : Tr_D(u) = \psi \right\},$$

for every $r > 1/2$, and note that we have the following dense and compact embeddings $\mathbb{V}_{r_1} \subset \mathbb{V}_{r_2}$, for any $r_1 > r_2 > 1/2$. The linear subspace \mathbb{V}_r is densely and compactly embedded into \mathbb{X}^2 , for any $r > 1/2$. We emphasize that \mathbb{V}_r is not a product space and that, due to the boundedness of the trace operator Tr_D , \mathbb{V}_r is topologically isomorphic to $H^r(\Omega)$ in the obvious way.

We begin by stating all the hypotheses on f and g that we need. We assume that $g \in L^2(\Omega)$ and the following conditions for $f \in C^1(\mathbb{R}, \mathbb{R})$ hold:

$$f'(y) > -c_f, \text{ for all } y \in \mathbb{R}, \quad (2.2)$$

$$\eta_1 |y|^p - C_f \leq f(y) y \leq \eta_2 |y|^p + C_f, \quad (2.3)$$

for some $\eta_1, \eta_2 > 0$, $C_f \geq 0$ and $p > 2$.

We have the following rigorous notion of weak solution to (1.1), (1.4), with initial condition $u(0) = u_0$, as in [22].

Definition 2.1 *The pair $U(t) = \begin{pmatrix} u(t) \\ \psi(t) \end{pmatrix}$ is said to be a weak solution if $\psi(t) = Tr_D(u)$ for almost all $t \in (0, T)$, for any $T > 0$, and U fulfills*

$$\begin{aligned} U &\in C([0, T]; \mathbb{X}^2) \cap L^\infty(0, T; \mathbb{V}_1) \cap L^p(\Omega \times (0, T)), \\ u &\in H_{loc}^1(0, \infty; L^2(\Omega)), \quad \psi \in H_{loc}^1(0, \infty; L^2(\Gamma)), \\ \partial_t U &\in L^2(0, T; \mathbb{V}_1^*), \end{aligned} \quad (2.4)$$

such that the identity

$$\langle \partial_t U, \Xi \rangle_{\mathbb{X}^2} + \nu \langle \nabla u, \nabla \sigma \rangle_2 + \langle f(u) - \lambda u, \sigma \rangle_2 = \langle g, \sigma \rangle_2,$$

holds almost everywhere in $(0, T)$, for all $\Xi = \begin{pmatrix} \sigma \\ \varpi \end{pmatrix} \in \mathbb{V}_1$. Moreover, we have, in the space \mathbb{X}^2 ,

$$U(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} =: U_0, \quad (2.5)$$

where $u(0) = u_0$ almost everywhere in Ω , and $v(0) = v_0$ almost everywhere in Γ . Note that in this setting, v_0 need not be the trace of u_0 at the boundary. Thus, in this context equation (1.4) is interpreted as an additional parabolic equation, acting now on the boundary Γ .

The following result is a direct consequence of results contained in [22, Section 2]. The proof is based on the application of a Galerkin approximation scheme which is not standard due to the nature of the boundary conditions (see, also, [5]).

Theorem 2.2 *Let the assumptions of (2.2), (2.3) be satisfied. For any given initial data $U_0 \in \mathbb{X}^2$, the problem (1.1), (1.4), (2.5) has a unique weak solution which depends continuously on the initial data in a Lipschitz way. The following estimate holds:*

$$\begin{aligned} & \|U(t)\|_{\mathbb{X}^2}^2 + \int_t^{t+1} \left(\|U(s)\|_{\mathbb{V}_1}^2 + \|u(s)\|_{L^p(\Omega)}^p \right) ds \\ & \leq c \left(\|U(0)\|_{\mathbb{X}^2}^2 e^{-\rho t} + c \left(1 + \|g\|_{L^2(\Omega)}^2 \right) \right), \end{aligned} \quad (2.6)$$

for all $t \geq 0$, where the positive constants c, ρ are independent of time and initial data.

As a consequence, problem (1.1), (1.4), (2.5) defines a (nonlinear) continuous semigroup \mathcal{S}_t acting on the phase-space \mathbb{X}^2 ,

$$\mathcal{S}_t : \mathbb{X}^2 \rightarrow \mathbb{X}^2, \quad t \geq 0,$$

given by

$$\mathcal{S}_t U_0 = U(t).$$

Theorem 2.3 *Let f satisfy assumptions (2.2), (2.3), let $g \in L^\infty(\Omega)$ and $\Gamma \in \mathcal{C}^2$. Then, $\{\mathcal{S}_t\}$ possesses the connected global attractor \mathcal{A}_W , which is a bounded subset of $\mathbb{V}_2 \cap \mathbb{X}^\infty$. As a consequence, the global attractor contains only strong solutions.*

Proof. The existence of an absorbing set in $\mathbb{V}_1 \cap L^p(\Omega)$ and, hence, the existence of the global attractor $\mathcal{A}_W \subset \mathbb{V}_1$ follows from [22, Theorem 2.8 and Corollary 3.11]. We will now show that the attractor is bounded in \mathbb{X}^∞ , and also in \mathbb{V}_2 . All the calculations below are formal. However, they can be rigorously justified by means of the approximation procedures devised in [22] and [16] (cf. [5] also). From now on, c will denote a positive constant that is independent of time and initial data, which only depends on the other structural parameters of the problem, that is, $|\Omega|$, $|\Gamma|$, η_i , ν , b , $\|g\|_\infty$ and n . Such a constant may vary even from line to line.

Step 1. We will first establish the existence of a bounded absorbing set in \mathbb{X}^∞ . First note that by (2.6), there is a constant $C_0 > 0$, independent of time and initial data, such that for any bounded subset B of \mathbb{X}^2 , $\exists \tau = \tau(\|B\|_{\mathbb{X}^2}) > 0$ with

$$\sup_{t \geq \tau} \|U(t)\|_{\mathbb{X}^2} \leq C_0. \quad (2.7)$$

We shall now perform an Alikakos-Moser iteration argument. We multiply (1.1) by $|u|^{r_k-2} u$, $r_k := 2^k$, $k \geq 1$, and integrate over Ω . We obtain

$$\begin{aligned} & \frac{1}{r_k} \frac{d}{dt} \|u\|_{r_k}^{r_k} + \langle f(u), |u|^{r_k-2} u \rangle_2 + \nu \int_{\Omega} \nabla u \cdot \nabla (|u|^{r_k-2} u) dx \\ &= \nu \int_{\Gamma} \partial_{\mathbf{n}} u |\psi|^{r_k-2} \psi dS + \langle \lambda u + g, |u|^{r_k-2} u \rangle_2. \end{aligned} \quad (2.8)$$

Similarly, we multiply (1.4) by $|\psi|^{r_k-2} \psi/b$ and integrate over Γ . We have

$$\frac{1}{br_k} \frac{d}{dt} \|\psi\|_{r_k, \Gamma}^{r_k} + \nu \int_{\Gamma} \partial_{\mathbf{n}} u |\psi|^{r_k-2} \psi dS = 0. \quad (2.9)$$

Adding the equalities (2.8), (2.9), we deduce

$$\begin{aligned} & \frac{1}{r_k} \frac{d}{dt} (\|U\|_{\mathbb{X}^{r_k}}^{r_k}) + \langle f(u), |u|^{r_k-2} u \rangle_2 + \nu \int_{\Omega} \nabla u \cdot \nabla (|u|^{r_k-2} u) dx \\ & = \langle \lambda u + g, |u|^{r_k-2} u \rangle_2. \end{aligned} \quad (2.10)$$

A simple manipulation of the third integral in (2.10), and employing assumption (2.3) on f , we readily get the estimate:

$$\begin{aligned} & \frac{d}{dt} (\|U\|_{\mathbb{X}^{r_k}}^{r_k}) + \eta_1 r_k \|u\|_{r_k+p-2}^{r_k+p-2} + \nu (2^k - 1) 2^{2-k} \left\| \nabla |u|^{2^{k-1}} \right\|_2^2 \\ & \leq r_k \langle \lambda u + g + C_f, |u|^{r_k-2} u \rangle_2. \end{aligned} \quad (2.11)$$

Next, using the fact that $|y|^{r_k-2} \leq |y|^{r_k} + 1$, for all $k \geq 1$ and $y \in \mathbb{R}$, we estimate the last term on the right-hand side of (2.11),

$$\langle \lambda u + g + C_f, |u|^{r_k-2} u \rangle_2 \leq c (\|u\|_{r_k}^{r_k} + 1), \quad (2.12)$$

for some positive constant c that depends on λ and the L^∞ -norm of g , but is independent of k . On the other hand, it follows from Gagliardo-Nirenberg inequality, and Young's inequality for $\varepsilon \in (0, 1)$, that

$$\|v\|_2 \leq c \|v\|_{H^1(\Omega)}^{n/(n+2)} \|v\|_1^{1-n/(n+2)} \leq \varepsilon \|v\|_{H^1(\Omega)} + c\varepsilon^{-n/2} \|v\|_1, \quad (2.13)$$

which implies

$$\|\nabla v\|_2^2 \geq \frac{1-\varepsilon}{\varepsilon} \|v\|_2^2 - c\varepsilon^{-n/2-1} \|v\|_1^2.$$

Note that the estimate (2.13) remains valid if one replaces the $L^2(\Omega)$ and $L^1(\Omega)$ -norms by the $L^2(\Gamma)$ and $L^1(\Gamma)$ -norms, respectively, and n by $n-1$, respectively. Setting $v = |u|^{r_{k-1}}$ in the above inequality, noting that $(2^k - 1)2^{2-k} \geq 2$, for each k , and the fact that Tr_D maps $H^1(\Omega)$ boundedly into $L^2(\Gamma)$, we can estimate the gradient term in (2.11) in terms of

$$c \frac{1-\varepsilon}{\varepsilon} \left(\|u\|_{r_k}^{r_k} + \|\psi\|_{r_k, \Gamma}^{r_k} \right) - c\varepsilon^{-n/2-1} \left(\| |u|^{r_{k-1}} \|_1^2 + \| |\psi|^{r_{k-1}} \|_{1, \Gamma}^2 \right).$$

(see, e.g., [35, Chapter 5]). This estimate together with (2.11), (2.12) yield

$$\begin{aligned} & \frac{d}{dt} (\|U\|_{\mathbb{X}^{r_k}}^{r_k}) + c \left(\nu \frac{1-\varepsilon}{\varepsilon} - r_k \right) \left(\|u\|_{r_k}^{r_k} + \|\psi\|_{r_k, \Gamma}^{r_k} \right) \\ & \leq c\varepsilon^{-n/2-1} \left(\| |u|^{r_{k-1}} \|_1^2 + \| |\psi|^{r_{k-1}} \|_{1, \Gamma}^2 \right) + cr_k, \end{aligned} \quad (2.14)$$

for all $k \geq 1$, where $c > 0$ is independent of k .

We shall now make use of an iterative argument to deduce the existence of a bounded absorbing set in \mathbb{X}^{r_k} , for all $k \geq 1$. Thus, noting that $r_k \leq (r_k)^{n/2+1}$, then choosing $\varepsilon = \delta/r_k$ with small $\delta = \delta(\nu) > 0$ such that

$$\left(\nu \frac{1-\varepsilon}{\varepsilon} - r_k \right) \geq r_k,$$

and setting

$$\mathcal{Y}_k(t) := \int_{\Omega} |u(t, \cdot)|^{r_k} dx + \int_{\Gamma} |\psi(t, \cdot)|^{r_k} \frac{dS}{b} = \|U\|_{\mathbb{X}^{r_k}}^{r_k}, \quad (2.15)$$

from (2.14) we derive the following estimate:

$$\partial_t \mathcal{Y}_k(t) + cr_k \mathcal{Y}_k(t) \leq c(r_k)^{n/2+1} (\mathcal{Y}_{k-1}(t) + 1)^2. \quad (2.16)$$

Let us now take two positive constants t, μ such that $t - \mu/r_k > 0$, for all $k \geq 1$. Their precise values will be chosen later. We claim that

$$\mathcal{Y}_k(t) \leq M_k(t, \mu) := c(r_k)^{n/2+1} \left(\sup_{s \geq t - \mu/r_k} \mathcal{Y}_{k-1}(s) + 1 \right)^2 \quad (2.17)$$

holds for \mathcal{Y}_k , defined by (2.15) and $k \geq 1$. To this end, let $\zeta(s)$ be a positive function $\zeta : \mathbb{R}_+ \rightarrow [0, 1]$ such that $\zeta(s) = 0$ for $s \in [0, t - \mu/r_k]$, $\zeta(s) = 1$ if $s \in [t, +\infty)$ and $|d\zeta/ds| \leq Cr_k$, if $s \in (t - \mu/r_k, t)$. We define $Z_k(s) = \zeta(s) \mathcal{Y}_k(s)$ and notice that

$$\frac{d}{ds} Z_k(s) \leq cr_k Z_k(s) + \zeta(s) \frac{d}{ds} \mathcal{Y}_k(s).$$

Combining this estimate with (2.16) and noticing that $Z_k \leq \mathcal{Y}_k$, we deduce the following estimate for Z_k :

$$\frac{d}{ds} Z_k(s) + cr_k Z_k(s) \leq M_k(t, \mu), \text{ for all } s \in [t - \mu/r_k, +\infty). \quad (2.18)$$

Integrating (2.18) with respect to s from $t - \mu/r_k$ to t and taking into account the fact that $Z_k(t - \mu/r_k) = 0$, we obtain that $\mathcal{Y}_k(t) = Z_k(t) \leq M_k(t, \mu) (1 - e^{-C\mu})$, which proves the claim (2.17).

Let now $\tau' > \tau > 0$ be given with τ as in (2.7), and define $\mu = 2(\tau' - \tau)$, $t_0 = \tau'$ and $t_k = t_{k-1} - \mu/r_k$, $k \geq 1$. Using (2.17), we have

$$\sup_{t \geq t_{k-1}} \mathcal{Y}_k(t) \leq c(r_k)^{n/2+1} \left(\sup_{s \geq t_k} \mathcal{Y}_{k-1}(s) + 1 \right)^2, \quad k \geq 1. \quad (2.19)$$

Note that from (2.7), we have $(\sup_{s \geq t_1 = \tau} \mathcal{Y}_1(s) + 1) \leq C_0 + 1 =: \overline{C}$. Thus, we can iterate in

(2.19) with respect to $k \geq 1$ and obtain that

$$\begin{aligned} \sup_{t \geq t_{k-1}} \mathcal{Y}_k(t) &\leq \left[c(r_k)^{n/2+1} \right] \left[c(r_{k-1})^{n/2+1} \right]^2 \cdot \dots \cdot \left[c(r_1)^{n/2+1} \right]^{2^k} (\overline{C})^{r_k} \\ &\leq c^{A_k} 2^{B_k n/2+1} (\overline{C})^{r_k}, \end{aligned}$$

where

$$A_k := 1 + 2 + 2^2 + \dots + 2^k \leq 2^k \sum_{i=1}^{\infty} \frac{1}{2^i} \quad (2.20)$$

and

$$B_k := k + 2(k-1) + 2^2(k-2) + \dots + 2^k \leq 2^k \sum_{i=1}^{\infty} \frac{i}{2^i}. \quad (2.21)$$

Therefore,

$$\sup_{t \geq t_0} \mathcal{Y}_k(t) \leq \sup_{t \geq t_{k-1}} \mathcal{Y}_k(t) \leq c^{A_k} 2^{B_k(n/2+1)} \overline{C}^{r_k}. \quad (2.22)$$

Since the series in (2.20) and (2.21) are convergent, we can take the r_k -root on both sides of (2.22) and let $k \rightarrow +\infty$. We deduce

$$\sup_{t \geq t_0 = \tau'} \|U(t)\|_{\mathbb{X}^\infty} \leq \lim_{k \rightarrow +\infty} \sup_{t \geq t_0} (\mathcal{Y}_k(t))^{1/r_k} \leq C_1, \quad (2.23)$$

for some positive constant C_1 independent of t , k , U and initial data.

Step 2. We claim that there is a positive constant C_2 , independent of time and initial data, and there exists $\tau'' > 0$ such that

$$\|U(t)\|_{\mathbb{V}_2} \leq C_2, \quad \text{for all } t \geq \tau''. \quad (2.24)$$

Before we prove (2.24), let us recall the following estimate (see [22, Theorems 3.5, 3.10]):

$$\begin{aligned}
& \sup_{t \geq \tau_0} \left(\|U(t)\|_{V_1}^2 + \|\partial_t u(t)\|_2^2 + \frac{1}{b} \|\partial_t \psi(t)\|_{2,\Gamma}^2 \right) \\
& + \sup_{t \geq \tau_0} \int_t^{t+1} \|\partial_t u(s)\|_{H^1(\Omega)}^2 ds \\
& \leq C_3,
\end{aligned} \tag{2.25}$$

for some positive constant C_3 that is independent of time and the initial data. In order to deduce (2.24) from (2.25) and (2.23), we need to differentiate (1.1) and (1.4) with respect to time. This yields

$$\partial_t^2 u = \nu \Delta \partial_t u - f'(u) \partial_t u + \lambda \partial_t u, \quad (\partial_t^2 \psi + \nu b \partial_{\mathbf{n}}(\partial_t u))|_{\Gamma} = 0. \tag{2.26}$$

Then, we multiply the first equation of (2.26) by $\partial_t^2 u(t)$ and integrate over Ω , using the boundary condition of (2.26). After standard transformations, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (\|\nabla \partial_t u(t)\|_2^2) + \|\partial_t^2 u(t)\|_2^2 + \frac{1}{b} \|\partial_t^2 \psi(t)\|_{2,\Gamma}^2 \\
& = - \left\langle (f'(u(t)) - \lambda) \partial_t u(t), \partial_t^2 u(t) \right\rangle_2.
\end{aligned}$$

Using Hölder and Young inequalities, we have

$$\begin{aligned}
& \frac{d}{dt} (\|\nabla \partial_t u(t)\|_2^2) + \|\partial_t^2 u(t)\|_2^2 + \frac{2}{b} \|\partial_t^2 \psi(t)\|_{2,\Gamma}^2 \\
& \leq c \left(\|f'(u(t)) \partial_t u(t)\|_2^2 + \|\partial_t u(t)\|_2^2 \right) \\
& \leq Q(\|u(t)\|_{\infty}) \|\partial_t u(t)\|_2^2,
\end{aligned}$$

for some positive nondecreasing function Q that depends only on f and c . This estimate

yields, owing to (2.23), (2.25),

$$\frac{d}{dt} \|\nabla \partial_t u(t)\|_2^2 \leq c.$$

Then, we can apply the so-called uniform Gronwall's lemma (see, e.g., [46, Chapter III, Lemma 1.1]) to find a time $\tau_1 \geq 1$, depending on τ_0 and τ , such that

$$\|\nabla \partial_t u(t)\|_2^2 \leq c, \quad \text{for all } t \geq \tau_1. \quad (2.27)$$

Therefore, (2.27) and (2.25) allow us to deduce from (1.1) and (1.4), via standard elliptic regularity, the following estimate

$$\|u(t)\|_{H^2(\Omega)}^2 \leq c, \quad \forall t \geq \tau_1. \quad (2.28)$$

Summing up, we conclude by observing that (2.24) follows from (2.28) and the boundedness of the trace map $Tr_D : H^2(\Omega) \rightarrow H^{3/2}(\Gamma)$. This completes the proof of the theorem. ■

Remark 2.4 *The proof of Theorem 2.3 shows how to get an absorbing set in \mathbb{V}_2 . Because of this, we can also prove the existence of a global attractor for the dynamical system $(\{\mathcal{S}_t\}_{t \geq 0}, \mathbb{V}_1)$.*

Theorem 2.5 *If Ω is a bounded C^∞ -domain, and f, g are C^∞ functions, then the global attractor \mathcal{A}_W is a bounded subset of \mathbb{V}_k , for every $k \geq 1$. In particular, if $U \in \mathcal{A}_W$ then $u \in C^\infty(\overline{\Omega})$.*

The proof of this result is standard and follows by successive time differentiation of the equations in (2.26) and an induction argument. We omit the details.

To prove the finite dimensionality of the global attractor \mathcal{A}_W , we can proceed in two different ways. One way is to establish the existence of a more refined object called exponential attractor \mathcal{E}_W , whose existence proof is often based on proper forms of the so-called

squeezing/smoothing property for the differences of solutions. This can be done by assuming smoother nonlinearities, i.e., $f \in C^2(\mathbb{R})$ (see, e.g., [16, 17]). This has been carried out in [16], and references therein, for a system of reaction-diffusion equations with dynamic boundary conditions of the form (1.4), without relating the attractor dimension to the physical parameters of the problem. However, since we wish to find explicit estimates of fractal or/and Hausdorff dimension of \mathcal{A}_W , we shall employ the classical machinery for proving the finite dimensionality of the global attractor \mathcal{A}_W . This is based on the so-called volume contraction arguments and requires the associated solution semigroup \mathcal{S}_t to be (uniformly quasi-) differentiable with respect to the initial data, at least on the attractor (see, e.g., [3]).

We give without proof the following result, which follows as a consequence of the boundedness of \mathcal{A}_W into $\mathbb{V}_2 \cap \mathbb{X}^\infty$.

Proposition 2.6 *Provided that $f \in C^2(\mathbb{R})$ satisfies the conditions (2.2) and (2.3), the flow \mathcal{S}_t generated by the reaction-diffusion equation (1.1) and dynamic boundary condition (1.4) is uniformly differentiable on \mathcal{A}_W , with differential*

$$\mathbf{L}(t, U(t)) : \Theta = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathbb{X}^2 \mapsto V = \begin{pmatrix} v \\ \varphi \end{pmatrix} \in \mathbb{X}^2, \quad (2.29)$$

where V is the unique solution to

$$\begin{aligned} \partial_t v &= \nu \Delta v - f'(u(t))v + \lambda v, \quad (\partial_t \varphi + \nu b \partial_{\mathbf{n}} v)|_{\Gamma} = 0, \\ V(0) &= \Theta. \end{aligned} \quad (2.30)$$

Furthermore, $\mathbf{L}(t, U(t))$ is compact for all $t > 0$.

The main result of this section is

Theorem 2.7 *Let the assumptions of Proposition 2.6 be satisfied. The fractal dimension of*

\mathcal{A}_W admits the estimate

$$\dim_F \mathcal{A}_W \leq c_0 \left(1 + \frac{c_f + \lambda}{C_W(\Omega, \Gamma) \nu} \right)^{n-1}, \text{ for } n \geq 2 \quad (2.31)$$

and

$$\dim_F \mathcal{A}_W \leq c_0 \left(1 + \frac{c_f + \lambda}{C_D(\Omega) \nu} \right)^{1/2}, \text{ for } n = 1, \quad (2.32)$$

where c_0 depends on the shape of Ω only. The positive constants C_W, C_D depend only on n, Ω, Γ, b and are given in the Appendix.

Proof. In order to deduce (2.31)-(2.32), it is sufficient (see, e.g., [6, Chapter III, Definition 4.1]) to estimate the j -trace of the operator

$$\mathbf{L}(t, U(t)) = \begin{pmatrix} \nu \Delta - f'(u(t)) + \lambda I & 0 \\ -b\nu \partial_{\mathbf{n}} & 0 \end{pmatrix}.$$

We have

$$\begin{aligned} \text{Trace}(\mathbf{L}(t, U(t)) Q_m) &= \sum_{j=1}^m \langle \mathbf{L}(t, U(t)) \varphi_j, \varphi_j \rangle_{\mathbb{X}^2} \\ &= \sum_{i=1}^m \langle \nu \Delta \varphi_j, \varphi_j \rangle_2 - \sum_{i=1}^m \langle \nu \partial_{\mathbf{n}} \varphi_j, \varphi_j \rangle_{2, \Gamma} \\ &\quad - \sum_{i=1}^m \langle f'(u(t)) \varphi_j, \varphi_j \rangle_2 + \sum_{i=1}^m \lambda \langle \varphi_j, \varphi_j \rangle_2, \end{aligned}$$

where the set of vector-valued functions $\varphi_j \in \mathbb{X}^2 \cap \mathbb{V}_1$ is an orthonormal basis in $Q_m \mathbb{X}^2$. Since the family φ_j is orthonormal in $Q_m \mathbb{X}^2$, using assumption (2.2) on f (i.e., $f'(y) \geq -c_f$, for all $y \in \mathbb{R}$), we find

$$\text{Trace}(\mathbf{L}(t, U) Q_m) \leq -\nu \sum_{i=1}^m \|\nabla \varphi_j\|_2^2 + (c_f + \lambda) m.$$

Let $n \geq 2$. From (5.12) (see Appendix, Proposition 5.5), we obtain

$$\begin{aligned} \text{Trace}(\mathbf{L}(t, U) Q_m) &\leq -\nu c_1 C_W(\Omega, \Gamma) m^{\frac{1}{n-1}+1} + (c_1 \nu C_W(\Omega, \Gamma) + c_f + \lambda) m \\ &=: \rho(m). \end{aligned}$$

The function $\rho(y)$ is concave. The root of the equation $\rho(d) = 0$ is

$$d^* = \left(1 + \frac{c_f + \lambda}{\nu c_1 C_W(\Omega, \Gamma)}\right)^{n-1}.$$

Thus, we can apply [6, Corollary 4.2 and Remark 4.1] to deduce that $\dim_F \mathcal{A}_W \leq d^*$, from which (2.31) follows. The case $n = 1$ is similar. ■

Remark 2.8 *Concerning the reaction-diffusion equation (1.1), we can also handle dynamic boundary conditions that involve surface diffusion:*

$$\partial_t u - \alpha \Delta_\Gamma u + b \nu \partial_{\mathbf{n}} \phi = 0, \quad \text{on } \Gamma, \quad (2.33)$$

where $\alpha > 0$ and Δ_Γ is the Laplace-Beltrami operator on Γ . Our method of establishing upper bounds, comparable to the bounds (2.31)-(2.32), for the dimension of the global attractor can be also extended to this case as well. The details will appear elsewhere.

3 Lower bounds on the dimension

Lower bounds on the dimension of the global attractor are usually based on the observation that the unstable manifold of any equilibrium of the system is always contained in the global attractor (see, e.g., [3]). Thus, a lower bound on the dimension of the attractor \mathcal{A}_W can be found by analyzing the dimension of an unstable manifold associated with a constant equilibrium Z for (1.1), (1.4). We begin by assuming that g is constant, for the sake of

simplicity. Steady-state solutions of (1.1), (1.4) satisfy

$$L_0(u) := \nu \Delta u - f(u) + \lambda u - g = 0, \quad (\partial_{\mathbf{n}} u)|_{\Gamma} = 0.$$

We seek a solution of this system $U = \begin{pmatrix} u \\ Tr_D(u) \end{pmatrix} \in \mathbb{X}^2$ which coincides with a constant vector $Z = \mathbf{c} = \begin{pmatrix} c \\ c \end{pmatrix}$, c is a constant. Such a stationary solution satisfies the equation $\bar{L}_0(z) := -f(z) + \lambda z - g = 0$. Since

$$f(y) y \geq \eta_1 |y|^p - C_f, \quad \text{for } p > 2,$$

we have $\bar{L}_0(z) z \leq -\tilde{\eta}_1 |z|^p + \tilde{C}_f$, for some positive constants $\tilde{\eta}_1, \tilde{C}_f$. Thus, $\bar{L}_0(z) z < 0$ on the interval $I_R = (-R, R)$, if R is large enough. It follows that $\bar{L}_0(z) = 0$ has at least one solution $Z = Z(\lambda)$ (see, e.g., [6, Chapter III]). By the implicit function theorem, this solution is of order $1/\lambda$ for sufficiently large λ .

Now fix this solution. In order to find a lower bound on the dimension of the global attractor \mathcal{A}_W , it suffices to establish a lower bound for $\dim E_+(Z)$, where $E_+(Z)$ is an invariant subspace of $\mathbf{L}(Z)$, which corresponds to

$$\mathbf{L}(Z) W = \begin{pmatrix} \nu \Delta w - f'(z) w + \lambda w \\ -b \nu \partial_{\mathbf{n}} w \end{pmatrix}$$

with $\sigma(\mathbf{L}(Z)) \subset \{\zeta : \zeta > 0\}$. We note that $(\mathbf{L}(Z), D(\mathbf{L}(Z)))$ is self-adjoint on X^2 with spectrum contained in $(-\infty, c_f + \lambda]$.

The main result of this section is the following.

Theorem 3.1 *Let $f \in C^2(\mathbb{R})$ satisfy assumptions (2.2)-(2.3). There exist a positive constant c_0 , depending on f, g and the shape of Ω , independent of $\lambda, \nu, b, |\Omega|, |\Gamma|$, such that*

$$\dim_F \mathcal{A}_W \geq \dim_H \mathcal{A}_W \geq \dim E_+(Z) \geq c_0 \left(\frac{\lambda}{C_W(\Omega, \Gamma) \nu} \right)^{n-1},$$

for $n \geq 2$. In one space dimension, the same estimate is valid with C_W replaced by C_D and $n - 1$, replaced by $1/2$, respectively.

Proof. For a fixed constant solution $Z = \mathbf{c}$ of $\bar{L}_0(z) = 0$ and sufficiently large $\lambda \geq 1$, we have $\chi(\lambda) := -f'(z) + \lambda > 0$.

Let $\{\varphi_j(x)\}_{j \in \mathbb{N}_0}$ be an orthonormal basis in \mathbb{X}^2 consisting of eigenfunctions of the Wentzell Laplacian Δ_W (see Appendix, Theorem 5.3),

$$\Delta_W \varphi_j = \Lambda_j \varphi_j, \quad j \in \mathbb{N}_0, \quad \varphi_j \in D(\Delta_W) \cap C(\bar{\Omega}) \quad (3.1)$$

such that

$$0 = \Lambda_0 < \Lambda_1 \leq \Lambda_2 \leq \dots \leq \Lambda_j \leq \Lambda_{j+1} \leq \dots$$

We shall seek for eigenvectors $W_j = \begin{pmatrix} w_j \\ \text{Tr}_D(w_j) \end{pmatrix} \in \mathbb{X}^2$, of the form $w_j(x) = \varphi_j(x) p_j$, $p_j \in \mathbb{R}$, satisfying equation

$$\mathbf{L}(Z) W_j = \zeta_j W_j, \quad W_j \in D(\mathbf{L}(Z)) := D(\Delta_W). \quad (3.2)$$

Note that for $W_j \in D(\mathbf{L}(Z)) \subset \mathbb{V}_1$, the trace of w_j makes sense as an element of $H^{1/2}(\Gamma)$. Substituting such w_j into (3.2), taking into account (3.1) and the fact that

$$\mathbf{L}(Z) W_j = -\nu \Delta_W W_j + \Pi_\lambda W_j, \quad \Pi_\lambda W_j := \begin{pmatrix} \chi(\lambda) w_j \\ 0 \end{pmatrix},$$

we obtain the equation

$$(-\nu \Lambda_j I + \Pi_\lambda) p_j = \zeta_j p_j, \quad \Pi_\lambda = \begin{pmatrix} \chi(\lambda) & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.3)$$

A nonzero p_j exists if $\zeta = \zeta_j$ is a root of the equation

$$\det(-\nu\Lambda_j I + \Pi_\lambda - \zeta I) = 0, \quad \zeta > 0. \quad (3.4)$$

When $\nu = 0$, this equation has at least one root $\zeta > 0$ since $\chi = \chi(\lambda) > 0$ (in fact, $\zeta = \chi(\lambda)$). Therefore, there exists $\delta > 0$ such that when $\nu\Lambda_j < \delta$, the equation (3.4) has a root $\zeta_j = \zeta_j(\nu)$ with $\zeta_j > 0$. Therefore, to any such root ζ_j , we can assign a nontrivial p_j , which is a solution of (3.3), and thus an eigenvector $W_j = \begin{pmatrix} w_j \\ \text{Tr}_D w_j \end{pmatrix}$, $w_j = \varphi_j p_j$. Let us now compute how many j 's satisfy the inequality $\nu\Lambda_j < \delta$. The asymptotic behavior of Λ_j is $\Lambda_j \sim C_W(\Omega, \Gamma) j^{1/(n-1)}$ as $j \rightarrow \infty$ (see, Appendix, Theorem 5.4). The last inequality certainly holds when

$$1 \leq j \leq c_1 \delta^{n-1} (C_W \nu)^{1-n} = c_2 \left(\frac{1}{C_W \nu} \right)^{n-1}, \quad \text{for } n \geq 2$$

and

$$1 \leq j \leq c_1 \delta^{1/2} (C_D \nu)^{-1/2} = c_2 \left(\frac{1}{C_D \nu} \right)^{1/2}, \quad \text{for } n = 1.$$

The positive constants c_1, c_2 depend on λ . In order to get more explicit estimates for c_1, c_2 , it is left to remark that equation (3.4) may be rewritten in the form

$$\det(-\nu\Lambda_j \lambda^{-1} I + \lambda^{-1} \Pi_\lambda - \zeta_1 I) = 0$$

with $\zeta_1 = \lambda^{-1} \zeta$, and to observe that a solution of this equation clearly exists if $\nu\Lambda_j \lambda^{-1} \leq \delta$, for sufficiently large λ and small δ . Employing the asymptotic formula for Λ_j once again, we find

$$1 \leq j \leq c'_1 \delta^{n-1} \lambda^{n-1} (C_W \nu)^{1-n} = c'_2 \left(\frac{\lambda}{C_W \nu} \right)^{n-1}, \quad \text{for } n \geq 2$$

and

$$1 \leq j \leq c'_1 \delta^{1/2} \lambda^{1/2} (C_D \nu)^{-1/2} = c'_2 \left(\frac{\lambda}{C_D \nu} \right)^{1/2}, \text{ for } n = 1.$$

It follows that

$$\dim E_+(Z(\lambda)) \geq c'_2 \left(\frac{\lambda}{C_W \nu} \right)^{n-1}, \text{ for } n \geq 2$$

and

$$\dim E_+(Z(\lambda)) \geq c'_2 \lambda^{1/2} (C_D \nu)^{-1/2},$$

in one space dimension. The proof is complete. ■

4 Concluding remarks

In the textbook literature on theoretical geophysics, it was traditional to use a Robin boundary condition with a nonlinear heat equation to describe temperature variations at the upper surface of the ocean [28, 29]. But it was recognized that this was not always the physically correct boundary condition [40]. Among its applicability to a wide range of phenomena, including phase-transitions in fluids, and so on [16, 42], the reaction-diffusion system (1.1)-(1.4) has important applications in climatology and is essentially used to determine large and rapid temperature changes in the ocean's surface as a response to changes into deep water formations [40]. In this paper, we provide explicit bounds for the dimension of the attractor for this system and study the effect of the dynamic term $b^{-1} \partial_t u$, representing change in thermal energy in an infinitesimal layer near the surface. Unlike the previous results, the dimension of the attractor is proportional to the surface area $|\Gamma|$, for large domains Ω and fixed parameters ν , λ and b . Moreover, all the constants involved in our estimates are given in an explicit form. We also observe that in the case without $b^{-1} \partial_t u$ in (1.4), i.e., $b = +\infty$, the dimension of the attractor is much larger (and proportional to the volume $|\Omega|$ of Ω) than the dimension of the global attractor for the same system when $0 < b \neq +\infty$.

Thus, we observe that the addition of the dynamic term $b^{-1}\partial_t u$, $b > 0$ drastically changes the situation. This is a remarkable fact that can have a profound effect onto the long-term dynamics of other systems that are subject to dynamic boundary conditions of this form. We will investigate these effects for other systems, such as the Bénard problem for nonlinear heat conduction, in forthcoming papers. Finally, we note that it is also possible to extend the results of this paper to the case when the boundary Γ consists of two disjoint open subsets Γ_1 and Γ_2 , each $\bar{\Gamma}_i \cap \Gamma_i$ is a S -null subset of Γ and $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$ with $\Gamma_1 \subsetneq \Gamma$, such that u satisfies a Dirichlet boundary condition on Γ_1 and a dynamic boundary condition on Γ_2 . We will come back to this issue in a forthcoming article.

5 Appendix

In this section, we shall recall several important results concerning a certain realization of $L = \nu\Delta$ with the Wentzell boundary condition (1.3). We have the following.

Theorem 5.1 *Let Ω be a bounded open set of \mathbb{R}^n with Lipschitz boundary Γ . Assume that $b > 0$ and $0 \leq q \in L^\infty(\Omega)$. Define the operator Δ_W on \mathbb{X}^2 , by*

$$\Delta_W \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} -\Delta u_1 + q(x)u_1 \\ b\partial_{\mathbf{n}}u_1 \end{pmatrix}, \quad (5.1)$$

with

$$D(\Delta_W) := \left\{ U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{V}_1 : -\Delta u_1 \in L^2(\Omega), \partial_{\mathbf{n}}u_1 \in L^2\left(\Gamma, \frac{dS}{b}\right) \right\}. \quad (5.2)$$

Then, $(\Delta_W, D(\Delta_W))$ is self-adjoint on \mathbb{X}^2 . Moreover, the resolvent operator $(I + \Delta_W)^{-1} \in \mathcal{L}(\mathbb{X}^2)$ is compact.

We refer the reader to [4, 18, 19] for an extensive survey of recent results concerning the "Wentzell" Laplacian Δ_W .

The eigenvalue problem associated with the operator Δ_W is given by $\Delta_W \varphi = \Lambda \varphi$; this leads to the following spectral problem for the perturbed Laplacian

$$-\Delta \varphi + q(x) \varphi = \Lambda \varphi \text{ in } \Omega, \quad (5.3)$$

with a boundary condition that depends on the eigenvalue Λ explicitly:

$$b \partial_{\mathbf{n}} \varphi = \Lambda \varphi \text{ on } \Gamma. \quad (5.4)$$

Such a function φ will be called an eigenfunction associated with Λ and the set of all eigenvalues Λ of (5.3)-(5.4) will be denoted by Λ_W . Let φ_j and $\Lambda_{W,j}$, $j \in J$, denote all the eigenfunctions and eigenvalues of (5.3)-(5.4). We have the following (see, e.g., [2, 45]).

Theorem 5.2 *Let $q \geq 0$ with $\int_{\Omega} q(x) dx > 0$. Then, there exists a sequence of numbers*

$$0 < \Lambda_{W,1} \leq \Lambda_{W,2} \leq \dots \leq \Lambda_{W,j} \leq \Lambda_{W,j+1} \leq \dots, \quad (5.5)$$

converging to $+\infty$, with the following properties:

(a) *The spectrum of Δ_W is given by*

$$\sigma(\Delta_W) = \{\Lambda_{W,j}\}_{j \in \mathbb{N}},$$

and each number $\Lambda_{W,j}$, $j \in \mathbb{N}$, is an eigenvalue for Δ_W of finite multiplicity.

(b) *There exists a countable family of orthonormal eigenfunctions for Δ_W which spans \mathbb{X}^2 .*

More precisely, there exists a collection of functions $\{\varphi_j\}_{j \in \mathbb{N}}$ with the following properties:

$$\begin{aligned} \varphi_j &\in D(\Delta_W) \text{ and } \Delta_W \varphi_j = \Lambda_{W,j} \varphi_j, \quad j \in \mathbb{N}, \\ \langle \varphi_j, \varphi_k \rangle_{\mathbb{X}^2} &= \delta_{jk}, \quad j, k \in \mathbb{N}, \\ \mathbb{X}^2 &= \overline{\oplus \text{lin. span } \{\varphi_j\}_{j \in \mathbb{N}}} \text{ (orthogonal direct sum)}. \end{aligned} \quad (5.6)$$

(c) If Γ is Lipschitz, then every eigenfunction $\varphi_j \in \mathbb{V}_1$, and in fact $\varphi_j \in C(\overline{\Omega}) \cap C^\infty(\Omega)$, for every j . If Γ is of class C^2 , then every eigenfunction $\varphi_j \in \mathbb{V}_1 \cap C^2(\overline{\Omega})$, for every j .

(d) The following min-max principle holds:

$$\Lambda_{W,j} = \min_{\substack{Y_j \subset \mathbb{V}_1, \\ \dim Y_j = j}} \max_{0 \neq \varphi \in Y_j} R_W(\varphi, \varphi), \quad j \in \mathbb{N}, \quad (5.7)$$

where the Rayleigh quotient R_W , for the perturbed Wentzell operators, is given by

$$R_W(\varphi, \varphi) := \frac{\|\nabla \varphi\|_2^2 + \langle q(x) \varphi, \varphi \rangle_2}{\|\varphi\|_{\mathbb{X}^2}^2}, \quad 0 \neq \varphi \in \mathbb{V}_1. \quad (5.8)$$

Concerning the case $q \equiv 0$, we have the following.

Theorem 5.3 *Let $q \equiv 0$. Then, there exists a sequence of numbers*

$$0 = \Lambda_{W,0} < \Lambda_{W,1} \leq \Lambda_{W,2} \leq \dots \leq \Lambda_{W,j} \leq \Lambda_{W,j+1} \leq \dots,$$

converging to $+\infty$, with the following properties:

(a) The spectrum of Δ_W is given by

$$\sigma(\Delta_W) = \{\Lambda_{W,j}\}_{j \in \mathbb{N} \cup \{0\}},$$

and each number $\Lambda_{W,j}$, $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, is an eigenvalue for Δ_W of finite multiplicity. The

eigenvalue $\Lambda_{W,0}$ is simple and its associated eigenfunction is of constant sign.

(b) There exists a countable family of orthonormal eigenfunctions for Δ_W which spans \mathbb{X}^2 . More precisely, the same conclusion (b) of Theorem 5.2 holds in this case as well. Finally, both conclusions (c) and (d) in Theorem 5.2 hold in the case $q \equiv 0$ as well.

The asymptotic behavior of the eigenvalues $\Lambda_{W,j}$, as $j \rightarrow \infty$, was established in [13, 14]. We refer the reader to [18] for more details about the Wentzell Laplacian and other generalizations. Let $J = \mathbb{N}_0$ or \mathbb{N} , according to whether $q = 0$ or $q > 0$ respectively. Set

$$C_D(\Omega) := \frac{(2\pi)^2}{(v_n |\Omega|)^{2/n}} \text{ and } C_S(\Gamma) = \frac{2\pi}{(v_{n-1} |\Gamma|)^{1/(n-1)}}.$$

Here v_n denotes the volume of the unit ball in \mathbb{R}^n , and we recall that $|\Omega|$ stands for the n -dimensional Euclidean volume of Ω , while $|\Gamma|$ stands for the usual $(n-1)$ -dimensional Lebesgue surface measure on Γ .

We summarize these results in the following.

Theorem 5.4 *The eigenvalue sequence $\{\Lambda_{W,j}\}_{j \in J}$ of the (un)perturbed Wentzell Laplacian Δ_W satisfies:*

(i) For $n \geq 2$, we have

$$\Lambda_{W,j} = C_W(\Omega, \Gamma) j^{1/(n-1)} + o(j^{1/(n-1)}), \text{ as } j \rightarrow +\infty, \quad (5.9)$$

for some

$$C_W(\Omega, \Gamma) \in \begin{cases} bC_S(\Gamma) [2^{-1/(n-1)}, 1], & \text{for } n \geq 3, \\ \left[\frac{C_D(\Omega)C_S(\Gamma)}{2(b^{-1}C_D(\Omega)+C_S(\Gamma))}, \min\{C_D(\Omega), bC_S(\Gamma)\} \right], & \text{for } n = 2. \end{cases} \quad (5.10)$$

(ii) For $n = 1$, we have

$$\Lambda_{W,j} = C_D(\Omega) j^2 + o(j^2), \text{ as } j \rightarrow +\infty. \quad (5.11)$$

The following version of the Lieb–Thirring inequality is essential.

Proposition 5.5 *Let ω_j , $1 \leq j \leq m$, be a finite family of \mathbb{V}_1 , which is orthonormal in \mathbb{X}^2 .*

We have

$$\sum_{i=1}^m \|\nabla \omega_j\|_2^2 \geq c_1 C_W(\Omega, \Gamma) \left(m^{\frac{1}{n-1}+1} - m \right). \quad (5.12)$$

The constant $c_1 > 0$ depends only on n and the shape of Ω , and is independent of the size of Ω , Γ , of m , and of the ω_j 's.

Proof. Let $B_W := \Delta_W + C_W(\Omega, \Gamma)I$ and let $D(B_W) = D(\Delta_W)$. By Theorems 5.1, 5.2, B_W is a linear positive unbounded self-adjoint operator on \mathbb{X}^2 , such that B_W^{-1} is compact. Thus, we can apply the abstract result of [46, Chapter VI, Lemma 2.1] to deduce that

$$\begin{aligned} \sum_{i=1}^m (\|\nabla \omega_j\|_2^2 + C_W \|\omega_j\|_{\mathbb{X}^2}^2) &= \sum_{i=1}^m \langle B_W \omega_j, \omega_j \rangle_{\mathbb{X}^2} \\ &\geq \Lambda_{W,1}(B_W) + \Lambda_{W,2}(B_W) + \dots + \Lambda_{W,m}(B_W) \\ &\geq C_W (1^{1/(n-1)} + 2^{1/(n-1)} + \dots + m^{1/(n-1)}) \\ &\geq c_0 C_W m^{\frac{1}{n-1}+1}, \end{aligned} \quad (5.13)$$

since, by (5.9)-(5.11), $\Lambda_{W,j}(B_W) \geq C_W(\Omega, \Gamma) j^{1/(n-1)}$, for all j , and some positive constant c_0 (indeed, we have $\Lambda_{W,j}(B_W) = \Lambda_{W,j}(\Delta_W) + C_W$). Thus, the proof of (5.12) follows immediately from (5.13). ■

References

- [1] T. Aiki, Multi-dimensional two-phase Stefan problems with nonlinear dynamic boundary conditions, in: *Nonlinear Analysis and Applications*, Warsaw, 1994, in: *GAKUTO Internat. Ser. Math. Sci. Appl.*, vol. 7, Gakkōtoshō, Tokyo, 1996, pp. 1–25.
- [2] C. Bandle, J. Von Below, W. Reichel, Parabolic problems with dynamical boundary conditions: eigenvalue expansions and blow up, *Rend. Lincei Mat. Appl.* 17 (2006), 35-67.
- [3] A. Babin, M. Vishik, *Attractors of Evolutionary Equations*, Moscow: Nauka, 1989.
- [4] G.M. Coclite, A. Favini, C.G. Gal, G.R. Goldstein, J.A. Goldstein, E. Obrecht, S. Romanelli, The role of Wentzell boundary conditions in linear and nonlinear analysis, In: S. SIVASUNDARAN. *Advances in Nonlinear Analysis: Theory, Methods and Applications*. vol. 3, p. 279-292, Cambridge, Cambridge Scientific Publishers Ltd., ISBN/ISSN: 1-904868-68-2.
- [5] C. Cavaterra, C. G. Gal, M. Grasselli and A. Miranville, Phase-field systems with nonlinear coupling and dynamic boundary conditions, *Nonlinear Anal.*, 72 (2010), 2375–2399
- [6] V. Chepyzhov, M. Vishik, *Attractors for Equations of Mathematical Physics*, Providence, RI: AMS, 2002.
- [7] G. Duvaut, P.-L. Lions, *Les Inéquations en Mécanique et en Physique*, Dunod, Paris, 1972.
- [8] R. Dillon, P. K. Maini, H. G. Othmer, Pattern formation in generalized Turing systems, I, Steady-state patterns in systems with mixed boundary conditions, *J. Math. Biol.* 32 (1994), 345–393.

- [9] W. Feller, The parabolic differential equations and associated semi-groups of transformations, *Ann. of Math.* 55 (1952), 468-519.
- [10] W. Feller, Generalized second order differential operators and their lateral conditions, *Illinois J. Math.* 1 (1957), 459–504.
- [11] A. Favini, G.R. Goldstein, J.A. Goldstein, S.Romanelli, The heat equation with general Wentzell boundary conditions, *J. Evol. Eq.* 2 (2002), 1-19.
- [12] J. Filo, S. Luckhaus, Modelling surface runoff and infiltration of rain by an elliptic-parabolic equation coupled with a first-order equation on the boundary, *Arch. Rational Mech. Anal.* 146 (1999) 157–182.
- [13] G. Francois, Comportement spectral asymptotique provenant de problèmes parabolique sous conditions au bord dynamiques, *Doctoral Thesis, ULCO, Calais, 2002.*
- [14] G. Francois, Spectral asymptotics stemming from parabolic equations under dynamical boundary conditions, *Asymptot. Anal.* 46 (2006), no. 1, 43–52.
- [15] C. G. Gal, Robust exponential attractors for a conserved Cahn-Hilliard model with singularly perturbed boundary conditions, *Commun. Pure Appl. Anal.* 7 (2008), 819–836.
- [16] C. G. Gal and M. Grasselli, The nonisothermal Allen-Cahn equation with dynamic boundary conditions, *Discrete Contin. Dyn. Syst.*, 22 (2008), 1009-1040.
- [17] C. G. Gal, M. Grasselli, On the asymptotic behavior of the Caginalp system with dynamic boundary conditions, *Commun. Pure Appl. Anal.* 8 (2009), no. 2, 689–710.
- [18] C. G. Gal, M. Grasselli, Dissipative Dynamical Systems and Dynamic boundary conditions, book in preparation.

- [19] C.G. Gal, G.R. Goldstein, J.A. Goldstein, S. Romanelli, M. Warma, Fredholm alternative, semilinear elliptic problems, and Wentzell boundary conditions, submitted.
- [20] C. G. Gal, A. Miranville, Robust exponential attractors and convergence to equilibria for non-isothermal Cahn-Hilliard equations with dynamic boundary conditions, *Discrete Contin. Dyn. Syst. Ser. S* 2 (2009), no. 1, 113–147.
- [21] C. G. Gal, A. Miranville, Uniform global attractors for non-isothermal viscous and non-viscous Cahn-Hilliard equations with dynamic boundary conditions, *Nonlinear Anal. Real World Appl.* 10 (2009), no. 3, 1738–1766.
- [22] C. G. Gal, M. Warma, Well-posedness and the global attractor of some quasilinear parabolic equations with nonlinear dynamic boundary conditions, *Differential Integral Equations*, 23 (2010), 327-358.
- [23] G. Galiano, J. Velasco, A dynamic boundary value problem arising in the ecology of mangroves, *Nonlinear Anal. Real World Appl.* 7 (2006), 1129–1144.
- [24] M. Grasselli, A. Miranville, G. Schimperna, The Caginalp phasefield system with coupled dynamic boundary conditions and singular potentials, *Discrete Contin. Dyn. Syst.* 28 (2010), no. 1, 67–98.
- [25] G. Gilardi, Gianni, A. Miranville, G. Schimperna, Long time behavior of the Cahn-Hilliard equation with irregular potentials and dynamic boundary conditions, *Chin. Ann. Math. Ser. B* 31 (2010), no. 5, 679–712.
- [26] G. R. Goldstein, Derivation and physical interpretation of general boundary conditions, *Adv. Differential Equations* 11 (2006), 457–480.
- [27] G. R. Goldstein, General boundary conditions for parabolic and hyperbolic operators, in *Interplay Between (C_0) -Semigroups and PDEs: Theory and Applications* (ed. by S.

- Romanelli, R. M. Mininni and S. Lucente), Ist. Naz. Alta Matem., Rome, Italy (2004), 91-112.
- [28] J.L. Lions, O.P. Manley, R. Temam, S. Wang, Physical interpretation of the attractor dimension for the primitive equations of atmospheric circulation, *J. Atmospheric Sci.* 54 (1997), no. 9, 1137–1143.
- [29] J.L. Lions, R. Temam, S.H. Wang, On the equations of the large-scale ocean, *Nonlinearity* 5 (1992), no. 5, 1007–1053.
- [30] E. J. Holder, D. G. Schaeffer, Boundary condition and mode jumping in the von Karman equations, *SIAM J. Math. Anal.* 15 (1984), 446–458.
- [31] N. Igbida, Hele-Shaw type problems with dynamical boundary conditions, *J. Math. Anal. Appl.* 335 (2007), 1061–1078.
- [32] J. Hale, *Asymptotic Behavior of Dissipative Systems*, Math. Surveys and Monographs vol 25, Providence, RI: A M S, 1987.
- [33] J. Hale, C. Rocha, Interaction of diffusion and boundary conditions, *Nonlinear Anal.* 11 (1987), no. 5, 633–649.
- [34] J. Hale, C. Rocha, Varying boundary conditions with large diffusivity, *J. Math. Pures Appl.* 66 (1987), no. 2, 139–158.
- [35] M. Meyries, Maximal regularity in weighted spaces, nonlinear boundary conditions, and global attractors, PhD thesis, 2010.
- [36] Z. Mei, F. Theil, Variation of bifurcations along a homotopy from Neumann to Dirichlet problems, *Nonlinear Anal.* 27 (1996), 1381–1395.

- [37] A. Miranville, S. Zelik, The Cahn-Hilliard equation with singular potentials and dynamic boundary conditions, *Discrete Contin. Dyn. Syst.* 28 (2010), 275–310.
- [38] A. Miranville, S. Zelik, Exponential attractors for the Cahn-Hilliard equation with dynamic boundary conditions, *Math. Methods Appl. Sci.* 28 (2005), 709–735.
- [39] H. W. March and W. Weaver, The diffusion problem for a solid in contact with a stirred fluid, *Physical Review*, 31 (1928), 1072-1082.
- [40] M. Morantine, R.G. Watts, Rapid climate change and the deep ocean, *Clim. Change* 16 (1990), 83-97.
- [41] J.F. Rodrigues, V.A. Solonnikov, F. Yi, On a parabolic system with time derivative in the boundary conditions and related free boundary problems, *Math. Ann.* 315 (1999), no. 1, 61–95.
- [42] T.-Z. Qian, X. Wang, P. Sheng, A variational approach to moving contact line hydrodynamics, *J. Fluid Mechanics* 564, 2006, 333-360.
- [43] Chih-Wen Shih, Influence of boundary conditions on pattern formation and spatial chaos in lattice systems, *SIAM J. Appl. Math.* 61 (2000), 335–368.
- [44] D. Schaeffer, M. Golubitsky, Boundary conditions and mode jumping in the buckling of a rectangular plate, *Comm. Math. Phys.*, 69 (1979), 209–236.
- [45] J.L. Vazquez, E. Vitillaro, Heat equation with dynamical boundary conditions of reactive type, *Comm. Partial Differential Equations* 33 (2008), 561–612.
- [46] R. Temam, *Infinite Dimensional Dynamical Systems in Physics and Mechanics*, 2nd edn, NewYork: Springer, 1997.

- [47] A. D. Wentzell, On boundary conditions for multi-dimensional diffusion processes, *Theory Probab. Appl.* 4 (1959) 164–177.