Metric-induced wrinkling of a thin elastic sheet

Peter Bella* and Robert V. Kohn†

June 16, 2014

Abstract

We study the wrinkling of a thin elastic sheet caused by a prescribed non-Euclidean metric. This is a model problem for the patterns seen, for example, in torn plastic sheets and the leaves of plants. Following the lead of other authors we adopt a variational viewpoint, according to which the wrinkling is driven by minimization of an elastic energy subject to appropriate constraints and boundary conditions. We begin with a broad introduction, including a discussion of key examples (some well-known, others apparently new) that demonstrate the overall character of the problem. We then focus on how the minimum energy scales with respect to the sheet thickness h, for certain classes of displacements. Our main result is that when the deformations are subject to certain hypotheses, the minimum energy is of order $h^{4/3}$. We also show that when the deformations are subject to more restrictive hypotheses, the minimum energy is strictly larger – of order h; it follows that energy minimization in the more restricted class is not a good model for the applications that motivate this work. Our results do not explain the cascade of wrinkles seen in some experimental and numerical studies; and they leave open the possibility that an energy scaling law better than $h^{4/3}$ could be obtained by considering a larger class of deformations.

1 Introduction

In the last few years the wrinkling and folding of thin elastic sheets has attracted a lot of attention in both the mathematics and physics communities. Wrinkled configurations can be viewed as (local) minimizers of a suitable elastic energy, consisting of a non-convex "membrane energy" plus a higher-order singular perturbation representing "bending energy." Though the physically relevant wrinkled configurations are local minimizers, we can begin to understand their character by focusing on the minimum value of the elastic energy, and the properties of low-energy deformations. In this paper we focus on the *scaling law* of the minimum energy.

Our work is motivated by an intriguing body of literature on the complex patterns that occur near the edges of leaves and flowers. The proposal of [1, 21, 23] is that these patterns can be explained by energy minimization, for a thin sheet with a non-Euclidean metric coming from the process of growth. A similar explanation has been offered for the wrinkling seen in torn plastic sheets [27, 26, 28]. In both situations the essential physics can be captured by considering a metric that prefers stretching in one direction (i.e. a metric of the form $dx^2 + m^2(x)dy^2$ with $m(x) > 1$, where x is roughly speaking the distance to the wrinkled edge). The goal of the present paper is to explore the consequences of energy minimization for metrics of this type.

We begin with a broad introduction, including a review of some other work on wrinkling in thin sheets, and discussion of some key examples that demonstrate the overall character of the problem. We then turn, in Section 1.5 and beyond, to a more mathematical discussion. We formulate an energy minimization problem

*Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany. Email: bella@mis.mpg.de. This work was begun while PB was a PhD student at the Courant Institute of Mathematical Sciences. Support from NSF grant DMS-0807347 is gratefully acknowledged.

†Courant Institute of Mathematical Sciences, New York University, New York, NY. Email: kohn@cims.nyu.edu. Support is gratefully acknowledged from NSF grants DMS-0807347 and OISE-0967140.

that is, roughly speaking, a geometrically nonlinear version of one previously studied numerically by Audoly and Boudaoud [2]. Our main result is that for a sheet of thickness h, and imposing certain restrictions on the admissible deformations,

(a) the minimum energy is of order $h^{4/3}$ (Theorem 1).

We also consider minimization of the energy in a more restricted class of deformations, showing that

(b) the minimum energy in the restricted class is of order h (Theorem 2).

It follows from (a) and (b) that energy minimization in the more restricted class is not a good model for the applications that motivate this work.

Our results do not explain the cascade of wrinkles seen in some experimental and numerical studies; and they leave open the possibility that an energy scaling law better than $h^{4/3}$ could be obtained by considering a class of deformations larger than one associated with (a).

1.1 The connection with leaves, flowers, and torn plastic

As already mentioned, our investigation is motivated by work in the physics literature on leaves, flowers, and torn plastic. Let us explain the connection (for additional detail and references see [3, Chapter 10]). We focus in this paper on metrics of the form $dx^2 + m^2dy^2$, where $m = m(x) \geq 1$ is a function of x and $0 < x < L$. The edge $x = L$ will be "free." In torn plastic, $x = L$ represents the edge created by the tear; it is where the wrinkling is greatest. Moreover the wrinkling is roughly uniform at a given distance from the edge. In this setting, the non-Euclidean metric is created by the large stresses introduced when the sheet is torn; since the stress depends on the distance from the tear, so does the metric. In leaves and flowers the situation is similar: thewrinkling is typically greatest at the free boundary, which grew most recently; moreover it is roughly uniform at a given distance from the edge. In this case the non-Euclidean metric is created by the growth process, which seems to accumulate intrinsic length monotonically. Our model, though greatly simplified, captures the essential characteristic of both problems: for our sheet to be isometrically embedded in $\mathbb{R}^3$, the line at constant distance from the free boundary would have to be stretched uniformly, by an amount that depends on the distance to the boundary. Though the function $m(x)$ is apparently monotonically increasing in x for leaves, flowers, and torn plastic, our results do not require monotonicity.

A key contribution is that of Audoly and Boudaoud [2], who considered essentially the same problem within a Föppl–von Kármán setting. Their numerical results suggest that, at least for some choices of the metric m, the wavelength λ of wrinkling at the free boundary scales like $h^{2/5}\ell^{3/5}$, where ℓ is a length scale determined by $m'(L)$. Their numerical results also reveal a "cascade" of wrinkles, i.e. the wavelength $\lambda(x)$ of wrinkling depends on x. Such a cascade is consistent with the system's apparent preference for small-wavelength wrinkling near the free boundary (due to the character of the metric) and large-wavelength wrinkling away from the free boundary (to avoid bending energy). See [2] for more quantitative (but still heuristic) arguments along these lines.

Another key contribution is that of Sharon, Roman, and Swinney [28], concerning experimental observations of metric-driven wrinkling in torn plastic sheets. They suggest that the wavelength of wrinkling at *any* x should scale like $\lambda(x) \sim h^{\alpha}(x)\ell^{1-\alpha}(x)$, where $\ell(x)$ is a length scale determined by $m'(x)$, and $h(x)$ is the sheet thickness (not constant in the experiments). For a particular experiment involving torn plastic sheets this law was confirmed, with $\alpha \approx 3/10$. (Sharon et al suggest, however, that the value of α may not be universal; rather, it could depend on the metric.) While the experiments of [28] reveal a "cascade" not unlike that seen numerically in [2], the explanation offered by [28] is rather different, since it involves the pointwise spatial variation of the length scale $\ell(x)$. Another significant difference between the two papers concerns whether or not the length scale of wrinkling $\lambda(x)$ tends to 0 as $h \to 0$ with x held fixed: Audoly and Boudaoud suggest not, while Sharon, Roman, and Swinney suggest so.

The overall goal of our work is to shed some light on the issues raised by [2] and [28]. Our main result, Theorem 1, identifies the minimum energy scaling law within a certain class of deformations. Moreover, its proof shows that one can achieve the optimal scaling *without* a cascade, using wrinkles with a wavelength of order $h^{1/3}$. This certainly does not explain the numerically and experimentally observed cascades. But it does suggest that either

(i) the class of deformations considered in Theorem 1 is too restrictive; or else

(ii) the origin of the cascade lies not in the scaling of the minimum energy with respect to h, but rather in the prefactor (i.e., the cascade might achieve energy $\sim C_{casc}h^{4/3}$ with C_{casc} substantially lower than the prefactor of the no-cascade construction in this paper).

Option (i) is certainly possible (see the final Remark in Section 1.5). However option (ii) seems consistent with the idea – very similar to [28] – that the preferred wavelength of wrinkling at x may be $h^{1/3}\ell^{2/3}(x)$, where ℓ is a length scale associated with the metric.

Our problem shares some elements with the metric-driven wrinkling seen in disks with constant negative curvature [17]. In that setting, as in ours: (1) the energy has membrane and bending terms, and the membrane term involves a non-Euclidean metric; and (2) the experimentally-observed folding patterns become increasingly complex as $h \to 0$. Mathematically, however, that problem seems quite different from ours. Indeed, for a constant-negative-curvature disk there exist deformations with vanishing membrane energy and finite bending energy [12]. As a result, the minimum energy is of order h^2 and minimizers do not depend strongly on h. The increasing complexity seen experimentally [17] as $h \to 0$ remains a puzzle.

Is our viewpoint the "right one" for modeling leaves, flowers, and torn plastic? There is room for disagreement. A different viewpoint is taken by Lewicka, Mahadevan and Pakzad in [20]. Like us, they study thin plates with prescribed non-Euclidean metrics. But unlike ours, their metrics depend on the thickness h. In fact, the metric in [20] is a perturbation of the Euclidean metric of order h to a power. This produces behavior very different from what we expect here. Indeed, in the setting of [20] minimizers converge strongly as $h \to 0$ to the solution of a (von Kármán-like) limit problem. In our setting, by contrast, we expect the configuration to be increasingly complex as $h \to 0$ – so it should not converge strongly, and there could be no "limit problem." It is reasonable to ask which viewpoint is better for modeling leaves, flowers, and torn plastic. We do not know the answer. In our view, both viewpoints deserve investigation.

1.2 Background and context

The question whether a 2D surface with a given metric can be isometrically embedded in $\mathbb{R}^3$ has a long history. A very non-intuitive positive result goes back to the 50's, when Nash and Kuiper [19, 24] showed that any "short" embedding can be approximated by a C^1 isometric embedding. This implies, in our setting, that for $m > 1$ the metric $dx^2 + m^2dy^2$ is realizable by a C^1 surface that is L^∞ close to a plane. But this famous result gives only a C^1 surface, whose bending energy is typically infinite. Since our elastic energy includes a bending term, the result of Nash and Kuiper is of little use.

There is a large (and growing) literature on pattern formation in thin sheets with the standard Euclidean metric. While a systematic review is beyond the scope of this article, we briefly mention a few themes:

(i) The *crumpling* or folding of thin sheets has received a lot of attention; a nice review can be found in [30]. In such problems the boundary conditions play a crucial yet hard-to-quantify role; for example, in deforming a piece of paper one easily finds boundary conditions that seem to require point singularities. Since our focus is on energy scaling laws, it is natural to highlight the work of Venkataramani [29], who gave the first lower bound for the energy of a "ridge." A notable recent result is that of Conti and Maggi [9], who showed that for "short" deformations (those that do not increase lengths) the minimum of the elastic energy per unit thickness scales at most as $h^{5/3}$.

(ii) *Tension-induced wrinkling* has also received a lot of attention. An early, classic study was [8]; some recent contributions are [4, 10, 11, 25, 16, 13]. In this setting a local state of uniaxial tension determines the wrinkling direction. The energy scaling law gives the *excess* energy due to positive thickness, and is related to the amplitude and length scale of the wrinkles.

The problem considered in this paper is more like (i) than (ii), since there is no tensile direction to guide the geometry of the wrinkling. A third familiar theme, somewhat closer to our problem, is

(iii) the analysis of *blisters* formed when a compressed thin film breaks away from its substrate. The associated energy minimization problem amounts to a study of metric-induced wrinkling, with an isotropic target metric $\mathrm{d}s^2 = \lambda(\mathrm{d}x^2 + \mathrm{d}y^2)$ (here $\lambda > 1$) and the boundary condition that the deformation be the identity at the edge of the blister. The minimum elastic energy scales linearly with h; this was proved in [5, 14] using a geometrically linear Föppl–von Kármán model, and in [6] using a fully nonlinear elastic model.

The boundary condition plays a crucial role in topic (iii) (blisters): roughly speaking, the energy is of order h because the boundary condition and the metric are incompatible. We already mentioned the importance of the boundary condition for crumpling. The boundary conditions play a determining role in the present paper too – though in a way rather different from these examples. Indeed, the metric we consider is locally realizable by a smooth surface (see Section 1.4), so one might expect the energy scaling law to be h^2. But Theorem 1 shows that the minimum energy is much larger, of order $h^{4/3}$. Evidently, our boundary conditions and restrictions on the deformation prevent the existence of an isometric embedding with finite bending energy.

1.3 The elastic energy

Turning now to a more quantitative discussion, we begin by discussing our "elastic energy" functional. It is

$$E_h(u) := \iint_\Omega \left|Du^T(x,y)Du(x,y) - g(x,y)\right|^2 + h^2 \left|D^2u(x,y)\right|^2 \,\mathrm{d}x\,\mathrm{d}y. \tag{1}$$

Here $\Omega \subset \mathbb{R}^2$ is the coordinate domain, and $u : \Omega \to \mathbb{R}^3$ is an embedding of Ω into $\mathbb{R}^3$, whose image is the midplane of our sheet. The parameter $h > 0$ represents the thickness of the sheet, and g is the prescribed metric. We call the first term $\left|Du^T(x,y)Du(x,y) - g(x,y)\right|^2$ the *membrane* energy; it favors isometries. We call the second term $h^2\left|D^2u(x,y)\right|^2$ the *bending* energy.

The functional (1) is a widely-used proxy for the elastic energy of a thin three-dimensional sheet (see e.g. [9, Section 1.3]). Let us briefly review the logic behind it. If $\Omega_h = \Omega \times (-h/2, h/2)$ is a thin three-dimensional sheet with midplane Ω, and $\tilde{u} : \Omega_h \to \mathbb{R}^3$ is an embedding of the sheet in $\mathbb{R}^3$, the associated 3D elastic energy has the form

$$\tilde{E}_h(\tilde{u}) := \frac{1}{h}\iiint_{\Omega_h} W(D\tilde{u}(x,y,z), \tilde{g}(x,y,z))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

where $\tilde{g}$ is the target metric, $W(F,g)$ is the elastic energy density, and we have divided by h to get the energy per unit thickness. Now assume that $\tilde{g}(x,y,z)$ is a diagonal matrix with entries $(1, m^2(x), 1)$, and take $\tilde{u}$ to be given by the Kirchhoff–Love ansatz:

$$\tilde{u}(x,y,z) := u(x,y) + z \cdot n(x,y),$$

where $u : \Omega \to \mathbb{R}^3$ is the mapping of the midplane and $n(x,y) = \frac{\partial_x u \times \partial_y u}{|\partial_x u \times \partial_y u|}$ is the unit normal of the surface $u(\Omega)$. If the elastic energy satisfies $W(F,g) \geq C|F^TF - g|^2$ and some modest additional conditions, the 3D elastic energy is bounded above and below by constant multiples of

$$\iint_\Omega \left|Du^T(x,y)Du(x,y) - g(x,y)\right|^2 + h^2\left|\Pi(x,y)\right|^2 \,\mathrm{d}x\,\mathrm{d}y,$$

where $g(x,y)$ is the diagonal 2×2 matrix with entries 1 and $m^2(x)$, and Π is the second fundamental form of the surface $u(\Omega)$ (so that $|\Pi|^2 = (n\cdot\partial_{xx}u)^2 + 2(n\cdot\partial_{xy}u)^2 + (n\cdot\partial_{yy}u)^2$). Our functional (1) is obtained by replacing the physically-accurate bending term $|\Pi|^2$ by the sum of all the second derivatives $|D^2u|^2 = |\partial_{xx}u|^2 + 2|\partial_{xy}u|^2 + |\partial_{yy}u|^2$. We do this for simplicity, and because we believe it does not change the essential character of the problem. (In some related settings it is known that the replacement of $|\Pi|^2$ by $|D^2u|^2$ does not change the energy scaling law, see e.g. [9, Section 6] and [4, 7]). Note that an upper bound on E_h certainly provides an upper bound on the physical energy (within the Kirchhoff-Love ansatz).

We briefly discuss the choice of the domain Ω and boundary conditions. Motivated by the experiments with torn plastic and wavy leaves, it would be natural to work in an infinite strip $[0, L] \times \mathbb{R}$, and to require that the boundary $x = 0$ be straight (i.e. that $u(0, y) = (0, y, 0)$). But it is inconvenient to work with an infinite domain; therefore we will mainly consider only part of the strip, taking $\Omega = [0, L] \times [0, 1]$, and using the following "periodic-like" boundary condition at $y = 0, 1$:

$$u(x, y) - (0, y, 0) \text{ is 1-periodic in } y. \tag{2}$$

We will always take the boundary $x = L$ to be free, since it is where we expect the wrinkling to be greatest.

1.4 Examples

We now discuss three examples that reveal important features of the problem. They show, in particular, that the choice of boundary conditions plays a crucial role in determining the energy scaling law.

Example 1: *The trumpet.* This example, discussed at length by Marder and Papanicolaou [22], does not satisfy the proposed boundary conditions $u(0, y) = (0, y, 0)$ and (2); indeed, we present it partly to demonstrate the need for such conditions.

In this example, lines in the y-direction are mapped to circles. As the variable x increases, these circles get larger to accommodate the increase of m (so the shape resembles a trumpet if m is monotone). We see from the parametrization

$$u^1(x, y) = \left(f(x), \frac{m(x)}{2\pi}\cos(2\pi y), \frac{m(x)}{2\pi}\sin(2\pi y) \right)$$

that if $\partial_x f(x) = \sqrt{1 - \frac{(\partial_x m)^2}{4\pi^2}}$, then the trumpet has zero membrane energy. (Obviously, this example requires $|\partial_x m| \leq 2\pi$.) Since u^1 does not depend on h, the elastic energy $E_h(u^1)$ scales like h^2. This example shows that the metric g we consider is realizable. In particular, to see increasing complexity as $h \to 0$ we must restrict the class of admissible deformations by imposing boundary conditions incompatible with the trumpet.

Example 2: *The helix and its generalization, the pseudohelix* . If the function m has a special form

$$\frac{\sqrt{m^2(x) - 1}}{x} = P, \tag{3}$$

where $P > 0$ is a constant, we define

$$u^2(x, y) = (x\cos(Py), y, x\sin(Py)), \tag{4}$$

and get

$$E_h(u^2) = Ch^2.$$

This example is prominent in the recent study [18], which examines why a narrow-leaved kelp forms helices while some wider-leaved kelps have approximately-flat leaves with wavy edges. One possible explanation is that it is easier for a narrow leaf to (approximately) satisfy condition (3).

The helix example can be generalized as follows. Instead of (4) consider

$$u^2(x, y) = (r(x)\cos(Py + p(x)), y + t(x), r(x)\sin(Py + p(x))),$$

where r, p, t are some functions which satisfy $r(0) = 0$, $p(0) = 0$, $t(0) = 0$. We have

$$\begin{aligned}
\partial_x u^2 &= (r' \cos(Py + p) - r \sin(Py + p)p', t', r' \sin(Py + p) + r \cos(Py + p)p'),\\
\partial_y u^2 &= (-r \sin(Py + p)P, 1, r \cos(Py + p)P).
\end{aligned}$$

For the membrane energy to vanish we require

$$\begin{aligned}
1 &= \left|\partial_x u^2\right|^2 = (r')^2 + r^2(p')^2 + (t')^2,\\
0 &= \partial_x u^2 \cdot \partial_y u^2 = r^2 P p' + t',\\
m^2(x) &= \left|\partial_y u^2\right|^2 = r^2 P^2 + 1.
\end{aligned}$$

From the last equation we derive $r(x) = P^{-1}\sqrt{m^2(x) - 1}$. Then the middle one implies $t' = -r^2 P p' = -P^{-1}(m^2(x) - 1)p'$. Assuming $m(0) = 1$, $m'(0) = 0$, and $m \in \mathcal{C}^2([0, L])$, one can prove using the first equation that there exist P and $p(x)$ such that the membrane energy vanishes. This example – a *pseudohelix* – shows that under very mild assumptions on m there is a deformation satisfying $u(0, y) = (0, y, 0)$ and the periodic-like condition (2) whose energy is of order h^2. Of course, the pseudohelix is topologically nontrivial (it winds around the y axis).

Example 3: *An example with interpenetration.* If we impose the constraint $u(x, y) \cdot e_1 = x$ then the image of u cannot wind around the y axis. Our next example shows that even with this constraint, examples with a scaling law approaching $h^{4/3}$ are readily available if we permit interpenetration. In view of this example, in the rest of the paper we will impose conditions that prevent interpenetration.

We now describe the construction. Away from $x = 0$, we set

$$u^3(x, y) = (x, \gamma^1(x, y), \gamma^2(x, y)),$$

where γ is $h^{1/3}$-periodic in the y-direction. One period of the graph of $\gamma(x, .)$ for a given x looks like Figure 1.

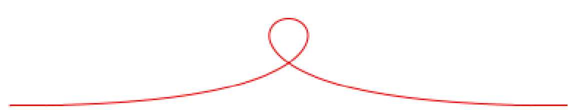

Figure 1: Building block of the interpenetrated solution

The loop on the curve shown in the figure must "grow" with increasing x, and must shrink to a point as $x \to 0$. It can be chosen so that u^3 satisfies both $|\partial_y u^3| = m(x)$ and $\partial_x u^3 \perp \partial_y u^3$ (we omit the details, since they are routine and uninformative, and nothing else in the paper depends on this example). Note that, as seen from the picture, the curve is penetrating itself. Since the loop is shrinking to a point as $x \to 0$, we would have $|\partial_{yy} u^3| \to \infty$ as $x \to 0$. To avoid this, one must introduce a "boundary layer" near $x = 0$. When the dust clears, one gets by this method an example with $E_h \sim C(\epsilon)h^{4/3-\epsilon}$ for any $\epsilon > 0$ if interpenetration is allowed.

Synthesis. Our work is motivated by the complex patterns seen in leaves, flowers, and torn plastic sheets. We believe they can be modeled via minimization of the elastic energy (1). Our examples show that if minimization of (1) is to produce increasing complexity as $h \to 0$ then we must impose conditions eliminating trumpets and pseudohelices from consideration. They also indicate the importance of prohibiting interpenetration.

1.5 Main results

We turn now to the formulation of our main results. To eliminate trumpets from consideration, we shall henceforth restrict our attention to deformations

$$u(x,y) = (u^x(x,y), u^y(x,y), u^z(x,y))$$

satisfying the boundary conditions

$$\begin{gathered} u(0,y) = (0,y,0), \quad y \in [0,1], \\ u(x,y) - (0,y,0) \text{ is 1-periodic in } y. \end{gathered} \tag{5}$$

To eliminate pseudohelices, we shall impose the further condition that

$$|u^x(x,y) - x| \leq \alpha h^{2/3}, \tag{6}$$

for some constant $\alpha > 0$ independent of h. For Theorem 1 we also impose the following additional assumptions:

$$\det D(u^x, u^y) \geq 0 \quad \text{and} \quad \partial_y u^y \geq 0; \tag{7}$$

note that these prohibit interpenetration. Our first main result is that under some mild conditions on the metric, the minimum energy scales like $h^{4/3}$:

Theorem 1. *Let $\Omega = (0,L) \times (0,1)$, and assume the function m (which determines the metric) is the restriction to $(0,L)$ of a function (also denoted m) in $\mathcal{C}^2([-1,L]) \cap W^{3,2}(0,L)$ with $m(x) = 1$ for $x \leq 0$ and $m(x) > 1$ in $(0,L]$. Then there exist constants $0 < C_0 < C_1$ (depending on Ω, m, and α) with the following property: for any $0 < h < 1$ there exists a deformation $u_h : \Omega \to \mathbb{R}^3$ which satisfies (5-7), and has*

$$E_h(u_h) \leq C_1 h^{4/3}.$$

Moreover, for any $u : \Omega \to \mathbb{R}^3$ which satisfies (5-7) we have a lower bound for the energy

$$E_h(u) \geq C_0 h^{4/3}.$$

It is natural to ask whether (6) can be replaced by the more rigid condition

$$u^x(x,y) = x. \tag{8}$$

(Note that the deformation of Example 3 has this property.) In fact this condition is too restrictive, in the sense that it changes the minimum energy scaling law from $h^{4/3}$ to h:

Theorem 2. *Let $\Omega = (0,L)\times(0,1)$, and assume m is the restriction to $(0,L)$ of a function in $\mathcal{C}^2([-1,L])$ with $m(x) = 1$ for $x \leq 0$, $m > 1$ in $(0,L]$, and $m(x_1) > 1.546264$ for some $x_1 \in (0,L]$. To prevent interpenetration and other nonphysical behavior, we restrict attention to deformations that satisfy, in addition to (5) and (8),*

$$\kappa^{-1} \leq |\partial_x u(x,y)| \leq \kappa, \quad \kappa^{-1} \leq |\partial_y u(x,y)| \leq \kappa, \tag{9}$$

$$\partial_y u^y(x,y) \geq 0, \tag{10}$$

where κ is a fixed (sufficiently large) constant. Then there exists $C_0 = C_0(\Omega, m, \kappa)$ such that

$$C_0 h \leq E_h(u)$$

for any u satisfying the stated restrictions, and any $0 < h < 1$. Also, there exists $C_1 = C_1(\Omega, m, \kappa)$ such that for any $0 < h < 1$ there exists a u_h satisfying the stated restrictions for which

$$E_h(u_h) \leq C_1 h.$$

In brief: when u is restricted by (8) and there exists a point where $m > 1.546264$, the minimum energy scales linearly in h.

Condition (9) is natural from a physical viewpoint, since most materials will break when the strain gets too large. Condition (10) serves to rule out interpenetration. The assumption that $m > 1.546264$ at some point is required by the proof of the lower bound. We believe that this hypothesis is an artifact of the proof, i.e. that the lower bound should remain true without it.

Remark. *Comparing Theorems 1 and 2, we see that relaxing the assumption $u_h(x, y) \cdot e_1 = x$ just a little bit dramatically changes the scaling law. This is reminiscent of the relationship between [15] and [5, 6, 14] in the literature on blisters in compressed thin films. Indeed, in [15] Jin and Sternberg found an energy scaling law under an assumption closely analogous to* (8)*. Soon afterward, Ben Belgacem, Conti, De Simone and Müller [5] and Jin and Sternberg [14] showed that relaxing the assumption changed the energy scaling law. The papers just cited use the von Kármán framework, but similar results were later shown in a fully nonlinear setting [6].*

Remark. *In light of the preceding remark, it is natural to ask whether our assumption (6) may still be too strong. This assumption seems to be crucial for our proof of the $h^{4/3}$ lower bound. We imposed (6) to exclude examples like the pseudohelices discussed in Section 1.4. But it remains an important open question whether weakening (6) would permit a physically plausible wrinkling pattern with an energy scaling law better than $h^{4/3}$. In particular, it remains open whether a pattern with a "cascades of wrinkles" like those seen numerically in [2] and experimentally in [28] might have been unintentionally excluded by our hypothesis (6).*

The plan for the rest of the paper is as follows. In Section 2 we construct a test function that demonstrates the upper bound assertion of Theorem 2. In Section 3 we use a modified version of that construction to prove the upper bound assertion of Theorem 1. Then in Section 4 we prove the lower bound assertion of Theorem 1. We finish with Section 5 where we prove the lower bound assertion of Theorem 2.

Throughout this article, a "constant" C may depend on the prescribed data (for example L, κ, α, or m). However our constants will never depend on h, nor on the particular deformation u under consideration.

Acknowledgement: We are grateful to Stefan Müller for pointing out that the assumption $u_h(x, y) \cdot e_1 = x$ is too rigid, and for suggesting the ansatz we use to prove the $h^{4/3}$ upper bound of Theorem 1. We also thank two anonymous referees for their insightful comments and suggestions.

2 The upper bound half of Theorem 2

The goal of this section is to prove the upper bound asserted by Theorem 2. Our task is thus to construct, for any $h > 0$, a deformation u_h satisfying (8) as well as the other assumptions of Theorem 2 such that $E_h(u_h) \leq Ch$.

Laying necessary groundwork, we first define a "sinusoidal" curve in the plane with given arclength $m \geq 1$, and prove some estimates for its dependence on m.

Lemma 2.1. *For any constant $m \geq 1$ there exists a smooth 1-periodic function $y \mapsto v(m, y) = (v_1, v_2)(m, y)$ such that $v(m, 0) = 0$, $\int_0^1 |\partial_{yy} v(m, y)|^2 \, \mathrm{d}y < \infty$, and*

$$\left| \frac{\partial v}{\partial y}(m, y) + e_1 \right| = m, \quad y \in [0, 1]. \tag{11}$$

Moreover, our choice of v has the property that

$$v_1(m, y) = (m - 1)\phi(m, y) \quad \textit{and} \quad v_2(m, y) = \sqrt{m - 1}\,\psi(m, y) \tag{12}$$

where ϕ and ψ are smooth functions of m and y.

Proof. For $m \geq 1$ we consider $\lambda(m) \geq 0$ such that the graph of the function $f_m(t) = (t, \lambda(m) \sin(2\pi t))$ from $t = 0$ to $t = 1$ has exactly length m:

$$\int_0^1 |f_m'(t)| dt = m.$$

It is easy to see that there exists a smooth function θ on $[1,\infty)$ such that

$$\lambda(m) = \sqrt{m-1}\,\theta(m). \tag{13}$$

We define $F(a) := \frac{1}{m}\int_0^a |f_m'(t)|dt$ and set

$$v(m,y) := f_m(F^{-1}(y)) - (y,0).$$

Then

$$|\partial_y v(m,y) + e_1| = m$$

and $v(m,\cdot)$ is a smooth 1-periodic function. Finally, (12) follows from (13). $\square$

We turn now to the construction of u_h. Given $0<h<1$ let

$$n := \lfloor h^{-1/2} \rfloor,$$

where the square brackets denote the integer part. We define an auxiliary metric $\tilde m$ by

$$\tilde m(x) := \begin{cases} 1+\varphi_h^2(x) & x\in[0,h^{2/3}], \\ m(x-h^{2/3}) + h^{4/3} & x \in [h^{2/3}, L], \end{cases}$$

where

$$\varphi_h(x) := h^{2/3}\varphi(h^{-2/3}x) \tag{14}$$

and φ is a smooth increasing function which satisfies $\varphi(0)=0, \varphi(1)=1, \varphi'(1)=\varphi''(1)=0$, and $|\varphi'| \le 2, |\varphi''| \le 4$. We see that $\tilde m$ is a C^2 function on $[0,L]$ and satisfies $\tilde m(0)=1$.

We consider v from Lemma 2.1 extended periodically for all $y\in[0,\infty)$ and define

$$u_h(x,y) := \left(x, y + \frac{1}{n}v_1(\tilde m(x), ny), \frac{1}{n}v_2(\tilde m(x), ny)\right).$$

This deformation obviously satisfies the prescribed boundary conditions (5).

An easy computation gives

$$\begin{aligned} \partial_x u_h(x,y) &= \left(1, \frac{1}{n}\partial_m v_1(\tilde m(x), ny)\tilde m'(x), \frac{1}{n}\partial_m v_2(\tilde m(x), ny)\tilde m'(x)\right), \\ \partial_y u_h(x,y) &= \left(0, 1+\partial_y v_1(\tilde m(x), ny), \partial_y v_2(\tilde m(x), ny)\right). \end{aligned}$$

We see from (11) that $|\partial_y u_h(x,y)| = \tilde m(x)$. We compute $E_h(u_h)$:

$$\begin{aligned} E_h(u_h) = &\iint_\Omega \left||\partial_y u_h|^2 - m(x)\right|^2 + 2\left|(\partial_x u_h, \partial_y u_h)\right|^2 \\ &\qquad + \left||\partial_x u_h|^2 - 1\right|^2 + h^2\left|D^2 u_h(x,y)\right|^2 \,\mathrm{d}x\,\mathrm{d}y \\ = &\iint_\Omega \left|\tilde m^2(x) - m^2(x)\right|^2 + 2\left|\frac{\tilde m'(x)}{n}\left(\partial_m v_1(1+\partial_y v_1) + \partial_m v_2 \partial_y v_2\right)\right|^2 \\ &\qquad + \left(\frac{\tilde m'(x)}{n}\right)^4 |\partial_m v|^4 \,\mathrm{d}x\,\mathrm{d}y \\ &+h^2 \iint_\Omega n^{-2}\left|(\tilde m')^2\partial_{mm}v + \tilde m''\partial_m v\right|^2 + 2|\tilde m'\partial_{my}v|^2 + n^2|\partial_{yy}v|^2 \,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{15}$$

We want to estimate all the terms on the RHS. The first term is easy to estimate since m is $\mathcal{C}^2$. Indeed, since for $x\in[0,h^{2/3}]$ we can find $\zeta\in[0,x]$ such that $m(x) = m(0) + m'(\zeta)x$, for $x\in[0,h^{2/3}]$ we get

$|\tilde{m}^2(x) - m^2(x)| = \left|1 + h^{4/3}\varphi^2 - (1 + m'(\zeta)x)\right| \cdot |\tilde{m}(x) + m(x)| \le C|h^{4/3} + x| \le Ch^{2/3}$. If $x \ge h^{2/3}$, we get $|\tilde{m}^2(x) - m^2(x)| = \left|h^{4/3} + m(x - h^{2/3}) - m(x)\right| \cdot |\tilde{m}(x) + m(x)| \le C|h^{4/3} + h^{2/3}m'(\zeta)| \le C'h^{2/3}$. Hence

$$\iint_\Omega \left|\tilde{m}^2(x) - m^2(x)\right|^2 \,\mathrm{d}x\,\mathrm{d}y \le C \iint_\Omega \left(h^{2/3}\right)^2 \,\mathrm{d}x\,\mathrm{d}y \le C'h^{4/3}. \tag{16}$$

To estimate terms involving v_1 and v_2 we first need estimates for derivatives of v_1 and v_2. Using (12) we see

$$\begin{aligned} |v_1| &\le Cm, \\ |\partial_m v_1| &= |\phi + (m-1)\partial_m\phi| \le Cm, \\ |\partial_{mm} v_1| &= |2\partial_m\phi + (m-1)\partial_{mm}\phi| \le Cm. \end{aligned} \tag{17}$$

The expression for v_2 involves $\sqrt{m-1}$, and so we obtain

$$\begin{aligned} |v_2| &\le C(m-1)^{1/2}, \\ |\partial_m v_2| &= \left|(m-1)^{-1/2}\psi/2 + (m-1)^{1/2}\partial_m\psi\right| \le C\left((m-1)^{1/2} + (m-1)^{-1/2}\right), \\ |\partial_{mm} v_2| &= \left|-\frac{3}{4}(m-1)^{-3/2}\psi + (m-1)^{-1/2}\partial_m\psi + (m-1)^{1/2}\partial_{mm}\psi\right| \\ &\le C\left((m-1)^{1/2} + (m-1)^{-3/2}\right). \end{aligned} \tag{18}$$

Remark. *We point out that since v_1 and v_2 depend on y only through the smooth functions ϕ and ψ, the previous estimates stay true even after we differentiate the LHS with respect to y. (Of course, the constants in the differentiated bounds will be different.) For example we have $|\partial_{my}v_2| \le C\left((m-1)^{1/2} + (m-1)^{-1/2}\right)$ and $|\partial_y v_1| \le Cm$.*

Combining (17), (18), and the previous remark gives

$$|\partial_m^k \partial_y^l v| \le C\left(m + (m-1)^{1/2-k}\right) \tag{19}$$

for $k \ge 0, l \ge 0, k + l \le 2$.

To estimate the terms involving v_1 and v_2 it is convenient to write Ω as a union of two parts: $\Omega_1 = [0, h^{2/3}] \times [0,1]$ and $\Omega_2 = [h^{2/3}, L] \times [0,1]$. We remind the reader that in Ω_1 we have $\tilde{m}(x) = 1 + \varphi_h^2(x)$, and so

$$u_h(x,y) = (x, y + n^{-1}\varphi_h^2(x)\phi(\tilde{m}(x), ny), n^{-1}\varphi_h(x)\psi(\tilde{m}(x), ny)). \tag{20}$$

Then

$$\begin{aligned} &\iint_{\Omega_1} 2\left|\frac{\tilde{m}'(x)}{n}\left(\partial_m v_1(1 + \partial_y v_1) + \partial_m v_2 \partial_y v_2\right)\right|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &\qquad\qquad \le Cn^{-2}\iint_{\Omega_1} \left|\partial_m v_1(1 + \partial_y v_1) + \partial_m v_2 \partial_y v_2\right|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &\qquad\qquad \le Cn^{-2}\iint_{\Omega_1} \tilde{m}^2 + ((\tilde{m}-1)^{1/2} + (\tilde{m}-1)^{-1/2})(\tilde{m}-1)^{1/2} \,\mathrm{d}x\,\mathrm{d}y \\ &\qquad\qquad \le C'n^{-2}|\Omega_1| = C'h^{5/3}. \end{aligned} \tag{21}$$

To estimate the next term in (15) we use that in Ω_1 we have $\tilde{m}'(x) = 2\varphi_h\varphi_h'$ and by (19) $|\partial_m v| \le C\left(1 + (\tilde{m}-1)^{-1/2}\right) \le C(1 + \varphi_h^{-1})$. Therefore

$$\begin{aligned} \iint_{\Omega_1} \left(\frac{\tilde{m}'(x)}{n}\right)^4 |\partial_m v|^4 \,\mathrm{d}x\,\mathrm{d}y &\le Cn^{-4}\iint_{\Omega_1} \left((\tilde{m}')^4 + |\varphi_h\varphi_h'\varphi_h^{-1}|^4\right) \,\mathrm{d}x\,\mathrm{d}y \\ &\le Cn^{-4}\iint_{\Omega_1} (1 + |\varphi_h'|^4) \,\mathrm{d}x\,\mathrm{d}y \le Ch^2|\Omega_1| = Ch^{8/3}. \end{aligned} \tag{22}$$

To finish the estimate in Ω_1 it remains to estimate the bending part of the integral. Using (20) we see

$$|\partial_{xx}u_h| = n^{-1}\left|\left(2(\partial_x\varphi_h)^2\phi + 2\varphi_h\partial_{xx}\varphi_h\phi + 4\varphi_h\partial_x\varphi_h\partial_x\phi + \varphi_h^2\partial_{xx}\phi,\ \partial_{xx}\varphi_h\psi + 2\partial_x\varphi_h\partial_x\psi + \varphi_h\partial_{xx}\psi\right)\right|.$$

From (14) we know that for $x \in \Omega_1$:

$$|\varphi_h(x)| \leq h^{2/3}, \quad |\partial_x\varphi_h(x)| \leq C, \quad |\partial_{xx}\varphi_h(x)| \leq Ch^{-2/3}.$$

Since ϕ and ψ are smooth functions of $\tilde{m}$ (and $\tilde{m}$ is $\mathcal{C}^2$), we get that $|\partial_{xx}u_h| \leq Cn^{-1}h^{-2/3} \leq Ch^{-1/6}$ in Ω_1. The same way we estimate $\partial_{xy}u_h$ and $\partial_{yy}u_h$ and obtain:

$$|\partial_{xy}u_h| \leq C, \qquad |\partial_{yy}u_h| \leq Ch^{1/6},$$

and so

$$h^2\iint_{\Omega_1}|D^2u_h|^2\,\mathrm{d}x\,\mathrm{d}y \leq Ch^2|\Omega_1|\left(h^{-1/6}\right)^2 \leq Ch^{7/3}. \tag{23}$$

Now we will estimate the same terms integrated over Ω_2. Arguing as for (21), the first term satisfies:

$$\iint_{\Omega_2} 2\left|\frac{\tilde{m}'(x)}{n}\left(\partial_m v_1(1+\partial_y v_1) + \partial_m v_2\partial_y v_2\right)\right|^2\,\mathrm{d}x\,\mathrm{d}y$$
$$\leq Cn^{-2}\iint_{\Omega_2}\tilde{m}^2 + ((\tilde{m}-1)^{1/2} + (\tilde{m}-1)^{-1/2})(\tilde{m}-1)^{1/2}\,\mathrm{d}x\,\mathrm{d}y \leq C'n^{-2}|\Omega_2| = C'h. \tag{24}$$

To deal with the next term we observe that since $\tilde{m} \in \mathcal{C}^2$ and $\tilde{m} \geq 1$ we have that

$$\left|\frac{\tilde{m}'(x)}{\sqrt{\tilde{m}(x)-1}}\right| \leq \sqrt{2||\tilde{m}''||_{L^\infty}}, \qquad \text{for } x \in [0, L/2]. \tag{25}$$

Indeed, using Taylor expansion at x, for any $x' \in [0, L]$ we can find ζ between x and x' such that

$$1 \leq \tilde{m}(x') = \tilde{m}(x) + \tilde{m}'(x)(x'-x) + \tilde{m}''(\zeta)(x'-x)^2/2.$$

We choose $x' = x - \tilde{m}'(x)/||\tilde{m}''||_{L^\infty}$. Observe that

$$\frac{|\tilde{m}'(x)|}{||\tilde{m}''||_{L^\infty}} = \frac{\left|\int_0^x \tilde{m}''\right|}{||\tilde{m}''||_{L^\infty}} \leq x$$

implies $x' \in [0, 2x] \subset [0, L]$. Then

$$\tilde{m}(x) - 1 \geq \frac{(\tilde{m}'(x))^2}{||\tilde{m}''||_{L^\infty}} - \frac{(\tilde{m}'(x))^2}{2||\tilde{m}''||_{L^\infty}} = \frac{(\tilde{m}'(x))^2}{2||\tilde{m}''||_{L^\infty}}.$$

Inequality (25) then follows by taking square root of the previous relation. We now observe that since $m(x) > 1$ in $(0, L]$ (in particular, $m'/\sqrt{m-1}$ is a continuous and bounded function on $[L/2, L]$) we have

$$\left|\frac{\tilde{m}'(x)}{\sqrt{\tilde{m}(x)-1}}\right| \leq C(m), \qquad \text{for } x \in [0, L]. \tag{26}$$

Using (19) and (26) we obtain

$$\iint_{\Omega_2}\left(\frac{\tilde{m}'(x)}{n}\right)^4|\partial_m v|^4\,\mathrm{d}x\,\mathrm{d}y \leq Cn^{-4}\iint_{\Omega_2}\left|\frac{\tilde{m}'(x)}{\sqrt{\tilde{m}(x)-1}}\right|^4 + \tilde{m}^4(\tilde{m}')^4\,\mathrm{d}x\,\mathrm{d}y \leq C'n^{-4} = h^2. \tag{27}$$

It remains to estimate the bending energy in Ω_2. By virtue of (26):

$$\begin{aligned}|\partial_{xx}u_h|^2 &= n^{-2}\left|\partial_{mm}v\cdot(\tilde m')^2+\partial_m v\cdot\tilde m''\right|^2\\ &\le Cn^{-2}\left(\left(\frac{|\tilde m'|}{(\tilde m-1)^{1/2}}\right)^4\frac{1}{\tilde m-1}+\frac{1}{\tilde m-1}(\tilde m'')^2+\tilde m^2\right)\\ &\le C'n^{-2}\left(\frac{1}{\tilde m-1}+\tilde m^2\right).\end{aligned}$$

Since $\tilde m\ge 1+h^{4/3}$, we have $|\partial_{xx}u_h|^2\le Cn^{-2}h^{-4/3}\le Ch^{-1/3}$. In a similar way we obtain

$$\begin{aligned}|\partial_{xy}u_h|^2\le |m'\partial_{my}v|^2\le C\left(\frac{|\tilde m'|^2}{\tilde m-1}+1\right)\le C',\\ |\partial_{yy}u_h|^2\le n^2|\partial_{yy}v|^2\le Cn^2\tilde m\le C'h^{-1}.\end{aligned}$$

The bending energy $h^2\iint_{\Omega_2}|D^2u_h|^2\,\mathrm{d}x\,\mathrm{d}y$ is therefore bounded by Ch.

Adding together (16), (21), (22), (23), (24), (27), and the last estimate we obtain the desired bound Ch for (15).

Remark. *The dominant terms in the preceding calculation were the $x-y$ component of the membrane energy, $2\left|(\partial_x u_h,\partial_y u_h)\right|^2$, and the $y-y$ component of the bending energy, $h^2|\partial_{yy}u_h|^2$. The former was of order n^{-2} (see* (15) *and* (24)*) while the latter was of order h^2n^2. Minimization of their sum $n^{-2}+h^2n^2$ was what led to the choice $n\sim h^{-1/2}$ and an overall energy of order h. Preparing for the next section, we note that the $x-x$ component of the membrane energy was of order $h^{4/3}$ (see* (15) *and* (16)*), while the $y-y$ component of the membrane energy was of order n^{-4} (see* (15) *and* (27)*).*

3 The upper bound half of Theorem 1

In the last section we constructed a deformation with energy of order h that satisfies the rather rigid condition (8). In this section we show that if (8) is weakened to (6) then a modified construction achieves a substantially smaller energy, of order $h^{4/3}$. The basic idea is this: in the construction of Section 2, the $x-y$ component of the membrane energy was larger than the $x-x$ and $y-y$ components. This was a consequence of the essentially two-dimensional character of the wrinkling permitted by hypothesis (8). To obtain a better construction (one with a smaller energy), we will let $u^x(x,y)$ vary slightly in y (satisfying (6) but not (8)). By taking advantage of this variation we will reduce the energy of the $x-y$ membrane term, making it at most the size of the other membrane terms ($h^{4/3}+n^{-4}$). The dominant bending term will still be $h^2|\partial_{yy}u_h|^2$, which scales like h^2n^2. Since $\min_n\{n^{-4}+h^2n^2\}\sim h^{4/3}$ is achieved when $n\sim h^{-1/3}$, we will get a total energy of order $h^{4/3}$ using wrinkling on scale $h^{1/3}$. We thank Stefan Müller for pointing out that a construction like this is possible.

We will use the auxiliary metric $\tilde m$ from the previous section:

$$\tilde m(x):=\begin{cases}1+\varphi_h^2(x) & x\in[0,h^{2/3}],\\ m(x-h^{2/3})+h^{4/3} & x\in[h^{2/3},L],\end{cases}$$

where $\varphi_h(x):=h^{2/3}\varphi(h^{-2/3}x)$. We look for u_h in the form

$$u_h(x,y):=\left(x+\frac{1}{n^2}w(x,ny),y+\frac{1}{n}\tilde v_1(x,ny),\frac{1}{n}\tilde v_2(x,ny)\right),$$

where $n:=\lfloor h^{-1/3}\rfloor$, and $w,\tilde v$ are 1-periodic functions in y so that

$$\begin{aligned}\left(|\partial_y u_h(x,y)|^2-\tilde m^2(x)\right)^2=\left(\frac{\partial_y w}{n}\right)^4,\\ \partial_x u_h(x,y)\cdot\partial_y u_h(x,y)=\frac{1}{n^3}\partial_x w(x,y)\ \partial_y w(x,y).\end{aligned}\tag{28}$$

This can be achieved by looking for 1-periodic functions $w, \tilde{v}$ which satisfy

$$
\begin{aligned}
|\,(\partial_y \tilde{v}_1(x,y)+1, \partial_y \tilde{v}_2(x,y))\,| &= \tilde{m}(x), \\
\partial_y w(x,y) &= -(\partial_y \tilde{v}(x,y)+e_1)\cdot \partial_x \tilde{v}(x,y).
\end{aligned} \tag{29}
$$

We observe that periodicity (and in fact existence) of w is equivalent to the fact that the integral of the second relation in y vanishes. Hence the above conditions are equivalent to:

$$
|\partial_y \tilde{v}(x,y)+e_1| = \tilde{m}(x), \tag{30}
$$

$$
\int_0^1 (\partial_y \tilde{v}(x,y)+e_1)\cdot \partial_x \tilde{v}(x,y)\,\mathrm{d}y = 0. \tag{31}
$$

Now let

$$
\tilde{v}_1(x,y) := v_1(\tilde{m}(x),y)+a(x), \qquad \tilde{v}_2(x,y) := v_2(\tilde{m}(x),y),
$$

where v_1, v_2 were defined in Lemma 2.1 and the value of $a(x)$ will be clear soon. We see that (30) follows from (11) and the definition of $\tilde{v}$. Condition (31) can be written as

$$
\int_0^1 \left[(\partial_y v_1+1)\partial_m v_1 + \partial_y v_2 \partial_m v_2\right]\partial_x \tilde{m}(x)\,\mathrm{d}y + \partial_x a(x)\int_0^1 (\partial_y v_1+1)\,\mathrm{d}y = 0.
$$

Since v is 1-periodic in y we have $\int_0^1 \partial_y v_1\,\mathrm{d}y = 0$, so the preceding equation is equivalent to

$$
\partial_x a(x) = -\partial_x \tilde{m}(x).\left(\int_0^1 (\partial_y v_1(\tilde{m}(x),y)+1)\partial_m v_1(\tilde{m}(x),y) + \partial_y v_2(\tilde{m}(x),y)\partial_m v_2(\tilde{m}(x),y)\,\mathrm{d}y\right). \tag{32}
$$

We set $a(0)=0$ and use (32) to define it for $x>0$. Then (28) is automatically satisfied.

Now we define w and check that u_h satisfies the boundary conditions (5). Equation (29) defines w up to a function of x. Let us therefore set $w(x,0)=0$. Since $\tilde{m}(0)=1$, we have that $v_1(0,y)=v_2(0,y)=w(0,y)=0$ and so the boundary conditions (5) are satisfied.

It remains to estimate the energy $E_h(u_h)$. First we show smoothness of a by computing the terms on the RHS of (32). It follows from (12) that

$$
\begin{aligned}
(\partial_y v_1(\tilde{m},y)+1)\partial_m v_1(\tilde{m},y) &= ((\tilde{m}-1)\partial_y \phi(\tilde{m},y)+1)(\phi(\tilde{m},y)+(\tilde{m}-1)\partial_m \phi) \\
&= \Phi(\tilde{m},y),
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_y v_2(\tilde{m},y)\partial_m v_2(\tilde{m},y) &= \sqrt{\tilde{m}-1}\,\partial_y \psi(\tilde{m},y)\Big(\psi(\tilde{m},y)/2\sqrt{\tilde{m}-1}+\sqrt{\tilde{m}-1}\,\partial_m \psi\Big) \\
&= \partial_y \psi(\tilde{m},y)\,(\psi(\tilde{m},y)/2+(\tilde{m}-1)\partial_m \psi) = \Psi(\tilde{m},y),
\end{aligned}
$$

where Φ and Ψ are smooth functions of m and y. Then by virtue of (32) $a(x)$ is a smooth function of the metric $\tilde{m}(x)$. Similarly, it follows from (29) that

$$
w(x,y) = \Theta(\tilde{m}(x),y)\partial_x \tilde{m}(x),
$$

where $\Theta(m,y)$ is a smooth function of m and y. Indeed, using that $\partial_x a$ is a (smooth in $\tilde{m}$) multiple of $\partial_x \tilde{m}$ (see (32)) we have

$$
-\partial_y w(x,y) = (\partial_y v_1+1)(\partial_m v_1 \partial_x \tilde{m}(x)+\partial_x a(x)) + \partial_y v_2 \partial_m v_2 \partial_x \tilde{m}(x) = -\partial_y \Theta(\tilde{m}(x),y)\partial_x \tilde{m}(x),
$$

for a suitably defined function Θ. (Note that on the left hand side v_1 and v_2 are evaluated at $(\tilde{m}(x),y)$.)

Before estimating the energy we observe that since w is bounded (uniformly in h) and $n=\lfloor h^{-1/3}\rfloor$, $|u_h^x - x| = n^{-2}|w| \le Ch^{2/3}$ and so u_h satisfies (6).

We are now ready to estimate the elastic energy. We will see that many terms are identical with the terms in (15), and so our main work will be to estimate the new terms involving w or a. By virtue of (28):

$$
\begin{aligned}
E_h(u_h) = &\iint_\Omega \left| |\partial_x u_h|^2 - 1 \right|^2 + 2\,|(\partial_x u_h, \partial_y u_h)|^2 \\
&+ \left| |\partial_y u_h|^2 - m^2(x) \right|^2 + h^2 \left| D^2 u_h(x,y) \right|^2 \, \mathrm{d}x\,\mathrm{d}y \\
\leq & C \iint_\Omega \left\{ \left[\frac{|\tilde{m}'|}{n} |\partial_m v| \right]^4 + \left(\frac{|\partial_x w|^2}{n^4} + \frac{|\partial_x w|^4}{n^8} + \frac{|a'|^4}{n^4} \right) \right\} \\
&+ 2 \left\{ \frac{1}{n^6} |\partial_x w \cdot \partial_y w|^2 \right\} + \left\{ \left[\left| \tilde{m}^2(x) - m^2(x) \right|^2 \right] + \left(\frac{\partial_y w}{n} \right)^4 \right\} \\
&+ h^2 \left(\left\{ \left| \frac{\partial_{xx} w}{n^2} \right|^2 + \frac{|\partial_{xx} a|^2}{n^2} + \left[\frac{1}{n^2} \left| \partial_{xx} (v_1, v_2)(\tilde{m}(x), ny) \right|^2 \right] \right\} \right. \\
&\qquad + \left\{ \left| \frac{\partial_{xy} w}{n} \right|^2 + \left[\left| \partial_{xy} (v_1, v_2)(\tilde{m}(x), ny) \right|^2 \right] \right\} \\
&\qquad \left. + |\partial_{yy} w|^2 + \left[n^2 \left| \partial_{yy} (v_1, v_2)(\tilde{m}(x), ny) \right|^2 \right] \right) \mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{33}
$$

The terms in square brackets are identical with some terms in (15) (they can be estimated the same way as before), while the other terms are new. Using the estimates done in Section 2 with $n = \lfloor h^{-1/3} \rfloor$ we estimate the terms in square brackets by $C(n^{-4} + h^2 n^2) = C' h^{4/3}$.

To finish the proof we need to estimate the remaining terms. Since $m \in \mathcal{C}^2 \cap W^{3,2}$ and w is one derivative less smooth than m, we immediately see that all the integrals involving m are finite and have prefactors at least $n^{-4} = h^{4/3}$ or h^2. Finally, a is a smooth function of m and the term involving a has prefactor $n^{-4} = h^{4/3}$. Hence we see that all the terms in (33) are bounded by $Ch^{4/3}$, which concludes the proof.

4 The lower bound half of Theorem 1

In this section we prove the lower bound assertion of Theorem 1. In outline, the argument has four main steps:

STEP 1: *If a deformation u has small energy, then its out-of-plane component u^z must be small in L^2.* Explaining briefly: if the membrane term were uniformly small, then each line segment $[0, L] \times \{y\}$ would not be stretched much. Since (6) gives $u^x(0, y) \approx 0$ and $u^x(L, y) \approx L$, the smallness of u^z along this line segment would follow from Pythagoras' theorem. In practice the membrane term is only small on average, so the proof given below uses an integrated form of this argument.

STEP 2: *If u has small energy then the deformation is almost planar, in the sense that Du^z is small in L^2.* Indeed, if the bending energy is small then we have an estimate for $D^2 u^z$ in L^2. The smallness of Du^z follows, by interpolation and using the conclusion of Step 1.

STEP 3: *If $u = (u^x, u^y, u^z)$ has small energy, then the associated planar deformation $(u^x, u^y, 0)$ also has small energy.* Indeed, setting $u^z = 0$ reduces the bending energy; if Du^z is small then it does not increase the membrane term very much.

STEP 4: *A planar deformation cannot have small energy.* This is of course expected: since $m(x) > 1$, each line segment $\{x\} \times [0, 1]$ should have to wrinkle out-of-plane to accommodate the prescribed metric and boundary conditions.

We shall actually prove a little more than the lower bound assertion of Theorem 1: we will show the existence of a constant $C_0 > 0$ such that

$$\int_\Omega \left|Du(x,y)^T Du(x,y) - g(x)\right|^2 + h^2|D^2u^z(x,y)|^2 \geq C_0 h^{4/3} \tag{34}$$

whenever u which satisfies (5), (6), (7) and $h < 1$. This is stronger than the assertion of Theorem 1, because the bending term of (34) involves only u^z.

It is sufficient to prove (34) when h is sufficiently small, i.e. for $h < h_0$. Indeed, given such a result, we have for $h_0 \leq h \leq 1$ that

$$\begin{aligned}\int_\Omega &\left|Du(x,y)^T Du(x,y) - g(x)\right|^2 + h^2|D^2u^z(x,y)|^2 \\ &\geq \int_\Omega \left|Du(x,y)^T Du(x,y) - g(x)\right|^2 + h_0^2|D^2u^z(x,y)|^2 \geq C_0 h_0^{4/3} = C_0\left(\frac{h_0}{h}\right)^{4/3} h^{4/3} = \widetilde{C}_0 h^{4/3}\end{aligned}$$

with $\widetilde{C}_0 = C_0 h_0^{-4/3}$.

To show that (34) holds when h and C_0 are sufficiently small, we may argue by contradiction. Our task is thus to show that if $\delta \leq 1$ is sufficiently small, and if $h < h_0(\delta)$, then the existence of u satisfying (5), (6), (7), and

$$\int_\Omega \left|Du(x,y)^T Du(x,y) - g(x)\right|^2 + h^2|D^2u^z(x,y)|^2 \leq \delta h^{4/3} \tag{35}$$

leads to a contradiction.

Pursuing the plan indicated at the beginning of the section, our first goal is to estimate the L^2 norm of u^z. From (35) we see immediately that

$$\begin{aligned}\int_\Omega (\partial_x u^x)^2 - 1 + (\partial_x u^y)^2 + (\partial_x u^z)^2 &\leq |\Omega|^{1/2}\left(\int_\Omega \left((\partial_x u^x)^2 - 1 + (\partial_x u^y)^2 + (\partial_x u^z)^2\right)^2\right)^{1/2} \\ &\leq |\Omega|^{1/2}(\delta h^{4/3})^{1/2} \leq C\delta^{1/2}h^{2/3}.\end{aligned} \tag{36}$$

For $x = L$ and any $y \in [0,1]$ condition (6) implies

$$(L - \alpha h^{2/3}) \leq u^x(L,y) = \int_0^L \partial_x u^x(t,y)\,\mathrm{d}t \leq L^{1/2}\left(\int_0^L (\partial_x u^x)^2\right)^{1/2},$$

and so

$$L^2 - 2\alpha L h^{2/3} \leq (L - \alpha h^{2/3})^2 \leq L\int_0^L (\partial_x u^x)^2.$$

By dividing both sides by L and by moving L under the integral we get $\int_0^L (\partial_x u^x)^2 - 1\,\mathrm{d}x \geq -2\alpha h^{2/3}$, whence (integrating in y)

$$\int_\Omega (\partial_x u^x)^2 - 1 \geq -2\alpha h^{2/3}.$$

We combine this relation with (36) to get

$$\int_\Omega (\partial_x u^y)^2 + (\partial_x u^z)^2 \leq C\delta^{1/2}h^{2/3} + 2\alpha h^{2/3} \leq C' h^{2/3}.$$

For any $x_0 \in [0, L]$ we see from the previous estimate that

$$\begin{aligned}\int_0^1 (u^z(x_0,y))^2\,\mathrm{d}y &= \int_0^1 \left(u^z(x_0,y) - u^z(0,y)\right)^2\,\mathrm{d}y \\ &\leq \int_0^1\left(\int_0^{x_0}\partial_x u^z\,\mathrm{d}x\right)^2\mathrm{d}y \leq x_0\int_\Omega (\partial_x u^z)^2\,\mathrm{d}x\,\mathrm{d}y \leq Ch^{2/3}.\end{aligned}$$

Integrating the previous relation in x_0 then implies

$$\int_\Omega (u^z)^2 \, \mathrm{d}x \, \mathrm{d}y \leq C h^{2/3}.$$

Step 1 is now complete.

Our next goal is an estimate for Du^z. From (35), the bending energy $\int_\Omega |D^2 u^z|^2$ is smaller than $\delta h^{-2/3}$. Using the interpolation inequality

$$\|Df\|^2_{L^2(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)} \|D^2 f\|_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)} \right)$$

with $f = u^z$, we conclude that

$$\int_\Omega |Du^z|^2 \leq C \left(\delta^{1/2} + h^{2/3} \right) \leq 2C\delta^{1/2} \tag{37}$$

provided $h^{2/3} \leq \delta^{1/2}$. This completes Step 2.

Our next task is to assess the energy of the associated planar deformation $\overline{u} = (u^x, u^y, 0)$, which means estimating $|D\overline{u}^T D\overline{u} - g|^2$. For the first entry we have:

$$((\partial_x \overline{u})^2 - g_{11})^2 = ((\partial_x u^x)^2 + (\partial_x u^y)^2 - g_{11})^2 = ((\partial_x u)^2 - g_{11} - (\partial_x u^z)^2)^2 \leq 2((\partial_x u)^2 - g_{11})^2 + 2(\partial_x u^z)^4. \tag{38}$$

To proceed we will use the following lemma:

Lemma 4.1. *If*

$$\int_\Omega |\partial_x u^z|^2 \, \mathrm{d}x \, \mathrm{d}y \leq \epsilon_1 \quad \textit{and} \quad \int_\Omega |(\partial_x u)^2 - g_{11}|^2 \, \mathrm{d}x \, \mathrm{d}y \leq \epsilon_2$$

then

$$\int_\Omega |\partial_x u^z|^4 \, \mathrm{d}x \, \mathrm{d}y \leq (1 + 2\|g_{11}\|^2_{L^\infty})\epsilon_1 + 2\epsilon_2.$$

Proof. We define a set

$$M := \{(x, y) \in \Omega : |\partial_x u^z(x, y)| \geq 1\}$$

and observe that

$$|M| \leq \epsilon_1.$$

We know that pointwise

$$(\partial_x u^z)^4 \leq (\partial_x u)^4 = \left((\partial_x u)^2 - g_{11} + g_{11}\right)^2 \leq 2 \left[\left((\partial_x u)^2 - g_{11}\right)^2 + g_{11}^2 \right].$$

Therefore we have

$$\begin{aligned}
\int_\Omega |\partial_x u^z|^4 \, \mathrm{d}x \, \mathrm{d}y &= \int_{M^c} |\partial_x u^z|^4 \, \mathrm{d}x \, \mathrm{d}y + \int_M |\partial_x u^z|^4 \, \mathrm{d}x \, \mathrm{d}y \leq \epsilon_1 + \int_M |\partial_x u^z|^4 \, \mathrm{d}x \, \mathrm{d}y \\
&\leq \epsilon_1 + 2 \left(\int_M \left((\partial_x u)^2 - g_{11}\right)^2 + g_{11}^2 \, \mathrm{d}x \, \mathrm{d}y \right) \leq \epsilon_1 + 2\epsilon_2 + 2|M| \|g_{11}\|^2_{L^\infty} \leq (1 + 2\|g_{11}\|^2_{L^\infty})\epsilon_1 + 2\epsilon_2.
\end{aligned}$$

□

Using Lemma 4.1 together with (37) and (35) (with $\epsilon_1 = C\delta^{1/2}$ and $\epsilon_2 = \delta h^{4/3}$) we see that

$$\int_\Omega |\partial_x u^z|^4 \leq C\delta^{1/2} + 2\delta h^{4/3},$$

and so in conjunction with (38) we obtain

$$\int_\Omega \left|(\partial_x \overline{u})^2 - g_{11}\right|^2 \leq 2\int_\Omega \left|(\partial_x u)^2 - g_{11}\right|^2 + C\delta^{1/2} + 4\delta h^{4/3}.$$

The other two entries in $|D\overline{u}^T D\overline{u} - g|^2$ can be estimated in a similar way. Then using $\delta \leq 1$ and $h \leq 1$ we obtain

$$\int_\Omega |D\overline{u}^T D\overline{u} - g|^2 \, \mathrm{d}x \, \mathrm{d}y \leq C(\delta^{1/2} + \delta h^{4/3}) \leq C'\delta^{1/2}. \tag{39}$$

Step 3 is now complete.

Our final task is to show that (if δ and h are sufficiently small) these estimates lead to a contradiction. To this end, we choose a rectangle ω where $m(x)$ is not close to 1; since $m > 1$ in Ω by hypothesis, we may (and shall) take $\omega := [L/2, L] \times [0,1]$). We have

$$\int_\omega \det D\overline{u} = \int_\omega \partial_x(u^x \partial_y u^y) - \partial_y(u^x \partial_x u^y) = \int_0^1 u^x \partial_y u^y \, dy \Big|_{x=L/2}^{x=L}$$

since u^x and $u^y - y$ are periodic in y. Combining this with (6) and the nonnegativity of $\partial_y u^y$ (from (7)), we find that

$$\int_\omega \det D\overline{u} \leq L/2 + 2\alpha h^{2/3}. \tag{40}$$

We now use a different argument to show that the area of $\overline{u}(\omega)$ has to be larger. From (7) we have $\det D\overline{u} + \det\sqrt{g} \geq 1$, so

$$\int_\omega |\det D\overline{u} - \det\sqrt{g}| \leq \int_\omega |(\det D\overline{u} - \det\sqrt{g})(\det D\overline{u} + \det\sqrt{g})\,| = \int_\omega \left|(\det D\overline{u})^2 - \det g\right|. \tag{41}$$

It is easy to verify the following simple lemma:

Lemma 4.2. *There exists a constant $C > 0$ such that for any two matrices $A, B \in \mathbb{R}^{2\times 2}$*

$$|\det A - \det B| \leq C\left(|A - B| + |A - B|^2\right).$$

Using the lemma we get

$$|(\det D\overline{u})^2 - \det g| \leq C\left(|D\overline{u}^T D\overline{u} - g| + |D\overline{u}^T D\overline{u} - g|^2\right),$$

and so by using (41) and (39) together with Hölder's inequality we obtain

$$\int_\omega |\det D\overline{u} - \det\sqrt{g}| \leq C\left(\delta^{1/2} + \delta^{1/4}\right) \leq C'\delta^{1/4},$$

and therefore

$$\int_\omega \det D\overline{u} \geq \int_\omega m(x) - C\delta^{1/4}. \tag{42}$$

Since $\min_{L/2\leq x\leq L} m(x) > 1$ and $|\omega| = L/2$, $\int_\omega m(x)$ is strictly greater than $L/2$, so (40) and (42) contradict one another when δ is sufficiently small and $h < h_0(\delta)$. This contradiction completes the proof of the lower bound half of Theorem 1.

5 The lower bound half of Theorem 2

The goal of this section is to prove the lower bound asserted by Theorem 2. Thus we must show that under the assumptions of Theorem 2, for any deformation u which satisfies (5), (8), (9), and (10),

$$E_h(u) \geq C_0 h, \tag{43}$$

where $C_0 > 0$ doesn't depend on u or h.

To prove the lower bound (43) let us assume the contrary. Then for any $\delta > 0$ there exist h and an admissible u such that

$$E_h(u) \leq \delta h. \tag{44}$$

In fact, we should write u_δ and h_δ instead of u and h. Nevertheless, to simplify the notation we will not use δ as a subscript. Since both terms in the energy are non-negative, relation (44) implies

$$\iint_\Omega \left|Du^T Du - g\right|^2 \,\mathrm{d}x\,\mathrm{d}y \leq \delta h \tag{45}$$

$$\iint_\Omega \left|D^2 u\right|^2 \,\mathrm{d}x\,\mathrm{d}y \leq \frac{\delta}{h}. \tag{46}$$

Under the assumptions (8), (9) and (10), we can write

$$\partial_y u(x,y) = (m(x) + \epsilon(x,y))\,(0, \cos(\theta(x,y)), \sin(\theta(x,y)))$$

and

$$\partial_x u(x,y) = \alpha(x,y)\partial_y u(x,y) + \beta(x,y)\partial_y^\perp u(x,y) + (1,0,0), \tag{47}$$

where $\partial_y^\perp u = (0, -\partial_y u^z, \partial_y u^y)$ and $\theta(x,y) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Using these two relations we get

$$\begin{aligned}
\partial_x(\partial_y u) &= \frac{\partial_x(m+\epsilon)}{m+\epsilon}\,\partial_y u + \partial_y^\perp u\,\partial_x\theta \\
\partial_{yy} u &= \frac{\partial_y \epsilon}{m+\epsilon}\,\partial_y u + \partial_y^\perp u\,\partial_y\theta \\
\partial_y(\partial_x u) &= \partial_y\alpha\,\partial_y u + \alpha\,\partial_{yy}u + \partial_y\beta\,\partial_y^\perp u + \beta\,\partial_y(\partial_y^\perp u) = \\
&= \partial_y\alpha\partial_y u + \alpha\frac{\partial_y\epsilon}{m+\epsilon}\partial_y u + \alpha\partial_y^\perp u\partial_y\theta + \partial_y\beta\partial_y^\perp u + \beta\frac{\partial_y\epsilon}{m+\epsilon}\partial_y^\perp u - \beta\partial_y u\partial_y\theta.
\end{aligned} \tag{48}$$

Collecting terms in the relation $\partial_x\partial_y u = \partial_y\partial_x u$, we obtain

$$\frac{\partial_x(m+\epsilon)}{m+\epsilon} = \partial_y\alpha + \alpha\,\frac{\partial_y\epsilon}{m+\epsilon} - \beta\,\partial_y\theta \tag{49}$$

$$\partial_x\theta = \alpha\,\partial_y\theta + \partial_y\beta + \beta\,\frac{\partial_y\epsilon}{m+\epsilon}. \tag{50}$$

Let x be fixed and integrate (49) with respect to y over the interval $I=(0,1)$:

$$
\begin{aligned}
\alpha(x,1)-\alpha(x,0) &= \int_I \partial_y\alpha \,\mathrm{d}y \\
&= \int_I \frac{\partial_x(m+\epsilon)}{m+\epsilon}\,\mathrm{d}y - \int_I \alpha\frac{\partial_y\epsilon}{m+\epsilon}\,\mathrm{d}y + \int_I \beta\,\partial_y\theta\,\mathrm{d}y \\
&= \int_I \frac{\partial_x(m+\epsilon)}{m+\epsilon}\,\mathrm{d}y - \int_I \alpha\frac{\partial_y\epsilon}{m+\epsilon}\,\mathrm{d}y + (\beta\,\theta)\,|_I - \int_I \partial_y\beta\,\theta\,\mathrm{d}y \\
&\overset{(50)}{=} \int_I \frac{\partial_x(m+\epsilon)}{m+\epsilon}\,\mathrm{d}y - \int_I \alpha\frac{\partial_y\epsilon}{m+\epsilon}\,\mathrm{d}y + (\beta\,\theta)\,|_I \\
&\qquad - \int_I \theta\partial_x\theta\,\mathrm{d}y + \int_I \alpha\,\theta\partial_y\theta\,\mathrm{d}y + \int_I \beta\frac{\partial_y\epsilon}{m+\epsilon}\theta\,\mathrm{d}y \\
&= \partial_x\left(\int_I \ln(m+\epsilon)\,\mathrm{d}y\right) - \frac{1}{2}\partial_x\left(\int_I |\theta|^2\,\mathrm{d}y\right) + R,
\end{aligned}
\tag{51}
$$

where $R=-\int_I \alpha\frac{\partial_y\epsilon}{m+\epsilon}\,\mathrm{d}y+\int_I \alpha\,\theta\partial_y\theta\,\mathrm{d}y+\int_I \beta\,\theta\frac{\partial_y\epsilon}{m+\epsilon}\,\mathrm{d}y$.

The last equality in (51) holds because the boundary term $\beta\,\theta|_I$ disappears. Indeed, by (5) both $\partial_x u$ and $\partial_y u$ are 1-periodic in y. Hence θ and $\beta=\partial_x u\partial_y^\perp u/|\partial_y u|^2$ are also 1-periodic in y, which implies $(\beta\theta)|_I=0$.

Since

$$\max_{x\in[0,L]} m(x) > 1.546264$$

and m is continuous, there exists an interval (l_0,l_1) in which $m>1.546264$. Using (45) we can find $X\in(l_0,l_1)$ such that

$$\int_0^1 \left||\partial_y u(X,y)|^2-m^2(X)\right|^2\,\mathrm{d}y \le \frac{\delta h}{l_1-l_0}.$$

Set $M:=\frac{\delta h}{l_1-l_0}$. Now we integrate equation (51) with respect to x over the interval $[0,X]$ to get

$$
\begin{aligned}
\int_0^X \alpha(x,1)-\alpha(x,0)\,\mathrm{d}x &= \int_I \left(\ln(m+\epsilon)(X,y)-\ln(m+\epsilon)(0,y)\right)\,\mathrm{d}y \\
&\quad -\frac{1}{2}\int_I \left(|\theta(X,y)|^2-|\theta(0,y)|^2\right)\,\mathrm{d}y + \int_0^X R\,\mathrm{d}x.
\end{aligned}
$$

Since $u(0,y)=(0,y,0)$, we have that $\ln(m+\epsilon)(0,y)=0$ and $\theta(0,y)=0$. Therefore the relation simplifies to

$$\int_0^X \alpha(x,1)-\alpha(x,0)\,\mathrm{d}x = \int_0^1 \ln(m+\epsilon)(X,y)\,\mathrm{d}y - \frac{1}{2}\int_0^1 |\theta(X,y)|^2\,\mathrm{d}y + \int_0^X R\,\mathrm{d}x. \tag{52}$$

We need to show smallness of the remainder term

$$\int_0^X R\,\mathrm{d}x = -\iint_{\Omega'} \alpha\frac{\partial_y\epsilon}{m+\epsilon}\,\mathrm{d}x\,\mathrm{d}y + \iint_{\Omega'} \alpha\,\theta\partial_y\theta\,\mathrm{d}x\,\mathrm{d}y + \iint_{\Omega'} \beta\,\theta\frac{\partial_y\epsilon}{m+\epsilon}\,\mathrm{d}x\,\mathrm{d}y, \tag{53}$$

where $\Omega'=[0,X]\times[0,1]$.

Lemma 5.1. *Under the assumptions of Theorem 2 we have*

$$\left|\int_0^X R\,\mathrm{d}x\right| \le C\delta.$$

Proof. Multiplying (47) by $\partial_y u$ we get

$$\alpha\,(m+\epsilon)^2 = (\partial_x u, \partial_y u).$$

Then the first term in (53) reads:

$$\begin{aligned}
\left|\iint_{\Omega'} \alpha \frac{\partial_y \epsilon}{m+\epsilon}\,\mathrm{d}x\,\mathrm{d}y\right| &= \left|\iint_{\Omega'} (\partial_x u, \partial_y u)\frac{\partial_y \epsilon}{(m+\epsilon)^3}\,\mathrm{d}x\,\mathrm{d}y\right| \\
&= \left|\frac{1}{2}\iint_{\Omega'} (\partial_x u, \partial_y u)\partial_y\left(\frac{1}{(m+\epsilon)^2} - \frac{1}{m^2}\right)\,\mathrm{d}x\,\mathrm{d}y\right| \\
&\overset{\text{using b.c.}}{=} \left|\frac{1}{2}\iint_{\Omega'} \partial_y\left(\partial_x u, \partial_y u\right)\left(\frac{1}{(m+\epsilon)^2} - \frac{1}{m^2}\right)\,\mathrm{d}x\,\mathrm{d}y\right| \\
&\leq \frac{1}{2}\left(\iint_{\Omega'} |\partial_y(\partial_x u, \partial_y u)|^2\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}\left(\iint_{\Omega'}\left|\frac{2m\epsilon+\epsilon^2}{(m+\epsilon)^2 m^2}\right|^2\mathrm{d}x\,\mathrm{d}y\right)^{1/2}. \qquad (54)
\end{aligned}$$

We use $\partial_y(\partial_x u, \partial_y u) = (\partial_{xy} u, \partial_y u) + (\partial_x u, \partial_{yy} u)$ to rewrite the first term in (54). We have

$$\iint_{\Omega'} |(\partial_{xy} u, \partial_y u)|^2\,\mathrm{d}x\,\mathrm{d}y \leq ||\partial_y u||^2_{L^\infty}\iint_{\Omega'} |\partial_{xy} u|^2\,\mathrm{d}x\,\mathrm{d}y \leq C\frac{\delta}{h},$$

where we used (9) and (46), and similarly $\iint |(\partial_x u, \partial_{yy} u)|^2\,\mathrm{d}x\,\mathrm{d}y \leq C\frac{\delta}{h}$. From (45) we get

$$\iint_{\Omega'} \left|(m+\epsilon)^2 - m^2\right|^2 = \iint_{\Omega'} \left||\partial_y u|^2 - m^2\right|^2 \leq \delta h$$

and so

$$\iint_{\Omega'} \left|\frac{2m\epsilon+\epsilon^2}{(m+\epsilon)^2 m^2}\right|^2 \leq C\iint_{\Omega'} \left|(m+\epsilon)^2 - m^2\right|^2 \leq C\delta h,$$

where we used that $m^2 \geq 1$ and $m+\epsilon = |\partial_y u| \geq 1/\kappa$. Using these estimates we get from (54) that

$$\left|\iint_{\Omega'} \alpha\frac{\partial_y \epsilon}{m+\epsilon}\,\mathrm{d}x\,\mathrm{d}y\right| \leq \sqrt{C\frac{\delta}{h}}\sqrt{C\delta h} = C\delta.$$

From

$$|\partial_y \theta| = \left|\frac{(\partial_{yy} u, \partial_y^\perp u)}{(m+\epsilon)^2}\right| \leq C|\partial_{yy} u|$$

we obtain for the middle term in (53)

$$\left|\iint_{\Omega'} \alpha\,\theta\partial_y\theta\right| \leq C\left(\iint_{\Omega'} |\alpha|^2\right)^{1/2}\frac{\pi}{2}\left(\iint_{\Omega'} |\partial_{yy} u|^2\right)^{1/2} \overset{(46)}{\leq} C\sqrt{\delta h}\sqrt{\frac{\delta}{h}} = C\delta,$$

where we used

$$\iint_{\Omega'} |\alpha|^2 = \iint_{\Omega'} \frac{|\partial_x u \partial_y u|^2}{|\partial_y u|^4} \leq \kappa^4\iint_{\Omega'} |\partial_x u \partial_y u|^2 \overset{(45)}{\leq} C\delta h.$$

The last term in (53) can be bound the same way as the first one. Combining these three estimates we get the result. ☐

Now we will show an estimate for the middle term on the RHS of (52):

Lemma 5.2. *Under the assumptions of Theorem 2 we have that*

$$\int_0^1 |\theta(X,y)|^2\,\mathrm{d}y \leq \frac{\pi^2}{4}\left(\frac{m(X)+\sqrt{M}-1}{m(X)}\right).$$

Proof. Since $|\theta| \le \frac{\pi}{2}$, we have that
$$\cos\theta \le 1 - \frac{4}{\pi^2}\theta^2.$$
To see this, we need simply to show that $g(\theta) = 1 - \frac{4}{\pi^2}\theta^2 - \cos\theta \ge 0$. Since this function is even, we can consider only $\theta \ge 0$. We know that $g(0) = g(\pi/2) = 0$. By computing first derivative $g'(\theta) = \sin\theta - \frac{8}{\pi^2}\theta$ we observe that g is increasing on a small interval $(0,\tau)$ (for some $0 < \tau \le \pi/2$) and then becomes decreasing for all $\theta > \tau$. Therefore $g \ge 0$ in $[0, \pi/2]$.

Integrating the proved inequality with respect to y over the interval $(0,1)$ we get that
$$\int \cos\theta \, \mathrm{d}y \le 1 - \frac{4}{\pi^2}\int \theta^2 \, \mathrm{d}y.$$
On the other hand, we know
$$\begin{aligned}
1 &= \int_0^1 \partial_y u^y(X,y)\,\mathrm{d}y = \int_0^1 (m(X)+\epsilon(X,y))\cos\theta(X,y)dy \\
&\le m(X)\int_0^1 \cos\theta(X,y)\,\mathrm{d}y + \int_0^1 |\epsilon(X,y)|\,\mathrm{d}y \\
&\le m(X)\left(1 - \frac{4}{\pi^2}\int_0^1 \theta^2(X,y)\,\mathrm{d}y\right) + \int_0^1 |\epsilon(X,y)|\,\mathrm{d}y \\
&\le m(X)\left(1 - \frac{4}{\pi^2}\int_0^1 \theta^2(X,y)\,\mathrm{d}y\right) + \sqrt{M},
\end{aligned} \tag{55}$$
where we used
$$\left(\int_0^1 |\epsilon|\,\mathrm{d}y\right)^2 \le \int_0^1 |\epsilon|^2\,\mathrm{d}y \le \int_0^1 |\epsilon|^2|2m+\epsilon|^2\,\mathrm{d}y = \int_0^1 |(m+\epsilon)^2 - m^2|^2\,\mathrm{d}y \le M. \tag{56}$$
By manipulating (55) we get the conclusion. □

We would like to use relation (52) to show that $\int_0^X \alpha(x,1) - \alpha(x,0)\,\mathrm{d}x$ has to be of order 1. To achieve this goal, we will use the previous lemmas and the following one:

Lemma 5.3. *Under the hypotheses of Theorem 2 we have*
$$\int_0^1 \ln(m(X)+\epsilon(X,y))\,\mathrm{d}y \ge \ln(m(X)) - \frac{\sqrt{M}}{m(X)}.$$
Proof. We have
$$\int_0^1 |\ln(m(X)+\epsilon(X,y)) - \ln(m(X))|\,\mathrm{d}y = \int_0^1 \left|\ln\left(\frac{m+\epsilon}{m}\right)\right|\,\mathrm{d}y \le \int_0^1 \frac{|\epsilon|}{m}\,\mathrm{d}y \overset{(56)}{\le} \frac{\sqrt{M}}{m}.$$
□

From Lemmas 5.1, 5.2 and 5.3 and (52) it follows that
$$\begin{aligned}
\int_0^X \alpha(x,1) - \alpha(x,0)\,\mathrm{d}x &\ge \ln(m(X)) - \frac{\sqrt{M}}{m(X)} - \frac{\pi^2}{8}\left(1 - \frac{1-\sqrt{M}}{m(X)}\right) - C\delta \\
&\ge \ln(m(X)) - \frac{\pi^2}{8}\left(1 - \frac{1}{m(X)}\right) - C\delta.
\end{aligned}$$

The last expression is strictly greater than 0 provided

$$\ln(m(X)) > \frac{\pi^2}{8}\left(1 - \frac{1}{m(X)}\right)$$

and δ is sufficiently small. This condition depends solely on the value $m(X)$. It is easy to compute that it is true at least for $m(X) \geq 1.54626325$. Assuming this, there exists h sufficiently small such that

$$\int_0^X \alpha(x,1) - \alpha(x,0)\,\mathrm{d}x \geq \tau > 0.$$

To finish the proof of the lower bound half of Theorem 2 we observe that by (5) both $\partial_x u$ and $\partial_y u$ are 1-periodic in y, hence α is also 1-periodic in y and so $\int_0^X \alpha(x,1) - \alpha(x,0)\,\mathrm{d}x = 0$ – a contradiction with the previous estimate.

References

[1] B. Audoly and A. Boudaoud, *'Ruban à godets': an elastic model for ripples in plant leaves*, C.R. Mecanique **330** (2002), 831–836.

[2] ______, *Self-similar structures near boundaries in strained systems*, Phys. Rev. Lett. **91** (2003), 086105.

[3] B. Audoly and Y. Pomeau, *Elasticity and geometry*, Oxford University Press, Oxford, 2010, From hair curls to the non-linear response of shells, With a foreword by John W. Hutchinson. MR 2677203

[4] P. Bella and R. V. Kohn, *Wrinkles as the result of compressive stresses in an annular thin film*, Comm. Pure Appl. Math. **67** (2014), no. 5, 693–747.

[5] H. Ben Belgacem, S. Conti, A. DeSimone, and S. Müller, *Rigorous bounds for the Föppl-von Kármán theory of isotropically compressed plates*, J. Nonlinear Sci. **10** (2000), no. 6, 661–683.

[6] ______, *Energy scaling of compressed elastic films—three-dimensional elasticity and reduced theories*, Arch. Ration. Mech. Anal. **164** (2002), no. 1, 1–37.

[7] J. Brandman, R. V. Kohn, and H.-M. Nguyen, *Energy scaling laws for conically constrained thin elastic sheets*, J. Elasticity **113** (2013), no. 2, 251–264.

[8] E. Cerda and L. Mahadevan, *Geometry and physics of wrinkling*, Phys. Rev. Lett. **90** (2003), no. 7, 074302.

[9] S. Conti and F. Maggi, *Confining thin elastic sheets and folding paper*, Arch. Ration. Mech. Anal. **187** (2008), no. 1, 1–48.

[10] B. Davidovitch, R. D. Schroll, and E. Cerda, *Nonperturbative model for wrinkling in highly bendable sheets*, Phys. Rev. E **85** (2012), 066115.

[11] B. Davidovitch, R. D. Schroll, D. Vella, M. Adda-Bedia, and E. Cerda, *Prototypical model for tensional wrinkling in thin sheets*, Proceedings of the National Academy of Sciences **108** (2011), no. 45, 18227–18232.

[12] J. Gemmer and S. Venkataramani, *Shape selection in non-euclidean plates*, Physica D **240** (2011), 1536–1552.

[13] T. J. Healey, Q. Li, and R-B Cheng, *Wrinkling behavior of highly stretched rectangular elastic films via parametric global bifurcation*, J. Nonlinear Sci. **23** (2013), no. 5, 777–805.

[14] W. Jin and P. Sternberg, *Energy estimates for the von Kármán model of thin-film blistering*, J. Math. Phys. **42** (2001), no. 1, 192–199.

[15] ———, *In-plane displacements in thin-film blistering*, Proc. Roy. Soc. Edinburgh Sect. A **132** (2002), no. 4, 911–930. MR 1926922 (2003f:74020)

[16] T-Y Kim, E. Puntel, and E. Fried, *Numerical study of the wrinkling of a stretched thin sheet*, International Journal of Solids and Structures **49** (2012), no. 5, 771–782.

[17] Y. Klein, S. Venkataramani, and E. Sharon, *Experimental study of shape transitions and energy scaling in thin non-euclidean plates*, Phys. Rev. Lett. **106** (2011), 118303.

[18] M. A. R. Koehl, W. K. Silk, H. Liang, and L. Mahadevan, *How kelp produce blade shapes suited to different flow regimes: A new wrinkle*, Integr. Comp. Biol. **48** (2008), no. 6, 834–851.

[19] N. H. Kuiper, *Isometric and short imbeddings*, Nederl. Akad. Wetensch. Proc. Ser. A. 62 = Indag. Math. **21** (1959), 11–25.

[20] M. Lewicka, L. Mahadevan, and M. R. Pakzad, *The Föppl-von Kármán equations for plates with incompatible strains*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **467** (2011), no. 2126, 402–426. MR 2748099 (2012a:74064)

[21] M. Marder, *The shape of the edge of a leaf*, Foundations of Physics **33** (2003), no. 12, 1743–1768.

[22] M. Marder and N. Papanicolaou, *Geometry and elasticity of strips and flowers*, J. Stat. Phys. **125** (2006), no. 5-6, 1069–1096.

[23] M. Marder, E. Sharon, S. Smith, and B. Roman, *Theory of edges of leaves*, Europhys. Lett. **62** (2003), no. 4, 498–504.

[24] J. Nash, *C^1 isometric imbeddings*, The Annals of Mathematics **60** (1954), no. 3, 383–396.

[25] E. Puntel, L. Deseri, and E. Fried, *Wrinkling of a stretched thin sheet*, Journal of Elasticity **105**, no. 1-2, 137–170.

[26] E. Sharon, M. Marder, and H. L. Swinney, *Leaves, flowers and garbage bags: Making waves*, American Scientist **92** (2004), no. 3, 254.

[27] E. Sharon, B. Roman, M. Marder, G.-S. Shin, and H. L. Swinney, *Mechanics: Buckling cascades in free sheets*, Nature **419** (2002), 579.

[28] E. Sharon, B. Roman, and H. L. Swinney, *Geometrically driven wrinkling observed in free plastic sheets and leaves*, Phys. Rev. E **75** (2007), no. 4, 046211.

[29] S.C. Venkataramani, *Lower bounds for the energy in a crumpled elastic sheet—a minimal ridge*, Nonlinearity **17** (2004), no. 1, 301–312.

[30] T. A. Witten, *Stress focusing in elastic sheets*, Reviews of Modern Physics **79** (2007), no. 2, 643–675.