

Finite Dimensionality of the Global Attractor for the Solutions to 3D Primitive Equations with Viscosity

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Abstract

A new method is presented to prove finiteness of the fractal and Hausdorff dimensions of the global attractor for the strong solutions to the 3D Primitive Equations with viscosity, which is applicable to even more general situations than the recent result of [7] in the sense that it removes *all* extra technical conditions imposed by previous analyses. More specifically, for finiteness of the dimensions of the global attractor, we only need the heat source $Q \in L^2$ which is exactly the condition for the existence of global strong solutions and the existence of the global attractor of these solutions; while the best existing result, which was obtained very recently in [7], still needs the extra condition that $\partial_z Q \in L^2$ for finiteness of the dimensions of the global attractor. Moreover, the new method can be applied to cases with more complicated boundary conditions which present essential difficulties for previous methods.

Keywords: 3D viscous Primitive Equations, global attractor, fractal dimension, Hausdorff dimension, regularity.

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1 Introduction

Given a bounded domain $D \subset \mathbb{R}^2$ with smooth boundary ∂D , we consider the following system of viscous Primitive Equations (PEs) of Geophysical Fluid Dynamics in the cylinder $\Omega = D \times (-h, 0) \subset \mathbb{R}^3$, where h is a positive constant, see e.g. [17] and the references therein:

Conservation of horizontal momentum:

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v + w \frac{\partial v}{\partial z} + \nabla p + f v^\perp + L_1 v = 0;$$

Hydrostatic balance:

$$\partial_z p + \theta = 0;$$

Continuity equation:

$$\nabla \cdot v + \partial_z w = 0;$$

Heat conduction:

$$\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta + w \frac{\partial \theta}{\partial z} + L_2 \theta = Q.$$

The unknowns in the above system of 3D viscous PEs are the fluid velocity field $(v, w) = (v_1, v_2, w) \in \mathbb{R}^3$ with $v = (v_1, v_2)$ and $v^\perp = (-v_2, v_1)$ being horizontal, the temperature θ and the pressure p . The Coriolis rotation frequency $f = f_0(\beta + y)$ in the β -plane approximation and the heat source Q are given. For the issue concerned in this article, Q is assumed to be independent of t . In the above equations and in this article, ∇ and Δ denote the horizontal gradient and Laplacian:

$$\nabla := (\partial_x, \partial_y) \equiv (\partial_1, \partial_2), \quad \Delta := \partial_x^2 + \partial_y^2 \equiv \sum_{i=1}^2 \partial_i^2.$$

The viscosity and the heat diffusion operators L_1 and L_2 are given respectively as follows:

$$L_i := -\nu_i \Delta - \mu_i \frac{\partial^2}{\partial z^2}, \quad i = 1, 2,$$

where the positive constants ν_1, μ_1 are the horizontal and vertical viscosity coefficients and the positive constants ν_2, μ_2 are the horizontal and vertical heat diffusivity coefficients.

The boundary of Ω is partitioned into three parts: $\partial\Omega = \Gamma_u \cup \Gamma_b \cup \Gamma_s$, where

$$\begin{aligned}\Gamma_u &:= \{(x, y, z) \in \overline{\Omega} : z = 0\}, \\ \Gamma_b &:= \{(x, y, z) \in \overline{\Omega} : z = -h\}, \\ \Gamma_s &:= \{(x, y, z) \in \overline{\Omega} : (x, y) \in \partial D\}.\end{aligned}$$

Consider the following boundary conditions of the PEs as in [2] and [6]:

$$\begin{aligned}\text{on } \Gamma_u &: \quad \frac{\partial v}{\partial z} = h\tau, \quad w = 0, \quad \frac{\partial \theta}{\partial z} = -\alpha(\theta - \Theta), \\ \text{on } \Gamma_b &: \quad \frac{\partial v}{\partial z} = 0, \quad w = 0, \quad \frac{\partial \theta}{\partial z} = 0, \\ \text{on } \Gamma_s &: \quad v \cdot n = 0, \quad \frac{\partial v}{\partial n} \times n = 0, \quad \frac{\partial \theta}{\partial n} = 0,\end{aligned}$$

where $\tau(x, y)$ and $\Theta(x, y)$ are respectively the wind stress and typical temperature distribution on the surface of the ocean, n is the normal vector of Γ_s and α is a non-negative constant. The above system of PEs will be solved with suitable initial conditions.

We assume that Q , τ and Θ are independent of time. Notice that results similar to those to be presented here for the autonomous case can still be obtained for the non-autonomous case with proper modifications. For the autonomous case, assuming some natural compatibility conditions on τ and Θ , one can further set $\tau = 0$ and $\Theta = 0$ without losing generality. See [2] for a detailed discussion on this issue.

Setting $\tau = 0$, $\Theta = 0$ and using the fact that

$$\begin{aligned}w(x, y, z, t) &= - \int_{-h}^z \nabla \cdot v(x, y, \xi, t) d\xi, \\ p(x, y, z, t) &= p_s(x, y, t) - \int_{-h}^z \theta(x, y, \xi, t) d\xi,\end{aligned}$$

one obtains the following equivalent formulation of the system of PEs:

$$\begin{aligned}\frac{\partial v}{\partial t} + L_1 v + (v \cdot \nabla)v - \left(\int_{-h}^z \nabla \cdot v(x, y, \xi, t) d\xi \right) \frac{\partial v}{\partial z} \\ + \nabla p_s(x, y, t) - \int_{-h}^z \nabla \theta(x, y, \xi, t) d\xi + f v^\perp = 0.\end{aligned}\tag{1.1}$$

$$\frac{\partial \theta}{\partial t} + L_2 \theta + v \cdot \nabla \theta - \left(\int_{-h}^z \nabla \cdot v(x, y, \xi, t) d\xi \right) \frac{\partial \theta}{\partial z} = Q; \quad (1.2)$$

$$\frac{\partial v}{\partial z} \Big|_{z=0} = \frac{\partial v}{\partial z} \Big|_{z=-h} = 0, \quad v \cdot n \Big|_{\Gamma_s} = 0, \quad \frac{\partial v}{\partial n} \times n \Big|_{\Gamma_s} = 0, \quad (1.3)$$

$$\left(\frac{\partial \theta}{\partial z} + \alpha \theta \right) \Big|_{z=0} = \frac{\partial \theta}{\partial z} \Big|_{z=-h} = 0, \quad \frac{\partial \theta}{\partial n} \Big|_{\Gamma_s} = 0, \quad (1.4)$$

$$v(x, y, z, 0) = v_0(x, y, z), \quad \theta(x, y, z, 0) = \theta_0(x, y, z). \quad (1.5)$$

We remark that the expressions of w and p via integrating the continuity equation and the hydrostatic balance equation were already used in [13] dealing with the Primitive Equations for large scale oceans. See also [12] for a similar treatment of the Primitive Equations for atmosphere.

Notice that the effect of the *salinity* is omitted in the above 3D viscous PEs for brevity of presentation. However, our results in this article are still valid when the effect of salinity is included. Notice also that the right-hand side term of (1.1) is set as 0. This is just for brevity of presentation and it is not technically essential here: if it is replaced by a non-zero given external force $R \in L^2(\Omega)$, the results of this paper are still valid.

To the best of our knowledge, the mathematical framework of the viscous primitive equations for the large scale ocean was first formulated in [13]; the notions of weak and strong solutions were defined and existence of weak solutions was proved. Uniqueness of weak solutions is still unresolved yet. Existence of strong solutions *local in time* and their uniqueness were obtained in [5] and [17]. Existence of strong solutions *global in time* was proved independently in [2] and [9]. See also [10] for dealing with some other boundary conditions. In [6], existence of the global attractor for the strong solutions of the system is proved in the functional space of strong solutions.

This article focus on the study of finiteness of the dimensions of the global attractor for the strong solutions of the system of 3D viscous PEs. This result was previously announced in [6]. For simplicity of discussion, we set the right-hand side of (1.1) as zero. If this term is replaced by a time-independent term R , no essential change is needed to be made on

our analysis. For the main results of this article, one just need to add the assumption that $R \in L^2$.

In a recent work [3], finiteness of the dimensions of the global attractor for the strong solutions for the 3D viscous PEs is obtained under *periodic* boundary conditions. The proof given in [3] of the Ladyzhenskaya squeezing property for the semigroup is based on the fact that, for the case of *periodic* boundary conditions, the L^2 norm of the second order spacial derivatives of the solutions is bounded uniformly in time and uniformly on the global attractor. This can be proved using previous results and analysis for uniform boundedness given in [6] as outlined in [3] and [14], since one can freely integrate by parts without having any boundary term. However, for the case with non-periodic boundary conditions such as given by (1.3) and (1.4), some complicated boundary terms can not be avoided with the integration by parts following the strategy of [3] and [14]. These boundary terms present essential difficulties for *a priori* estimates in the H^2 norm. To deal with these difficulties, a more recent work [7] provides an involved analysis, which proves the existence of a bounded absorbing ball in H^2 and the uniform boundedness of the H^2 norm of the solutions. Indeed, before the work of [7], it was not known whether or not uniform boundedness for the solutions in H^2 is still valid, no matter how smooth the initial data and the right-hand side terms are. As an application of the uniform H^2 boundedness, it is proved in [7] that the Ladyzhenskaya squeezing property of the semigroup is indeed still valid for the solutions on the global attractor in the case of the non-periodic boundary conditions as considered here. Thus, the problem of finiteness of the dimension of the global attractor for the strong solutions of the system (1.1)-(1.5) measured in the space of strong solutions is positively resolved.

Compared with [3] and [14], the analysis of [7] achieves two improvements. Firstly, the new analysis is applicable to both the non-periodic case and the periodic case, while the previous methods do not seem to apply to non-periodic boundary conditions, such as (1.3) and (1.4). Secondly, the new result of [7] requires less demanding conditions than in previous works. More specifically, [7] requires only that $Q, \partial_z Q \in L^2$, instead of requiring

that $Q \in H^1$ and $\partial_z Q \in L^6$ in addition to periodicity as needed in [3]. See Theorem 2.3 in Section 2 of this article for details.

However, recalling the main result of [6] (see e.g. Theorem 2.2 in Section 2 of this article), we notice that $Q \in L^2$ is enough for the global existence of strong solutions and for the existence of the global attractor \mathcal{A} for strong solutions. So, a very natural question out of curiosity is that whether or not the additional condition $\partial_z Q \in L^2$ imposed in Theorem 2.3 is essentially necessary for the dimensions of the global attractor to be finite. Another issue involved with Theorem 2.3 is that the condition $\alpha = 0$ seems indispensable for the boundary conditions in the analysis of [7]. Resolving these questions and difficulties is the main concern of this article.

From the geophysics context and the anisotropic mathematical structure of the 3D Primitives equations, it may seem that the condition $\partial_z Q \in L^2$ might be quite natural in physics and essential in mathematics. A little surprisingly, it will be proved in this article that this condition together with the condition $\alpha = 0$ can indeed be completely dropped out. This main result of our current article, Theorem 5.2, will be proved in Section 5. To achieve our goal, a new way is discovered to prove finiteness of the Hausdorff and fractal dimensions of the global attractor for the strong solutions to the 3D Primitive Equations with viscosity. An interesting aspect is that it does *not* need the uniform boundedness of the H^2 norms of the solutions, as required by previous methods. What will be used instead are some uniform continuity properties for the solutions on the global attractor \mathcal{A} . This new idea helps us to successfully remove the extra conditions $\partial_z Q \in L^2$ and $\alpha = 0$ and thus resolve *completely* the problem of finiteness of dimensions of the global attractor for the strong solutions of the system (1.1)-(1.5). The new approach presented in this paper has its *advantage* over the previous one used in [7], as demonstrated further by the additional example given in Section 6.

The rest of this article is organized as follows:

In Section 2, we give the notations, briefly review the background results and present the problems to be studied and recall some important facts crucial to later analysis. In Section 3, we prove our first main result, Theorem 3.1, on the uniform boundedness of (u_t, θ_t) in L^2 and the existence

of a bounded absorbing ball for (u_t, θ_t) , which will be needed to prove our next second main result Theorem 4.1. In Section 4, we state and prove Theorem 4.1 about some uniform continuity properties for the solutions on the global attractor, which will be crucial for our proof of our final result Theorem 5.2. In Section 5, as an application of the previous results, we finally prove Theorem 5.2, the main result about the finiteness of the Hausdorff and fractal dimensions of the global attractor as measured in V , the space of strong solutions. In Section 6, we briefly mention the case with “physical boundary conditions” (6.1) on the velocity field v , for which the new method used in this paper can be easily applied to obtain same conclusions as for the case with v satisfying the boundary conditions (1.3). However, it seems rather difficult for the approach of [7] to deal with the case of “physical boundary conditions”.

2 Preliminaries

We recall that D is a bounded smooth domain in \mathbb{R}^2 and $\Omega = D \times [0, -h]$, where h is a positive constant. We denote by $L^p(\Omega)$ and $L^p(D)$ ($1 \leq p < +\infty$) the classic L^p spaces with the norms:

$$\|\phi\|_p = \begin{cases} (\int_{\Omega} |\phi(x, y, z)|^p dx dy dz)^{\frac{1}{p}}, & \forall \phi \in L^p(\Omega); \\ (\int_D |\phi(x, y)|^p dx dy)^{\frac{1}{p}}, & \forall \phi \in L^p(D). \end{cases}$$

Denote by $H^m(\Omega)$ and $H^m(D)$ ($m \geq 1$) the classic Sobolev spaces for square-integrable functions with square-integrable derivatives up to order m . We do not distinguish the notations for vector and scalar function spaces, which are self-evident from the context. For simplicity, we may use $d\Omega$ to denote $dx dy dz$ and dD to denote $dx dy$, or we may simply omit them when there is no confusion. Using the Hölder inequality, it is easy to show that, for $\varphi \in L^p(\Omega)$,

$$\|\bar{\varphi}\|_{L^p(\Omega)} = h^{\frac{1}{p}} \|\bar{\varphi}\|_{L^p(D)} \leq \|\varphi\|_p, \quad \forall p \in [1, +\infty], \quad (2.1)$$

where $\bar{\varphi}$ is defined as the vertical average of φ :

$$\bar{\varphi}(x, y) = h^{-1} \int_{-h}^0 \varphi(x, y, z) dz.$$

Define the function spaces H and V as follows:

$$H := H_1 \times H_2 := \{v \in L^2(\Omega)^2 \mid \nabla \cdot \bar{v} = 0, \quad \bar{v} \cdot n|_{\Gamma_s} = 0\} \times L^2(\Omega),$$

$$V := V_1 \times V_2 := \{v \in H^1(\Omega)^2 \mid \nabla \cdot \bar{v} = 0, \quad v \cdot n|_{\Gamma_s} = 0\} \times H^1(\Omega).$$

Define the bilinear forms: $a_i : V_i \times V_i \rightarrow \mathbb{R}$, $i = 1, 2$ as follows:

$$\begin{aligned} a_1(v, u) &= \int_{\Omega} (\nu_1 \nabla v_1 \cdot \nabla u_1 + \nu_1 \nabla v_2 \cdot \nabla u_2 + \mu_1 v_z \cdot u_z) d\Omega; \\ a_2(\theta, \eta) &= \int_{\Omega} (\nu_2 \nabla \theta \cdot \nabla \eta + \mu_2 \theta_z \eta_z) d\Omega + \alpha \int_{\Gamma_u} \theta \eta dx dy. \end{aligned}$$

Let V_i' ($i = 1, 2$) denote the dual space of V_i . We define the linear operators $A_i : V_i \mapsto V_i'$, $i = 1, 2$ as follows:

$$\langle A_1 v, u \rangle = a_1(v, u), \quad \forall v, u \in V_1; \quad \langle A_2 \theta, \eta \rangle = a_2(\theta, \eta), \quad \forall \theta, \eta \in V_2,$$

where $\langle \cdot, \cdot \rangle$ is the corresponding scalar product between V_i' and V_i . We also use $\langle \cdot, \cdot \rangle$ to denote the inner products in H_1 and H_2 . Define:

$$D(A_i) = \{\phi \in V_i, A_i \phi \in H_i\}, \quad i = 1, 2.$$

Since A_i^{-1} is a self-adjoint compact operator in H_i , by the classic spectral theory, the power A_i^s can be defined for any $s \in \mathbb{R}$. Then $D(A_i)' = D(A_i^{-1})$ is the dual space of $D(A_i)$ and $V_i = D(A_i^{\frac{1}{2}})$, $V_i' = D(A_i^{-\frac{1}{2}})$. Moreover,

$$D(A_i) \subset V_i \subset H_i \subset V_i' \subset D(A_i)',$$

where the embeddings above are all compact. Define the norm $\|\cdot\|_{V_i}$ by:

$$\|\cdot\|_{V_i}^2 = a_i(\cdot, \cdot) = \langle A_i \cdot, \cdot \rangle = \left\langle A_i^{\frac{1}{2}} \cdot, A_i^{\frac{1}{2}} \cdot \right\rangle, \quad i = 1, 2.$$

The Poincaré inequalities are valid. There is a constant $c > 0$, such that for any $\phi = (\phi_1, \phi_2) \in V_1$ and $\psi \in V_2$

$$c\|\phi\|_2 \leq \|\phi\|_{V_1}, \quad c\|\psi\|_2 \leq \|\psi\|_{V_2}. \quad (2.2)$$

Therefore, there exist constants $c > 0$ and $C > 0$ such that for any $\phi = (\phi_1, \phi_2) \in V_1$ and $\psi \in V_2$,

$$c\|\phi\|_{V_1} \leq \|\phi\|_{H^1(\Omega)} \leq C\|\phi\|_{V_1}, \quad c\|\psi\|_{V_2} \leq \|\psi\|_{H^1(\Omega)} \leq C\|\psi\|_{V_2}.$$

Notice that, in the above first inequality, we have written $\|\phi\|_{H^1(\Omega)}$ instead of $\|\phi\|_{H^1(\Omega)^2}$. We could also simply write $\|\phi\|_{H^1}$. Here and later on as well, we do not distinguish the notations for vector and scalar function spaces which are self-evident from the context. In this article, we use c and C to denote generic positive constants, the values of which may vary from one place to another.

Recall the following definitions of weak and strong solutions:

Definition 2.1 *Suppose $Q \in L^2(\Omega)$, $(v_0, \theta) \in H$ and $T > 0$. The pair (v, θ) is called a weak solution of the 3D viscous PEs (1.1)-(1.5) on the time interval $[0, T]$ if it satisfies (1.1)-(1.2) in the weak sense, and also*

$$(v, \theta) \in C([0, T]; H) \cap L^2(0, T; V), \quad \partial_t(v, \theta) \in L^1(0, T; V').$$

Moreover, if $(v_0, \theta_0) \in V$, a weak solution (v, θ) is called a strong solution of (1.1)-(1.5) on the time interval $[0, T]$ if, in addition, it satisfies

$$(v, \theta) \in C([0, T]; V) \cap L^2(0, T; D(A_1) \times D(A_2)).$$

The following theorem on global existence and uniqueness for the strong solutions was proved in [2]. See also a related result in [9].

Theorem 2.1 *Suppose $Q \in H^1(\Omega)$. Then, for every $(v_0, \theta_0) \in V$ and $T > 0$, there exists a unique strong solution (v, θ) on $[0, T]$ to the system of 3D viscous PEs, which depends on the initial data continuously in H .*

Remark 2.1 *It is easy to see from the proof of Theorem 2.1 given in [2] that the condition $Q \in H^1(\Omega)$ can be relaxed to $Q \in L^6(\Omega)$. Notice that there are gaps between Definition 2.1 and Theorem 2.1 for the condition on Q , for the continuity of the strong solution with respect to time and for the continuous dependence of the strong solution with respect to initial data.*

We now recall the following result proven in [6] for the existence of global attractor \mathcal{A} for the strong solutions of the 3D viscous PEs (1.1)-(1.5).

Theorem 2.2 *Suppose that $Q \in L^2(\Omega)$ is independent of time. Then the solution operator $\{S(t)\}_{t \geq 0}$ of the 3D viscous PEs (1.1)-(1.5): $S(t)(v_0, \theta_0) =$*

$(v(t), \theta(t))$ defines a semigroup in the space V for $t \in \mathbb{R}_+$. Moreover, the following statements are valid:

1. For any $(v_0, \theta_0) \in V$, $t \mapsto S(t)(v_0, \theta_0)$ is continuous from \mathbb{R}_+ into V .
2. For any $t > 0$, $S(t)$ is a continuous and compact map in V .
3. $\{S(t)\}_{t \geq 0}$ possesses a global attractor \mathcal{A} in the space V . The global attractor \mathcal{A} is compact and connected in V and it is the minimal bounded attractor in V in the sense of the set inclusion relation; \mathcal{A} attracts all bounded subsets of V in the norm of V .

Recall also the following result proved in [7] for finiteness of the Hausdorff and fractal dimensions of the global attractor \mathcal{A} as obtained in Theorem 2.2:

Theorem 2.3 *Suppose $\alpha = 0$ and $Q, Q_z \in L^2(\Omega)$. Then the global attractor \mathcal{A} has finite Hausdorff and fractal dimensions measured in the V space.*

The main goal of this article is to prove finiteness of the Hausdorff and fractal dimensions of the global attractor \mathcal{A} in the space V for any $\alpha \geq 0$ and for any $Q \in L^2(\Omega)$, thus dropping all the *extra* assumptions imposed in Theorem 2.3. The formal statement of this main result of this article, Theorem 5.2, and its proof will be presented in Section 5.

We recall the following lemma which will be of critical usefulness for the *a priori* estimates in the following sections. See [1], and also [6], for a proof.

Lemma 2.1 *Suppose that $\nabla v, \varphi \in H^1(\Omega)$, $\psi \in L^2(\Omega)$. Then, there exists a constant $C > 0$ independent of v, φ, ψ and h , such that*

$$\left| \left\langle \left(\int_{-h}^z \nabla \cdot v(x, y, \xi) d\xi \right) \varphi, \psi \right\rangle \right| \leq C \|\nabla v\|_2^{\frac{1}{2}} \|\nabla v\|_{H^1}^{\frac{1}{2}} \|\varphi\|_2^{\frac{1}{2}} \|\varphi\|_{H^1}^{\frac{1}{2}} \|\psi\|_2.$$

Similarly, one can prove the following Lemma 2.2, which will be of critical usefulness in proving our first main result Theorem 3.1.

Lemma 2.2 *Suppose that $\nabla v, \varphi, \nabla \varphi, \psi, \nabla \psi \in L^2(\Omega)$. Then, there exists a constant $C > 0$ independent of v, φ, ψ and h , such that*

$$\left| \left\langle \left(\int_{-h}^z \nabla \cdot v(x, y, \xi) d\xi \right) \varphi, \psi \right\rangle \right| \leq C \|\nabla v\|_2 \|\varphi\|_2^{\frac{1}{2}} \|\nabla \varphi\|_2^{\frac{1}{2}} \|\psi\|_2^{\frac{1}{2}} \|\nabla \psi\|_2^{\frac{1}{2}}.$$

We recall the following formulation of the uniform Gronwall lemma, the proof of which can be found in [15].

Lemma 2.3 (Uniform Gronwall Lemma) *Let g , h and y be three non-negative locally integrable functions on $(t_0, +\infty)$ such that*

$$\frac{dy}{dt} \leq gy + h, \quad \forall t \geq t_0,$$

and

$$\int_t^{t+r} g(s)ds \leq a_1, \quad \int_t^{t+r} h(s)ds \leq a_2, \quad \int_t^{t+r} y(s)ds \leq a_3, \quad \forall t \geq t_0,$$

where r , a_1 , a_2 and a_3 are positive constants. Then

$$y(t+r) \leq \left(\frac{a_3}{r} + a_2 \right) e^{a_1}, \quad \forall t \geq t_0.$$

3 The bounded absorbing ball for $(\partial_t v, \partial_t \theta)$ in the space $H_1 \times H_2$

The existence of a bounded absorbing ball for $(\partial_t v, \partial_t \theta)$ in the space $H_1 \times H_2$, and the uniform boundedness of $\|\partial_t v\|_2$ and $\|\partial_t \theta\|_2$ for $t \in [0, +\infty)$, has been proved in [7] for the case with $\alpha = 0$ and $Q, \partial_z Q \in L^2$. In the following, we prove the same result for the case $\alpha \geq 0$ under the condition that $Q \in L_2$ only. Notice that the extra condition $\partial_z Q \in L^2$ is not needed in the following Theorem 3.1, which is our first main result. It will be used for the proof of our final main result of this article.

Theorem 3.1 *Suppose $Q \in L^2(\Omega)$ and $\alpha \geq 0$.*

For any $(v_0, \theta_0) \in V_1 \times V_2$ and $(\partial_t v(0), \partial_t \theta(0)) \in H_1 \times H_2$, there exists a unique solution (v, θ) of (1.1)-(1.5) such that

$$(\partial_t v, \partial_t \theta) \in L^\infty(0, +\infty; H_1 \times H_2).$$

Moreover, there exists a bounded absorbing ball for $(\partial_t v, \partial_t \theta)$ in the space of $H_1 \times H_2$.

Proof:

We prove the case $\alpha > 0$. The case $\alpha = 0$ is similar.

We only need to prove the existence of a bounded absorbing ball for $(\partial_t v, \partial_t \theta)$ in the space of $H_1 \times H_2$. The uniform boundedness of $\|\partial_t v\|_2, \|\partial_t \theta\|_2$ for t in $[0, +\infty)$ will then follow easily and the uniqueness of the solution is obvious.

Denote

$$u := v_t = \partial_t v, \quad \zeta := \theta_t = \partial_t \theta.$$

Notice that u and ζ in the proof of Theorem 3.1 are different from those in the proof of Theorem 5.2 of Section 5.

The proof of Theorem 3.1 is divided into two steps.

Step 1. We prove that the time average of $\|(v_t, \theta_t)\|_2^2$ is uniformly bounded with respect to t .

Taking the inner product of (1.1) with v_t and using the boundary conditions (1.3), we find

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v\|_{V_1}^2 + \|u\|_2^2 &= -\langle v \cdot \nabla v, u \rangle - \langle w v_z, u \rangle \\ &\quad + \left\langle \int_{-h}^z \nabla \theta(x, y, \xi, t) d\xi, u \right\rangle - \langle f v^\perp, u \rangle, \end{aligned} \quad (3.1)$$

where we have used the following calculations

$$\begin{aligned} \int_{\Omega} v_t \cdot \Delta v &= \int_{-h}^0 \left[\int_{\partial D} v_t \cdot \frac{\partial v}{\partial n} - \frac{1}{2} \int_D \partial_t (|\nabla v|^2) \right] = -\frac{1}{2} \frac{d}{dt} \|\nabla v\|_2^2, \\ \int_{\Omega} v_t \cdot \partial_z^2 v &= \int_D \left[v_t \cdot v_z \Big|_{z=-h}^0 - \int_{-h}^0 \partial_t (v_z) \cdot v_z \right] = -\frac{1}{2} \frac{d}{dt} \|v_z\|_2^2, \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega} \nabla p_s(x, y, t) \cdot v_t &= \int_{-h}^0 \left[\int_{\partial D} p(x, y, t) v_t \cdot n - \int_D p_s(x, y, t) \nabla \cdot v_t \right] \\ &= - \int_D p_s(x, y, t) \left(\int_{-h}^0 \nabla \cdot v(x, y, z, t) dz \right)_t dx dy \\ &= \int_D p_s(x, y, t) w_t(x, y, 0, t) dx dy = 0, \end{aligned}$$

since w satisfies the boundary condition $w(x, y, 0, t) = 0$ for $(x, y) \in D$.

From (3.1), we easily obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|v\|_{V_1}^2 + \|u\|_2^2 \\
& \leq \|v\|_6 \|\nabla v\|_3 \|u\|_2 + \|wv_z\|_2 \|u\|_2 + C(\|\nabla\theta\|_2 + \|v\|_2) \|u\|_2 \\
& \leq C_\varepsilon (\|v\|_{H^1}^2 \|\nabla v\|_2 \|\nabla v\|_{H^1} + \|wv_z\|_2^2 + \|\nabla\theta\|_2^2 + \|v\|_2^2) + \varepsilon \|u\|_2^2 \\
& \leq C_\varepsilon (\|v\|_{H^1}^2 \|\nabla v\|_2 \|\nabla v\|_{H^1} + \|\nabla v\|_2 \|\nabla v\|_{H^1} \|v_z\|_2 \|v_z\|_{H^1}) \\
& \quad + C_\varepsilon (\|\nabla\theta\|_2^2 + \|v\|_2^2) + \varepsilon \|u\|_2^2,
\end{aligned}$$

where in the last inequality above, Lemma 2.1 is used. Choosing $\varepsilon = \frac{1}{2}$, we obtain

$$\frac{d}{dt} \|v\|_{V_1}^2 + \|u\|_2^2 \leq Ch_1(t), \quad (3.2)$$

with

$$h_1(t) := \|v\|_{H^1}^2 \|\nabla v\|_2 \|\nabla v\|_{H^1} + \|\nabla v\|_2 \|\nabla v\|_{H^1} \|v_z\|_2 \|v_z\|_{H^1} + \|\nabla\theta\|_2^2 + \|v\|_2^2.$$

Taking the inner product of (1.2) with θ_t and using (1.4) yields

$$\begin{aligned}
& \frac{\nu_2}{2} \frac{d}{dt} \|\nabla\theta\|_2^2 + \frac{\mu_2}{2} \frac{d}{dt} \|\theta_z\|_2^2 + \frac{\mu_2\alpha}{2} \frac{d}{dt} \|\theta(z=0)\|_2^2 + \|\zeta\|_2^2 \\
& = -\langle v \cdot \nabla\theta, \zeta \rangle - \langle w\theta_z, \zeta \rangle + \langle Q, \zeta \rangle,
\end{aligned} \quad (3.3)$$

where we have used the following calculations:

$$\begin{aligned}
& \int_\Omega \theta_t \Delta\theta = \int_{-h}^0 \left[\int_{\partial D} \theta_t \frac{\partial\theta}{\partial n} - \frac{1}{2} \int_D \partial_t (|\nabla\theta|^2) \right] = -\frac{1}{2} \frac{d}{dt} \|\nabla\theta\|_2^2, \\
& \int_\Omega \theta_t \partial_z^2 \theta = \int_D \left[\theta_t \theta_z \Big|_{z=-h}^0 - \int_{-h}^0 \partial_t (\theta_z) \theta_z \right] = -\alpha \frac{d}{dt} \|\theta(z=0)\|_2^2 - \frac{1}{2} \frac{d}{dt} \|\theta_z\|_2^2.
\end{aligned}$$

By (3.3), we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\theta\|_{V_2}^2 + \|\zeta\|_2^2 \\
& \leq \|v\|_6 \|\nabla\theta\|_3 \|\zeta\|_2 + \|w\theta_z\|_2 \|\zeta\|_2 + \|Q\|_2 \|\zeta\|_2 \\
& \leq C(\|v\|_6^2 \|\nabla\theta\|_3^2 + \|w\theta_z\|_2^2 + \|Q\|_2^2) + \frac{1}{2} \|\zeta\|_2^2 \\
& \leq C(\|v\|_{H^1}^2 \|\nabla\theta\|_2 \|\nabla\theta\|_{H^1} + \|\nabla v\|_2 \|\nabla v\|_{H^1} \|\theta_z\|_2 \|\theta_z\|_{H^1} + \|Q\|_2^2) + \frac{1}{2} \|\zeta\|_2^2,
\end{aligned}$$

where in the last inequality we have used Lemma 2.1. Therefore,

$$\frac{d}{dt} \|\theta\|_{V_2}^2 + \|\zeta\|_2^2 \leq Ch_2(t), \quad (3.4)$$

with

$$h_2(t) := \|v\|_{H^1}^2 \|\nabla\theta\|_2 \|\nabla\theta\|_{H^1} + \|\nabla v\|_2 \|\nabla v\|_{H^1} \|\theta_z\|_2 \|\theta_z\|_{H^1} + \|Q\|_2^2.$$

Integrating (3.2) and (3.4) with respect to t yields

$$\|v(t+1)\|_{V_1}^2 + \int_t^{t+1} \|u(\tau)\|_2^2 d\tau \leq \|v(t)\|_{V_1}^2 + C \int_t^{t+1} h_1(\tau) d\tau, \quad (3.5)$$

$$\|\theta(t+1)\|_{V_2}^2 + \int_t^{t+1} \|\zeta(\tau)\|_2^2 d\tau \leq \|\theta(t)\|_{V_2}^2 + C \int_t^{t+1} h_2(\tau) d\tau. \quad (3.6)$$

Notice that the previous uniform *a priori* estimates in [6] yield uniform boundedness of

$$\|v(t)\|_{V_1}, \quad \|\theta(t)\|_{V_2}, \quad \int_t^{t+1} h_1(\tau) d\tau \quad \text{and} \quad \int_t^{t+1} h_2(\tau) d\tau,$$

with respect to $t > 0$ and a bounded absorbing set in \mathbb{R}_+ for each of the above four terms. With these uniform estimates, we can conclude Step 1 from (3.5) and (3.6).

Step 2. We now prove the existence of a bounded absorbing ball for (v_t, θ_t) in H and the uniform boundedness of $\|(v_t, \theta_t)\|_2$ for $t \in [0, +\infty)$.

By (1.1) and (1.2), we have

$$\begin{aligned} u_t + L_1 u + (u \cdot \nabla)v + (v \cdot \nabla)u + w_t v_z + w u_z \\ + \nabla(p_s)_t - \int_{-h}^z \nabla\zeta(x, y, \xi, t) d\xi + f u^\perp = 0, \end{aligned} \quad (3.7)$$

$$\zeta_t + L_2 \zeta + u \cdot \nabla\theta + v \cdot \nabla\zeta + w_t \theta_z + w \zeta_z = 0. \quad (3.8)$$

Taking the inner product of (3.7) with u and using the boundary conditions (1.3) and (1.4), we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|u\|_2^2 + \nu_1 \|\nabla u\|_2^2 + \mu_1 \|u_z\|_2^2 \\
&= - \langle (u \cdot \nabla)v, u \rangle - \langle (v \cdot \nabla)u, u \rangle - \langle w_t v_z, u \rangle - \langle w u_z, u \rangle \\
&\quad - \langle \nabla(p_s)_t, u \rangle + \left\langle \int_{-h}^z \nabla \zeta, u \right\rangle - \langle f u^\perp, u \rangle \\
&= - \langle (u \cdot \nabla)v, u \rangle - \langle w_t v_z, u \rangle + \left\langle \int_{-h}^z \nabla \zeta, u \right\rangle =: I_1 + I_2 + I_3,
\end{aligned} \tag{3.9}$$

where we have used the fact that

$$\langle \nabla(p_s)_t, u \rangle = \langle f u^\perp, u \rangle = \langle (v \cdot \nabla)u, u \rangle + \langle w u_z, u \rangle = 0.$$

Taking the inner product of (3.8) with ζ and using (1.4), we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\zeta\|_2^2 + \nu_2 \|\nabla \zeta\|_2^2 + \mu_2 \|\zeta_z\|_2^2 + \mu_2 \alpha \|\zeta(z=0)\|_2^2 \\
&= - \langle u \cdot \nabla \theta, \zeta \rangle - \langle v \cdot \nabla \zeta, \zeta \rangle - \langle w_t \theta_z, \zeta \rangle - \langle w \zeta_z, \zeta \rangle \\
&= - \langle u \cdot \nabla \theta, \zeta \rangle - \langle w_t \theta_z, \zeta \rangle =: J_1 + J_2,
\end{aligned} \tag{3.10}$$

where we have used the fact that

$$\langle v \cdot \nabla \zeta, \zeta \rangle + \langle w \zeta_z, \zeta \rangle = 0.$$

Notice that first,

$$\begin{aligned}
|I_1| &\leq \|\nabla v\|_2 \|u\|_4^2 \leq C \|\nabla v\|_2 \|u\|_2^{\frac{1}{2}} \|u\|_{H^1}^{\frac{3}{2}} \\
&\leq C \|\nabla v\|_2 \|u\|_2^{\frac{1}{2}} (\|u\|_2^{\frac{3}{2}} + \|\nabla u\|_2^{\frac{3}{2}} + \|u_z\|_2^{\frac{3}{2}}) \\
&\leq (C \|\nabla v\|_2 + C_\varepsilon \|\nabla v\|_2^4) \|u\|_2^2 + \varepsilon (\|\nabla u\|_2^2 + \|u_z\|_2^2),
\end{aligned}$$

and similarly,

$$\begin{aligned}
|J_1| &\leq \|\nabla \theta\|_2 \|u\|_4 \|\zeta\|_4 \leq \frac{1}{2} \|\nabla \theta\|_2 (\|u\|_4^2 + \|\zeta\|_4^2) \\
&\leq (C \|\nabla \theta\|_2 + C_\varepsilon \|\nabla \theta\|_2^4) (\|u\|_2^2 + \|\zeta\|_2^2) \\
&\quad + \varepsilon (\|\nabla u\|_2^2 + \|u_z\|_2^2 + \|\nabla \zeta\|_2^2 + \|\zeta_z\|_2^2).
\end{aligned}$$

Secondly, by Lemma 2.2, we have

$$\begin{aligned}
|I_2| &= \left| \left\langle \left(\int_{-h}^z \nabla \cdot u \right) v_z, u \right\rangle \right| \\
&\leq C \|\nabla u\|_2 \|v_z\|_2^{\frac{1}{2}} \|\nabla v_z\|_2^{\frac{1}{2}} \|u\|_2^{\frac{1}{2}} \|\nabla u\|_2^{\frac{1}{2}} \\
&= C \|v_z\|_2^{\frac{1}{2}} \|\nabla v_z\|_2^{\frac{1}{2}} \|u\|_2^{\frac{1}{2}} \|\nabla u\|_2^{\frac{3}{2}} \\
&\leq C_\varepsilon \|v_z\|_2^2 \|\nabla v_z\|_2^2 \|u\|_2^2 + \varepsilon \|\nabla u\|_2^2,
\end{aligned}$$

and similarly,

$$\begin{aligned}
|J_2| &= \left| \left\langle \left(\int_{-h}^z \nabla \cdot u \right) \theta_z, \zeta \right\rangle \right| \\
&\leq C \|\nabla u\|_2 \|\theta_z\|_2^{\frac{1}{2}} \|\nabla \theta_z\|_2^{\frac{1}{2}} \|\zeta\|_2^{\frac{1}{2}} \|\nabla \zeta\|_2^{\frac{1}{2}} \\
&\leq C_\varepsilon \|\theta_z\|_2^2 \|\nabla \theta_z\|_2^2 \|\zeta\|_2^2 + \varepsilon \|\nabla \zeta\|_2^2 + \varepsilon \|\nabla u\|_2^2.
\end{aligned}$$

Finally, we find

$$\begin{aligned}
|I_3| &= \left| \left\langle \int_{-h}^z \nabla \zeta, u \right\rangle \right| = \left| \left\langle \int_{-h}^z \zeta, \nabla \cdot u \right\rangle \right| \\
&\leq \|\zeta\|_2 \|\nabla u\|_2 \leq C_\varepsilon \|\zeta\|_2^2 + \varepsilon \|\nabla u\|_2^2.
\end{aligned}$$

Now, inserting the above estimates on the I_i 's and J_i 's into (3.9) and (3.10) and choosing $\varepsilon > 0$ sufficiently small, we obtain

$$y'(t) + \gamma(\|u\|_{V_1}^2 + \|\nabla_3 \zeta\|_{V_2}^2) \leq Cg(t)y(t), \quad (3.11)$$

where $\gamma = \min\{\nu_1, \nu_2, \mu_1, \mu_2\}$ and

$$\begin{aligned}
y(t) &:= \|u(t)\|_2^2 + \|\zeta(t)\|_2^2 \\
g(t) &:= 1 + \|\nabla v\|_2^4 + \|\nabla \theta\|_2^4 + \|v_z\|_2^2 \|\nabla v_z\|_2^2 + \|\theta_z\|_2^2 \|\nabla \theta_z\|_2^2.
\end{aligned}$$

Now, we can apply the Uniform Gronwall Lemma to (3.11) and using the previous *a priori* uniform estimates proved in [6] and in Step 1, we obtain the existence of a bounded absorbing set for y in \mathbb{R}_+ and the uniform boundedness of $y(t)$ for $t \in \mathbb{R}_+$. This finishes Step 2.

□

4 Uniform Continuity on the Global Attractor

In this section, as an application of Theorem 3.1, we prove the following theorem about a few important uniform continuity properties for the solutions on the global attractor, which will be very crucial in the analysis in Section 5. Notice that for simplicity of presentation, we do not formulate the theorem in its most complete or sharpest form. We just present the version of the results sufficient for our purpose in proving our final main result, Theorem 5.2.

Theorem 4.1 *Suppose $Q \in L^2$ and $\alpha \geq 0$.*

Let $(v_0, \theta_0) \in \mathcal{A}$. Then there is a constant $C > 0$, which is independent of (v_0, θ_0) and $T > 0$, such that

$$\| \|v(T)\|_{V_1}^2 - \|v(0)\|_{V_1}^2 \| \leq CT^{\frac{1}{2}}, \quad \| \|\theta(T)\|_{V_2}^2 - \|\theta(0)\|_{V_2}^2 \| \leq CT^{\frac{1}{2}},$$

$$\int_0^T (\|v_z(t)\|_{V_1}^2 + \|\nabla v(t)\|_{V_1}^2) dt \leq C(T^{\frac{1}{2}} + T),$$

and

$$\int_0^T (\|\Delta\theta(t)\|_2^2 + \|\theta_z(t)\|_{V_2}^2) dt \leq C(T^{\frac{1}{2}} + T).$$

Proof:

It should be pointed out that it is proved in [6] that, for any $(v_0, \theta_0) \in V_1 \times V_2$, the above two integrals are indeed uniformly bounded with respect to T , and if $(v_0, \theta_0) \in \mathcal{A}$ the bounds of the above two integrals are independent of both $T > 0$ and initial data. What we are interested to here is that, as $T \rightarrow 0+$, these integrals are not only continuous with respect to T , i.e. they go to zero as T goes to 0; they are indeed *uniformly* continuous with respect to T and *initial data*.

Notice first that, by Theorem 2.2 and a well known lemma in [16], we have

$$\begin{aligned} \|v(T)\|_{V_1}^2 - \|v(0)\|_{V_1}^2 &= \|A_1^{\frac{1}{2}}v(T)\|_2^2 - \|A_1^{\frac{1}{2}}v(0)\|_2^2 \\ &= 2 \int_0^T \left\langle (A_1^{\frac{1}{2}}v)_t(t), A_1^{\frac{1}{2}}v(t) \right\rangle dt \\ &= 2 \int_0^T \langle v_t(t), A_1v(t) \rangle dt. \end{aligned}$$

Therefore,

$$\begin{aligned}
|\|v(T)\|_{\dot{V}_1}^2 - \|v(0)\|_{\dot{V}_1}^2| &\leq 2 \int_0^T \|v_t(t)\|_2 \|A_1 v(t)\|_2 dt \\
&\leq 2 \sup_{t \in (0, T)} \|v_t(t)\|_2 \int_0^T \|A_1 v(t)\|_2 dt \\
&\leq 2 \sup_{t \in (0, T)} \|v_t(t)\|_2 T^{\frac{1}{2}} \left(\int_0^T \|A_1 v(t)\|_2^2 dt \right)^{\frac{1}{2}} \\
&\leq CT^{\frac{1}{2}}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
|\|\theta(T)\|_{\dot{V}_2}^2 - \|\theta(0)\|_{\dot{V}_2}^2| &\leq 2 \sup_{t \in (0, T)} \|\theta_t(t)\|_2 T^{\frac{1}{2}} \left(\int_0^T \|A_2 \theta(t)\|_2^2 dt \right)^{\frac{1}{2}} \\
&\leq CT^{\frac{1}{2}}.
\end{aligned}$$

Next, from [2], we know that

$$\frac{d}{dt} \|v_z\|_2^2 + \nu_1 \|\nabla v_z\|^2 + \nu \|v_{zz}\|_2^2 \leq C(\|v\|_6^4 \|v_z\|_2^2 + \|\theta\|_2^2).$$

Therefore,

$$\begin{aligned}
\int_0^T \|v_z(t)\|_{\dot{V}_1}^2 dt &\leq |\|v_z(T)\|_2^2 - \|v_z(0)\|_2^2| \\
&\quad + C \int_0^T (\|v(t)\|_6^4 \|v_z(t)\|_2^2 + \|\theta(t)\|_2^2) dt \\
&\leq CT^{\frac{1}{2}} + CT.
\end{aligned}$$

Similarly, from [2], we know that

$$\frac{d}{dt} \|\nabla v\|_2^2 + \|\nabla v\|_{\dot{V}_1}^2 \leq C(\|v\|_6^4 \|\nabla v\|_2^2 + \|\nabla v_z\|_2^2 \|v_z\|_2^2 \|\nabla v\|_2^2 + \|\nabla \theta\|_2^2)$$

Note that in [2], the second term on the right-hand side of the above inequality is incorrectly written with $\|\nabla v_z\|_2^2$ being replaced by $\|\nabla v\|_2^2$.

Therefore,

$$\begin{aligned}
\int_0^T \|\nabla v(t)\|_{V_1}^2 dt &\leq \left| \|\nabla v(T)\|_2^2 - \|\nabla v(0)\|_2^2 \right| \\
&\quad + C \int_0^T (\|v\|_6^4 \|\nabla v\|_2^2 + \|\nabla v_z\|_2^2 \|v_z\|_2^2 \|\nabla v\|_2^2 + \|\nabla \theta\|_2^2) dt \\
&\leq CT^{\frac{1}{2}} + CT + C \int_0^T \|\nabla v_z\|_2^2 dt \\
&\leq CT^{\frac{1}{2}} + CT.
\end{aligned}$$

Finally, by (1.2) and (1.4), we have

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} (\|\theta_z\|_2^2 + \|\nabla \theta\|_2^2 + \alpha \|\theta(z=0)\|_2^2) \\
&\quad + \nu_2 \|\Delta \theta\|_2^2 + (\nu_2 + \mu_2) \|\nabla \theta_z\|_2^2 + \mu_2 \|\theta_{zz}\|_2^2 \\
&\quad + \alpha (\nu_2 + \mu_2) \|\nabla \theta(z=0)\|_2^2 \\
&= \langle v \cdot \nabla \theta, \Delta \theta + \theta_{zz} \rangle + \langle w \partial_z \theta, \Delta \theta + \theta_{zz} \rangle + \langle Q, \Delta \theta + \theta_{zz} \rangle \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

We estimate the I_i 's, $i = 1, 2, 3$, as following.

$$\begin{aligned}
|I_1| &\leq \|v\|_6 \|\nabla \theta\|_3 (\|\Delta \theta\|_2 + \|\theta_{zz}\|_2) \\
&\leq C \|v\|_6^2 \|\nabla \theta\|_3^2 + \varepsilon (\|\Delta \theta\|_2^2 + \|\theta_{zz}\|_2^2) \\
&\leq C \|v\|_6^2 \|\nabla \theta\|_2 \|\nabla \theta\|_{V_2} + \varepsilon (\|\Delta \theta\|_2^2 + \|\theta_{zz}\|_2^2).
\end{aligned}$$

By Lemma 2.1,

$$\begin{aligned}
|I_2| &\leq C \|\nabla v\|_2^{\frac{1}{2}} \|\nabla^2 v\|_2^{\frac{1}{2}} \|\partial_z \theta\|_2^{\frac{1}{2}} \|\nabla \partial_z \theta\|_2^{\frac{1}{2}} (\|\Delta \theta\|_2 + \|\theta_{zz}\|_2) \\
&\leq C \|\nabla v\|_2 \|\nabla^2 v\|_2 \|\partial_z \theta\|_2 \|\nabla \partial_z \theta\|_2 + \varepsilon (\|\Delta \theta\|_2^2 + \|\theta_{zz}\|_2^2) \\
&\leq C \|\nabla v\|_2^2 \|\partial_z \theta\|_2^2 \|\nabla^2 v\|_2^2 + \varepsilon (\|\nabla \partial_z \theta\|_2^2 + \|\Delta \theta\|_2^2 + \|\theta_{zz}\|_2^2).
\end{aligned}$$

Finally,

$$|I_3| \leq \|Q\|_2 (\|\Delta \theta\|_2 + \|\theta_{zz}\|_2) \leq C \|Q\|_2^2 + \varepsilon (\|\Delta \theta\|_2^2 + \|\theta_{zz}\|_2^2).$$

Hence,

$$\begin{aligned}
&\frac{d}{dt} \|\theta\|_{V_2}^2 + \|\Delta \theta\|_2^2 + \|\theta_z\|_{V_2}^2 \\
&\leq C (\|v\|_6^2 \|\nabla \theta\|_2 \|\nabla \theta\|_{V_2} + \|\nabla v\|_2^2 \|\theta_z\|_2^2 \|\nabla^2 v\|_2^2 + \|Q\|_2^2)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_0^T (\|\Delta\theta\|_2^2 + \|\theta_z\|_{V_2}^2) &\leq \left| \|\theta(T)\|_{V_2}^2 - \|\theta(0)\|_{V_2}^2 \right| + C\|Q\|_2^2 T \\
&\quad + C \int_0^T \|\nabla\theta\|_{V_2} dt + C \int_0^T \|\nabla^2 v\|_2^2 dt \\
&\leq CT^{\frac{1}{2}} + CT + CT^{\frac{1}{2}} \left(\int_0^T \|\nabla\theta\|_{V_2}^2 dt \right)^{\frac{1}{2}} \\
&\quad + C \int_0^T \|\nabla v\|_{V_1}^2 dt \\
&\leq C(T^{\frac{1}{2}} + T).
\end{aligned}$$

□

5 Dimensions of the Global Attractor

Recall the following result due to Ladyzhenskaya, see [11].

Theorem 5.1 *Let X be a Hilbert space, $S : X \mapsto X$ be a map and $\mathcal{A} \subset X$ be a compact set such that $S(\mathcal{A}) = \mathcal{A}$. Suppose that there exist $l \in [1, +\infty)$ and $\delta \in (0, 1)$, such that $\forall a_1, a_2 \in \mathcal{A}$,*

$$\begin{aligned}
\|S(a_1) - S(a_2)\|_X &\leq l\|a_1 - a_2\|_X, \\
\|Q_N[S(a_1) - S(a_2)]\|_X &\leq \delta\|a_1 - a_2\|_X,
\end{aligned}$$

where Q_N is the projection in X onto some subspace $(X_N)^\perp$ of co-dimension $N \in \mathbb{N}$. Then

$$d_H(\mathcal{A}) \leq d_F(\mathcal{A}) \leq N \frac{\ln\left(\frac{8Ga^2l^2}{1-\delta^2}\right)}{\ln\left(\frac{2}{1+\delta^2}\right)},$$

where $d_H(\mathcal{A})$ and $d_F(\mathcal{A})$ are the Hausdorff and fractal dimensions of \mathcal{A} respectively and Ga is the Gauss constant:

$$Ga := \frac{1}{2\pi} \beta\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{2}{\pi} \int_0^1 \frac{dx}{\sqrt{1-x^4}} = 0.8346268\dots$$

Now we use Theorem 5.1 to prove our main result of this paper:

Theorem 5.2 *Suppose $Q \in L^2(\Omega)$ and $\alpha \geq 0$. Then the global attractor \mathcal{A} has finite Hausdorff and fractal dimensions measured in the V space.*

Remark 5.1 *It is proved in [7] that for $Q, \partial_z Q \in L^2(\Omega)$ and $\alpha = 0$, the global attractor \mathcal{A} has finite Hausdorff and fractal dimensions measured in the V space. In the proof of [7], H^2 uniform boundedness on the global attractor is used. In the following, we give a different proof which does not use H^2 uniform boundedness. Thus, it extends the result of finite dimensionality of \mathcal{A} to the more general cases.*

Proof:

Suppose $(v^{(i)}, \theta^{(i)})$, $i = 1, 2$, are two strong solutions to the 3D viscous PEs, with the initial data $(v_0^{(i)}, \theta_0^{(i)}) \in \mathcal{A}$, where \mathcal{A} is the global attractor obtained in Theorem 2.2. Without losing generality, we can assume that $(v_0^{(i)}, \theta_0^{(i)}) \in D(A_1) \times D(A_2)$ for $i = 1, 2$. Therefore $\|v_t^i(0)\|_2, \|\theta_t^i(0)\|_2$ are finite for $i = 1, 2$. Since $(v_0^{(i)}, \theta_0^{(i)}) \in \mathcal{A}$, by Theorem 3.1, without losing generality, we can assume that $\|v_t^i(t)\|_2, \|\theta_t^i(t)\|_2$ are uniformly bounded for $t \in [0, +\infty)$, with the bounds uniform on the global attractor \mathcal{A} . Moreover, $(v^{(i)}(t), \theta^{(i)}(t)) \in D(A_1) \times D(A_2)$ for any $t \geq 0$.

Denote:

$$u = v^{(1)} - v^{(2)}, \quad \eta = \theta^{(1)} - \theta^{(2)}, \quad q(x, y, t) = p^{(1)}(x, y, t) - p^{(2)}(x, y, t).$$

Notice that here u is different from that in Section 3.

We have the following equations for u and η :

$$\begin{aligned} & \partial_t u + L_1 u + (u \cdot \nabla) v^{(1)} + (v^{(2)} \cdot \nabla) u \\ & - \left(\int_{-h}^z \nabla \cdot u(x, y, \xi, t) d\xi \right) \partial_z v^{(1)} - \left(\int_{-h}^z \nabla \cdot v^{(2)}(x, y, \xi, t) d\xi \right) \partial_z u \\ & + \nabla q(x, y, t) - \int_{-h}^z \nabla \eta(x, y, \xi, t) d\xi + f u^\perp = 0, \end{aligned}$$

$$\begin{aligned} & \partial_t \eta + L_2 \eta + u \cdot \nabla \theta^{(1)} + v^{(2)} \cdot \nabla \eta \\ & - \left(\int_{-h}^z \nabla \cdot u(x, y, \xi, t) d\xi \right) \partial_z \theta^{(1)} - \left(\int_{-h}^z \nabla \cdot v^{(2)}(x, y, \xi, t) d\xi \right) \partial_z \eta = 0. \end{aligned}$$

Let $\{\lambda_k\}_1^\infty$ be the set of eigenvalues of A_1 and $\{\sigma_k\}_1^\infty$ be the set of eigenvalues of A_2 . Let $P_{1,n}$ be the orthogonal projector in H_1 on the subspace of spanned by the first n eigenvectors associated with $\lambda_1, \dots, \lambda_n$, and $P_{2,n}$ be the orthogonal projector in H_2 on the subspace of H_2 spanned by the first n eigenvectors associated with $\sigma_1, \dots, \sigma_n$. Let $Q_{i,n} = I - P_{i,n}$, $i = 1, 2$. By the classic spectral theory of compact operators, we know that

$$\lim_{n \rightarrow \infty} \lambda_n = \lim_{n \rightarrow \infty} \sigma_n = +\infty.$$

Now, we consider the following equations for $Q_{1,n}u$ and $Q_{2,n}\eta$:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|Q_{1,n}u\|_{V_1}^2 + \|A_1 Q_{1,n}u\|_2^2 \\ &= - \left\langle (u \cdot \nabla)v^{(1)}, A_1 Q_{1,n}u \right\rangle - \left\langle (v^{(2)} \cdot \nabla)u, A_1 Q_{1,n}u \right\rangle \\ & \quad + \left\langle \left(\int_{-h}^z \nabla \cdot u(x, y, \xi, t) d\xi \right) \partial_z v^{(1)}, A_1 Q_{1,n}u \right\rangle \\ & \quad + \left\langle \left(\int_{-h}^z \nabla \cdot v^{(2)}(x, y, \xi, t) d\xi \right) \partial_z u, A_1 Q_{1,n}u \right\rangle \\ & \quad + \left\langle \int_{-h}^z \eta(x, y, \xi, t) d\xi, A_1 Q_{1,n}u \right\rangle - \left\langle fu^\perp, A_1 Q_{1,n}u \right\rangle \\ & \equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned} \tag{5.1}$$

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|Q_{2,n}\eta\|_{V_2}^2 + \|A_2 Q_{2,n}\eta\|_2^2 \\ &= - \left\langle (u \cdot \nabla)\theta^{(1)}, A_2 Q_{2,n}\eta \right\rangle - \left\langle (v^{(2)} \cdot \nabla)\eta, A_2 Q_{2,n}\eta \right\rangle \\ & \quad + \left\langle \left(\int_{-h}^z \nabla \cdot u(x, y, \xi, t) d\xi \right) \partial_z \theta^{(1)}, A_2 Q_{2,n}\eta \right\rangle \\ & \quad + \left\langle \left(\int_{-h}^z \nabla \cdot v^{(2)}(x, y, \xi, t) d\xi \right) \partial_z \eta, A_2 Q_{2,n}\eta \right\rangle \\ & \equiv J_1 + J_2 + J_3 + J_4 \end{aligned} \tag{5.2}$$

First, we estimate the right-hand side of (5.1) term by term.

$$\begin{aligned} I_1 &\leq \|u\|_6 \|\nabla v^{(1)}\|_3 \|A_1 Q_{1,n}u\|_2 \\ &\leq C \|u\|_{V_1} \|v^{(1)}\|_{V_1}^{\frac{1}{2}} \|A_1 v^{(1)}\|_2^{\frac{1}{2}} \|A_1 Q_{1,n}u\|_2 \\ &\leq C \|u\|_{V_1}^2 \|v^{(1)}\|_{V_1} \|A_1 v^{(1)}\|_2 + \frac{1}{12} \|A_1 Q_{1,n}u\|_2^2. \end{aligned}$$

Using the Agmon inequality, we obtain

$$\begin{aligned}
I_2 &\leq \|v^{(2)}\|_\infty \|\nabla u\|_2 \|A_1 Q_{1,n} u\|_2 \\
&\leq C \|v^{(2)}\|_{V_1}^{\frac{1}{2}} \|A_1 v^{(2)}\|_2^{\frac{1}{2}} \|u\|_{V_1} \|A_1 Q_{1,n} u\|_2 \\
&\leq C \|v^{(2)}\|_{V_1} \|A_1 v^{(2)}\|_2 \|u\|_{V_1}^2 + \frac{1}{12} \|A_1 Q_{1,n} u\|_2^2.
\end{aligned}$$

By Lemma 2.1, we have

$$\begin{aligned}
I_3 &\leq C \|\nabla u\|_2^{\frac{1}{2}} \|\nabla u\|_{H^1}^{\frac{1}{2}} \|\partial_z v\|_2^{\frac{1}{2}} \|\partial_z v\|_{H^1}^{\frac{1}{2}} \|A_1 Q_{1,n} u\|_2 \\
&\leq C \|v_z^{(1)}\|_2 \|v_z^{(1)}\|_{H^1} \|u\|_{V_1} \|A_1 u\|_2 + \frac{1}{12} \|A_1 Q_{1,n} u\|_2^2.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
I_4 &\leq C \|\nabla v^{(2)}\|_2^{\frac{1}{2}} \|\nabla^2 v^{(2)}\|_2^{\frac{1}{2}} \|u_z\|_2^{\frac{1}{2}} \|\nabla u_z\|_2^{\frac{1}{2}} \|A_1 Q_{1,n} u\|_2 \\
&\leq C \|\nabla v^{(2)}\|_2 \|\nabla^2 v^{(2)}\|_2 \|u_z\|_2 \|\nabla u_z\|_2 + \frac{1}{12} \|A_1 Q_{1,n} u\|_2^2.
\end{aligned}$$

The last two terms can be estimated as follows:

$$\begin{aligned}
I_5 &\leq C \|\eta\|_{V_2} \|A_1 Q_{1,n} u\|_2 \leq C \|\eta\|_{V_2}^2 + \frac{1}{12} \|A_1 Q_{1,n} u\|_2^2, \\
I_6 &\leq C \|u\|_{V_1} \|A_1 Q_{1,n} u\|_2 \leq C \|u\|_{V_1}^2 + \frac{1}{12} \|A_1 Q_{1,n} u\|_2^2.
\end{aligned}$$

Next, we estimate the right-hand side of (5.2) term by term. Similar to the estimate for I_1 , we have

$$\begin{aligned}
J_1 &\leq \|u\|_6 \|\nabla \theta^{(1)}\|_3 \|A_2 Q_{2,n} \eta\|_2 \\
&\leq C \|u\|_{V_1} \|\theta^{(1)}\|_{V_2}^{\frac{1}{2}} \|A_2 \theta^{(1)}\|_2^{\frac{1}{2}} \|A_2 Q_{2,n} \eta\|_2 \\
&\leq C \|\theta^{(1)}\|_{V_2} \|A_2 \theta^{(1)}\|_2 \|u\|_{V_1}^2 + \frac{1}{8} \|A_2 Q_{2,n} \eta\|_2^2.
\end{aligned}$$

Similar to the estimate for I_2 , we have

$$\begin{aligned}
J_2 &\leq \|v^{(2)}\|_\infty \|\nabla \eta\|_{V_2} \|A_2 Q_{2,n} \eta\|_2 \\
&\leq C \|v^{(2)}\|_{V_1}^{\frac{1}{2}} \|A_1 v^{(2)}\|_2^{\frac{1}{2}} \|\eta\|_{V_2} \|A_2 Q_{2,n} \eta\|_2 \\
&\leq C \|v^{(2)}\|_{V_1} \|A_1 v^{(2)}\|_2 \|\eta\|_{V_2}^2 + \frac{1}{8} \|A_2 Q_{2,n} \eta\|_2^2.
\end{aligned}$$

Similar to the estimates for I_3 and I_4 , we have

$$\begin{aligned} J_3 &\leq C \|\nabla u\|_2^{\frac{1}{2}} \|\nabla^2 u\|_2^{\frac{1}{2}} \|\theta_z^{(1)}\|_2^{\frac{1}{2}} \|\nabla \theta_z^{(1)}\|_2^{\frac{1}{2}} \|A_2 Q_{2,n} \eta\|_2 \\ &\leq C \|\theta_z^{(1)}\|_2 \|\nabla \theta_z^{(1)}\|_2 \|\nabla u\|_2 \|\nabla^2 u\|_2 + \frac{1}{8} \|A_2 Q_{2,n} \eta\|_2^2, \end{aligned}$$

and

$$\begin{aligned} J_4 &\leq C \|v^{(2)}\|_{V_1}^{\frac{1}{2}} \|A_1 v^{(1)}\|_2^{\frac{1}{2}} \|\eta_z\|_2^{\frac{1}{2}} \|\eta_z\|_{H^1}^{\frac{1}{2}} \|A_2 Q_{2,n} \eta\|_2 \\ &\leq C \|v^{(2)}\|_{V_1} \|A_1 v^{(2)}\|_2 \|\eta_z\|_2 \|\eta_z\|_{H^1} + \frac{1}{8} \|A_2 Q_{2,n} \eta\|_2^2 \\ &\leq C \|v^{(2)}\|_{V_1} \|A_1 v^{(2)}\|_2 \|\eta\|_{V_2} \|A_2 \eta\|_2 + \frac{1}{8} \|A_2 Q_{2,n} \eta\|_2^2. \end{aligned}$$

Now adding (5.1) and (5.2) and using the estimates for I_i ($i = 1, \dots, 6$) and J_j ($j = 1, \dots, 4$), and denoting

$$\begin{aligned} y(t) &:= \|u(t)\|_{V_1}^2 + \|\eta(t)\|_{V_2}^2, \\ z(t) &:= \|Q_{1,n} u(t)\|_{V_1}^2 + \|Q_{2,n} \eta(t)\|_{V_2}^2, \end{aligned}$$

we have, noticing that $(v^{(i)}(t), \theta^{(i)}(t)) \in \mathcal{A}$, for $i = 1, 2$,

$$\begin{aligned} z'(t) &+ \|A_1 Q_{1,n} u\|_2^2 + \|A_2 Q_{2,n} \eta\|_2^2 \\ &\leq C y(t) (1 + \|A_1 v^{(1)}\|_2 + \|A_1 v^{(2)}\|_2 + \|A_2 \theta^{(1)}\|_2) \\ &\quad + C \|u\|_{V_1} \|A_1 u\|_2 (\|\nabla v_z^{(1)}\|_2 + \|\nabla \theta_z^{(1)}\|_2 + \|\nabla^2 v^{(2)}\|_2) \\ &\quad + C \|\eta\|_{V_1} \|A_2 \eta\|_2 \|\nabla^2 v^{(2)}\|_2, \end{aligned} \tag{5.3}$$

where C is a generic positive constant independent of t , n and the initial data. Denote $\rho_n := \min(\lambda_n^2, \sigma_n^2)$. Then, it follows from (5.3) that

$$\begin{aligned} z'(t) + \rho_n z(t) &\leq C y(t) (1 + \|A_1 v^{(1)}\|_2 + \|A_1 v^{(2)}\|_2 + \|A_2 \theta^{(1)}\|_2) \\ &\quad + C y^{\frac{1}{2}}(t) (\|A_1 u\|_2^2 + \|A_2 \eta\|_2^2)^{\frac{1}{2}} \\ &\quad \times (\|\nabla v_z^{(1)}\|_2 + \|\nabla \theta_z^{(1)}\|_2 + \|\nabla^2 v^{(2)}\|_2). \end{aligned} \tag{5.4}$$

Now, integrating (5.4) with respect to $t \in [0, T]$, we obtain

$$\begin{aligned}
z(T) &\leq e^{-\rho_n T} z(0) + C e^{-\rho_n T} \int_0^T e^{\rho_n t} y(t) dt \\
&\quad + C e^{-\rho_n T} \int_0^T e^{\rho_n t} y(t) (\|A_1 v^{(1)}\|_2 + \|A_1 v^{(2)}\|_2 + \|A_2 \theta^{(1)}\|_2) dt \\
&\quad + C e^{-\rho_n T} \int_0^T e^{\rho_n t} y^{\frac{1}{2}}(t) (\|A_1 u(t)\|_2^2 + \|A_2 \eta(t)\|_2^2)^{\frac{1}{2}} \\
&\quad \times (\|\nabla v_z^{(1)}\|_2 + \|\nabla \theta_z^{(1)}\|_2 + \|\nabla^2 v^{(2)}\|_2) dt \\
&=: Z_1 + Z_2 + Z_3 + Z_4.
\end{aligned}$$

First, notice that

$$Z_1 \leq e^{-\rho_n T} y(0).$$

Next, recall that it is proved in [6] that for strong solutions $(v^{(i)}(t), \theta^{(i)}(t)) \in V$, $i = 1, 2$, there exists a positive continuous non-decreasing function $K(t)$, independent of the initial data, such that

$$y(t) + \int_0^t (\|A_1 u(\tau)\|_2^2 + \|A_2 \eta(\tau)\|_2^2) d\tau \leq K(t) y(0), \quad \forall t \geq 0. \quad (5.5)$$

Therefore,

$$Z_2 \leq CK(T) y(0) e^{-\rho_n T} \int_0^T e^{\rho_n t} dt \leq CK(T) \rho_n^{-1} y(0)$$

and

$$\begin{aligned}
Z_3 &\leq CK(T) y(0) e^{-\rho_n T} \int_0^T e^{\rho_n t} (\|A_1 v^{(1)}\|_2 + \|A_1 v^{(2)}\|_2 + \|A_2 \theta^{(1)}\|_2) dt \\
&\leq CK(T) y(0) e^{-\rho_n T} \left(\int_0^T e^{2\rho_n t} dt \right)^{\frac{1}{2}} \\
&\quad \times \left[\int_0^T (\|A_1 v^{(1)}\|_2^2 + \|A_1 v^{(2)}\|_2^2 + \|A_2 \theta^{(1)}\|_2^2) dt \right]^{\frac{1}{2}} \\
&\leq CK(T) y(0) e^{-\rho_n T} \left(\int_0^T e^{2\rho_n t} dt \right)^{\frac{1}{2}} \leq CK(T) \rho_n^{-\frac{1}{2}} y(0),
\end{aligned}$$

by the uniform boundedness of $\int_0^T (\|A_1 v^{(1)}\|_2^2 + \|A_1 v^{(2)}\|_2^2 + \|A_2 \theta^{(1)}\|_2^2) dt$ on the global attractor \mathcal{A} .

The estimate of Z_4 is more complicated and needs a special treatment.

$$\begin{aligned}
Z_4 &\leq CK^{\frac{1}{2}}(T)y^{\frac{1}{2}}(0) \int_0^T (\|A_1u(t)\|_2^2 + \|A_2\eta(t)\|_2^2)^{\frac{1}{2}} \\
&\quad \times (\|\nabla v_z^{(1)}\|_2 + \|\nabla\theta_z^{(1)}\|_2 + \|\nabla^2v^{(2)}\|_2) dt \\
&\leq CK^{\frac{1}{2}}(T)y^{\frac{1}{2}}(0) \left(\int_0^T \|A_1u\|_2^2 + \|A_2\eta\|_2^2 dt \right)^{\frac{1}{2}} \\
&\quad \times \left[\int_0^T (\|\nabla v_z^{(1)}\|_2^2 + \|\nabla\theta_z^{(1)}\|_2^2 + \|\nabla^2v^{(2)}\|_2^2) dt \right]^{\frac{1}{2}} \\
&\leq CK(T)(T^{\frac{1}{2}} + T)^{\frac{1}{2}}y(0),
\end{aligned}$$

where we have used (5.5) and uniform continuity property as proved in Theorem 4.1.

Combining the estimates of the Z_i 's, $i = 1, \dots, 4$, we obtain

$$z(T) \leq \left[e^{-\rho_n T} + C_1K(T)\rho_n^{-1} + C_2K(T)\rho_n^{-1/2} + C_3K(T)(T^{\frac{1}{2}} + T)^{\frac{1}{2}} \right] y(0),$$

where C_i 's, $i = 1, 2, 3$, are all constants independent of T , n and initial data.

Notice that

$$\lim_{T \rightarrow 0^+} K(T)(T^{\frac{1}{2}} + T)^{\frac{1}{2}} = 0.$$

Therefore, for any $\delta \in (0, 1)$, we can choose $T > 0$ sufficiently small and uniform for any pair of $(v^{(i)}, \theta^{(i)}) \in \mathcal{A}$, $i = 1, 2$, such that

$$C_3K(T)(T^{\frac{1}{2}} + T)^{\frac{1}{2}} \leq \frac{\delta}{2},$$

and then we choose n sufficiently large, which is uniform for any pair of $(v^{(i)}, \theta^{(i)}) \in \mathcal{A}$, $i = 1, 2$, such that

$$e^{-\rho_n T} + C_1K(T)\rho_n^{-1} + C_2K(T)\rho_n^{-1/2} \leq \frac{\delta}{2}.$$

Thus,

$$z(T) \leq \delta y(0).$$

This proves the expected squeezing property of $S(T)$ for any $T > 0$ that is sufficiently small.

Notice that (5.5) gives the Lipschitz continuity of the solution map $S(t)(v_0, \theta_0) := (v(t), \theta(t))$ in the space V for any fixed $t \in [0, \infty)$. Thus, Theorem 5.2 follows immediately from Theorem 5.1.

□

Remark 5.2 *It is easy to see, by a continuation argument, that the squeezing property of $S(T)$ is indeed still valid for any $T > 0$.*

6 The case with “physical boundary conditions”

The previous sections of this paper were finished in December 2014 and the original manuscript was submitted for publication by then. This short section is now added to include the case with “physical boundary conditions”. This is the case that the velocity field v satisfies, instead of (1.3), the following boundary conditions:

$$(\alpha_v v + v_z)|_{z=0} = v|_{(x,y) \in \partial D} = v|_{z=-h} = 0, \quad (6.1)$$

where α_v is a non-negative real constant independent of v , to be distinguished from the non-negative real constant α in (1.4).

It can be shown that *the main results of this paper are still valid for this case*. The detailed proofs are very similar to those given in the previous sections and thus are omitted here. However, it is *not* clear if the method of [7] can be used for this case. This is an additional example showing the advantage of the new approach as given in this paper when compared with the one presented in our previous [7].

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