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COMPLETELY INTEGRABLE DIFFERENTIAL SYSTEMS ARE ESSENTIALLY LINEAR

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Abstract. Let $\dot{x} = f(x)$ be a C^k autonomous differential system with $k \in \mathbb{N} \cup \{\infty, \omega\}$ defined in an open subset Ω of $\mathbb{R}^n$. Assume that the system $\dot{x} = f(x)$ is C^r completely integrable, i.e. there exist $n-1$ functionally independent first integrals of class C^r with $2 \le r \le k$. If the divergence of system $\dot{x} = f(x)$ is non–identically zero, then any Jacobian multiplier is functionally independent of the $n-1$ first integrals. Moreover the system $\dot{x} = f(x)$ is C^{r-1} orbitally equivalent to the linear differential system $\dot{y} = y$ in a full Lebesgue measure subset of Ω. For Darboux and polynomial integrable polynomial differential systems we characterize their type of Jacobian multipliers.

1. Introduction and statement of results

Consider a C^k autonomous differential system

$$\dot{x} = f(x), \qquad x \in \Omega \subset \mathbb{R}^n, \tag{1}$$

where $k \in \mathbb{N} \cup \{\infty, \omega\}$, the dot denotes derivative with respect to the independent variable t, Ω is an open subset of $\mathbb{R}^n$, and $f(x) = (f_1(x), \dots, f_n(x)) \in C^k(\Omega)$. Recall that $\mathbb{N}$ is the set of positive integers, and C^∞ and C^ω are respectively the sets of infinitely smooth functions and analytic functions.

A function $H(x)$ is a *first integral* of system (1) if it is continuous and defined in a full Lebesgue measure subset Ω_1 of Ω, and it is not locally constant on any positive Lebesgue measure subset of Ω_1; moreover $H(x)$ is constant along each orbit of system (1) in Ω_1. System (1) is C^r *completely integrable*, if it has $n-1$ functionally independent C^r first integrals in Ω with $1 \le r \le k$. Recall that k functions $H_1(x), \dots, H_k(x)$ are *functionally independent* in Ω if their gradients $\nabla H_1, \dots, \nabla H_k$ have rank k in a full Lebesgue measure subset of Ω.

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In this paper we denote by $\mathcal{X}$ the vector field associated to system (1). We will use ∂_i to denote the partial derivative with respect to x_i for $i = 1, \ldots, n$. By convention $C^{r-1} = C^r$ if $r = \infty$ or ω; and $\mathrm{div}\mathcal{X} = \mathrm{div} f = \partial_1 f_1 + \ldots + \partial_n f_n$.

A C^1 function J is a *Jacobian multiplier* of system (1) if it is defined in a full Lebesgue measure subset $\Omega^* \subset \Omega$, and satisfies

$$\mathrm{div}(Jf) \equiv 0, \quad i.e., \quad \mathcal{X}(J) = -J\mathrm{div}\mathcal{X}, \qquad x \in \Omega^*.$$

If system (1) is two dimensional, a Jacobian multiplier is called an *integrating factor*.

We extend the usual ordering of $\mathbb{N}$ to the set $\mathbb{N} \cup \{\infty, \omega\}$ as follows: for all $k \in \mathbb{N}$ we have $k < \infty < \omega$. Two differential systems $\dot{x} = f(x)$ and $\dot{y} = g(y)$ are C^r *orbitally equivalent* in a region $\mathcal{M}$ with $1 \le r \le \omega$ if there exists a C^r invertible transformation $y = \Phi(x)$ defined on $\mathcal{M}$ such that $\Phi_* f \circ \Phi^{-1}(y) = q(y)g(y)$, where Φ_* is the tangent map of Φ and $q(y)$ is a non–vanishing scalar function defined on $\mathcal{M}$.

The next is our first main result.

Theorem 1. *Let $\dot{x} = f(x)$ be the C^k autonomous differential system (1) with $k \in (\mathbb{N} \setminus \{1\}) \cup \{\infty, \omega\}$ defined in Ω. Assume that system (1) is C^r completely integrable in Ω with $2 \le r \le k$, $\mathrm{div}\,\mathcal{X} \not\equiv 0$, and that the Lebesgue measure of the set of its singularities is zero. Let $H_1(x), \ldots, H_{n-1}(x)$ be $n-1$ functionally independent C^r first integrals. Then the following statements hold.*

- (a) *System (1) has always a C^{r-1} Jacobian multiplier defined in a full Lebesgue measure subset of Ω.*
- (b) *If $J(x)$ is a C^{r-1} Jacobian multiplier of system (1), then $J(x)$ is functionally independent of $H_1(x), \ldots, H_{n-1}(x)$.*
- (c) *There exists a full Lebesgue measure subset $\Omega_0 \subset \Omega$ in which system (1) is C^{r-1} orbitally equivalent to the linear differential system*

$$\dot{y} = y. \tag{2}$$

We remark that statement (a) is not new, it can be obtained from [20, Theorem 1.1]. We include its proof here for completeness. Statement (b) is new. Statement (c) generalizes and improves Theorem 1 of [11] extending it from two dimensional differential systems to any finite dimensional differential systems.

The flow box theorem states the existence of $n-1$ functionally independent first integrals in a neighborhood of a regular point of the differential system $\dot{x} = f(x)$ by making it diffeomorphic to the differential system $(\dot{y}_1, \dot{y}_2, \ldots, \dot{y}_n) = (1, 0, \ldots, 0)$. While Theorem 1 under the assumptions of the existence of $n-1$ functionally independent first integrals and the divergence non–identically zero for the C^k differential system $\dot{x} = f(x)$ defined in an open subset Ω of $\mathbb{R}^n$, shows that the system is diffeomorphic to the linear differential system $(\dot{y}_1, \ldots, \dot{y}_n) = (y_1, \ldots, y_n)$ in an open and dense subset of Ω.

We want to note that completely integrable differential systems defined in an open subset Ω of $\mathbb{R}^n$ with divergence identically zero in general cannot be diffeomorphic in an open and dense subset of Ω to the linear differential system $\dot{y} = y$. For instance this is the case of superintegrable Hamiltonian systems when n is even, due to the Liouville–Arnold Theorem, see for more details [1, 2, 4, 12] and the references therein.

In the next proposition we characterize the zero Lebesgue measure subset mentioned in statement (c) of Theorem 1.

Proposition 2. *Under the assumptions of Theorem 1, we can assume without loss of generality that*

$$D(x) = \det \begin{pmatrix} \partial_1 H_1(x) & \ldots & \partial_{n-1} H_1(x) \\ \vdots & \vdots & \vdots \\ \partial_1 H_{n-1}(x) & \ldots & \partial_{n-1} H_{n-1}(x). \end{pmatrix} \not\equiv 0.$$

Then the zero Lebesgue measure subset $\Omega \setminus \Omega_0$ *of statement* (c) *of Theorem 1 is*

$$\{x \in \Omega |\, D(x)\, f_n(x)\, \mathrm{div} f(x) = 0\}\,.$$

Theorem 1 shows the existence of Jacobian multipliers for completely integrable differential systems. We now characterize the class of the Jacobian multipliers of these integrable differential systems.

Let $\mathbb{C}[x]$ be the ring of all polynomials in the variables $(x_1, \ldots, x_n) = x$ with coefficients in $\mathbb{C}$. A function $H(x)$ is of *Darboux type* if it is of the form

$$f_1^{k_1}(x) \ldots f_r^{k_r}(x) \exp\left(\frac{g(x)}{h(x)}\right),$$

where $f_i, g, h \in \mathbb{C}[x]$ with g and h coprime, and $k_i \in \mathbb{C}$ for $i = 1, \ldots, r$. Recall that the notion of Darboux function was considered by Darboux [8, 9] in 1878 for studying the existence of first integrals through invariant algebraic curves (or surfaces or hypersurfaces) of polynomial differential systems. A *Darboux first integral* of the system $\dot{x} = f(x)$ is a first integral of Darboux type. A polynomial differential system

$\dot{x} = f(x)$ in $\mathbb{R}^n$ or $\mathbb{C}^n$ is *Darboux integrable* if it has $n-1$ functionally independent Darboux first integrals.

Let $\mathbb{C}(x)$ be the field of all rational functions in the variables x with coefficients in $\mathbb{C}$. A function is *Liouvillian* if it belongs to the *Liouvillian field extension* of $\mathbb{C}(x)$, for more details on the Liouvillian field extension see for instance [16]. A polynomial differential system $\dot{x} = f(x)$ in $\mathbb{R}^n$ or $\mathbb{C}^n$ is *Liouvillian integrable* if it has $n-1$ functionally independent Liouvillian first integrals.

Our next result provides the class of functions where belong the Jacobian multipliers of an integrable polynomial differential system.

Theorem 3. *Assume that system* (1) *with* $f = (f_1, \ldots, f_n)$ *is an* n*–dimensional polynomial differential system with* $f_1, \ldots, f_n$ *relatively prime. Then the following statements hold.*

- (a) *If system* (1) *is Liouvillian integrable, then it has a Darboux Jacobian multiplier.*
- (b) *If system* (1) *is Darboux integrable, then it has a rational Jacobian multiplier.*
- (c) *If system* (1) *is polynomial integrable, then it has a polynomial Jacobian multiplier.*

Statement (a) with $n = 2$ is due to Singer [16], Christopher [6] provided a different proof, see also [7, 14]. Statement (a) with $n > 2$ was proved recently by Zhang in [19]. We include this statement here for completeness. Statement (b) was proved in [5] for $n = 2$. The proof of statement (c) with $n = 2$ follows from [5, 10].

This paper is organized as follows. In the next section we will prove our results. In section 3 we present an application of Theorem 1.

2. Proof of the main results

For proving Theorem 1 we need the following results. The first one is due to Olver see [13, Theorem 2.16], it reveals the essential property of functional dependence.

Theorem 4. *Assume that* $M \subset \mathbb{R}^n$ *is a* C^∞ *manifold, and* $g_1, \ldots, g_k$ *are real* C^1 *functions on* M*. Then* $g_1, \ldots, g_k$ *are functionally dependent on* M *if and only if for all* $x \in M$*, there exists a neighborhood* U *of* x *and a* C^1 *real function* $F(z_1, \ldots, z_k)$ *in* k *variables such that*

$$F(g_1(x), \ldots, g_k(x)) \equiv 0, \qquad x \in U.$$

The second result characterizes a first integral of system (1) in function of $n-1$ functionally independent first integrals.

Lemma 5. *Assume that an n–dimensional C^1 autonomous differential system $\dot{x} = f(x)$ has $n-1$ functionally independent C^1 first integrals in Ω, namely $H_1(x), \dots, H_{n-1}(x)$. If H is a C^1 first integral of system (1) defined in an open subset $\Omega_0 \subset \Omega$, then for any $x_0 \in \Omega_0$ there exists a neighborhood $U_0 \subset \Omega_0$ of x_0, and a C^1 function Φ in $n-1$ variables, such that*

$$H(x) = \Phi(H_1(x), \dots, H_{n-1}(x)), \quad x \in \Omega_0.$$

Proof. The proof can be obtained from Theorem 4. See also for instance, Theorem 2.17 of [13] or Proposition 3 of [21, Chapter 1]. □

The third result provides a method for constructing first integrals using Jacobian multipliers.

Lemma 6. *If $J_1(x)$ and $J_2(x)$ are Jacobian multipliers of system $\dot{x} = f(x)$ in Ω, then $J_1(x)/J_2(x)$ is a first integral of the system in $\Omega \setminus \{J_2(x) = 0\}$.*

Proof. The proof follows from direct calculations and it is easy, see for instance [3]. □

We must mention that the idea of the proof of statement (a) of Theorem 1 partially comes from [19, 20], and the proof of statement (c) partially comes from [11].

Proof of Theorem 1. Since $H_1(x), \dots, H_{n-1}(x)$ are C^r first integrals of system (1) in Ω, by definition we have

$$\begin{aligned} &\partial_1 H_1 f_1 + \dots + \partial_{n-1} H_1 f_{n-1} + \partial_n H_1 f_n = 0, \\ &\qquad\qquad\qquad\vdots \\ &\partial_1 H_{n-1} f_1 + \dots + \partial_{n-1} H_{n-1} f_{n-1} + \partial_n H_{n-1} f_n = 0. \end{aligned} \tag{3}$$

Since $H_1, \dots, H_{n-1}$ are functionally independent in Ω we can assume without loss of generality that

$$D(\mathcal{H}) := \det\left(\partial_1 \mathcal{H}, \cdots, \partial_{n-1} \mathcal{H}\right) \neq 0, \qquad x \in \Omega_0 \subset \Omega,$$

where Ω_0 is a full Lebesgue measure subset of Ω, and

$$\begin{aligned} \mathcal{H} &:= (H_1, \dots, H_{n-1})^T, \\ \partial_i \mathcal{H} &:= (\partial_i H_1, \dots, \partial_i H_{n-1})^T, \qquad i = 1, \dots, n. \end{aligned}$$

where T denotes the transpose of a matrix

For $i = 1, \dots, n-1$, set

$$D_i(\mathcal{H}) := \det\left(\partial_1 \mathcal{H}, \cdots, \partial_{i-1} \mathcal{H}, \partial_n \mathcal{H}, \partial_{i+1} \mathcal{H}, \cdots, \partial_{n-1} \mathcal{H}\right).$$

Using Cramer's rule to solve (3) with respect to $f_1, \ldots, f_{n-1}$, we get

$$f_i(x) = -\frac{D_i(\mathcal{H})}{D(\mathcal{H})} f_n(x), \quad i = 1, \ldots, n-1. \tag{4}$$

It follows that

$$D(\mathcal{H})(f_1(x), \ldots, f_{n-1}(x)) = -(D_1(\mathcal{H}), \ldots, D_{n-1}(\mathcal{H})) f_n(x). \tag{5}$$

Set

$$Q(x) = \frac{D(\mathcal{H})}{f_n(x)}. \tag{6}$$

It is clear that Q is defined in a full Lebesgue measure subset $\Omega_Q \subset \Omega_0 \subset \Omega$, and is a C^{r-1} function in Ω_Q. Moreover we get from (5) that

$$D_i(\mathcal{H})(x) = -Q(x) f_i(x), \quad i = 1, \ldots, n-1. \tag{7}$$

We claim that $Q(x)$ is a Jacobian multiplier of system (1) in Ω_0. We now prove this claim. It follows from (6) and (7) that

$$\sum_{i=1}^{n-1} \partial_i(Q f_i) + \partial_n(Q f_n) = -\sum_{i=1}^{n-1} \partial_i D_i(\mathcal{H}) + \partial_n D(\mathcal{H}). \tag{8}$$

Next we only need to prove the right hand side of (8) is identically zero. Using the derivative of a determinant we get easily that

$$\begin{aligned} \partial_n D(\mathcal{H}) &= \sum_{i=1}^{n-1} \det\left(\partial_1 \mathcal{H}, \ldots, \partial_{i-1}\mathcal{H}, \partial_n \partial_i \mathcal{H}, \partial_{i+1}\mathcal{H}, \ldots, \partial_{n-1}\mathcal{H}\right), \\ \partial_i D_i(\mathcal{H}) &= \det\left(\partial_1 \mathcal{H}, \ldots, \partial_{i-1}\mathcal{H}, \partial_i \partial_n \mathcal{H}, \partial_{i+1}\mathcal{H}, \ldots, \partial_{n-1}\mathcal{H}\right) \\ &\quad + \sum_{j=1, j\neq i}^{n-1} \det\left(\partial_1 \mathcal{H}, \ldots, \partial_{j-1}\mathcal{H}, \partial_i \partial_j \mathcal{H}, \partial_{j+1}\mathcal{H}, \ldots, \partial_{i-1}\mathcal{H},\right. \\ &\qquad\qquad \left.\partial_n \mathcal{H}, \partial_{i+1}\mathcal{H}, \ldots, \partial_{n-1}\mathcal{H}\right). \end{aligned}$$

Hence we have

$$\begin{aligned} &\partial_n D(\mathcal{H}) - \sum_{i=1}^{n-1} \partial_i D_i(\mathcal{H}) \\ &\quad = -\sum_{i=1}^{n-1} \sum_{j=1, j\neq i}^{n-1} \det\left(\partial_1 \mathcal{H}, \ldots, \partial_{j-1}\mathcal{H}, \partial_i \partial_j \mathcal{H}, \partial_{j+1}\mathcal{H}, \ldots, \partial_{i-1}\mathcal{H},\right. \\ &\qquad\qquad \left.\partial_n \mathcal{H}, \partial_{i+1}\mathcal{H}, \ldots, \partial_{n-1}\mathcal{H}\right) \\ &\quad = 0, \end{aligned}$$

because in the last equality we have used the fact that

$$\begin{aligned}&\det(\partial_1\mathcal{H},\dots,\partial_{j-1}\mathcal{H},\partial_i\partial_j\mathcal{H},\partial_{j+1}\mathcal{H},\dots,\partial_{i-1}\mathcal{H},\ \partial_n\mathcal{H},\partial_{i+1}\mathcal{H},\dots,\partial_{n-1}\mathcal{H})\\ &\quad+\det(\partial_1\mathcal{H},\dots,\partial_{j-1}\mathcal{H},\partial_n\mathcal{H},\partial_{j+1}\mathcal{H},\dots,\partial_{i-1}\mathcal{H},\ \partial_j\partial_i\mathcal{H},\partial_{i+1}\mathcal{H},\dots,\partial_{n-1}\mathcal{H})\\ &\quad=0.\end{aligned}$$

Now it follows from (8) that

$$\sum_{i=1}^{n-1}\partial_i(Qf_i)+\partial_n(Qf_n)\equiv 0.$$

That is $Q(x)$ is a Jacobian multiplier, and consequently statement (a) follows.

For proving (b) we first prove that $Q(x)$ is functionally independent of $H_1(x),\dots,H_{n-1}(x)$. Since

$$A:=\det\begin{pmatrix}\partial_1 Q & \cdots & \partial_{n-1}Q & \partial_n Q\\ \partial_1 H_1 & \cdots & \partial_{n-1}H_1 & \partial_n H_1\\ \vdots & \vdots & \vdots & \vdots\\ \partial_1 H_{n-1} & \cdots & \partial_{n-1}H_{n-1} & \partial_n H_{n-1}\end{pmatrix}=\sum_{i=1}^{n}\partial_i Q\, Q_i^*,$$

where $Q_i^*=(-1)^{1+i}M_{1i}$ with M_{1i} the minor defined to be the determinant of the $(n-1)\times(n-1)$–matrix that results from the matrix by removing the first row and the ith column. Clearly

$$D(\mathcal{H})=M_{1n},\quad D_i(\mathcal{H})=(-1)^{n-1-i}M_{1i},\quad i=1,\dots,n-1.$$

So we have

$$\begin{aligned}A&=\sum_{i=1}^{n-1}\partial_i Q(-1)^n D_i(\mathcal{H})+\partial_n Q(-1)^{n+1}D(\mathcal{H})\\ &=\sum_{i=1}^{n-1}\partial_i Q(-1)^n(-Qf_i)+\partial_n Q(-1)^{n+1}Qf_n=(-1)^{n+1}Q\mathcal{X}(Q),\end{aligned}$$

where in the second equality we have used (6) and (7). Recall that $\mathcal{X}$ is the vector field associated to system (1).

Since Q is not a first integral of system (1), it follows that $\mathcal{X}(Q)=Q\mathrm{div}\mathcal{X}$ is a nonzero function on Ω_Q. This implies that A can only vanish in a zero Lebesgue measure subset of Ω. This shows that $Q,H_1,\dots,H_{n-1}$ are functionally independent on Ω.

By assumption $J(x)$ is a C^{r-1} Jacobian multiplier, then by Lemma 6 we obtain that Q/J is a first integral of system (1). Then we get from Lemma 5 that Q/J can be locally expressed as

$$Q/J=\Phi(H_1,\dots,H_{n-1}),\tag{9}$$

where Φ is a C^1 function. If J is functionally dependent of $H_1, \ldots, H_{n-1}$, we get from Theorem 4 that J can be locally expressed as a C^1 function of $H_1, \ldots, H_{n-1}$. This together with (9) shows that Q is functionally dependent of $H_1, \ldots, H_{n-1}$, a contradiction. This proves that J is functionally independent of $H_1, \ldots, H_{n-1}$, so statement (b) follows.

By statement (a) system (1) has a C^{r-1} Jacobian multiplier $J(x)$. From (b) it follows that the functions J, H_1, $\ldots$, H_{n-1} are functionally independent. So there exists a full Lebesgue measure subset $\Omega_0 \subset \Omega$ such that ∇J, ∇H_1, $\ldots$, ∇H_{n-1} have rank n at all points of Ω_0. Taking the invertible change of variables

$$y_n = J(x), \quad y_i = J(x)H_i, \quad i = 1, \ldots, n-1, \qquad x \in \Omega_0,$$

we have

$$\begin{aligned} \dot{y}_n &= \dot{J} = J\mathrm{div} f = y_n \,\mathrm{div} f, \\ \dot{y}_i &= \dot{J}H_i + J\dot{H} = \dot{J}H_i = J\mathrm{div} f H_i = y_i \,\mathrm{div} f. \end{aligned}$$

This proves that system (1) is C^{r-1} orbitally equivalent to the linear system (2). Hence statement (c) follows. This completes the proof of the theorem. $\square$

Proof of Proposition 2. We note that $D(x) = D(\mathcal{H})(x)$. The latter was defined in the proof of Theorem 1. By assumption it follows that $D(x)$ is a C^1 function and does not vanish on a full Lebesgue measure subset of Ω. From (4) of the proof of Theorem 1 we have that $f_n(x)$ does not vanish on a full Lebesgue measure subset of Ω. Otherwise all f_i's are almost zero, and so system (1) has a positive Lebesgue measure subset of singularities, a contradiction.

Choose the Jacobian multiplier $Q = D(x)/f_n(x)$, then from the proof of Theorem 1 we get that

$$\begin{aligned} \det \begin{pmatrix} \partial_1 Q & \cdots & \partial_n Q \\ \partial_1(QH_1) & \cdots & \partial_n(QH_1) \\ \vdots & \vdots & \vdots \\ \partial_1(QH_{n-1}) & \cdots & \partial_n(QH_{n-1}) \end{pmatrix} &= (-1)^{n+1} Q^n \mathcal{X}(Q) \\ &= (-1)^{n+1} \left(\frac{D(x)}{f_n(x)} \right)^{n+1} \mathrm{div}\mathcal{X}. \end{aligned}$$

This shows that the transformation from system $\dot{x} = f(x)$ to $\dot{y} = y$ defined by

$$y_1 = Q(x)H_1(x), \; \ldots, \; y_{n-1} = Q(x)H_{n-1}(x), \; y_n = Q(x),$$

is a C^{r-1} diffeomorphism on $\Omega_0 := \{x \in \Omega : \, D(x)\, f_n(x)\, \mathrm{div}\mathcal{X}(x) \neq 0\}$. Obviously $\Omega \backslash \Omega_0 = \{x \in \Omega : \, D(x)\, f_n(x)\, \mathrm{div}\mathcal{X}(x) = 0\}$ is a zero Lebesgue measure subset of Ω. This proves the proposition. □

Proof of Theorem 3. Recall that statement (a) was proved in [16, 19]. We now prove statements (b) and (c).

Let $H_1(x), \ldots, H_{n-1}(x)$ be $n-1$ functionally independent Darboux first integrals of the polynomial differential system (1). Taking $G_i(x) = \log H_i(x)$, $i = 1, \ldots, n-1$. Then $G_1(x), \ldots, G_{n-1}(x)$ are also functionally independent first integrals of system (1).

We assume without loss of generality that

$$\mathcal{G}(x) := \det \begin{pmatrix} \partial_1 G_1(x) & \cdots & \partial_{n-1} G_1(x) \\ \vdots & \vdots & \vdots \\ \partial_1 G_{n-1}(x) & \cdots & \partial_{n-1} G_{n-1}(x) \end{pmatrix},$$

is not zero in $\mathbb{R}^n$ except perhaps a zero Lebesgue measure subset. Then we get from the proof of Proposition 2 that $f_n \neq 0$ in a full Lebesgue measure subset of Ω. Since $H_i(x)$ is a Darboux function, we assume that it is of the form

$$H_i(x) = g_{i1}^{k_{i1}} \ldots g_{ir_i}^{k_{ir_i}} \exp\left(\frac{q_i(x)}{h_i(x)}\right),$$

where g_{ij}, q_i, h_i are polynomials, and $k_{ij} \in \mathbb{C}$, $j = 1, \ldots, r_i$. Computing the partial derivative of $G_i(x) = \log H_i(x)$ with respect to x_s for $s = 1, \ldots, n$, we get

$$\partial_s G_i(x) = \sum_{j=1}^{r_i} k_{ij} \frac{\partial_s g_{ij}}{g_{ij}} + \frac{h_i \partial_s q_i - q_i \partial_s h_i}{h_i^2}.$$

This is a rational function. So $\mathcal{G}(x)$ is also a rational function.

By Theorem 1 and its proof it follows that system (1) has the rational Jacobian multiplier $Q(x) = \mathcal{G}(x)/f_n(x)$, because f_n is a polynomial. Hence statement (b) follows.

Finally we prove statement (c). Let $H_1, \ldots, H_{n-1}$ be $n-1$ functionally independent polynomial first integrals of the polynomial differential system (1). Here we will use the notations defined in the proof of Theorem 1. We assume that $D(\mathcal{H}) := \det(\partial_1 \mathcal{H}, \cdots, \partial_{n-1}\mathcal{H}) \neq 0$ in $\mathbb{R}^n$ except perhaps a zero Lebesgue measure subset. Recall that $\mathcal{H} = (H_1, \ldots, H_{n-1})^T$ and $\partial_i \mathcal{H} = (\partial_i H_1, \ldots, \partial_i H_{n-1})^T$, $i = 1, \ldots, n$. The proof of Theorem 1 shows that

$$D(\mathcal{H})(f_1(x), \ldots, f_{n-1}(x)) = -(D_1(\mathcal{H}), \ldots, D_{n-1}(\mathcal{H})) f_n(x).$$

Since $D(\mathcal{H}), D_1(\mathcal{H}), \ldots, D_{n-1}(\mathcal{H})$ are polynomials, and $f_1(x), \ldots, f_n(x)$ are relatively prime polynomials, it verifies that $f_n(x)$ divides $D(\mathcal{H})(x)$.

Hence system (1) has the polynomial Jacobian multiplier

$$Q(x) = D(\mathcal{H})(x)/f_n(x).$$

This proves statement (c), and consequently the theorem. $\square$

3. An application of Theorem 1

Consider the differential system

$$\dot{x} = -y - z, \quad \dot{y} = x, \quad \dot{z} = xz, \tag{10}$$

which is the only completely integrable case of the Rössler differential system constructed by Rössler [15] in 1976. This unique integrable Rössler differential system was first proved in [18]. Recently system (10) was studied from the Poisson dynamics point of view, see [17].

We can check that system (10) has the two functionally independent first integrals

$$H_1(x,y,z) = \frac{1}{2}(x^2 + y^2) + z, \qquad H_2(x,y,z) = e^{-y} z,$$

and the Jacobian multiplier $J = e^{-y}$. We can check that the transformation of variables

$$u = J(x,y,z), \quad v = J(x,y,z)H_1(x,y,z), \quad w = J(x,y,z)H_2(x,y,z), \tag{11}$$

is invertible in the region $\Omega_0 := \{(x,y,z) \in \mathbb{R}^3 |\, x \neq 0\}$, because the Jacobian determinant of this transformation is xe^{-4y}. By Theorem 1, system (10) is transformed to

$$\dot{u} = u, \quad \dot{v} = v, \quad \dot{w} = w,$$

via the change of variables (11) in Ω_0. We note that the divergence of system (10) is x.

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