# The global regularity for the 3D continuously stratified inviscid quasi-geostrophic equations 

Dongho Chae<br>Department of Mathematics<br>Chung-Ang University<br>Seoul 156-756, Republic of Korea<br>email: dchae@cau.ac.kr<br>(to appear in J. Nonlinear Science)


#### Abstract

We prove the global well-posedness of the continuously stratified inviscid quasi-geostrophic equations in $\mathbb{R}^{3}$.


AMS Subject Classification Number: 35Q86, 35Q35, 76B03
keywords: stratified quasi-geostrophic equations, global regularity

## 1 Introduction

Let us consider the continuously stratified quasi-geostrophic equation for the stream function $\psi=\psi(x, y, z, t)$ on $\mathbb{R}^{3}$.

$$
\begin{align*}
& q_{t}+J(\psi, q)+\beta \psi_{x}=\nu \Delta q+\mathcal{F}  \tag{1.1}\\
& \text { with } \quad q:=\psi_{x x}+\psi_{y y}+F^{2} \psi_{z z}
\end{align*}
$$

Here, $F=L / L_{R}$ with $L$ the characteristic horizontal length of the flow and $L_{R}=$ $\sqrt{g H_{0}} / f_{0}$ the Rossby deformation radius, $H_{0}$ the typical depth of the fluid layer and $f_{0}$ the rotation rate of the fluid. On the other hand, $\nu$ is the viscosity, $\mathcal{F}$ is the external force, which will be set to zero for simplicity. In the above we used the notation, $J(f, g)=f_{x} g_{y}-f_{y} g_{x}$. The equation (1.1) is one of the basic equations in the geophysical fluid flows. For a physical meaning of it we mention that it can be derived from the Boussinesq equations(see [9, 6]). In Section 1.6 of [8] one can also see a very nice explanation of (1.1) in relation to the other models of the geophysical flows. Below we consider the inviscid case $\nu=0$, and set $\beta=1$ for convenience. The case $\nu>0$ is much easier to prove the global regularity. Below we introduce the notations

$$
v:=v(x, y, z, t)=\left(-\psi_{y}, \psi_{x}, \psi_{z}\right), \quad \bar{v}:=\left(-\psi_{y}, \psi_{x}, 0\right) .
$$

Rescaling in the $z$ variable as $z \rightarrow F^{-1} z$, we have $q=\Delta \psi$. Then the equation (1.1) in our case can be written as a Cauchy problem,

$$
\left\{\begin{array}{l}
q_{t}+(v \cdot \nabla) q=-v_{2}  \tag{1.2}\\
q=\Delta \psi \\
\left.v\right|_{t=0}=v_{0}
\end{array}\right.
$$

Comparing the system with the vorticity formulation of the 2D Euler equations,

$$
\begin{equation*}
\omega_{t}+(\bar{v} \cdot \nabla) \omega=0, \quad \omega=-\left(\partial_{x}^{2}+\partial_{y}^{2}\right) \psi, \quad v=\left(-\psi_{y}, \psi_{x}\right), \tag{1.3}
\end{equation*}
$$

One can observe a similarity with the correspondence $q \leftrightarrow \omega$. We note, however, that there exists an extra term, $v_{2}$ coupled, in the first equation of (1.2). Furthermore, more seriously, the relation between $\psi$ and $q$ is given by a full 3D Poisson equation in the second equation of (1.2), while in (1.3) the corresponding one is a 2 D equation. As far as the author knows the only mathematical result on the Cauchy problem of (1.2) is the local in time well-posedness due to Bennett and Kloeden([1). In the viscous case there is a study of the long time behavior of solutions of (1.1) by S . Wang([11). Actually the authors of [1] considered 3D periodic domain for the result, but since their proof used Kato's particle trajectory method([2]), it is straightforward to extend the result to the case of whole domain in $\mathbb{R}^{3}$ (see [7] in the case of 3D Euler equations on $\mathbb{R}^{3}$ ). In this paper our aim is to prove the following global regularity of solution to (1.2) for a given smooth initial data.
Theorem 1.1. Let $m>7 / 2$, and $v_{0} \in H^{m}\left(\mathbb{R}^{3}\right)$ be given. Then, for any $T>0$ there exists unique solution $v \in C\left([0, T) ; H^{m}\left(\mathbb{R}^{3}\right)\right)$ to the equation (1.2).

## 2 Proof of the main theorem

Proof of Theorem 1.1 The local well posedness of (1.2) for smooth $v_{0}$ is proved in [1], and therefore it suffices to prove the global in time a priori estimate. Namely, we will show that

$$
\begin{equation*}
\sup _{0<t<T}\|v(t)\|_{H^{m}} \leq C\left(\left\|v_{0}\right\|_{H^{m}}, T\right)<\infty \quad \forall T>0 \tag{2.1}
\end{equation*}
$$

for all $m>7 / 2$. Taking $L^{2}$ inner product (1.2) by $\psi$, and integrating by part, we obtain immediately

$$
\begin{equation*}
\|v(t)\|_{L^{2}}=\left\|v_{0}\right\|_{L^{2}}, \quad t>0 \tag{2.2}
\end{equation*}
$$

Similarly, taking $L^{2}$ inner product (1.2) by $q=\Delta \psi$, and integrating by part, we obtain immediately

$$
\begin{equation*}
\|q(t)\|_{L^{2}}=\left\|q_{0}\right\|_{L^{2}} \tag{2.3}
\end{equation*}
$$

Multiplying (1.2) by $q|q|^{4}$, and integrating over $\mathbb{R}^{3}$, we obtain after integration by part

$$
\begin{align*}
\frac{1}{6} \frac{d}{d t}\|q(t)\|_{L^{6}}^{6} & \leq \int_{\mathbb{R}^{3}}\left|v_{2}\left\|\left.q\right|^{5} d x \leq\right\| v_{2}\left\|_{L^{6}}\right\| q \|_{L^{6}}^{5}\right. \\
& \leq C\left\|\nabla v_{2}\right\|_{L^{2}}\|q\|_{L^{6}}^{5} \leq C\|q\|_{L^{2}}\|q\|_{L^{6}}^{5} \\
& =C\left\|q_{0}\right\|_{L^{2}}\|q\|_{L^{6}} \tag{2.4}
\end{align*}
$$

where we used the Sobolev inequality, $\|f\|_{L^{6}} \leq C\|\nabla f\|_{L^{2}}$ in $\mathbb{R}^{3}$, and the CalderonZygmund estimate(see [10]),

$$
\begin{equation*}
\|\nabla v\|_{L^{p}} \leq C_{p}\|q\|_{L^{p}}, \quad 1<p<\infty . \tag{2.5}
\end{equation*}
$$

From (2.4) we obtain

$$
\begin{equation*}
\|q(t)\|_{L^{6}} \leq\left\|q_{0}\right\|_{L^{6}}+C t\left\|q_{0}\right\|_{L^{2}} \tag{2.6}
\end{equation*}
$$

Hence, by the Gagliardo-Nirenberg inequality and the Calderon-Zygmund inequality we have

$$
\begin{align*}
\|v\|_{L^{\infty}} & \leq C\|D v\|_{L^{6}}^{\frac{3}{4}}\|v\|_{L^{2}}^{\frac{1}{4}} \leq C\|q\|_{L^{6}}^{\frac{3}{4}}\|v\|_{L^{2}}^{\frac{1}{4}} \leq C\|q\|_{L^{6}}+C\|v\|_{L^{2}} \\
& \leq C\left(\left\|q_{0}\right\|_{L^{6}}+t\left\|q_{0}\right\|_{L^{2}}\right)+C\left\|v_{0}\right\|_{L^{2}} \tag{2.7}
\end{align*}
$$

We introduce the particle trajectory $\{X(a, t)\}$ on the plane generated by $\tilde{v}:=\left(v_{1}, v_{2}\right)$,

$$
\frac{\partial X(a, t)}{\partial t}=\tilde{v}(X(a, t), t), \quad X(a, 0)=a \in \mathbb{R}^{2}
$$

We write (1.2) in the form

$$
\frac{\partial}{\partial t} q(X(a, t), z, t)=-v_{2}(X(a, t), z, t)
$$

which can be integrated in time as

$$
q(X(a, t), z, t)=q_{0}(a, z)-\int_{0}^{t} v_{2}(X(a, s), z, s) d s
$$

Thus, we have

$$
\begin{equation*}
\|q(t)\|_{L^{\infty}} \leq\left\|q_{0}\right\|_{L^{\infty}}+\int_{0}^{t}\|v(s)\|_{L^{\infty}} d s \leq C\left(1+t^{2}\right) \tag{2.8}
\end{equation*}
$$

where $C=C\left(\left\|q_{0}\right\|_{L^{6}},\left\|q_{0}\right\|_{L^{2}},\left\|v_{0}\right\|_{L^{2}}\right)$. Combining (2.3) and (2.8), using the standard $L^{p}$ interpolation, one has

$$
\begin{equation*}
\|q(t)\|_{L^{p}} \leq C_{1}\left(1+t^{2}\right), \quad \forall p \in[2, \infty] \tag{2.9}
\end{equation*}
$$

where $C_{1}=C_{1}\left(\left\|q_{0}\right\|_{L^{6}},\left\|q_{0}\right\|_{L^{2}},\left\|v_{0}\right\|_{L^{2}}\right)$. Taking $D=\left(\partial_{1}, \partial_{2}, \partial_{3}\right)$ on (1.2), one has

$$
D q_{t}+(D \bar{v} \cdot \nabla) q+(\bar{v} \cdot \nabla) D q=-D v_{2}
$$

Let $p \geq 2$. Multiplying this equation by $D q|D q|^{p-2}$ and integrating it over $\mathbb{R}^{3}$, we have after integration by part, and using the Hölder inequality and (2.5),

$$
\begin{align*}
\frac{1}{p} \frac{d}{d t}\|D q(t)\|_{L^{p}}^{p} & \leq\|D v\|_{L^{\infty}}\|D q\|_{L^{p}}^{p}+\left\|D v_{2}\right\|_{L^{p}}\|D q\|_{L^{p}}^{p-1} \\
& \leq\|D v\|_{L^{\infty}}\|D q\|_{L^{p}}^{p}+C\|q\|_{L^{p}}\|D q\|_{L^{p}}^{p-1} \tag{2.10}
\end{align*}
$$

from which we obtain, for $p>3$,

$$
\begin{align*}
\frac{d}{d t}\|D q(t)\|_{L^{p}} & \leq\|D v\|_{L^{\infty}}\|D q\|_{L^{p}}+C\|q\|_{L^{p}} \\
& \leq C\left\{1+\|D v\|_{B M O} \log \left(e+\left\|D^{2} v\right\|_{L^{p}}\right)\right\}\|D q\|_{L^{p}}+C\left\|_{q}\right\|_{L^{p}} \\
& \leq C\left\{1+\|q\|_{B M O} \log \left(e+\|D q\|_{L^{p}}\right)\right\}\|D\|_{L^{p}}+C\|q\|_{L^{p}} \\
& \leq C\left\{1+\|q\|_{L^{\infty}} \log \left(e+\|D q\|_{L^{p}}\right)\right\}\|D q\|_{L^{p}}+C\|q\|_{L^{p}} \\
& \leq C\left(1+\|q\|_{L^{\infty}}+\|q\|_{L^{p}}\right)\left(e+\|D q\|_{L^{p}}\right) \log \left(e+\|D q\|_{L^{p}}\right) \tag{2.11}
\end{align*}
$$

where we used the logarithmic Sobolev inequality,

$$
\begin{equation*}
\|f\|_{L^{\infty}} \leq C\left\{1+\|f\|_{B M O} \log \left(e+\|D f\|_{W^{k, p}}\right)\right\}, \quad k p>3 \tag{2.12}
\end{equation*}
$$

proved in [5], and the Calderon-Zygmund inequality. By Gronwall's inequality, we obtain from (2.11) that

$$
\begin{equation*}
e+\|D q(t)\|_{L^{p}} \leq\left(e+\left\|D q_{0}\right\|_{L^{p}}\right)^{\exp \left\{C \int_{0}^{t}\left(1+\|q(s)\|_{L^{\infty}}+\|q(s)\|_{L^{p}}\right) d s\right\}} . \tag{2.13}
\end{equation*}
$$

Taking into account (2.9) and (2.8), we find from (2.13) that

$$
\begin{equation*}
\sup _{0<t<T}\|D q(t)\|_{L^{p}} \leq C\left(v_{0}, T\right)<\infty \quad \forall T>0, \quad \forall p \in(3, \infty) \tag{2.14}
\end{equation*}
$$

Combining this with the Gagliardo-Nirenberg inequality and (2.5), we obtain

$$
\begin{align*}
\sup _{0<t<T}\|D v\|_{L^{\infty}} & \leq C \sup _{0<t<T}\left\|D^{2} v\right\|_{L^{4}} \leq C \sup _{0<t<T}\|D q\|_{L^{4}} \\
& \leq C\left(\left\|v_{0}\right\|_{W^{2,4}}, T\right)<\infty \quad \forall T>0 \tag{2.15}
\end{align*}
$$

For $p \in[2,3]$, one has from (2.10) that

$$
\begin{aligned}
\frac{d}{d t}\|D q(t)\|_{L^{p}} & \leq\|D v\|_{L^{\infty}}\|D q\|_{L^{p}}+C\|q\|_{L^{p}} \\
& \leq\left(\|D v\|_{L^{\infty}}+\|q\|_{L^{p}}+1\right)\left(\|D q\|_{L^{p}}+1\right)
\end{aligned}
$$

from which we obtain

$$
\begin{equation*}
\|D q(t)\|_{L^{p}}+1 \leq\left(\left\|D q_{0}\right\|_{L^{p}}+1\right) \exp \left\{C \int_{0}^{t}\left(\|D v(s)\|_{L^{\infty}}+\|q(s)\|_{L^{p}}+1\right) d s\right\} \tag{2.16}
\end{equation*}
$$

Hence, the estimates (2.9) and (2.15) imply that

$$
\begin{equation*}
\sup _{0<t<T}\|D q(t)\|_{L^{p}} \leq C\left(\left\|v_{0}\right\|_{W^{2, p}}, T\right)<\infty \quad \forall T>0 \tag{2.17}
\end{equation*}
$$

Taking $D^{2}$ on (1.1), we have

$$
D^{2} q_{t}+\left(D^{2} \bar{v} \cdot \nabla\right) q+2(D \bar{v} \cdot \nabla) D q+(v \cdot \nabla) D^{2} q=-D^{2} v_{2}
$$

Multiplying this by $D^{2} q\left|D^{2} q\right|$ and integrating it over $\mathbb{R}^{3}$, we have after integration by part, and using the Hölder inequality,

$$
\begin{align*}
& \frac{1}{3} \frac{d}{d t}\left\|D^{2} q(t)\right\|_{L^{3}}^{3} \leq\left\|D^{2} v \cdot D q\right\|_{L^{3}}\left\|D^{2} q\right\|_{L^{3}}^{2}+2\|D v\|_{L^{\infty}}\left\|D^{2} q\right\|_{L^{3}}^{3} \\
&+\left\|D^{2} v_{2}\right\|_{L^{3}}\left\|D^{2} q\right\|_{L^{3}}^{2} \\
& \leq C\left(\left\|D^{2} v\right\|_{B M O}\|D q\|_{L^{3}}+\left\|D^{2} v\right\|_{L^{3}}\|D q\|_{B M O}\right)\left\|D^{2} q\right\|_{L^{3}}^{2} \\
&+2\|D v\|_{L^{\infty}}\left\|D^{2} q\right\|_{L^{3}}^{3}+\left\|D^{2} v_{2}\right\|_{L^{3}}\left\|D^{2} q\right\|_{L^{3}}^{2} \\
& \leq C\|D q\|_{B M O}\|D q\|_{L^{3}}\left\|D^{2} q\right\|_{L^{3}}^{2} \\
& \quad+2\|D v\|_{L^{\infty}}\left\|D^{2} q\right\|_{L^{3}}^{3}+\|D q\|_{L^{3}}\left\|D^{2} q\right\|_{L^{3}}^{2} \\
& \leq C\|D q\|_{L^{3}}\left\|D^{2} q\right\|_{L^{3}}^{3}+2\|D v\|_{L^{\infty}}\left\|D^{2} q\right\|_{L^{3}}^{3} \\
&+\|D q\|_{L^{3}}\left\|D^{2} q\right\|_{L^{3}}^{2} \tag{2.18}
\end{align*}
$$

where we used the following bilinear estimate, proved in [4],

$$
\|f \cdot g\|_{L^{p}} \leq C_{p}\left(\|f\|_{L^{p}}\|g\|_{B M O}+\|f\|_{B M O}\|g\|_{L^{p}}\right), \quad p \in(1, \infty)
$$

and also the critical Sobolev inequality in $\mathbb{R}^{3}$,

$$
\|f\|_{B M O} \leq C\|\nabla f\|_{L^{3}} .
$$

From (2.18) one has

$$
\begin{align*}
\frac{d}{d t}\left\|D^{2} q(t)\right\|_{L^{3}} & \leq C\left(\|D q\|_{L^{3}}+\|D v\|_{L^{\infty}}\right)\left\|D^{2} q\right\|_{L^{3}}+\|D q\|_{L^{3}} \\
& \leq C\left(\|D q\|_{L^{3}}+\|D v\|_{L^{\infty}}+1\right)\left(\left\|D^{2} q\right\|_{L^{3}}+1\right) \tag{2.19}
\end{align*}
$$

from which one has

$$
\begin{equation*}
\left\|D^{2} q(t)\right\|_{L^{3}}+1 \leq\left(\left\|D^{2} q_{0}\right\|_{L^{3}}+1\right) \exp \left\{C \int_{0}^{t}\left(\|D q(s)\|_{L^{3}}+\|D v(s)\|_{L^{\infty}}+1\right) d s\right\} . \tag{2.20}
\end{equation*}
$$

Combining this with (2.17) and (2.15), we have

$$
\begin{equation*}
\sup _{0<t<T}\|D q(t)\|_{B M O} \leq C \sup _{0<t<T}\left\|D^{2} q(t)\right\|_{L^{3}} \leq C\left(\left\|v_{0}\right\|_{W^{3,3}}, T\right)<\infty \quad \forall T>0 \tag{2.21}
\end{equation*}
$$

Let $\alpha=\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right) \in(\mathbb{N} \cup\{0\})^{3}$ be a muti-index. Let $m>7 / 2$. Taking $D^{\alpha}$ on (1.2), and multiplying it by $D^{\alpha} q$, summing over $|\alpha| \leq m-1$, and integrating it over $\mathbb{R}^{3}$, we have

$$
\begin{align*}
\frac{1}{2} \frac{d}{d t}\|q(t)\|_{H^{m-1}}^{2} & =\sum_{|\alpha| \leq m-1}\left(D^{\alpha}(\bar{v} \cdot \nabla) q, D^{\alpha} q\right)_{L^{2}}-\sum_{|\alpha| \leq m-1}\left(D^{\alpha} v_{2}, D^{\alpha} q\right)_{L^{2}} \\
& =\sum_{|\alpha| \leq m}\left(D^{\alpha}(\bar{v} \cdot \nabla) q-(\bar{v} \cdot \nabla) D^{\alpha} q, D^{\alpha} q\right)_{L^{2}}-\sum_{|\alpha| \leq m-1}\left(D^{\alpha} v_{2}, D^{\alpha} q\right)_{L^{2}} \\
& \leq \sum_{|\alpha| \leq m-1}\left\|D^{\alpha}(\bar{v} \cdot \nabla) q-(\bar{v} \cdot \nabla) D^{\alpha} q\right\|_{L^{2}}\|q\|_{H^{m-1}}+\|v\|_{H^{m-1}}\|q\|_{H^{m-1}} \\
& \leq C\left(\|\nabla v\|_{L^{\infty}}+\|\nabla q\|_{L^{\infty}}\right)\left(\|q\|_{H^{m-1}}+\|v\|_{H^{m-1}}\right)\|q\|_{H^{m-1}}+\|v\|_{H^{m-1}}\|q\|_{H^{m-1}} \tag{2.22}
\end{align*}
$$

where we used the following commutator estimate,

$$
\sum_{|\alpha| \leq m}\left\|D^{\alpha}(f g)-f D^{\alpha} g\right\|_{L^{2}} \leq C_{m}\left(\|\nabla f\|_{L^{\infty}}\left\|D^{m-1} g\right\|_{L^{2}}+\left\|D^{m} f\right\|_{L^{2}}\|g\|_{L^{\infty}}\right)
$$

proved in [3]. We observe the following norm equivalence: there exists a constant $K$ independent of $v, q$ such that

$$
K^{-1}\left(\|v\|_{L^{2}}^{2}+\|q\|_{H^{m-1}}^{2}\right) \leq\|v\|_{H^{m}}^{2} \leq K\left(\|v\|_{L^{2}}^{2}+\|q\|_{H^{m-1}}^{2}\right),
$$

which is an immediate consequence of (2.5). Since $\|v(t)\|_{L^{2}}=\left\|v_{0}\right\|_{L^{2}}$ as in (2.2), one can add $\left\|v_{0}\right\|_{L^{2}}^{2}$ to $\|q\|_{H^{m-1}}^{2}$ in the left hand side of (2.22) to obtain

$$
\begin{align*}
\frac{d}{d t} Y(t) & \leq C\left(\|\nabla v\|_{L^{\infty}}+\|\nabla q\|_{L^{\infty}}+1\right) Y(t) \\
& \leq C\left(\|\nabla v\|_{B M O}+\|\nabla q\|_{B M O}+1\right) Y(t) \log Y(t) \tag{2.23}
\end{align*}
$$

where we set

$$
Y(t):=e+\|v(t)\|_{L^{2}}^{2}+\|q(t)\|_{H^{m}}^{2}
$$

and used the logarithmic Sobolev inequality in the form (2.12) for $m>7 / 2$. By Gronwall's inequality we have

$$
\begin{align*}
e+\|v(t)\|_{H^{m}}^{2} & \leq e+C\left(\|v(t)\|_{L^{2}}^{2}+\|\left. q(t)\right|_{H^{m-1}} ^{2}\right) \\
& \leq e+C\left(\left\|v_{0}\right\|_{L^{2}}^{2}+\left\|q_{0}\right\|_{H^{m-1}}^{2}\right)^{\exp }\left\{C \int_{0}^{t}\left(\|\nabla v(s)\|_{B M O}+\|\nabla q(s)\|_{B M O}+1\right) d s\right\} \\
& \leq e+\left(C\left\|v_{0}\right\|_{H^{m}}^{2}\right)^{\exp \left\{C \int_{0}^{t}\left(\|\nabla v(s)\|_{L^{\infty}}+\|\nabla q(s)\|_{B M O}+1\right) d s\right\}} . \tag{2.24}
\end{align*}
$$

The estimates (2.15) and (2.21), combined with (2.24), provides us with (2.1).

## Acknowledgements

The author would like to thank deeply to Prof. Shouhong Wang for suggesting the problem and helpful discussions. This research is supported partially by NRF Grants no.2006-0093854 and no.2009-0083521.

## References

[1] A. F. Bennett and P. E. Kloeden. The periodic quasigeostrophic equations: existence and uniqueness of strong solutions, Proc. Roy. Soc. Edinburgh, 91A, (1982), pp. 185-203.
[2] T. Kato, On classical solutions of the two-dimensional non-stationary Euler equations, Arch. Rational Mech. Anal., 25, (1967), pp. 188-200.
[3] S. Klainerman and A. Majda, Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids, Commun. Pure Appl. Math, 34, (1981), pp. 481-524.
[4] H. Kozono and Y. Taniuchi, The Bilinear estimates in BMO and the Navier-Stokes equations, Math. Z., 235(1), (2000), pp. 173-194.
[5] H. Kozono and Y. Taniuchi, Limiting case of the Sobolev inequalty in BMO with application to the Euler equations, Comm. Math. Phys. 214 (1), (2000), pp. 191200.
[6] A. Majda, Introduction to P.D.E's and Waves for the Atmosphere and Ocean, Courant Institute Lecture Note Series, 9, Amer. Math. Soc..
[7] A. Majda and A. Bertozzi, Vorticity and Incompressible Flow, Cambridge University Press, (2002), Cambridge.
[8] A. Majda and X. Wang, Nonlinear Dynamics and Statistical Theories for Basic Geophysical Flows, Cambridge University Press, (2006), Cambridge.
[9] J. Pedlosky, Geophysical Fluid Dynamics, 2nd Edition, Springer, (1987), New York.
[10] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, (1970), Princeton, NJ.
[11] S. Wang, Attractors for the 3D baroclinic quasi-geostrophic equations of largescale atmosphere, J. Math. Anal. Appl., 165, no. 1, (1992), pp. 266-283,

