The global regularity for the 3D continuously stratified inviscid quasi-geostrophic equations

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Abstract

We prove the global well-posedness of the continuously stratified inviscid quasi-geostrophic equations in \mathbb{R}^3 .

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1 Introduction

Let us consider the continuously stratified quasi-geostrophic equation for the stream function $\psi = \psi(x, y, z, t)$ on \mathbb{R}^3 .

$$q_t + J(\psi, q) + \beta \psi_x = \nu \Delta q + \mathcal{F}$$
with
$$q := \psi_{xx} + \psi_{yy} + F^2 \psi_{zz}.$$
(1.1)

Here, $F = L/L_R$ with L the characteristic horizontal length of the flow and $L_R = \sqrt{gH_0}/f_0$ the Rossby deformation radius, H_0 the typical depth of the fluid layer and f_0 the rotation rate of the fluid. On the other hand, ν is the viscosity, \mathcal{F} is the external force, which will be set to zero for simplicity. In the above we used the notation, $J(f,g) = f_x g_y - f_y g_x$. The equation (1.1) is one of the basic equations in the geophysical fluid flows. For a physical meaning of it we mention that it can be derived from the Boussinesq equations(see[9, 6]). In Section 1.6 of [8] one can also see a very nice explanation of (1.1) in relation to the other models of the geophysical flows. Below we consider the inviscid case $\nu = 0$, and set $\beta = 1$ for convenience. The case $\nu > 0$ is much easier to prove the global regularity. Below we introduce the notations

$$v := v(x, y, z, t) = (-\psi_y, \psi_x, \psi_z), \quad \bar{v} := (-\psi_y, \psi_x, 0).$$

Rescaling in the z variable as $z \to F^{-1}z$, we have $q = \Delta \psi$. Then the equation (1.1) in our case can be written as a Cauchy problem,

$$\begin{cases} q_t + (v \cdot \nabla)q = -v_2, \\ q = \Delta \psi, \\ v|_{t=0} = v_0. \end{cases}$$
(1.2)

Comparing the system with the vorticity formulation of the 2D Euler equations,

$$\omega_t + (\bar{v} \cdot \nabla)\omega = 0, \quad \omega = -(\partial_x^2 + \partial_y^2)\psi, \quad v = (-\psi_y, \psi_x), \tag{1.3}$$

One can observe a similarity with the correspondence $q \leftrightarrow \omega$. We note, however, that there exists an extra term, v_2 coupled, in the first equation of (1.2). Furthermore, more seriously, the relation between ψ and q is given by a full 3D Poisson equation in the second equation of (1.2), while in (1.3) the corresponding one is a 2D equation. As far as the author knows the only mathematical result on the Cauchy problem of (1.2) is the local in time well-posedness due to Bennett and Kloeden([1]). In the viscous case there is a study of the long time behavior of solutions of (1.1) by S. Wang([11]). Actually the authors of [1] considered 3D periodic domain for the result, but since their proof used Kato's particle trajectory method([2]), it is straightforward to extend the result to the case of whole domain in \mathbb{R}^3 (see [7] in the case of 3D Euler equations on \mathbb{R}^3). In this paper our aim is to prove the following global regularity of solution to (1.2) for a given smooth initial data.

Theorem 1.1. Let m > 7/2, and $v_0 \in H^m(\mathbb{R}^3)$ be given. Then, for any T > 0 there exists unique solution $v \in C([0,T); H^m(\mathbb{R}^3))$ to the equation (1.2).

2 Proof of the main theorem

Proof of Theorem 1.1 The local well posedness of (1.2) for smooth v_0 is proved in [1], and therefore it suffices to prove the global in time *a priori* estimate. Namely, we will show that

$$\sup_{0 < t < T} \|v(t)\|_{H^m} \le C(\|v_0\|_{H^m}, T) < \infty \quad \forall T > 0$$
(2.1)

for all m > 7/2. Taking L^2 inner product (1.2) by ψ , and integrating by part, we obtain immediately

$$\|v(t)\|_{L^2} = \|v_0\|_{L^2}, \quad t > 0.$$
(2.2)

Similarly, taking L^2 inner product (1.2) by $q = \Delta \psi$, and integrating by part, we obtain immediately

$$\|q(t)\|_{L^2} = \|q_0\|_{L^2}.$$
(2.3)

Multiplying (1.2) by $q|q|^4$, and integrating over \mathbb{R}^3 , we obtain after integration by part

$$\frac{1}{6} \frac{d}{dt} \|q(t)\|_{L^6}^6 \leq \int_{\mathbb{R}^3} |v_2| |q|^5 dx \leq \|v_2\|_{L^6} \|q\|_{L^6}^5 \\
\leq C \|\nabla v_2\|_{L^2} \|q\|_{L^6}^5 \leq C \|q\|_{L^2} \|q\|_{L^6}^5 \\
= C \|q_0\|_{L^2} \|q\|_{L^6}^5,$$
(2.4)

where we used the Sobolev inequality, $||f||_{L^6} \leq C ||\nabla f||_{L^2}$ in \mathbb{R}^3 , and the Calderon-Zygmund estimate(see [10]),

$$\|\nabla v\|_{L^p} \le C_p \|q\|_{L^p}, \quad 1 (2.5)$$

From (2.4) we obtain

$$\|q(t)\|_{L^6} \le \|q_0\|_{L^6} + Ct \|q_0\|_{L^2}.$$
(2.6)

Hence, by the Gagliardo-Nirenberg inequality and the Calderon-Zygmund inequality we have

$$\|v\|_{L^{\infty}} \leq C \|Dv\|_{L^{6}}^{\frac{3}{4}} \|v\|_{L^{2}}^{\frac{1}{4}} \leq C \|q\|_{L^{6}}^{\frac{3}{4}} \|v\|_{L^{2}}^{\frac{1}{4}} \leq C \|q\|_{L^{6}} + C \|v\|_{L^{2}}$$

$$\leq C (\|q_{0}\|_{L^{6}} + t \|q_{0}\|_{L^{2}}) + C \|v_{0}\|_{L^{2}}.$$
 (2.7)

We introduce the particle trajectory $\{X(a,t)\}$ on the plane generated by $\tilde{v} := (v_1, v_2)$,

$$\frac{\partial X(a,t)}{\partial t} = \tilde{v}(X(a,t),t), \quad X(a,0) = a \in \mathbb{R}^2,$$

We write (1.2) in the form

$$\frac{\partial}{\partial t}q(X(a,t),z,t) = -v_2(X(a,t),z,t),$$

which can be integrated in time as

$$q(X(a,t), z, t) = q_0(a, z) - \int_0^t v_2(X(a, s), z, s) ds.$$

Thus, we have

$$\|q(t)\|_{L^{\infty}} \le \|q_0\|_{L^{\infty}} + \int_0^t \|v(s)\|_{L^{\infty}} ds \le C(1+t^2),$$
(2.8)

where $C = C(||q_0||_{L^6}, ||q_0||_{L^2}, ||v_0||_{L^2})$. Combining (2.3) and (2.8), using the standard L^p interpolation, one has

$$||q(t)||_{L^p} \le C_1(1+t^2), \quad \forall p \in [2,\infty]$$
 (2.9)

where $C_1 = C_1(||q_0||_{L^6}, ||q_0||_{L^2}, ||v_0||_{L^2})$. Taking $D = (\partial_1, \partial_2, \partial_3)$ on (1.2), one has

$$Dq_t + (D\bar{v}\cdot\nabla)q + (\bar{v}\cdot\nabla)Dq = -Dv_2.$$

Let $p \geq 2$. Multiplying this equation by $Dq|Dq|^{p-2}$ and integrating it over \mathbb{R}^3 , we have after integration by part, and using the Hölder inequality and (2.5),

$$\frac{1}{p} \frac{d}{dt} \|Dq(t)\|_{L^{p}}^{p} \leq \|Dv\|_{L^{\infty}} \|Dq\|_{L^{p}}^{p} + \|Dv_{2}\|_{L^{p}} \|Dq\|_{L^{p}}^{p-1} \\
\leq \|Dv\|_{L^{\infty}} \|Dq\|_{L^{p}}^{p} + C\|q\|_{L^{p}} \|Dq\|_{L^{p}}^{p-1},$$
(2.10)

from which we obtain, for p > 3,

$$\frac{d}{dt} \|Dq(t)\|_{L^{p}} \leq \|Dv\|_{L^{\infty}} \|Dq\|_{L^{p}} + C\|q\|_{L^{p}} \\
\leq C\{1 + \|Dv\|_{BMO} \log(e + \|D^{2}v\|_{L^{p}})\} \|Dq\|_{L^{p}} + C\|q\|_{L^{p}} \\
\leq C\{1 + \|q\|_{BMO} \log(e + \|Dq\|_{L^{p}})\} \|Dq\|_{L^{p}} + C\|q\|_{L^{p}} \\
\leq C\{1 + \|q\|_{L^{\infty}} \log(e + \|Dq\|_{L^{p}})\} \|Dq\|_{L^{p}} + C\|q\|_{L^{p}} \\
\leq C\{1 + \|q\|_{L^{\infty}} + \|q\|_{L^{p}})(e + \|Dq\|_{L^{p}}) \log(e + \|Dq\|_{L^{p}}),$$
(2.11)

where we used the logarithmic Sobolev inequality,

$$||f||_{L^{\infty}} \le C\{1 + ||f||_{BMO} \log(e + ||Df||_{W^{k,p}})\}, \quad kp > 3$$
(2.12)

proved in [5], and the Calderon-Zygmund inequality. By Gronwall's inequality, we obtain from (2.11) that

$$e + \|Dq(t)\|_{L^p} \le (e + \|Dq_0\|_{L^p})^{\exp\left\{C\int_0^t (1+\|q(s)\|_{L^{\infty}} + \|q(s)\|_{L^p})ds\right\}}.$$
(2.13)

Taking into account (2.9) and (2.8), we find from (2.13) that

$$\sup_{0 < t < T} \|Dq(t)\|_{L^p} \le C(v_0, T) < \infty \quad \forall T > 0, \quad \forall p \in (3, \infty).$$
(2.14)

Combining this with the Gagliardo-Nirenberg inequality and (2.5), we obtain

$$\sup_{0 < t < T} \|Dv\|_{L^{\infty}} \leq C \sup_{0 < t < T} \|D^{2}v\|_{L^{4}} \leq C \sup_{0 < t < T} \|Dq\|_{L^{4}} \\
\leq C(\|v_{0}\|_{W^{2,4}}, T) < \infty \quad \forall T > 0.$$
(2.15)

For $p \in [2, 3]$, one has from (2.10) that

$$\begin{aligned} \frac{d}{dt} \|Dq(t)\|_{L^p} &\leq \|Dv\|_{L^{\infty}} \|Dq\|_{L^p} + C \|q\|_{L^p} \\ &\leq (\|Dv\|_{L^{\infty}} + \|q\|_{L^p} + 1)(\|Dq\|_{L^p} + 1), \end{aligned}$$

from which we obtain

$$\|Dq(t)\|_{L^p} + 1 \le (\|Dq_0\|_{L^p} + 1) \exp\left\{C\int_0^t (\|Dv(s)\|_{L^\infty} + \|q(s)\|_{L^p} + 1)ds\right\}.$$
 (2.16)

Hence, the estimates (2.9) and (2.15) imply that

$$\sup_{0 < t < T} \|Dq(t)\|_{L^p} \le C(\|v_0\|_{W^{2,p}}, T) < \infty \quad \forall T > 0.$$
(2.17)

Taking D^2 on (1.1), we have

$$D^2q_t + (D^2\bar{v}\cdot\nabla)q + 2(D\bar{v}\cdot\nabla)Dq + (v\cdot\nabla)D^2q = -D^2v_2.$$

Multiplying this by $D^2 q |D^2 q|$ and integrating it over \mathbb{R}^3 , we have after integration by part, and using the Hölder inequality,

$$\frac{1}{3} \frac{d}{dt} \|D^{2}q(t)\|_{L^{3}}^{3} \leq \|D^{2}v \cdot Dq\|_{L^{3}} \|D^{2}q\|_{L^{3}}^{2} + 2\|Dv\|_{L^{\infty}} \|D^{2}q\|_{L^{3}}^{3}
+ \|D^{2}v_{2}\|_{L^{3}} \|D^{2}q\|_{L^{3}}^{2}
\leq C(\|D^{2}v\|_{BMO}\|Dq\|_{L^{3}} + \|D^{2}v\|_{L^{3}} \|Dq\|_{BMO}) \|D^{2}q\|_{L^{3}}^{2}
+ 2\|Dv\|_{L^{\infty}} \|D^{2}q\|_{L^{3}}^{3} + \|D^{2}v_{2}\|_{L^{3}} \|D^{2}q\|_{L^{3}}^{2}
\leq C\|Dq\|_{BMO} \|Dq\|_{L^{3}} \|D^{2}q\|_{L^{3}}^{2}
+ 2\|Dv\|_{L^{\infty}} \|D^{2}q\|_{L^{3}}^{3} + \|Dq\|_{L^{3}} \|D^{2}q\|_{L^{3}}^{2}
\leq C\|Dq\|_{L^{3}} \|D^{2}q\|_{L^{3}}^{3} + 2\|Dv\|_{L^{\infty}} \|D^{2}q\|_{L^{3}}^{2}
\leq C\|Dq\|_{L^{3}} \|D^{2}q\|_{L^{3}}^{3} + 2\|Dv\|_{L^{\infty}} \|D^{2}q\|_{L^{3}}^{3}
+ \|Dq\|_{L^{3}} \|D^{2}q\|_{L^{3}}^{2},$$
(2.18)

where we used the following bilinear estimate, proved in [4],

$$||f \cdot g||_{L^p} \le C_p(||f||_{L^p} ||g||_{BMO} + ||f||_{BMO} ||g||_{L^p}), \quad p \in (1,\infty)$$

and also the critical Sobolev inequality in \mathbb{R}^3 ,

$$\|f\|_{BMO} \le C \|\nabla f\|_{L^3}.$$

From (2.18) one has

$$\frac{d}{dt} \|D^2 q(t)\|_{L^3} \leq C(\|Dq\|_{L^3} + \|Dv\|_{L^{\infty}}) \|D^2 q\|_{L^3} + \|Dq\|_{L^3} \\
\leq C(\|Dq\|_{L^3} + \|Dv\|_{L^{\infty}} + 1)(\|D^2 q\|_{L^3} + 1),$$
(2.19)

from which one has

$$\|D^{2}q(t)\|_{L^{3}} + 1 \leq (\|D^{2}q_{0}\|_{L^{3}} + 1) \exp\left\{C\int_{0}^{t} (\|Dq(s)\|_{L^{3}} + \|Dv(s)\|_{L^{\infty}} + 1)ds\right\}.$$
(2.20)

Combining this with (2.17) and (2.15), we have

$$\sup_{0 < t < T} \|Dq(t)\|_{BMO} \le C \sup_{0 < t < T} \|D^2q(t)\|_{L^3} \le C(\|v_0\|_{W^{3,3}}, T) < \infty \quad \forall T > 0.$$
(2.21)

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in (\mathbb{N} \cup \{0\})^3$ be a muti-index. Let m > 7/2. Taking D^{α} on (1.2), and multiplying it by $D^{\alpha}q$, summing over $|\alpha| \leq m-1$, and integrating it over \mathbb{R}^3 , we have

$$\frac{1}{2} \frac{d}{dt} \|q(t)\|_{H^{m-1}}^{2} = \sum_{|\alpha| \le m-1} (D^{\alpha}(\bar{v} \cdot \nabla)q, D^{\alpha}q)_{L^{2}} - \sum_{|\alpha| \le m-1} (D^{\alpha}v_{2}, D^{\alpha}q)_{L^{2}} \\
= \sum_{|\alpha| \le m} (D^{\alpha}(\bar{v} \cdot \nabla)q - (\bar{v} \cdot \nabla)D^{\alpha}q, D^{\alpha}q)_{L^{2}} - \sum_{|\alpha| \le m-1} (D^{\alpha}v_{2}, D^{\alpha}q)_{L^{2}} \\
\le \sum_{|\alpha| \le m-1} \|D^{\alpha}(\bar{v} \cdot \nabla)q - (\bar{v} \cdot \nabla)D^{\alpha}q\|_{L^{2}} \|q\|_{H^{m-1}} + \|v\|_{H^{m-1}} \|q\|_{H^{m-1}} \\
\le C(\|\nabla v\|_{L^{\infty}} + \|\nabla q\|_{L^{\infty}})(\|q\|_{H^{m-1}} + \|v\|_{H^{m-1}})\|q\|_{H^{m-1}} + \|v\|_{H^{m-1}} \|q\|_{H^{m-1}}, \tag{2.22}$$

where we used the following commutator estimate,

$$\sum_{|\alpha| \le m} \|D^{\alpha}(fg) - fD^{\alpha}g\|_{L^{2}} \le C_{m}(\|\nabla f\|_{L^{\infty}}\|D^{m-1}g\|_{L^{2}} + \|D^{m}f\|_{L^{2}}\|g\|_{L^{\infty}}),$$

proved in [3]. We observe the following norm equivalence: there exists a constant K independent of v, q such that

$$K^{-1}(\|v\|_{L^{2}}^{2} + \|q\|_{H^{m-1}}^{2}) \leq \|v\|_{H^{m}}^{2} \leq K(\|v\|_{L^{2}}^{2} + \|q\|_{H^{m-1}}^{2}),$$

which is an immediate consequence of (2.5). Since $||v(t)||_{L^2} = ||v_0||_{L^2}$ as in (2.2), one can add $||v_0||_{L^2}^2$ to $||q||_{H^{m-1}}^2$ in the left hand side of (2.22) to obtain

$$\frac{d}{dt}Y(t) \leq C(\|\nabla v\|_{L^{\infty}} + \|\nabla q\|_{L^{\infty}} + 1)Y(t) \\
\leq C(\|\nabla v\|_{BMO} + \|\nabla q\|_{BMO} + 1)Y(t)\log Y(t)$$
(2.23)

where we set

$$Y(t) := e + \|v(t)\|_{L^2}^2 + \|q(t)\|_{H^m}^2,$$

and used the logarithmic Sobolev inequality in the form (2.12) for m > 7/2. By Gronwall's inequality we have

$$e + \|v(t)\|_{H^{m}}^{2} \leq e + C(\|v(t)\|_{L^{2}}^{2} + \|q(t)\|_{H^{m-1}}^{2})$$

$$\leq e + C(\|v_{0}\|_{L^{2}}^{2} + \|q_{0}\|_{H^{m-1}}^{2})^{\exp\left\{C\int_{0}^{t}(\|\nabla v(s)\|_{BMO} + \|\nabla q(s)\|_{BMO} + 1)ds\right\}}$$

$$\leq e + (C\|v_{0}\|_{H^{m}}^{2})^{\exp\left\{C\int_{0}^{t}(\|\nabla v(s)\|_{L^{\infty}} + \|\nabla q(s)\|_{BMO} + 1)ds\right\}}.$$

$$(2.24)$$

The estimates (2.15) and (2.21), combined with (2.24), provides us with (2.1). \Box

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