On orbital instability of spectrally stable vortices of the NLS in the plane

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Abstract

We explain how spectrally stable vortices of the Nonlinear Schrödinger Equation in the plane can be orbitally unstable. This relates to the nonlinear Fermi golden rule, a mechanism which exploits the nonlinear interaction between discrete and continuous modes of the NLS.

1 Introduction

In this paper we consider the nonlinear Schrödinger equation (NLS):

$$\mathrm{i}u_t = -\Delta u + Vu + \beta(|u|^2)u,\ u(0,x) = u_0(x),\ (t,x) \in \mathbb{R} \times \mathbb{R}^d, \tag{1.1}$$

with V a real valued Schwartz function. We are interested in bound states, which are solutions of (1.1) of the form $u(t,x) = e^{\mathrm{i}\omega t}\phi_\omega(x)$ with $\omega \in \mathbb{R}$. When ϕ_ω is real valued and of fixed sign then we call $e^{\mathrm{i}\omega t}\phi_\omega$ a ground state. In all other cases we call it an excited state. In the $d = 2$ case on which we focus in this paper, and with $V(x) = V(|x|)$, an important class of excited states, which we call vortices, involves solutions of the form

$$u(t,x) = e^{\mathrm{i}\omega t}\phi_\omega(x) \text{ with } \phi_\omega(x) = e^{\mathrm{i}m \arg(x)}\psi_\omega(|x|) \text{ with } \psi_\omega : \mathbb{R}^2 \to \mathbb{R}, \tag{1.2}$$

with $\phi_\omega(x)$ smooth and rapidly decreasing to 0 at infinity and $m \geq 1$. In the sequel, we will always assume that there is a family of bound states ϕ_ω for ω in some open interval $\mathcal{O} \subseteq \mathbb{R}_+ = (0,\infty)$ (see (H4) in section 2). We will study the following classical notion of stability, [31, 81].

Definition 1.1 (Orbital stability)**.** A bound state $e^{\mathrm{i}\omega t}\phi_\omega$ of (1.1) is *orbitally stable* if

$$\forall\ \epsilon > 0,\ \exists\ \delta > 0 \text{ s.t. } \|\phi_\omega - u_0\|_{H^1} < \delta \Rightarrow \sup_{t>0} \inf_{s\in\mathbb{R}} \|e^{\mathrm{i}s}\phi_\omega - u(t)\|_{H^1} < \epsilon$$

where u is the solution of (1.1) with $u(0) = u_0$.

The orbital stability of bound states has been extensively studied, mainly using two tools: Lyapunov functions; linearized operators.

It is well known that (1.1) conserves the energy

$$E(u) := \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 + V|u|^2 + B(|u|^2)\, dx \tag{1.3}$$

where $B(0) = 0$ and $B'(s) = \beta(s)$, and the mass

$$Q(u) := 2^{-1} \int_{\mathbb{R}^2} |u|^2 dx \text{ (we will set } q(\omega) = Q(\phi_\omega) \text{ and } q'(\omega) = \frac{d}{d\omega} q(\omega)). \tag{1.4}$$

Using these conservations laws, if ϕ_ω is a strict local minimizer up to constant phase $e^{\mathrm{i}\theta}$ of E under the constraint $\|u\|_{L^2} = \|\phi_\omega\|_{L^2}$, then it has been shown that $e^{\mathrm{i}\omega t}\phi_\omega$ is orbitally stable, see [8, 31, 32, 80]. We are interested on standing waves not covered by this classical result. We will use the following notion.

Definition 1.2. A bound state $e^{\mathrm{i}\omega t}\phi_\omega$ is *not trapped by the energy* if for any $\varepsilon > 0$ there exists an u_ε satisfying $\|\phi_\omega - u_\varepsilon\|_{H^1} < \varepsilon$, $\|u_\varepsilon\|_{L^2} = \|\phi_\omega\|_{L^2}$ and $E(u_\varepsilon) < E(\phi_\omega)$.

The results [8, 31, 32, 80] do not cover the case when ϕ_ω is a local but not strict minimizer. They also leave unsolved the case when ϕ_ω is not trapped by the energy, which we will discuss here.

In order to study the stability, it is natural to consider the linearized operator $\mathcal{L}_\omega$ of ϕ_ω (see (2.7) for the explicit form). Indeed, if $\mathcal{L}_\omega$ has unstable modes (spectrum with positive real part), then $e^{\mathrm{i}\omega t}\phi_\omega$ is orbitally unstable. (Even though this may look trivial, it is not, and it was proved rather recently first in 2D by [56] and later in general by [28]). Classical papers proving orbital instability of solitary waves by first proving their spectral instability are [30, 32, 37, 38].

In the case of ground states, except for the degenerate cases when $q'(\omega) = 0$, only the above two cases (trapped by the energy or linearly unstable) occur. That is, if $q'(\omega) < 0$ then $\mathcal{L}_\omega$ has an unstable mode while if $q'(\omega) > 0$ it is trapped by the energy, [31, 32]. For the degenerate case $q'(\omega) = 0$, see [9, 49, 61].

Excited state are usually not trapped by the energy and furthermore there are cases when $\mathcal{L}_\omega$ has no unstable modes. For example, if $-\Delta + V$ has $\mathbf{n}$ simple negative eigenvalues $\{e_1 < e_2 < \cdots < e_{\mathbf{n}}(< 0)\}$ then, if $2e_j < e_1$ for some $j \geq 2$, the excited states bifurcating form e_j are not trapped by the energy and are spectrally stable. Even if spectrally stable they are orbitally unstable, [18]. Prior to [18] no systematic proof of this orbital instability was available. The series [26, 27, 34, 60, 71, 72, 73, 74, 75, 76], which stemmed from [7, 70], was able to treat only case $2e_2 > e_1$ where, generically, excited states are spectrally unstable, see [20, 34, 60, 76].

Vortices (1.2) of (1.1) in the important pure power case $\beta(|u|^2)u = -|u|^{p-1}u$ have been considered in [57, 58, 59] which have various instability results, always by proving first spectral instability. Another important example is given by [63, 65, 77] for vortices for the cubic-quintic nonlinearity

$$\mathrm{i}u_t = -\Delta u - (|u|^2 - |u|^4)u, \; u(0, x) = u_0(x), \; (t, x) \in \mathbb{R} \times \mathbb{R}^2, \tag{1.5}$$

(for a review paper see also [10], for work on spinning solitons in 3D see [55]; see also [5]). [63, 77] show numerically for some values $|m| \geq 1$ the existence of a critical value ω_{cr} such that for $\omega < \omega_{cr}$ the vortices are spectrally unstable and for $\omega \geq \omega_{cr}$ are spectrally stable. In the simulations in [63] the spectrally stable vortices for $\omega > \omega_{cr}$ appear stable while in [77] for $m = 3$ appear to slowly develop instabilities. This latter observation appears consistent with the more recent numerical observations in [22, 43], in turn based on the instability theory in [14]. The theory in [14] is centered on the notion of Krein signature, which we introduce in Lemma 2.4 (although the standard definition is in the proof of Lemma 7.1). In this paper, we generalize [14] by using a simple idea from [18]. We will need the following notion.

Definition 1.3 (Conditional asymptotic stability)**.** We say that a bound state $e^{\mathrm{i}\omega t}\phi_\omega$ is *conditionally asymptotically stable* if there exist constants $\epsilon_0 > 0$ and $C_0 > 0$ s.t. if $u \in C^0([0, \infty), H^1)$ is a solution of (1.1) with $\sup_{t\geq 0}\inf_{\vartheta\in\mathbb{R}}\|u(t)-e^{\mathrm{i}\vartheta}\phi_\omega\|_{H^1}<\epsilon<\epsilon_0$ then there exist $\omega_+\in\mathcal{O}$, $\theta\in C^1(\mathbb{R};\mathbb{R})$ and $h_+\in H^1$ with $\|h_+\|_{H^1}+|\omega_+-\omega|\leq C_0\epsilon$ such that

$$\lim_{t\to\infty}\|u(t)-e^{\mathrm{i}\theta(t)}\phi_{\omega_+}-e^{\mathrm{i}t\Delta}h_+\|_{H^1}=0. \tag{1.6}$$

Then we have the following orbital instability result.

Theorem 1.4. *Consider a bound state $e^{\mathrm{i}\omega t}\phi_\omega$ and suppose that Hypotheses (H1)–(H5) in Section 2 below are satisfied. Then if $e^{\mathrm{i}\omega t}\phi_\omega$ is not trapped by the energy and is conditionally asymptotically stable it is also orbitally unstable.*

Proof. Here the key hypothesis is (H5), i.e. $q'(\omega)\neq 0$ and hence either $q'(\omega)>0$ for all ω or $q'(\omega)<0$ for all ω. We prove the theorem by contradiction.
Assume that the statement is false and that there is a $\omega_1\in\mathcal{O}$ s.t. the standing wave $e^{\mathrm{i}\omega_1 t}\phi_{\omega_1}$ is orbitally stable. Since $e^{\mathrm{i}\omega_1 t}\phi_{\omega_1}$ is not trapped by the energy there are initial data u_0 arbitrarily close to ϕ_{ω_1} such that $E(u_0)<E(\phi_{\omega_1})$ and $Q(u_0)=Q(\phi_{\omega_1})$. We can apply the Conditional asymptotic stability, Definition 1.3, and conclude by the conservation of the mass that

$$q(\omega_1)=Q(u(t))=\lim_{t\to\infty}Q(u(t))=q(\omega_+)+2^{-1}\|h_+\|_{L^2}^2\geq q(\omega_+).$$

Similarly, by the conservation of the energy we have

$$E(\phi_{\omega_1})>E(u_0)=\lim_{t\to\infty}E(u(t))=E(\phi_{\omega_+})+2^{-1}\|\nabla h_+\|_{L^2}^2\geq E(\phi_{\omega_+}). \tag{1.7}$$

By $\nabla E(\phi_\omega)=-\omega\nabla Q(\phi_\omega)$ we have $\frac{d}{d\omega}E(\phi_\omega)=-\omega q'(\omega)$ (recall the notation $q(\omega):=Q(\phi_\omega)$ in (1.4)). On the other hand, by (H5) we know that $q'(\omega)$ has a fixed sign. So, since $\omega>0$, both $\omega\to E(\phi_\omega)$ and $\omega\to q(\omega)$ are strictly monotonic, one increasing and the other decreasing. This means that $q(\omega_1)\geq q(\omega_+)$ implies that $E(\phi_{\omega_1})\leq E(\phi_{\omega_+})$. But this contradicts $E(\phi_{\omega_1})>E(\phi_{\omega_+})$ in (1.7). This contradiction shows that $e^{\mathrm{i}\omega_1 t}\phi_{\omega_1}$ is not orbitally stable for any $\omega_1\in\mathcal{O}$. □

Remark 1.5. Notice that (after modifying the assumption (H4)) Theorem 1.4 holds for arbitrary dimension and we do not require that the nonlinear bound states are vortices.

Remark 1.6. We do not claim that conditional asymptotic stability is necessary for the instability of excited states.

Remark 1.7. For discrete NLS there are examples of bound states which are not trapped by the energy, are orbitally stable, but are not asymptotically stable, see [50, 51].

We now start the discussion on how to apply Theorem 1.4. We first discuss a sufficient condition for the standing waves to be not trapped by the energy. One natural way is to look at the Taylor expansion of the energy, which is often used in the study of orbital stability. Set $S_\omega(u)=E(u)+\omega Q(u)$ (notice that if $Q(u)=Q(v)$, $E(u)>E(v)$ is equivalent to $S_\omega(u)>S_\omega(v)$). Then, since $e^{\mathrm{i}\omega t}\phi_\omega$ is a bound state if and only if $\nabla S_\omega(\phi_\omega)=0$, we have $S_\omega(\phi_\omega+v)=S_\omega(\phi_\omega)+\frac{1}{2}\left\langle\nabla^2S_\omega(\phi_\omega)v,v\right\rangle+o(v^2)$. Roughly speaking, we have one constraint $Q(\phi_\omega+v)=Q(\phi_\omega)$ which may eliminate at most one negative direction of $\nabla^2S_\omega(\phi_\omega)$. Therefore, if $\nabla^2S_\omega(\phi_\omega)$ has more than two negative eigenvalues, ϕ_ω is not trapped by the energy. Further, for the case $\nabla^2S_\omega(\phi_\omega)$ has one negative eigenvalue, the trapping/nontrapping can be determined by the sign of $q'(\omega)$. That is, if $q'(\omega)>(<)0$ then $e^{\mathrm{i}\omega t}\phi_\omega$ is trapped (not trapped) by the energy.

Another viewpoint which we adopt in this paper is to study the linearized operator $\mathcal{L}_\omega=J\nabla^2S_\omega(\phi_\omega)$ given in (2.7), where $J=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$. Here, we have extended $\nabla^2S_\omega(\phi_\omega)$ to be a matrix

to make it self-adjont, see section 2. Since $\mathcal{L}_\omega$ has the form $J(-\Delta+\omega)+$"rapidly decaying potential", we see that $\mathrm{i}\sigma_{\mathrm{e}}(\mathcal{L}_\omega) = (-\infty, -\omega] \cup [\omega, \infty)$. Here $\sigma_{\mathrm{e}}(\mathcal{L}_\omega)$ is the essential spectrum of $\mathcal{L}_\omega$. Since we are only interested in the case that $\mathcal{L}_\omega$ is spectrally stable we assume $\sigma(\mathcal{L}_\omega) \subset \mathrm{i}\mathbb{R}$.
In general no eigenvalues are expected in the interior of the essential spectrum. References [2, 54] have computational proofs about the absence of embedded eigenvalues when ϕ_ω is a ground state for some equations.[63] does not discuss explicitly the issue of embedded eigenvalues but they seem to be excluded, see Remark 7.10 below for some further comments. [20] proves that in some *generic* sense embedded eigenvalues do not exist because they are unstable, see also [30, 76]. The proof is similar to the discussion of the well established instability of embedded eigenvalues in the case of self-adjoint operators, cfr. [40]. However it is not clear whether taking for example V generic in the cubic–quintic NLS would make $\mathcal{L}_\omega$ generic in the sense of [20]. We think that embedded eigenvalues (with some negative Krein signature) can exist, but that they are unstable (we conjecture the non existence of embedded eigenvalues of positive Krein signature). Hence the assumption of absence of embedded eigenvalues seems reasonable.
Similarly, the edge of the essential spectrum is shown in [19, 78] generically to be neither an eigenvalue nor resonance. Again, we did not check whether taking V generic makes the $\mathcal{L}_\omega$ associated to the cubic–quintic NLS generic in that sense, although this is even more likely than for the issue of embedded eigenvalues. In [63] there is no discussion about the edge of the essential spectrum where the presence of a resonance or an eigenvalue would affect the computations discussed in pp. 371–372. We think that assuming absence of eigenvalue or resonance at the edge is reasonable.
We will further assume that the kernel is spanned by $J\phi_\omega$ and all nonzero eigenvalues (which are assumed to be finitely many but in fact this can be proved to be the case, see the comments under (H11) in Sect. 2) have the same algebraic and geometric dimensions which are finite (in the presence of a nonzero imaginary eigenvalue whose two dimensions differ we would have a sort of linear instability, see Remark 7.10 for further comments). Now, let ξ be the eigenfunction of $\mathcal{L}_\omega$ associated to $\mathrm{i}\lambda$ with $\lambda > 0$ (for eigenfunctions of $\mathrm{i}\lambda$ with $\lambda < 0$, we have the symmetry. Thus it suffices to consider the positive case). Then, the "energy" of ξ is $\langle \nabla^2 S_\omega(\phi_\omega)\xi, \bar{\xi} \rangle = \Omega(\mathcal{L}_\omega \xi, \bar{\xi}) = \mathrm{i}\lambda\Omega(\xi, \bar{\xi})$, where $\Omega = \langle J^{-1}, \cdot, \cdot \rangle$ (see (2.6)). It is known that we can normalize ξ s.t. $\Omega(\xi, \bar{\xi}) = \mathrm{i}s$ for $s \in \{1, -1\}$. Therefore, if $s = 1$, ξ have a negative energy and if $s = -1$, ξ have a positive energy. Now, set $n(\nabla^2 S_\omega(\phi_\omega))$ be the number of negative eigenvalues of $\nabla^2 S_\omega(\phi_\omega)$. It is known that $n(\nabla^2 S_\omega(\phi_\omega))$ can be represented as

$$n(\nabla^2 S_\omega(\phi_\omega)) = p(q') + 2N_r^- + N_i + 2N_c,$$

where $p(q') = 1$ if $q' > 0$ and 0 if $q' < 0$, N_r, N_i, N_c are the number of real, imaginary and complex eigenvalues of $\mathcal{L}_\omega$ (See [20, 79]). In our situation, $N_i = N_c = 0$. Therefore, if $N_r^- = 0$, ϕ_ω will be trapped by the energy and if $N_r^- \geq 1$, the ϕ_ω is not trapped by the energy. Thus, by the above discussion, the trapping/nontrapping of the energy is determined by the existence of eigenvalue of $\mathcal{L}_\omega$ with negative energy. Here, we will give a direct proof of the nontrapping when there exists a negative energy eigenvalue.

Proposition 1.8. *Assume hypothesis (H1)–(H11) and (H14) in Section 2. Then the standing waves are not trapped by the energy.*

Proof. Here (H14) is the key hypothesis which is roughly stating that for any ω there is one eigenvalue $\mathrm{i}\lambda_j \in \mathbb{R}_+$ with negative Krein signature (i.e. the corresponding eigenfunction ξ_j satisfies $\Omega(\xi_j, \bar{\xi}_j) = 1$).

Fix $\omega_1 \in \mathcal{O}$ and choose $\alpha(\epsilon)$ to satisfy $Q\left((1-\alpha(\epsilon))\phi_{\omega_1} + \epsilon(\xi_j(\omega_1) + \bar{\xi}_j(\omega_1))\right) = Q(\phi_{\omega_1})$. Since $\xi_j(\omega), \bar{\xi}_j(\omega) \in N_g(\mathcal{L}^*_{\omega_1})^\perp$ (see (2.9)), we see that

$$Q\left((1-\alpha(\epsilon))\phi_{\omega_1} + \epsilon(\xi_j(\omega_1) + \bar{\xi}_j(\omega_1))\right) = (1-\alpha(\epsilon))^2 Q\left(\phi_{\omega_1}\right) + \epsilon^2 Q\left((\xi_j(\omega_1) + \bar{\xi}_j(\omega_1))\right).$$

Thus, we can conclude $\alpha(\epsilon) \sim \epsilon^2$. Consequently, we have

$$\begin{aligned}
&E\left((1-\alpha(\epsilon))\phi_{\omega_1} + \epsilon(\xi_j(\omega_1) + \bar{\xi}_j(\omega_1))\right) - E(\phi_{\omega_1}) \\
&= S_{\omega_1}\left((1-\alpha(\epsilon))\phi_{\omega_1} + \epsilon(\xi_j(\omega_1) + \bar{\xi}_j(\omega_1))\right) - S_{\omega_1}(\phi_{\omega_1}) \\
&= \frac{\epsilon^2}{2}\left\langle \nabla^2 S_{\omega_1}(\phi_{\omega_1})\left(\xi_j(\omega_1) + \bar{\xi}_j(\omega_1)\right), \xi_j(\omega_1) + \bar{\xi}_j(\omega_1)\right\rangle + o(\varepsilon^2).
\end{aligned}$$

Finally, since $\mathcal{L}_{\omega_1} = J\nabla^2 S_{\omega_1}(\phi_{\omega_1})$, we have

$$\begin{aligned}
&\left\langle \nabla^2 S_{\omega_1}(\phi_{\omega_1})\left(\xi_j(\omega_1) + \bar{\xi}_j(\omega_1)\right), \xi_j(\omega_1) + \bar{\xi}_j(\omega_1)\right\rangle = \Omega\left(\mathcal{L}_{\omega_1}\left(\xi_j(\omega_1) + \bar{\xi}_j(\omega_1)\right), \xi_j(\omega_1) + \bar{\xi}_j(\omega_1)\right) \\
&= \mathrm{i}\lambda_j \Omega(\xi_j(\omega_1), \bar{\xi}_j(\omega_1)) - \mathrm{i}\lambda_j \Omega(\bar{\xi}_j(\omega_1), \xi_j(\omega_1)) = -2\lambda_j < 0.
\end{aligned}$$

Therefore, we have the conclusion. $\square$

We now turn our attention to the conditional asymptotic stability. Its proof is very close to the proof of asymptotic stability of ground states in [12, 13] and we will need assumptions for the spectrum of $\mathcal{L}_\omega$ discussed above (i.e. absence of embedded eigenvalues, $\pm \mathrm{i}\omega$ are neither eigenvalues nor resonance, the generalized kernel is 2 dimensional, all nonzero eigenvalues have nonzero energy, see section 2). We will consider only the 2D case.

Theorem 1.9. *Assume hypothesis (H1)-(H13) in Section 2. Then, we have the conditional asymptotic stability.*

The main example for the theory developed in this paper that we have in mind is the cubic quintic NLS (1.5) and specifically perturbations obtained adding a radial potential $V(|x|)u$. As we mentioned above, in [63] it is shown that for any $m = 1, 2, 3, 4, 5$ there is a critical value ω_{cr} such that for $\omega < \omega_{cr}$ the vortices are spectrally unstable and for $\omega \geq \omega_{cr}$ are spectrally stable. [63] proves numerically that the linearization $\mathcal{L}_\omega$ has an eigenvalue of negative Krein signature, i.e. our hypothesis (H14) is satisfied. Indeed in [63] it is shown numerically that for (1.5) the spectral instability for $\omega < \omega_{cr}$ occurs because a pair of eigenvalues on $\mathrm{i}\mathbb{R}$ coalesce as $\omega \to \omega_{cr}^+$ and then for $\omega < \omega_{cr}$ form two eigenvalues which exit $\mathrm{i}\mathbb{R}$ in opposite directions. This can happen only if the eigenvalues for $\omega > \omega_{cr}$ do not have the same Krein signature. Thus for ω close to and larger than ω_{cr} at least one imaginary eigenvalue has negative signature. See below in Lemma 7.1 for a more precise discussion. Our results do not apply directly to the 2D cubic quintic NLS (1.5) because of its translation invariance. However, as we show in Section 7, when we add to (1.5) a small linear potential, then we obtain an equation which satisfies hypothesis (H14) and which is not translation invariant. When we take the small potential with a nondegenerate minimum at the origin, then for $\omega > \omega_{cr}$ we obtain spectrally stable vortices. Our hypotheses (H1)–(H7) are either obvious or we know they are true as a consequence of the numerical experiments in [63]. The hypotheses (H9)–(H13), while probably generically true, ought to be checked numerically. We will say more later about them, especially (H13), the most delicate and least analyzed. The conclusion that we can draw is that, assuming that indeed (H9)–(H13) are true, then for ω close to and larger than ω_{cr} the vortices of appropriate perturbations of the cubic quintic NLS (1.5) are not trapped by the energy and are conditionally asymptotically stable. And hence by Theorem 1.4 they are orbitally unstable.

We have seen that Theorem 1.4 is a simple consequence of non trapping by energy (by Proposition 1.8 a consequence of the existence of one eigenvalue of negative Krein signature) and of the Conditional asymptotic stability. The latter follows from (H1)–(H13) by Theorem 1.9. A more precise formulation of Theorem 1.9 is in Proposition 2.5. The Conditional asymptotic stability, like the asymptotic stability of ground states proved in [13], is due to a mechanism of loss of energy of discrete modes related to the Fermi golden rule (FGR) and to linear scattering of the continuous modes.

The FGR was first introduced in [7, 67, 70] and we will discuss it shortly. But first we discuss linear scattering, which is based on a number of results on the group $e^{t\mathcal{L}_\omega}$ associated to the linearization of the NLS at the vortex. The results needed here are quoted (here we focus on 2D) from [21] and require that $\mathcal{L}_\omega$ should not have eigenvalues (and resonances) in the essential spectrum, see hypotheses (H9)–(H10) below. The results in [21] allow to say that, for all practical purposes, the restriction of $e^{t\mathcal{L}_\omega}$ on the continuous spectrum part, behaves like $e^{\mathrm{i}t(-\Delta+V)}$ restricted to its continuous spectrum part. (H9)–(H10) are probably generically true, but nonetheless ought to be either proved or checked numerically on any given example. The fact that we cannot treat translation (the asymptotic stability result on moving solitons in [15] has not been proved in dimensions 1 and 2) depends on the specific way in which scattering of continuous modes is proved. Probably there is a simpler and more robust way to prove dispersion using virial inequalities and the theory of Martel and Merle [52, 53]. This should require fewer hypotheses and lead to similar results. This approach has been very successful in the context of KdV equations where it has improved the result by Pego and Weinstein [64]. The theory is proving successful also in different contexts, see [6, 11, 29, 47, 42]. However here we follow our standard approach and we use material in [21].

We now turn to the FGR. It can be easily seen under appropriate coordinate systems. It leads to nonlinear interactions between discrete an continuous modes of the NLS which are responsible for energy leaking out from the discrete modes.

When we discuss the FGR we need to separate two distinct issues, as we will see with a simple example below. One issue is the fact that certain coefficients of the system have a 2nd power structure. This has been proved in [13]. See also [12, 18] for generalizations and references. A separate issue, is whether or not these 2nd powers, which are non negative, are also strictly positive. There might be cases when this is not true, but in general we expect that they are strictly positive. We do not have the expertise to run numerical tests, but a simple model might clarify this point (for other examples of FGR see also the survey [82]).

For $(z,h) \in \mathbb{C} \times H^1(\mathbb{R}^2, \mathbb{C})$, consider the Hamiltonian

$$\mathcal{H}(z,h) = |z|^2 + \|\nabla h\|^2_{L^2} + |z|^2 \overline{z} \int_{\mathbb{R}^2} G(x) h(x) dx + |z|^2 z \int_{\mathbb{R}^2} \overline{G}(x) \overline{h}(x) dx, \tag{1.8}$$

where G is a $\mathbb{C}$-valued Schwartz function and the symplectic form $\mathrm{i}dz \wedge d\bar{z} + 2 \langle \mathrm{i}dh, dh \rangle$ where

$$\langle f, g \rangle = \mathrm{Re} \int_{\mathbb{R}^2} f(x) \overline{g}(x) dx = \int_{\mathbb{R}^2} (f_1(x) g_1(x) + f_2(x) g_2(x)) dx \text{ for } f, g : \mathbb{R}^2 \to \mathbb{C} (= \mathbb{R}^2) \; . \tag{1.9}$$

Then we have the Hamiltonian system

$$\mathrm{i}\dot{h} = -\Delta h + |z|^2 z \overline{G}, \tag{1.10}$$

$$\mathrm{i}\dot{z} = z + 2|z|^2 \int_{\mathbb{R}^2} h(x) G(x) dx + z^2 \int_{\mathbb{R}^2} \overline{h}(x) \overline{G}(x) dx. \tag{1.11}$$

The solution of the linearized equation around $(0,0)$ is $(z,h) = (e^{-\mathrm{i}t} z(0), e^{\mathrm{i}t\Delta} h(0))$. Therefore, at the linear level, we do not see the asymptotic stability of $(0,0)$. In the following, we sketch heuristically why the equilibrium $(0,0)$ is asymptotically stable thanks to the FGR and scattering of the continuous mode. First, if we assume $z(t)=e^{-\mathrm{i}t}z(0)$, then $-|z|^2zR^+_{-\Delta}(1)\bar{G}$ solves (1.10) (for the use of $R^+_{-\Delta}(1)=\lim_{\varepsilon\to 0^+}R_{-\Delta}(1+\mathrm{i}\varepsilon)$ see the remark on p. 30 [70]) . Therefore, it is reasonable to set

$$h=-|z|^2zR^+_{-\Delta}(1)\overline{G}+g \tag{1.12}$$

and think g is a remainder. In fact (1.12) is a normal form transformation intended to eliminate the term $|z|^2z\overline{G}$ from the r.h.s. in (1.10). g satisfies an analogous equation as h, but with a higher degree polynomial in $(z,\overline{z})$, and so it is smaller than h (this goes back to [7, 70])
When we substitute (1.12) in (1.11) and we ignore g, we get

$$\mathrm{i}\dot{z}=z-2|z|^4z\int_{\mathbb{R}^2}GR^+_{-\Delta}(1)\overline{G}dx-|z|^4z\int_{\mathbb{R}^2}\overline{G}R^-_{-\Delta}(1)Gdx.$$

We recall that $R^{\pm}_{-\Delta}(\lambda)=P.V.(-\Delta-\lambda)^{-1}\pm\mathrm{i}\pi\delta(-\Delta-\lambda)$ for any $\lambda>0$, where δ is the Dirac delta function and $P.V.x^{-1}$ is the Cauchy principal value. These can be given sense using the Fourier transform. Multiplying by $\overline{z}$ and taking imaginary part we get

$$\frac{d}{dt}|z|^2=-2\pi\mathfrak{c}|z|^6 \text{ with } \mathfrak{c}=\int_{\mathbb{R}^2}G\delta(-\Delta-1)\overline{G}dx\geq 0. \tag{1.13}$$

We conclude that $\mathfrak{c}\geq 0$ by the following formula, see ch.2 [23]:

$$\mathfrak{c}=\frac{1}{2}\int_{|\xi|=1}|\widehat{G}(\xi)|^2d\sigma(\xi). \tag{1.14}$$

This is the 2nd power structure discussed above. In the context of the study of ground states of the NLS, the analogue of this formula has been proved in [13], see formulas (6.13)–(6.14) and later in this section.
The next step is to ask whether in (1.14) we have not only $\mathfrak{c}\geq 0$ but rather the strict inequality $\mathfrak{c}>0$. Obviously, for generic G we have $\widehat{G}\Big|_{|\xi|=1}\neq 0$, and so $\mathfrak{c}>0$. Numerical computations for random choices of G would yield convincingly this fact. The strict inequality $\mathfrak{c}>0$ and the equation of z in (1.13) yield the explicit formula

$$|z(t)|^2=\frac{|z(0)|^2}{(1+4\pi\mathfrak{c}|z(0)|^4t)^{\frac{1}{2}}}.$$

On the other hand, h will scatter by linear mechanisms, thanks also to the fact that the forcing term $|z|^2z\overline{G}$ in (1.10) is in L^2 for time. (Combining this fact with Kato smoothing estimates such as Lemmas A.3 and A.4, we can show h has finite Strichartz norm, which implies the scattering).
Proceeding differently and as a reference for later comments, one could integrate (1.13) and write

$$|z(t)|^2+2\pi\mathfrak{c}\int_0^t|z(s)|^6ds=|z(0)|^2. \tag{1.15}$$

Our FGR hypothesis, stated explicitly in (6.15), is the same as assuming $\mathfrak{c}>0$ in model (1.8). The analogue of $\mathfrak{c}\geq 0$ instead is rigorously proved in (6.14). Numerical computations are likely to prove (6.15) true for generic equations exactly in the same way they would show that $\mathfrak{c}>0$ in the above model.

We now discuss the FGR, still heuristically, for a model closer to the one necessary to examine equation (1.1). The discussion is more complicated than for model (1.8), but will yield similar conclusions.

First of all, by a Noetherian reduction of coordinates related to the $U(1)$ invariance of the NLS, we will see that we reduce to an effective Hamiltonian of the form, for appropriate finite sums,

$$\mathcal{H}(z,h) = -\sum_{j=1}^{\mathbf{n}} s_j \lambda_j |z_j|^2 + \langle (-\Delta + \omega + \mathfrak{V})h, \sigma_1 h\rangle + \sum_{|\lambda\cdot(\mu-\nu)|>\omega} z^\mu \overline{z}^\nu \langle G_{\mu\nu}, \sigma_3\sigma_1 h\rangle, \tag{1.16}$$

where: $(z,h) \in \mathbb{C}^{\mathbf{n}} \times H^1(\mathbb{R}^2;\mathbb{C}^2)$ with $\overline{h} = \sigma_1 h$; the inner product $\langle f, g\rangle = \int {}^t f(x) g(x) dx$ is a bilinear map; $\mathrm{i}\lambda_j \in \mathrm{i}\mathbb{R}_+$ are eigenvalues of the linearization $\mathcal{L}_\omega$; $\lambda = (\lambda_1, ..., \lambda_{\mathbf{n}})$; $\mathfrak{V}(x)$ is smooth in x, rapidly convergent to 0 as $x \to \infty$ and is for every x a self adjoint 2×2 matrix; $\omega > 0$ and in the application we are thinking it is in $\mathcal{O}$ of (H4); μ, ν are multi-index such as $\mu = (\mu_1, \cdots, \mu_{\mathbf{n}})$ and $z^\mu = z_1^{\mu_1} \cdots z_{\mathbf{n}}^{\mu_{\mathbf{n}}}$ and similar for $\bar{z}^\nu$; the second (finite) sum is taken for multi-indices μ, ν; $G_{\mu\nu}(x)$ is Schwartz class in x with values in 2 components vectors;

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \tag{1.17}$$

the number $-s_j$ is the Krein signature of each $\mathrm{i}\lambda_j$, with at least one $s_j = 1$ (for the connection between Krein signature and number s_j see in the proof of Lemma 7.1).
Here $\overline{h} = \sigma_1 h$, the Hamiltonian $\mathcal{H}(z,h)$ is real valued and so $\overline{G}_{\mu\nu} = -\sigma_1 G_{\nu\mu}$. We consider the symplectic form

$$\sum_{j=1}^{\mathbf{n}} \mathrm{i}\ s_j\ dz_j \wedge d\overline{z}_j + \mathrm{i}\langle dh, \sigma_3\sigma_1 dh\rangle.$$

The equations are then of the form

$$\begin{aligned} &\mathrm{i}\dot{h} = \mathcal{K}h + \sum_{|\lambda\cdot(\alpha-\beta)|>\omega} z^\alpha \overline{z}^\beta G_{\alpha\beta} \text{ for } \mathcal{K} = \sigma_3(-\Delta + \omega + \mathfrak{V}) \\ &\mathrm{i}s_j \dot{z}_j = -s_j \lambda_j z_j + \sum_{|\lambda\cdot(\mu-\nu)|>\omega} \nu_j \frac{z^\mu \overline{z}^\nu}{\overline{z}_j} \langle G_{\mu\nu}, \sigma_3\sigma_1 h\rangle. \end{aligned} \tag{1.18}$$

Setting $h = -\sum z^\alpha \overline{z}^\beta R^+_{\mathcal{K}}(\lambda\cdot(\beta-\alpha))G_{\alpha\beta} + g$ like in (1.12), substituting and ignoring g (as we did earlier) we get

$$\mathrm{i}s_j \dot{z}_j = -s_j \lambda_j z_j - \sum_{\substack{|\lambda\cdot(\mu-\nu)|>\omega \\ |\lambda\cdot(\alpha-\beta)|>\omega}} \nu_j \frac{z^\mu \overline{z}^\nu z^\alpha \overline{z}^\beta}{\overline{z}_j} \langle G_{\mu\nu}, \sigma_3\sigma_1 R^+_{\mathcal{K}}(\lambda\cdot(\beta-\alpha))G_{\alpha\beta}\rangle.$$

As we will see, we can ignore the terms where $\lambda\cdot(\mu-\nu) \neq \lambda\cdot(\alpha-\beta)$. Furthermore, up to smaller terms that we ignore, we have

$$\mathrm{i}s_j \dot{z}_j = -s_j \lambda_j z_j - \sum_{L>\omega} \sum_{\lambda\cdot\nu = \lambda\cdot\alpha = L} \nu_j \frac{z^\alpha \overline{z}^\nu}{\overline{z}_j} \langle G_{0\nu}, \sigma_3\sigma_1 R^+_{\mathcal{K}}(-L)G_{\alpha 0}\rangle.$$

Let us write formally $R^+_{\mathcal{K}}(-L) = P.V.\frac{1}{\mathcal{K}+L} + \mathrm{i}\pi\delta(\mathcal{K}+L)$ (there is a distorted Fourier transform that allows to make sense of this). Then, using $G_{0\nu} = -\sigma_1 \overline{G}_{\nu 0}$,

$$\begin{aligned} \partial_t \sum_{j=1}^{\mathbf{n}} s_j \lambda_j |z_j|^2 &= -\pi \sum_{L>\omega} L \operatorname{Re} \left\langle \sum_{\lambda\cdot\nu=L} \overline{z}^{\nu} G_{0\nu}, \sigma_3\sigma_1 \delta(\mathcal{K}+L) \sum_{\lambda\cdot\alpha=L} z^{\alpha} G_{\alpha 0} \right\rangle \\ &= -\pi \sum_{L>\omega} L \langle \overline{G}_L(z), \sigma_3 \delta(\mathcal{K}+L) G_L(z) \rangle \text{ where } G_L(z) := \sum_{\lambda\cdot\alpha=L} z^{\alpha} G_{\alpha 0}. \end{aligned} \tag{1.19}$$

Furthermore, there exists $G_L^{(0)}(z)$ s.t.

$$\langle \overline{G}_L(z), \sigma_3 \delta(\mathcal{K}+L) G_L(z) \rangle = \langle \overline{G}_L^{(0)}(z), \begin{pmatrix} \overbrace{\delta(-\Delta+\omega+L)}^{0} & 0 \\ 0 & -\delta(\Delta-\omega+L) \end{pmatrix} G_L^{(0)}(z) \rangle.$$

Observe now that $\delta(\Delta-\omega+L) = \delta(-\Delta-(L-\omega))$ and that integrating in (1.19) we get

$$\begin{aligned} &\sum_{j=1}^{\mathbf{n}} s_j \lambda_j |z_j(t)|^2 - \pi \sum_{L>\omega} L \int_0^t \overline{\langle (G_L^{(0)}(z(s)))_2}, \delta(-\Delta-(L-\omega))(G_L^{(0)}(z(s)))_2 \rangle ds \\ &= \sum_{j=1}^{\mathbf{n}} s_j \lambda_j |z_j(0)|^2. \end{aligned} \tag{1.20}$$

Notice that inside the integral we have a similar 2nd power structure (for each L we have a positive quadratic form in the vector $(z^{\alpha})_{\lambda\cdot\alpha=L}$) to the (1.14) we found in model (1.8). In particular, if $\mathbf{n}=1$ in (1.16), then the time integral in (1.20) will be similar (possibly with different power) to the time integral in (1.15). So formula (1.20) is a generalization of (1.15). When the Krein signatures are all positive, that is if $s_j \equiv -1$, then (1.20) can be used to prove $z(t) \overset{t\to\infty}{\to} 0$ (and so the asymptotic stability of the standing wave). Indeed, starting from a $|z(0)|$ small, (1.20) is telling us that also $|z(t)|$ remains small. Furthermore, the fact that the integrals remain bounded as $t\to\infty$ can be used, along with the fact that $|\dot z(t)|$ remains bounded (which can be seen by the 2nd equation in (1.18)), to show that $z(t) \overset{t\to\infty}{\to} 0$. However, to make this rigorous we need to have something analogous to the inequality $\mathfrak{c} > 0$, which is exactly the meaning of (6.15).

If there are negative Krein signatures, and so some $s_j = 1$, obviously the proof of $z(t) \overset{t\to\infty}{\to} 0$ breaks down since in the l.h.s. of (1.20) there are terms with different signs whose size could become large even if the sum is small.

However, if we know that a solution $u(t)$ remains close to a standing wave, and consequently the corresponding $|z(t)|$ remains small, then (1.20) allows to prove $z(t) \overset{t\to\infty}{\to} 0$ because we know that the sum $\sum_j s_j \lambda_j |z_j(t)|^2$ remains small. This allows to control the integrals and, with hypothesis (H13), that is (6.15), to conclude $z(t) \overset{t\to\infty}{\to} 0$.

The validity of (6.15) (or of more explicit formulations of the formula) is an open question. Proofs related to special situations are in [3, 46, 1]. It is fair to expect that numerical experiments will confirm strict positivity for most examples, like for the $\mathfrak{c}$ in (1.14). Some numerical verifications are in [22, 43], but there is room for more systematic studies. These are absent in the literature not because some intrinsic difficulty, but simply because the ideas in [13, 14] are not well known.

Having given a general overview of the main concepts and results of this paper, we list now the content of the remaining section of the paper. In Section 2 we state hypotheses and give in Proposition 2.5 a statement that is more detailed than Theorem 1.9. In Sections 3–6 we describe the proof of Proposition 2.5. The proof is basically the same of the proofs of asymptotic stability of ground states of the NLS in [12, 13]. In Section 3, after introducing a natural system of coordinates related to the *modulation* we state in Proposition 3.4 a result on Darboux coordinates, whose proof is in [12] and which is a key step to the subsequent search of an effective Hamiltonian. We will only give a sketch here. The expansion in these new coordinates of the functional $K(u)$, defined in (3.14), is again given without proof since the proof is in [12]. In Section 4 we complexify $L^2(\mathbb{R}^2, \mathbb{R}^2)$ and consider the spectral decomposition of the linearization $\mathcal{L}_\omega$ in $L^2(\mathbb{R}^2, \mathbb{C}^2)$. This produces discrete coordinates $z = (z_1, ..., z_{\mathbf{n}}) \in \mathbb{C}^{\mathbf{n}}$ and a continuous coordinate $f \in L^2(\mathbb{R}^2, \mathbb{R}^2)$. We then state the expansion of the functional $K(u)$ in these coordinates, with the proofs in [12]. Next step is the search of an effective Hamiltonian by means of a Birkhoff normal forms argument. This is accomplished in in Proposition 4.3, which we give only a sketch of the proof, for which we refer again to [12]. Proposition 5.1 contains Strichartz and smoothing estimates satisfied by f, and the statement of $z(t) \overset{t\to\infty}{\to} 0$. We then show that Proposition 5.1 implies Proposition 2.5. By an elementary continuation argument Proposition 5.1 is in turn a consequence of Proposition 5.2, whose proof is in Section 6. In consists first in Strichartz and smoothing estimates for f. We then split f like in (1.12) as a term dependent only of z, which is the part of f which really affects the z's, and a g which is smaller, does not affect z substantially and can be treated as a reminder term. The estimates on f and on g are the same of [21]. Finally in Section 6 we return to the Fermi golden rule, explaining why $z(t) \overset{t\to\infty}{\to} 0$. All the estimates are proved in the literature, for example in [16]. Therefore, we limit ourselves at describing the structure of the argument. However we give a sketch of the proof for some important theorem (especially Darboux theorem and Birkhoff normal forms arguments) for reader's convenience.

In Section 7 we discuss the cubic quintic equation (1.5). We discuss how starting from the numerical observations in [63] it satisfies hypothesis (H14) for values $\omega > \omega_{cr}$ close to ω_{cr}. Since our theory does not apply to translation invariant equations like (1.5) we show that by adding a small radial potential with a non degenerate local minimum at 0 produces spectrally stable vortices which still satisfy (H14) because their linearization is a small perturbation of that of (1.5). At the end of Section 7 we also discuss the status of the other of the other hypotheses for equation (1.5) perturbed by adding a liner potential. Some of them follow from the computations in [63], the others ought to be checked numerically and in out opinion are plausibly true.

2 Hypotheses and statements

To begin with, assume the following hypotheses.

(H1) $\beta(0) = 0$, $\beta \in C^\infty(\mathbb{R}, \mathbb{R})$.

(H2) There exists a $p \in \mathbb{R}$ such that for every $k \geq 0$ there is a fixed C_k with

$$\left| \frac{d^k}{dv^k} \beta(v^2) \right| \leq C_k |v|^{p-k-1} \quad \text{if } |v| \geq 1. \tag{2.1}$$

(H3) $V(x)$ is smooth, non zero, real valued, and for any multi index α there are $C_\alpha > 0$ and $a_\alpha > 0$ such that $|\partial_x^\alpha V(x)| \leq C_\alpha e^{-a_\alpha |x|}$.

For $n \geq 1$ and $K = \mathbb{R}, \mathbb{C}$ then $\Sigma_n = \Sigma_n(\mathbb{R}^2, K^2)$ is the Banach space with

$$\|u\|^2_{\Sigma_n} := \sum_{|\alpha| \leq n} (\|x^\alpha u\|^2_{L^2(\mathbb{R}^2)} + \|\partial_x^\alpha u\|^2_{L^2(\mathbb{R}^2)}) < \infty. \tag{2.2}$$

We set $\Sigma_0 = L^2(\mathbb{R}^2, K^2)$. We define Σ_t by $\|u\|_{\Sigma_t} := \|(1-\Delta+|x|^2)^{\frac{t}{2}} u\|_{L^2} < \infty$ for $t \in \mathbb{R}$. For $t \in \mathbb{N}$ the two definitions are equivalent, see [15].

(H4) There exists an open interval $\mathcal{O} \subset \mathbb{R}_+$ such that

$$\Delta u - Vu - \omega u + \beta(|u|^2)u = 0 \quad \text{for } x \in \mathbb{R}^2, \tag{2.3}$$

admits a function $\omega \to \phi_\omega$ which for any k is in $C^2(\mathcal{O}, \Sigma_k(\mathbb{R}^2, \mathbb{C}))$. We also assume that $\phi_\omega(x) = e^{\mathrm{i} m \arg(x)} \psi_\omega(|x|)$ like in (1.2).

(H5) We have for $q(\omega) := 2^{-1}\|\phi_\omega\|^2_{L^2(\mathbb{R}^2)}$, see (1.4),

$$q'(\omega) \neq 0 \text{ for all } \omega \in \mathcal{O}. \tag{2.4}$$

Remark 2.1. Notice that by a standard bootstrapping argument we can relax hypothesis (H4) by only asking that $\omega \mapsto \phi_\omega$ be in $C^1(\mathcal{O}, H^1(\mathbb{R}^2, \mathbb{C}))$.

We identify $\mathbb{C} = \mathbb{R}^2$ setting $w_1 = \operatorname{Re} w$ and $w_2 = \operatorname{Im} w$ for $w \in \mathbb{C}$. In particular we identify the imaginary unit i with $-J$ where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.5}$$

and the bound state ϕ_ω with $\begin{pmatrix} \operatorname{Re} \phi_\omega \\ \operatorname{Im} \phi_\omega \end{pmatrix}$.

We consider the *strong* symplectic form defined

$$\Omega(X, Y) = \langle J^{-1}X, Y\rangle. \tag{2.6}$$

Definition 2.2. We denote by $\langle \ , \ \rangle$ also the bilinear form in $L^2(\mathbb{R}^2, \mathbb{C}^2)$ obtained extending (1.9). We extend Ω to $L^2(\mathbb{R}^2, \mathbb{C}^2)$ as a bilinear form.

For $F \in C^1(\mathbf{U}, \mathbb{R})$ with $\mathbf{U}$ an open subset of $H^1(\mathbb{R}^2, \mathbb{R}^2)$, the gradient $\nabla F(u)$ is defined by $\langle \nabla F(u), X\rangle = dF(u)X$, with $dF(u)$ the Frechét derivative at u. If $F \in C^2(\mathbf{U}, \mathbb{R})$ it remains defined $\nabla^2 F(u) \in C^0(\mathbf{U}, B(H^1(\mathbb{R}^2, \mathbb{R}^2), H^{-1}(\mathbb{R}^2, \mathbb{R}^2)))$ (with $B(\mathbb{X}, \mathbb{Y})$ the space of $\mathbb{R}$–linear bounded operators from a Banach space $\mathbb{X}$ to another a Banach space $\mathbb{Y}$).

The ϕ_ω are constrained critical points of E with associated Lagrange multiplier $-\omega$ so that $\nabla E(\phi_\omega) = -\omega \nabla Q(\phi_\omega)$. The *linearization* of the NLS at ϕ_ω is

$$\begin{aligned} &\mathcal{L}_\omega := J(\nabla^2 E(\phi_\omega) + \omega) = J(-\Delta + \omega + V) + J\mathcal{V}_\omega \text{ with} \\ &\mathcal{V}_\omega := \begin{pmatrix} \beta(|\phi_\omega|^2) + 2\beta'(|\phi_\omega|^2)(\operatorname{Re}\phi_\omega)^2 & \beta'(|\phi_\omega|^2)(\operatorname{Re}\phi_\omega)(\operatorname{Im}\phi_\omega) \\ \beta'(|\phi_\omega|^2)(\operatorname{Re}\phi_\omega)(\operatorname{Im}\phi_\omega) & \beta(|\phi_\omega|^2) + 2\beta'(|\phi_\omega|^2)(\operatorname{Im}\phi_\omega)^2 \end{pmatrix}. \end{aligned} \tag{2.7}$$

Since there is a natural identification $L^2(\mathbb{R}^2, \mathbb{C}^2) = L^2(\mathbb{R}^2, \mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C}$ and $H^2(\mathbb{R}^2, \mathbb{C}^2) = H^2(\mathbb{R}^2, \mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C}$, the operator $\mathcal{L}_\omega$ extends naturally in a $\mathbb{C}$ linear operator in $L^2(\mathbb{R}^2, \mathbb{C}^2)$ with domain $H^2(\mathbb{R}^2, \mathbb{C}^2)$ simply starting from formulas $\mathcal{L}_\omega(v \otimes_{\mathbb{R}} z) = (\mathcal{L}_\omega v) \otimes_{\mathbb{R}} z$.

Starting from $\overline{v \otimes_{\mathbb{R}} z} := v \otimes_{\mathbb{R}} \overline{z}$ a complex conjugation can be defined in $L^2(\mathbb{R}^2, \mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C}$, which should not be confused with the complex conjugation in the initial $L^2(\mathbb{R}^2, \mathbb{C})$. Using this complex conjugation we consider in $L^2(\mathbb{R}^2, \mathbb{C})$ the hermitian form $\langle f, \overline{g}\rangle$, where we recall that $\langle \ , \ \rangle$. is a bilinear form.
Notice that if $\mathcal{L}_\omega \xi = z\xi$ with $z \in \mathbb{C}$, then applying this complex conjugation we obtain $\mathcal{L}_\omega \overline{\xi} = \overline{z}\overline{\xi}$.

Notice also that as operators in $L^2(\mathbb{R}^2, \mathbb{R}^2)$ we have $J\mathcal{L}_\omega = -\mathcal{L}_\omega^* J$ and $\mathcal{L}_\omega J = -J\mathcal{L}_\omega^*$. This extends also for the operators in $L^2(\mathbb{R}^2, \mathbb{C}^2)$. From this we conclude that $\sigma(\mathcal{L}_\omega)$ is symmetric with respect to the coordinate axes.

We consider only standing waves which are *spectrally stable*. Specifically, we will assume

(H6) $\sigma(\mathcal{L}_\omega) \subset \mathrm{i}\mathbb{R}$ for all $\omega \in \mathcal{O}$.

Since (H4) implies that $\phi_\omega(x)$ is exponentially decreasing to 0 as $x \to \infty$, we know that always for the *essential spectrum* we have $\mathrm{i}\sigma_e(\mathcal{L}_\omega) = (-\infty, -\omega] \cup [\omega, \infty)$. Hence (H6) is all about the set of eigenvalues $\sigma_p(\mathcal{L}_\omega)$. We want our ϕ_ω's to be also *linearly stable*. This is somewhat ambiguous as in principle it should mean that $\|e^{t\mathcal{L}_\omega}\|_{L^2 \to L^2}$ is bounded, which is never true since $\mathcal{L}_\omega$ is not skew–adjoint and has a nontrivial Jordan block at 0. By linear stability we mean (H6) and the following two additional hypotheses.

(H7) For $N_g(L) := \cup_{j=1}^\infty \ker(L^j)$ we have

$$\ker \mathcal{L}_\omega = \mathrm{Span}\{J\phi_\omega\} \text{ and } N_g(\mathcal{L}_\omega) = \mathrm{Span}\{J\phi_\omega, \partial_\omega \phi_\omega\}. \tag{2.8}$$

(H8) For any eigenvalue $\mathbf{e} \in \sigma_p(\mathcal{L}_\omega) \backslash \{0\}$ the algebraic and geometric dimensions coincide.

Notice that $N_g(\mathcal{L}_\omega)$ for (1.5) has been computed in [63]. For what happens when a potential is added to (1.5) breaking the translation invariance see Lemma 7.4.

We assume the following hypotheses.

(H9) There are no eigenvalues of $\mathcal{L}_\omega$ contained in $\sigma_e(\mathcal{L}_\omega)$.

(H10) The points $\pm \mathrm{i}\omega$ are neither eigenvalues not resonances of $\mathcal{L}_\omega$, i.e. if $\mathcal{L}_\omega F = \pm \mathrm{i}\omega F$ in a distributional sense for an $F \in L^\infty(\mathbb{R}^2)$, then $F = 0$.

As we mentioned in the introductions oth conditions appear to be generically true.
We have the symmetry $J\mathcal{L}_\omega = -\mathcal{L}_\omega^* J$. Thus, (2.8) implies

$$N_g(\mathcal{L}_\omega^*) = \mathrm{Span}\{\phi_\omega, J^{-1}\partial_\omega \phi_\omega\}. \tag{2.9}$$

We have the following beginning of Jordan blocks decomposition, where we use the hermitian from $\langle f, \overline{g} \rangle$ to define the orthogonal spaces,

$$L^2(\mathbb{R}^2, \mathbb{C}^2) = N_g(\mathcal{L}_\omega) \oplus N_g^\perp(\mathcal{L}_\omega^*) \ . \tag{2.10}$$

[21], as consequence of [41], proves that $\|e^{t\mathcal{L}_\omega}|_{N_g^\perp(\mathcal{L}_\omega^*)}\|_{L^2 \to L^2}$ is bounded under (H6)–(H10). Appropriate dispersive and Strichatz estimates can be proved for the restriction $e^{t\mathcal{L}_\omega}\big|_{L^2_c(\omega)}$ for the space $L^2_c(\omega)$, see Lemma 2.3 below.

We assume existence of non zero eigenvalues.

(H11) For any $\omega \in \mathcal{O}$ there is a number $\mathbf{n} \geq 1$ and positive numbers $0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_{\mathbf{n}}$ such that $\sigma_p(\mathcal{L}_\omega)$ consists exactly of the numbers $\pm \mathbf{e}_j$ and 0, where we set $\mathbf{e}_j(\omega) := \mathrm{i}\lambda_j(\omega)$. We assume that there are fixed integers $\mathbf{n}_0 = 0 < \mathbf{n}_1 < ... < \mathbf{n}_{l_0} = \mathbf{n}$ such that $\lambda_j = \lambda_i$ exactly for i and j both in $(\mathbf{n}_l, \mathbf{n}_{l+1}]$ for some $l \leq l_0$. In this case $\dim \ker(\mathcal{L}_\omega - \mathrm{i}\lambda_j(\omega)) = \mathbf{n}_{l+1} - \mathbf{n}_l$. We denote by $N_j \in \mathbb{N}$ the number such that $N_j + 1 = \inf\{n \in \mathcal{N} : n\mathrm{i}\lambda_j \in \sigma_e(\mathcal{L}_\omega)\}$. We set $\mathbf{N} = \sup_j N_j$.

Notice that in (H11) we do not ask any more uniformity with respect to ω, as in [13, 15, 16, 17]. We remark that the fact that the sum of all the algebraic dimensions of the eigenvalues of $\mathcal{L}_\omega$ is finite can be proved from (H10) and from the fact that each $\phi_\omega(x)$ is in fact not only a Schwartz function in $\mathcal{S}(\mathbb{R}^2, \mathbb{C})$ but converges exponentially to 0 as $x \to \infty$, see [35]. The proof is standard, is similar to an argument in p.305 [66] and involves extending the resolvent beyond the resolvent set as a meromorphic function. Since we are in dimension 2, the discussion of what happens near $\pm \mathrm{i}\omega$ is more complicated than the 3D argument near 0 in [66], but nonetheless an accumulation of eigenvalues near $\pm \mathrm{i}\omega$ can be excluded using $\pm \mathrm{i}\omega$.
The following is a rather standard non–degeneracy hypothesis in the context of normal forms arguments.

(H12) For distinct $\lambda_{j_1} < ... < \lambda_{j_k}$ and $\mu \in \mathbb{Z}^k$ with $|\mu| \le 2\mathbf{N} + 3$, then

$$\mu_1 \lambda_{j_1} + \cdots + \mu_k \lambda_{j_k} = 0 \iff \mu = 0 \ .$$

It is plausible that (H12) is generically true.
Next we assume the Fermi golden rule which we will state explicitly later and on which we commented at length in the Introduction.

(H13) The Fermi golden rule Hypothesis (H13) in Section 6, see (6.15), holds.

So far the hypotheses (H1)–(H13) are similar to the analogous ones in [13]. In [13] though the main result is that the standing waves are (asymptotically) stable, while here we want to prove instability, that is the opposite. So we need an hypothesis which will generate orbital instability. To obtain this hypothesis we consider the signature, or Krein signature, see [44, 45].

Recall the extensions of $\langle \ , \ \rangle$ and Ω in $L^2(\mathbb{R}^2, \mathbb{C}^2)$ made in Def. 2.2. Recall that $\sigma_p(\mathcal{L}_\omega) = \sigma_p(\mathcal{L}_\omega^*)$. By general argument we have the following result.

Lemma 2.3. *The following spectral decomposition remains determined:*

$$\begin{aligned} & L^2(\mathbb{R}^2, \mathbb{C}^2) = L^2_d(\mathcal{L}_\omega) \oplus L^2_c(\mathcal{L}_\omega) \text{ where } L^2_c(\omega) = (L^2_d(\mathcal{L}^*_\omega))^\perp \text{ and for } L = \mathcal{L}_\omega, \mathcal{L}_\omega * \\ & L^2_d(L) := N_g(L) \oplus \widetilde{L}^2_d(L) \text{ with } \widetilde{L}^2_d(L) := \oplus_{\mathbf{e} \in \sigma_p(\mathcal{L}_\omega) \setminus \{0\}} \ker(L - \mathbf{e}). \end{aligned} \tag{2.11}$$

We denote by $P_c(\omega)$ the projection on $L^2_c(\omega)$ associated to (2.11).

The form Ω remains symplectic also in $\widetilde{L}^2_d(\mathcal{L}_\omega)$. The proof of the following lemma is elementary, see for example Lemma 5.2 [12].

Lemma 2.4. *For any $\omega \in \mathcal{O}$ and corresponding $\mathbf{n}$ in (H11) there are functions $\xi_j(\omega) \in \Sigma_k$ for any k and $j = 1, ..., \mathbf{n}$ such that the following facts hold.*

(1) $\xi_j(\omega) \in \ker(\mathcal{L}_\omega - \mathrm{i}\lambda_j(\omega))$ for all j.

(2) $\Omega(\xi_j(\omega), \xi_k(\omega)) = 0$ for all j and k and $\Omega(\xi_j(\omega), \overline{\xi}_k(\omega)) = \mathrm{i} s_j \delta_{jk}$ with $s_j \in \{1, -1\}$.

□

In the case of ground states, that is when $\phi_\omega(x) = \psi_\omega(|x|)$ with $\psi_\omega(|x|) > 0$ and $m = 0$, then $s_j \equiv -1$, see Lemma 2.7 [17]. Here, where $m \neq 0$, we assume instead what follows.

(H14) There exists at least one j s.t. $s_j = 1$.

We have already discussed, and we will say more in Section 7, that it has been shown numerically that this hypothesis occurs for spectrally stable vortices of the cubic–quintic NLS (1.5).

Theorem 1.9 is a consequence of the following proposition, which is a consequence of [12, 13, 16, 21].

Proposition 2.5. *Let $\omega_1 \in \mathcal{O}$ and assume (H1)–(H13). Then there exist constants $\epsilon_0 > 0$ and C_0 s.t. if $u \in C^0([0,\infty), H^1)$ is a solution of* (1.1) *with $\sup_{t\geq 0}\inf_{\vartheta\in\mathbb{R}} \|u(t) - e^{\mathrm{i}\vartheta}\phi_{\omega_1}\|_{H^1} < \epsilon < \epsilon_0$ then there exist $\omega_+ \in \mathcal{O}$, $\theta \in C^1(\mathbb{R};\mathbb{R})$ and $h_+ \in H^1$ with $\|h_+\|_{H^1} + |\omega_+ - \omega_1| \leq C_0\epsilon$ such that*

$$\lim_{t\to\infty} \|u(t) - e^{\mathrm{i}\theta(t)}\phi_{\omega_+} - e^{\mathrm{i}t\Delta}h_+\|_{H^1} = 0. \tag{2.12}$$

It is possible to write $u(t,x) = e^{\mathrm{i}\theta(t)}\phi_{\omega(t)} + A(t,x) + \widetilde{u}(t,x)$ with $|A(t,x)| \leq C_N(t)\langle x\rangle^{-N}$ for any N, with $\lim_{t\to\infty} C_N(t) = 0$, with $\lim_{t\to\infty}\omega(t) = \omega_+$, and such that for any admissible pair (q,p), i.e.

$$1/q + 1/p = 1/2\,, q > 2, \tag{2.13}$$

we have

$$\|\widetilde{u}\|_{L^q([0,\infty),W^{1,p})} \leq C_0\epsilon. \tag{2.14}$$

From Section 3 to Section 6 we focus on Proposition 2.5. There are various steps. We aim at showing that there exists an effective Hamiltonian of the form (1.16). This has to be found through a Birkhoff normal forms argument, see Theorem 11 [39]. In order to initiate the process we need to to find and appropriate system of Darboux coordinates.

3 Modulation and Darboux coordinates

Asymptotic (or conditional asymptotic) stability arguments require traditionally, since [69], the choice of appropriate *modulation* coordinates. Indeed, since we are discussing the stability of vortices, it is natural to express a solution $u(t)$ which is close to a vortex as a sum of a vortex plus an error and to frame stability in terms of what happens to the error. This and more is what modulation aims to do. The first step to define precisely this vaguely stated aim is the following standard Modulation Lemma.

Lemma 3.1 (Modulation Lemma)**.** *Fix $\underline{n} \in \mathbb{Z}$, $\omega_1 \in \mathcal{O}$ and $\Psi_1 = e^{-J\vartheta_1}\phi_{\omega_1}$, where $\mathcal{O}$ is given in (H4) and J is given in* (2.5)*. Then there exists a neighborhood $\mathcal{U}_{\underline{n}}$ of Ψ_1 in $\Sigma_{-\underline{n}}(\mathbb{R}^2,\mathbb{R}^2)$ and functions $\omega \in C^\infty(\mathcal{U}_{\underline{n}},\mathcal{O})$ and $\vartheta \in C^\infty(\mathcal{U}_{\underline{n}},\mathbb{R})$ s.t. $\omega(\Psi_1) = \omega_1$ and $\vartheta(\Psi_1) = \vartheta_1$ and s.t. $\forall u \in \mathcal{U}_{\underline{n}}$*

$$u = e^{-J\vartheta}(\phi_\omega + R) \text{ and } R \in N_g^\perp(\mathcal{L}_\omega^*). \tag{3.1}$$

We give a sketch of the proof. See also Lemma 2.4 [12] or Lemma 2.2 [16] for a detailed proof.

Proof. It suffices to apply the implicit function Theorem to

$$F_1(\vartheta,\omega,u) = \begin{pmatrix} \langle e^{J\vartheta}u - \phi_\omega, \phi_\omega\rangle \\ \langle e^{J\vartheta}u - \phi_\omega, J^{-1}\partial_\omega\phi_\omega\rangle \end{pmatrix}.$$

$\frac{\partial F_1}{\partial(\vartheta,\omega)}\Big|_{(\vartheta,\omega,u)=(\vartheta,\omega_1,e^{-J\vartheta}\phi_{\omega_1})}$ will be invertible because of (H5). ∎

The above Modulation Lemma is the starting point to find appropriate coordinates in the neighborhood of Ψ_1 in $H^1(\mathbb{R}^2,\mathbb{C})$. Solutions $u(t)$ starting close to Ψ_1 will admit a time dependent decomposition (3.1). If $u(t)$ stays close to the orbit of Ψ_1 for all time and scatters to a vortex, this will be equivalent at showing that $R(t)$ scatters to 0 as $t \to \infty$ and $\omega(t) \overset{t\to+\infty}{\to} \omega_+$ for some $\omega_+ \in \mathcal{O}$. This will be proved working on the parameters in the r.h.s. of (3.1).

Equation (1.1) can be expressed as $\dot u = J\nabla E(u)$. The following discussion is standard, and is only sketched in order to give an overview of the use of the parameters (ϑ,ω,R). By substituting the r.h.s. of (3.1) and using $\nabla E(\phi_\omega)=-\omega\phi_\omega$ we obtain after standard computations

$$(\omega-\dot\vartheta)J\phi_\omega+\dot\omega\partial_\omega\phi_\omega+\dot R=\mathcal{L}_\omega R+N(R)-(\omega-\dot\vartheta)JR, \tag{3.2}$$

where $N(R)$ contains terms which are quadratic or higher order for R small. Denote by $P_{N_g}(\omega)=P_{N_g(\mathcal{L}_\omega)}$ the projection on $N_g(\mathcal{L}_\omega)$ related to (2.10). Because $\langle\phi_\omega,\partial_\omega\phi_\omega\rangle=q'(\omega)$, we have

$$P_{N_g}(\omega)X=-(q'(\omega))^{-1}\left(\Omega(X,\partial_\omega\phi_\omega)J\phi_\omega\ +\Omega(J\phi_\omega,X)\partial_\omega\phi_\omega\right),\quad \forall\, X\in\mathcal{S}'(\mathbb{R}^2,\mathbb{C}^2). \tag{3.3}$$

Applying the projection

$$P(\omega):=1-P_{N_g}(\omega). \tag{3.4}$$

to (3.2) we obtain for R the following equation

$$\dot R=\mathcal{L}_\omega R+P(\omega)N(R)-(\omega-\dot\vartheta)P(\omega)JR. \tag{3.5}$$

Since, as is well known, the term $(\omega-\dot\vartheta)$ is higher order in R, we can think of (3.5) as a perturbation of $\dot R=\mathcal{L}_\omega R$. It is natural now to look at the rest of the spectrum of $\mathcal{L}_\omega$. The main difficulty is to show that the discrete components of R associated to the point spectrum of $\mathcal{L}_\omega$, which at a linear level want to oscillate like the $e^{-\mathrm{i}t}z(0)$ component of the linearization of (1.10)–(1.11), will lose their energy because of some friction originating from the nonlinear interaction with the continuous components of R. This effect will be captured by an argument similar to the Fermi golden rule discussed in the Introduction. For that argument to work we need to find an appropriate system of coordinates.

Lemma 3.1 does not provide coordinates. We co back to the projection $P_{N_g}(\omega)$. We have $\Omega(P_{N_g}(\omega)X,Y)=\Omega(X,P_{N_g}(\omega)Y)$. By (H4)–(H5) and (2.8) for $\mathcal{S}(\mathbb{R}^2,K^2)=\cap_{k\ge 0}\Sigma_k(\mathbb{R}^2,K^2)$ the space of Schwartz functions and for $\mathcal{S}'(\mathbb{R}^2,K^2)=\cup_{k\le 0}\Sigma_k(\mathbb{R}^2,K^2)$ the space of tempered distributions, we have

$$P_{N_g}(\omega)\in C^\infty(\mathcal{O},B(\mathcal{S}'(\mathbb{R}^2,K^2),\mathcal{S}(\mathbb{R}^2,K^2)))\text{ for }K=\mathbb{R},\mathbb{C}. \tag{3.6}$$

For $P(\omega)$ defined as in (3.4) we have $\omega\to P(\omega)P(\omega_1)\in C^\infty(\mathcal{O},B(\Sigma_k,\Sigma_k))$ for any k. By (H4) we have $P(\omega)\overset{\omega\to\omega_1}{\to}P(\omega_1)$ in the operator topology of $B(\Sigma_k,\Sigma_k)$. Thus, writing

$$P(\omega)P(\omega_1)=(1+(P(\omega)-P(\omega_1)))P(\omega_1),$$

we see that there exists an $a>0$ such that if $|\omega-\omega_1|<a$ the map $P(\omega)P(\omega_1)$ restricts into an isomorphism from $N_g^\perp(\mathcal{L}^*_{\omega_1})\cap X_k$ to $N_g^\perp(\mathcal{L}^*_\omega)\cap X_k$ for any $k\ge-\underline{n}$ with X_k equal either to H^k or to Σ_k. Hence for $k\ge-\underline{n}$ the map

$$\begin{aligned}&\mathbb{R}\times\{\omega:|\omega-\omega_1|<a\}\times(N_g^\perp(\mathcal{L}^*_{\omega_1})\cap X_k)\to X_k,\\&(\vartheta,\omega,r)\to u=e^{-J\vartheta}(\phi_\omega+P(\omega)r),\end{aligned} \tag{3.7}$$

is for $\|r\|_{X_k}<a$ a local C^∞ diffeomorphism in the image. Therefore, (ϑ,ω,r) in (3.7) provides an initial system of independent coordinates.

If we consider the function $Q=Q(u)$, the map $(\vartheta,\omega,r)\to(\vartheta,Q,r)$ is a local diffeomorphism because of the assumption (H5). Indeed, applying implicit function theorem to

$$F_2(Q,\rho,r,\omega)=Q(\phi_\omega-P_{N_g}(\omega)r)+\rho+\left\langle\phi_\omega-P_{N_g}(\omega)r,r\right\rangle-Q, \tag{3.8}$$

there exists $\omega(Q,\rho,r)$ which is a smooth function defined in the neighborhood of $(Q(\phi_{\omega_0}),0,0) \in \mathbb{R}\times\mathbb{R}\times\Sigma_{-\underline{n}}$. Notice that $F_2(Q,Q(r),r,\omega) = Q(\phi_\omega + P(\omega)r) - Q$. We have put an auxiliary variable ρ because if we directly put $\omega(Q,r)$ to be the implicit function of $\tilde{F}_2 = Q(\phi_\omega + P(\omega)r) - Q$, then we will only able to define ω in the neighborhood of $(Q(\phi_{\omega_0}),0)$ in $\mathbb{R}\times L^2$. Differentiating, (3.8) by Q,ρ,r, we have

$$\partial_Q\omega = -\partial_\rho\omega = A^{-1}, \tag{3.9}$$

$$\langle\nabla_r\omega, X\rangle = A^{-1}\left(\left\langle P(\omega)r, P_{N_g}(\omega)X\right\rangle + \left\langle P_{N_g}(\omega)r, X\right\rangle\right), \tag{3.10}$$

where $A = \langle\phi_\omega + P(\omega)r, \partial_\omega\phi_\omega + \partial_\omega P(\omega)r\rangle$.

We now expand Ω by using the coordinates (ϑ,Q,r). Notice that

$$X = duX = \partial_\vartheta u X_\theta + \partial_Q u X_Q + \langle\nabla_r u, X_r\rangle,$$

where $X_\vartheta = d\vartheta X, X_Q = dQX$ and $X_r = drX$. Then, after some cancelations, we obtain

$$\begin{aligned}\Omega = &-d\vartheta\wedge dQ + \Omega(P(\omega)dr, P(\omega)dr) \\ &+ A^{-1}dQ\wedge\Omega(\partial_\omega P(\omega)r, P(\omega)dr) + \langle\nabla_r\omega + \partial_q\omega r, dr\rangle\wedge\Omega(\partial_\omega P(\omega)r, P(\omega)dr).\end{aligned} \tag{3.11}$$

Notice that the coordinates (ϑ,Q,r) are not a system of Darboux coordinates for the symplectic form Ω.

We now prepare some notations.

- Let F be a Frechét differentiable function. Then, its hamiltonian vector field X_F is defined by $\Omega(X_F,Y) = dF(Y)$ for any given vector Y. In particular, we have $X_F = J\nabla F$.
- For F,G two scalar valued functions, we set the Poisson bracket by $\{F,G\} := dF(X_G)$.
- If $\mathcal{G}$ has values in a given Banach space $\mathbb{E}$ and G is a scalar valued function, then we set $\{\mathcal{G},G\} := d\mathcal{G}(X_G)$.

In the coordinate system (ϑ,Q,r) our NLS can be expressed as

$$\dot{Q} = \{Q,E\} = (X_E)_Q, \quad \dot{\vartheta} = \{\vartheta,E\} = (X_E)_\vartheta, \quad \dot{r} = \{r,E\} = (X_E)_r. \tag{3.12}$$

Further, comparing the coefficients of Y_ϑ in $\Omega(X_E,Y) = dEY$ by (3.11), we have $(X_E)_Q = dQX_E = 0$. Therefore, we have $\dot{Q} = 0$. Notice that this shows that with coordinates (ϑ,Q,r) we have achieved a *reduction of order* in the system, see [62] p. 412, effectively reducing to the variable r only.
In the sequel we choose ω_0 such that if u_0 is the initial value in (1.1), then

$$Q(\phi_{\omega_0}) = Q(u_0). \tag{3.13}$$

We consider (recall $q(\omega) = Q(\phi_\omega)$)

$$K(u) := E(u) - E(\phi_{\omega_0}) + \omega(u)(Q(u) - q(\omega_0)). \tag{3.14}$$

Then, since $(X_Q)_\vartheta = -1, (X_Q)_Q = 0$ and $(X_Q)_r = 0$, we see that (3.12) is equivalent to

$$\dot{Q} = 0\,, \quad \dot{\vartheta} = \{\vartheta,K\} + \omega\,, \quad \dot{r} = \{r,K\}.$$

See [12] Lemma 2.6 and Section 2.3, and it is important that $Q(u_0) = q(\omega_0)$.

In the sequel the changes of coordinates will differ from the identity transformation by perturbations that can be written in terms of the two classes of symbols which we introduce now.

Definition 3.2. For I an interval with 0 in the interior, $\mathcal{A} \subset \mathbb{R}\times\mathbb{R}\times(N_g^{\perp}(\mathcal{L}_{\omega_1}^*)\cap\Sigma_{-n})$ a neighborhood of $(q(\omega_1), 0, 0)$, we say that $\mathfrak{F} \in C^m(I \times \mathcal{A}, \mathbb{R})$ is $\mathcal{R}^{i,j}_{n,m}$ if there exists a $C > 0$ and a smaller neighborhood $\mathcal{A}'$ of of $(q(\omega_1), 0, 0)$ s.t.

$$|\mathfrak{F}(t, Q, \varrho, r)| \le C\|r\|^j_{\Sigma_{-n}}(\|r\|_{\Sigma_{-n}} + |\varrho| + |Q - q(\omega_1)|)^i \text{ in } I \times \mathcal{A}'. \tag{3.15}$$

We will write also $\mathfrak{F} = \mathcal{R}^{i,j}_{n,m}$ or $\mathfrak{F} = \mathcal{R}^{i,j}_{n,m}(t, Q, \varrho, r)$. Given a function $F : I \times \mathcal{U}_{\omega_1} \to \mathbb{R}$ for $\mathcal{U}_{\omega_1}$ a neighborhood of ϕ_{ω_1} in $L^2(\mathbb{R}^2, \mathbb{R}^2)$, we say that $F = \mathcal{R}^{i,j}_{n,m}$ if there is a $\mathcal{R}^{i,j}_{n,m}$ function s.t. $F(t, u) = \mathcal{R}^{i,j}_{n,m}(t, Q, Q(r), r)$.

We say $\mathfrak{F} = \mathcal{R}^{i,j}_{n,\infty}$ if $\mathfrak{F} = \mathcal{R}^{i,j}_{n,m}$ for all m. We say $F = \mathcal{R}^{i,j}_{\infty,m}$ if we can take n arbitrarily large. If $F = \mathcal{R}^{i,j}_{\infty,m}$ for any m, we set $F = \mathcal{R}^{i,j}$.

Definition 3.3. A $\mathfrak{T} \in C^m(I \times \mathcal{A}, \Sigma_n(\mathbb{R}^2, \mathbb{C}^2))$, with $I \times \mathcal{A}$ like above, is $\mathbf{S}^{i,j}_{n,m}$ and we write as above $\mathfrak{T} = \mathbf{S}^{i,j}_{n,m}$ or $\mathfrak{T} = \mathbf{S}^{i,j}_{n,m}(t, Q, \varrho, r)$, if there exists a $C > 0$ and a smaller neighborhood $\mathcal{A}'$ of $(p_0, p_0, 0)$ s.t.

$$\|\mathfrak{T}(t, Q, \varrho, r)\|_{\Sigma_n} \le C\|r\|^j_{\Sigma_{-n}}(\|r\|_{\Sigma_{-n}} + |\varrho| + |Q - q(\omega_1)|)^i \text{ in } I \times \mathcal{A}'. \tag{3.16}$$

We use notation $\mathfrak{T} = \mathbf{S}^{i,j}_{n,\infty}$, $\mathfrak{T} = \mathbf{S}^{i,j}_{\infty,m}$ and $\mathfrak{T} = \mathbf{S}^{i,j}$ as above. As above, given a function $T : I \times \mathcal{U}_{\omega_1} \to \Sigma_{-n}$ we write $F = \mathbf{S}^{i,j}_{n,m}$ if there is a $\mathbf{S}^{i,j}_{n,m}$ function s.t. $T(t, u) = \mathbf{S}^{i,j}_{n,m}(t, Q, Q(r), r)$.

Next we consider the following symplectic form:

$$\Omega_0 := -d\vartheta \wedge dQ + \Omega(dr\ , dr\ ). \tag{3.17}$$

This is how our symplectic Ω form should look in appropriate coordinates. Indeed in Section 3 [12] the following Darboux Theorem is proved.

Proposition 3.4 (Darboux Theorem)**.** *There is a local diffeomorphism $\mathcal{F}$ around ϕ_{ω_1} in $L^2(\mathbb{R}^2, \mathbb{R}^2)$ such that $\mathcal{F}^*\Omega = \Omega_0$ and which in the (ϑ, Q, r) coordinates is of the form*

$$\begin{aligned} &\vartheta' = \vartheta + \mathcal{R}^{0,2}(Q, Q(r), r)\,, \quad Q' = Q\,, \\ &r' = e^{J\mathcal{R}^{0,2}(Q,Q(r),r)}(r + \mathbf{S}^{1,1}(Q, Q(r), r)). \end{aligned} \tag{3.18}$$

□

Remark 3.5. Notice that the idea of taking as fixed point ϕ_{ω_1} rather than ϕ_{ω_0} as in [12], is taken from [4]. The proof of Prop. 3.4 is unaffected.

For the convenience of the readers we give a sketch of the proof here.

Sketch of the proof of Proposition 3.4. To make a change of coordinate to convert the symplectic form Ω into Ω_0 we need three steps. First, we find a 1-form Γ s.t. $\Omega - \Omega_0 = d\Gamma$. Next, we solve $i_{\mathcal{X}^s}\Omega_s = -\Gamma$, where $i_X\omega(Y) := \omega(X, Y)$ and $\Omega_s := \Omega_0 + s(\Omega - \Omega_0)$. Finally, let $\mathcal{Y}_s$ be the flow of $\frac{d}{ds}\mathcal{Y}_s = \mathcal{X}^s(\mathcal{Y}_s)$. Then, we have

$$\frac{d}{ds}\left(\mathcal{Y}_s^*\Omega_s\right) = \mathcal{Y}_s^*\left(\mathcal{L}_{\mathcal{X}^s}\Omega_s - \partial_s\Omega_s\right) = \mathcal{Y}_s^*(di_{\mathcal{X}^s}\Omega_s - d\Gamma) = 0.$$

Thus, $\mathcal{Y} := \mathcal{Y}^1$ gives us the desired transformation. This is a standard proof of Darboux theorem (see [39]).
In our situation, we have to care about the regularity of the transformation (or in other words, error from the identity). Therefore, we need to compute Γ rather explicitly.

First, we seek Γ. It suffices to find some Γ satisfying $\Gamma = B - B_0 + dC$, where

$$\begin{aligned} 2B_0 &= Qd\vartheta + \Omega(r, dr), \\ 2B &= \Omega(u, \cdot) \\ &= Qd\vartheta + \Omega(P(\omega)r, dr) + \Omega(\phi_\omega, P(\omega)dr) + \Omega(\phi_\omega + P(\omega)r, \partial_\omega \phi_\omega + \partial_\omega P(\omega)r)d\omega. \end{aligned}$$

It is elementary that $dB = \Omega$, $dB_0 = \Omega_0$. We have

$$\begin{aligned} 2(B - B_0) =& d\left(\Omega(\phi_\omega, P(\omega)r)\right) + \Omega(P(\omega)r, \partial_\omega P r)\partial_Q \omega dQ \\ &+ \Omega(-P_{N_g}(\omega)r + \Omega(P(\omega)r, \partial_\omega P(\omega)r)\partial_\rho \omega Jr + \Omega(P(\omega)r, \partial_\omega P(\omega)r)\nabla_r \omega, dr). \end{aligned}$$

Therefore, we can choose $\Gamma = \Omega(\Gamma_r, dr) + \Gamma_Q dQ$ as

$$\begin{aligned} 2\Gamma_r &= -P_{N_g}(\omega)r + \Omega(P(\omega)r, \partial_\omega P(\omega)r)\partial_\rho \omega Jr + \Omega(P(\omega)r, \partial_\omega P(\omega)r)\nabla_r \omega, \\ 2\Gamma_Q &= \Omega(P(\omega)r, \partial_\omega P r)\partial_Q \omega. \end{aligned}$$

Since $P_{N_g}(\omega)r = P_{N_g}(\omega)\left(P(\omega_0) - P(\omega)\right)r$ and $|\omega - \omega_0| \sim Q(r)$, we see $\Gamma_r = \mathbf{S}^{1,1} + \mathcal{R}^{0,2}Jr$ and $\Gamma_Q = \mathcal{R}^{0,2}$.
Next, we solve $i_{\mathcal{X}^s}\Omega_s = -\Gamma$. Since $s\left(\Omega(\mathcal{X}^s, \cdot) - \Omega_0(\mathcal{X}^s, \cdot)\right)$ can be handled as a perturbation, the main part of the equation will be $\Omega_0(\mathcal{X}^s, \cdot) = -\Gamma$. Therefore, we have $\mathcal{X}^s_Q = 0$, $\mathcal{X}^s_\theta = \mathcal{R}^{0,2}$ and $\mathcal{X}^s_r = \mathcal{S}^{1,1} + \mathcal{R}^{0,2}Jr$. Finally, solving $\frac{d}{ds}\mathcal{Y}_s = \mathcal{X}^s(\mathcal{Y}_s)$, we have the conclusion. $\square$

It is well known that normal forms processes are based on Taylor expansions of the Hamiltonian, see [39]. So we need an expansion of the functional K defined in (3.14) in terms of the Darboux coordinates. This is provided by the following, proved in Lemma 4.3 [12].

Lemma 3.6. *Consider an integer $L \in \mathcal{N}$ s.t. $L > p$ for p the exponent* (2.1) *in hypothesis (H2). For any preassigned (k, m) and in the coordinates (Q, ϑ, r) of* (3.18)*, K admits the expansion*

$$d(\omega) - d(\omega_0) - (\omega - \omega_0)q(\omega_0) + \frac{1}{2}\Omega(\mathcal{L}_\omega P(\omega)r, P(\omega)r) + \mathcal{R}^{1,2}_{k,m}(Q, Q(r), r) + E_P(P(\omega)r) + \boldsymbol{R}'',$$

$$\boldsymbol{R}'' := \sum_{d=2}^{L-1} \langle B_d(Q, Q(r), r), (P(\omega)r)^d \rangle + \int_{\mathbb{R}^2} B_L(x, r(x), Q, Q(r), r)(P(\omega)r)^L(x)dx \text{ with:} \tag{3.19}$$

- *$d(\omega) = E(\phi_\omega) + \omega q(\omega)$;*
- *$B_2(q(\omega_0), 0, 0) = 0$;*
- *$(P(\omega)r)^d(x)$ represent $d-$products of components of $P(\omega)r$;*
- *$B_d(\cdot, Q, \varrho, r) \in C^m(\mathcal{U}_{-k}, \Sigma_k(\mathbb{R}^2, B((\mathbb{R}^2)^{\otimes d}, \mathbb{R})))$ for $2 \le d \le 4$ with $\mathcal{U}_{-k} \subset \mathcal{P}^{-k}$ a neighborhood of $(q(\omega_1), 0, 0)$ in $\mathbb{R} \times \mathbb{R} \times (N_g^\perp(\mathcal{L}^*_{\omega_1}) \cap \Sigma_{-n})$;*
- *for $\zeta \in \mathbb{R}^2$ and $(Q, \varrho, r) \in \mathcal{U}_{-k}$ we have for $i + j \le m$*

$$\|\partial_r^j \partial^i_{\zeta, Q, \varrho} B_L(\cdot, \zeta, Q, \varrho, r)\|_{B(\Sigma^{\otimes j}_{-k}, \Sigma_k(\mathbb{R}^2, B((\mathbb{R}^2)^{\otimes L}, \mathbb{R})))} \le C_i. \tag{3.20}$$

Remark 3.7. We have $d(\omega) - d(\omega_0) - (\omega - \omega_0)q(\omega_0) = O(\omega - \omega_0)^2 = \mathcal{R}^{2,0}(Q(r)) + \mathcal{R}^{1,2}_{k,m}(Q, Q(r), r)$. In Lemma 4.3 [12] inequality (3.20) is stated for $|\zeta| \le \varepsilon$ for some small $\varepsilon > 0$, but in fact in the proof is unnecessary, thanks to (H2). Notice also that $L = 5$ in [12], but a similar proof holds for our choice of L.

Sketch of the proof. Notice that here we are just expanding $K(u) = S_\omega(u) - S_\omega(\phi_{\omega_0})$ such as

$$K(\phi_\omega + P(\omega)r) = S_\omega(\phi_\omega) + \left\langle \nabla^2 S_\omega(\phi_\omega)P(\omega)r, P(\omega)r \right\rangle + o\left((P(\omega)r)^2 \right) - d(\omega_0) + (\omega - \omega_0)q(\omega_0),$$

where we have used $\nabla S_\omega(\phi_\omega) = 0$. $\square$

If we expand $P(\omega)r = r + (P(\omega) - P(\omega_1))r$ we obtain what follows, see Lemma 4.4 [12].

Lemma 3.8. *The expansion of K in Lemma 3.6 can be rewritten as follows,with similar notation:*

$$\begin{aligned} K &= \mathcal{R}^{2,0}_{k,m}(Q, Q(r)) + 2^{-1}\Omega(\mathcal{L}_{\omega_1} r, r) + \mathcal{R}^{1,2}_{k,m}(Q, Q(r), r) + E_P(r) + \boldsymbol{R}', \\ \boldsymbol{R}' &:= \sum_{d=2}^{L-1} \langle B_d(Q, Q(r), r), r^d \rangle + \int_{\mathbb{R}^2} B_L(x, r(x), Q, Q(r), r) r^L(x) dx \ . \end{aligned} \tag{3.21}$$

4 Spectral coordinates associated to $\mathcal{L}_{\omega_1}$

Recall that $r \in N_g^\perp(\mathcal{L}^*_{\omega_1}) \cap L^2(\mathbb{R}^2, \mathbb{R}^2)$. We consider the spectral decomposition of r in terms of $\mathcal{L}_{\omega_1}$:

$$r = \sum_{j=1}^{\mathbf{n}} z_j \xi_j(\omega_1) + \sum_{j=1}^{\mathbf{n}} \overline{z}_j \overline{\xi}_j(\omega_1) + f \text{ where } f \in L^2_c(\omega_1) \text{ and also } f \in L^2(\mathbb{R}^2, \mathbb{R}^2). \tag{4.1}$$

This yields new coordinates $r \to (z, f)$ to replace r.
Correspondingly we have the expansion

$$\Omega(dr \ , dr \ ) = \sum_{j=1}^{\mathbf{n}} \mathrm{i} \ s_j \ dz_j \wedge d\overline{z}_j + \Omega(df \ , df \ ). \tag{4.2}$$

Equation $\dot{r} = \{r, K\}$ splits into

$$\mathrm{i}\dot{z}_j = s_j \frac{\partial}{\partial \overline{z}_j} K \, , \quad \dot{f} = \{f, K\}, \tag{4.3}$$

where we recall that $s_j \in \{1, -1\}$ and $s_j = 1$ for at least one j.

We have reduced our NLS to system (4.3). Obviously, having replaced r with (z, f), we need to rewrite the expansion of K in Lemma 3.8 in terms of (z, f). This is done in Lemma 5.4 [12], which we quote.

Lemma 4.1. *In the coordinate system (z, f) near $(0, 0)$ for any preassigned pair (k, m) we have an expansion*

$$K = \mathcal{R}^{2,0}_{k,m}(Q, Q(f)) + H_2' + \sum_{j=-1}^{4} \boldsymbol{R}_j + \mathcal{R}^{1,2}_{k,m}(Q, Q(f), f) \textit{ with what follows.} \tag{4.4}$$

(1) For $\varrho = Q(f)$

$$H_2' = -\sum_{j=1}^{\mathbf{n}} s_j \lambda_j |z_j|^2 + \sum_{\substack{|\mu+\nu|=2 \\ \mathbf{e}\cdot(\mu-\nu)=0}} \mathcal{R}^{1,0}_{k,m}(Q, \varrho) z^\mu \overline{z}^\nu + 2^{-1}\Omega(\mathcal{L}_{\omega_1} f, f). \tag{4.5}$$

(2) We have for $\varrho = Q(f)$:

$$\mathbf{R}_{-1} = \sum_{\substack{|\mu+\nu|=2 \\ \mathbf{e}\cdot(\mu-\nu)\neq 0}} \mathcal{R}^{1,0}_{k,m}(Q,\varrho) z^\mu \overline{z}^\nu + \sum_{|\mu+\nu|=1} z^\mu \overline{z}^\nu \langle J\mathbf{S}^{1,0}_{k,m}(Q,\varrho), f\rangle;$$

for $\mathbf{N}$ as in (H12), $\varrho = Q(f)$, $g_{\mu\nu}(Q,\varrho) = \mathcal{R}^{0,0}_{k,m}(Q,\varrho)$, $G_{\mu\nu}(Q,\varrho) = \mathbf{S}^{0,0}_{k,m}(Q,\varrho)$ and with the symmetries $g_{\nu\mu} = \overline{g}_{\mu\nu}$ and $G_{\nu\mu} = -\overline{G}_{\mu\nu}$, we have

$$\boldsymbol{R}_0 = \sum_{|\mu+\nu|=3}^{2\mathbf{N}+1} z^\mu \overline{z}^\nu g_{\mu\nu}(Q,\varrho); \quad \boldsymbol{R}_1 = \mathrm{i} \sum_{|\mu+\nu|=2}^{2\mathbf{N}} z^\mu \overline{z}^\nu \langle JG_{\mu\nu}(Q,\varrho), f\rangle;$$

$$\boldsymbol{R}_2 = \langle \mathbf{S}^{1,0}_{k,m}(Q,\varrho), f^2\rangle \text{ with } \mathbf{B}_2(q(\omega_1),0) = 0,$$

where $f^d(x)$ represents schematically $d-$products of components of f;

$$\boldsymbol{R}_3 = \sum_{\substack{|\mu+\nu|= \\ =2N+2}} z^\mu \overline{z}^\nu \mathcal{R}^{0,0}_{k,m}(Q,z,\varrho,f) + \sum_{\substack{|\mu+\nu|= \\ =2N+1}} z^\mu \overline{z}^\nu \langle J\mathbf{S}^{0,0}_{k,m}(Q,z,\varrho,f), f\rangle;$$

$$\boldsymbol{R}_4 = \sum_{d=2}^{L-1} \langle B_d(Q,z,\varrho,f), f^d\rangle + \int_{\mathbb{R}^2} B_L(x, f(x), Q, z, Q(f), f) f^L(x) dx + E_P(f),$$

where the B's are like in Lemma 3.6.

$\square$

Now we start discussing about the normal forms argument. It will consist in eliminating as many terms as possible from $\mathbf{R}_j$ with $j = -1, 0, 1$.
We set, for $\mathbf{n}$ the number associated to ω_1 in (H11),

$$\mathbf{e} = (\mathbf{e}_1, ..., \mathbf{e}_{\mathbf{n}}). \tag{4.6}$$

Some of the monomials in $\mathbf{R}_j$ with $j = 0, 1$ cannot be eliminated because they are *resonant*, that is of the following type.

Definition 4.2 (Normal Forms)**.** A function $Z(z,\varrho,f)$ is in normal form if $Z = Z_0 + Z_1$ where Z_0 and Z_1 are finite sums of the following type:

$$Z_1 = \mathrm{i} \sum_{\mathbf{e}\cdot(\mu-\nu)\in\sigma_e(\mathcal{L}_{\omega_1})} z^\mu \overline{z}^\nu \langle JG_{\mu\nu}(\varrho), f\rangle, \tag{4.7}$$

where $G_{\mu\nu} = \mathbf{S}^{0,0}_{k,m}(\varrho)$ for fixed $k, m \in \mathbb{N}$;

$$Z_0 = \sum_{\mathbf{e}(\omega_1)\cdot(\mu-\nu)=0} g_{\mu\nu}(\varrho) z^\mu \overline{z}^\nu, \tag{4.8}$$

and $g_{\mu\nu} = \mathcal{R}^{0,0}_{\infty,m}(\varrho)$. We assume furthermore that Z_0 and Z_1 are real valued for $f \in L^2(\mathbb{R}^2,\mathbb{R}^2)$, and hence $\overline{g}_{\mu\nu} = g_{\nu\mu}$ and $\overline{G}_{\mu\nu} = -G_{\nu\mu}$.

With an appropriate canonical change of coordinates (that is, it preserves the r.h.s. of (4.2)) the term $\mathbf{R}_{-1}$ and all non resonant terms in $\mathbf{R}_0$ and $\mathbf{R}_1$ cancel out. Indeed we have the following fact, which we quote from Theorem 6.4 [12].

Proposition 4.3 (Birkhoff normal forms)**.** *There is a canonical transformation* $(z, f) \overset{\mathcal{F}}{\to} (z', f')$ *where*

$$z' = z + \mathcal{R}^{0,2}(Q, z, Q(f), f)\,,$$
$$f = e^{J\mathcal{R}^{0,2}(Q,z,Q(f),f)}(f + \mathbf{S}^{1,1}(Q, z, Q(f), f)). \tag{4.9}$$

such that in the new coordinates (z, f) *we have*

$$K(Q, z, f) = \mathcal{R}^{2,0}_{k,m}(Q, Q(f)) + H_2' + \boldsymbol{R}_0 + \boldsymbol{R}_1 + \mathcal{R} \text{ with}$$
$$\mathcal{R} = \mathcal{R}^{1,2}_{k,m}(Q, Q(f), f) + \sum_{j=2}^{4} \boldsymbol{R}_j + \widehat{\boldsymbol{R}}_2(Q, z, \varrho, f), \tag{4.10}$$

with H_2' *and* $\boldsymbol{R}_j$ *like in Lemma and where we have:*

1. *(1) the term* $\boldsymbol{R}_{-1}$ *in* (4.4) *is here* $\boldsymbol{R}_{-1} = 0$*;*
2. *(2) all the nonzero terms in* $\boldsymbol{R}_0$ *with* $|\mu + \nu| \le 2\mathbf{N} + 1$ *are in normal form, that is* $\mathbf{e} \cdot (\mu - \nu) = 0$*, and are in* Z_0*;*
3. *(3) all the nonzero terms in* $\boldsymbol{R}_1$ *with* $|\mu+\nu| \le 2\mathbf{N}$ *are in normal form, that is* $\mathbf{e}\cdot(\mu-\nu) \in \sigma_e(\mathcal{H}_{p_0})$*, and are in* Z_1*;*
4. *(4) we have* $\widehat{\boldsymbol{R}}_2 \in C^m(\mathbb{U}, \mathbb{C})$ *for* $\mathbb{U} \subset \mathbb{R} \times \mathbb{C}^{\mathbf{n}} \times \mathbb{R} \times P_c\Sigma_{-k}$ *a neighborhood of* $(q(\omega_1), 0, 0, 0)$ *and*
$$|\widehat{\boldsymbol{R}}_2(Q, z, f, \varrho)| \le C(|Q - q(\omega_1)| + |z| + \|f\|_{\Sigma_{-k}})\|f\|^2_{\Sigma_{-k}}.$$

Sketch of the proof. Our canonical transformation will be generated by Hamiltonian functions of the following form:

$$\sum_{\substack{|\mu+\nu|=m \\ \mathbf{e}\cdot(\mu-\nu)\neq 0}} A_{\mu,\nu}(Q, \varrho) z^\mu \overline{z}^\nu + \sum_{\substack{|\mu+\nu|=m-1 \\ \mathbf{e}\cdot(\mu-\nu)\notin\sigma_e(\mathcal{L}_{\omega_1})}} z^\mu \overline{z}^\nu \Omega(B_{\mu,\nu}(Q, \varrho), f).$$

The Hamilton vector flow generated from this Hamiltonian vector field will be

$$z_j(s) \sim z_j - s\mathrm{i}s_j \left(\sum_{\substack{|\mu+\nu|=m \\ \mathbf{e}\cdot(\mu-\nu)\neq 0}} \nu_j A_{\mu,\nu}(Q, \varrho) \frac{z^\mu \overline{z}^\nu}{\bar{z}_j} + \sum_{\substack{|\mu+\nu|=m-1 \\ \mathbf{e}\cdot(\mu-\nu)\notin\sigma_e(\mathcal{L}_{\omega_1})}} \nu_j \frac{z^\mu \overline{z}^\nu}{\bar{z}_j} \Omega(B_{\mu,\nu}(Q, \varrho), f) \right),$$
$$f(s) \sim f + s \sum_{\substack{|\mu+\nu|=m-1 \\ \mathbf{e}\cdot(\mu-\nu)\notin\sigma_e(\mathcal{L}_{\omega_1})}} z^\mu \overline{z}^\nu B_{\mu,\nu}(Q, \varrho).$$

Thus,

$$-\sum_{j=1}^{\mathbf{n}} s_j \lambda_j |z_j(1)|^2 + 2^{-1}\Omega(\mathcal{L}_{\omega_1} f(1), f(1)) \sim -\sum_{j=1}^{\mathbf{n}} s_j \lambda_j |z_j|^2 + 2^{-1}\Omega(\mathcal{L}_{\omega_1} f, f)$$
$$+ \sum_{\substack{|\mu+\nu|=m \\ \mathbf{e}\cdot(\mu-\nu)\neq 0}} \mathbf{e}\cdot(\mu - \nu) A_{\mu,\nu}(Q, \varrho) z^\mu \overline{z}^\nu + \sum_{\substack{|\mu+\nu|=m-1 \\ \mathbf{e}\cdot(\mu-\nu)\notin\sigma_e(\mathcal{L}_{\omega_1})}} z^\mu \overline{z}^\nu \Omega((\mathcal{L}_{\omega_1} - \mathbf{e}\cdot(\mu - \nu))\, B_{\mu,\nu}(Q, \varrho), f).$$

By the above, we can erase the nonresonant terms. □

In (4.10) the functional K is written in a form which is essentially the same of (1.16). What follows in Sections 5–6 is rather close to the classical discussion in [7, 70]. The difference between these papers and [13, 3] lies in the fact that the latter two use in an essential form the Hamiltonian structure of the system to study higher order interactions between discrete and continuous modes. The ideas originate from [14]. The method [7, 70] does not work well with higher order interactions and requires very stringent restrictions on the spectrum of the linearization, which are completely eased in [13, 3].

As we have seen in the analysis of (1.16) the 2nd power, the structure of the Fermi golden rule will be easily seen in the framework provided by Proposition 4.3. However the informal analysis on (1.16) which we made in Section 1 has to be supplemented by a number of estimates, especially for the variable f. So we will need to write the equation for f and derive some estimates.

5 Equations

In the new coordinates (z,f) in Proposition 4.3 our NLS continues to be of the form (4.3). In particular we have

$$\begin{aligned}\dot f &= J\nabla_f \mathcal{R}^{2,0}_{k,m}(Q,Q(f)) + J\nabla_f H_2'(Q,z,Q(f),f) + J\nabla_f Z_0(Q,z,Q(f)) \\ &\quad + J\nabla_f Z_1(Q,z,Q(f),f) + J\nabla_f \mathcal{R}(Q,z,Q(f),f),\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}&\nabla_f(\mathcal{R}^{2,0}_{k,m}(Q,Q(f)) + H_2' + Z_0) = \mathcal{L}_{\omega_1} f + A'Jf, \\ &A' := \partial_{Q(f)} \mathcal{R}^{2,0}_{k,m}(Q,Q(f)) + \sum_{\substack{|\mu+\nu|\ge 2 \\ \mathbf{e}\cdot(\mu-\nu)=0}} \partial_{Q(f)} a_{\mu\nu}(Q,Q(f)) z^\mu \overline{z}^\nu,\end{aligned} \tag{5.2}$$

and similarly we split the 2nd line of (5.1) into

$$\begin{aligned}&A''Jf - \mathrm{i} \sum_{\mathbf{e}\cdot(\mu-\nu)\in\sigma_e(\mathcal{L}_{\omega_1})} z^\mu \overline{z}^\nu G_{\mu\nu}(Q,Q(f)) + J\nabla_f \mathcal{R}(Q,z,\varrho,f)_{|\varrho=Q(f)}, \\ &A'' := \partial_{Q(f)}[Z_1(Q,z,Q(f),f) + \mathcal{R}(Q,z,Q(f),f)].\end{aligned} \tag{5.3}$$

So finally we write the equation of f as

$$\dot f = \mathcal{L}_{\omega_1} f + AJf - \mathrm{i} \sum_{\mathbf{e}\cdot(\mu-\nu)\in\sigma_e(\mathcal{L}_{\omega_1})} z^\mu \overline{z}^\nu G_{\mu\nu}(Q,0) + \mathfrak{R} \text{ where} \tag{5.4}$$

$$A = A' + A'' \text{ and } \mathfrak{R} = \nabla_f \mathcal{R}(Q,z,\varrho,f)_{|\varrho=Q(f)} - \mathrm{i} \sum_{\mathbf{e}\cdot(\mu-\nu)\in\sigma_e(\mathcal{L}_{\omega_1})} z^\mu \overline{z}^\nu \left(G_{\mu\nu}(Q,Q(f)) - G_{\mu\nu}(Q,0)\right).$$

We write the equations for z as

$$s_j \mathrm{i}\dot z_j = \frac{\partial}{\partial \overline{z}_j}(H_2' + Z_0) + \mathrm{i} \sum_{\mathbf{e}\cdot(\mu-\nu)\in\sigma_e(\mathcal{L}_{\omega_1})} \nu_j \frac{z^\mu \overline{z}^\nu}{\overline{z}_j} \langle JG_{\mu\nu}(Q,Q(f)), f\rangle + \frac{\partial}{\partial \overline{z}_j}\mathcal{R}. \tag{5.5}$$

We set, for $\mathbf{n}$ the number associated to ω_1 in (H11),

$$\lambda = (\lambda_1, ..., \lambda_{\mathbf{n}}), \tag{5.6}$$

where the $\lambda_j = \lambda_j(\omega_1)$ are introduced in (H11).

Proposition 5.1. *For ϵ_0 sufficiently small there exists a $C >$ s.t., given a solution $u(t)$ of the NLS which satisfies $\sup_{t\geq 0}\inf_{\vartheta\in\mathbb{R}}\|u(t)-e^{\mathrm{i}\vartheta}\phi_{\omega_1}\|_{H^1}<\epsilon<\epsilon_0$, for $t\in I=[0,\infty)$ we have*

$$\|f\|_{L^p_t(I,W^{1,q}_x)}+\|f\|_{L^2_t(I,H^{1,-s}_x)}\leq C\epsilon \text{ for all admissible pairs } (p,q), \tag{5.7}$$

$$\|z^\mu\|_{L^2_t(I)}\leq C\epsilon \text{ for all multi indices } \mu \text{ with } \lambda\cdot\mu>\omega_1, \tag{5.8}$$

$$\|z_j\|_{W^{1,\infty}_t(I)}\leq C\epsilon \text{ for all } j\in\{1,\dots,\mathbf{n}\}, \tag{5.9}$$

$$\|\omega-\omega_1\|_{L^\infty_t(I)}\leq C\epsilon. \tag{5.10}$$

Furthermore

$$\lim_{t\to+\infty} z(t)=0. \tag{5.11}$$

Notice that the case $(p,q)=(\infty,2)$ in (5.7) and inequalities (5.9) and (5.10) are an easy consequence of $\sup_{t\geq 0}\inf_{\vartheta\in\mathbb{R}}\|u(t)-e^{\mathrm{i}\vartheta}\phi_{\omega_1}\|_{H^1}<\epsilon$.

Proposition 5.1 implies Proposition 2.5. For the proof see [16], in particular Section 12. First of all, by standard arguments there is a $f_+\in H^1(\mathbb{R}^2,\mathbb{C}^2)$ such that for the f in (5.7) and for the A' in (5.2) we have

$$\lim_{t\to+\infty}\|f(t)-e^{J(\omega_1 t+\int_0^t A'(s)ds)}e^{-Jt\Delta}f_+\|_{H^1}=0, \tag{5.12}$$

see Lemma 7.2 [16]. We can express the solution $u(t)$ as

$$\begin{aligned} &u(t)=e^{-J\vartheta(t)}(\phi_{\omega(t)}+P(\omega(t))(z_j'(t)\xi_j+\overline{z'}_j(t)\overline{\xi}_j+f'(t)) \text{ where}\\ &z'=z+\mathcal{R}^{0,2}_{k,m}(Q,z,Q(f),f) \text{ and } f'=e^{J\mathcal{R}^{0,2}_{k,m}(Q,z,Q(f),f)}(f+\mathbf{S}^{1,1}_{k,m}(Q,z,Q(f),f)).\end{aligned} \tag{5.13}$$

where (z,f) are the variables in Proposition 5.1 and (k,m) are arbitrary. This follows from the fact that composing the change of variables (3.18) and (4.9) yields a change of variables like in (4.9), see [12].

(5.11) and (5.12) imply $\lim_{t\to+\infty}\mathbf{S}^{1,1}_{k,m}=0$ in H^1 and $\lim_{t\to+\infty}\mathcal{R}^{0,2}_{k,m}=0$ in $\mathbb{R}$.

It is easy to see that when we plug (5.13) in $\dot u=J\nabla\mathbf{E}(u)$ we get

$$\dot f=J(-\Delta+\dot\vartheta-\dot{\mathcal{R}}^{0,2})f+G_1(u),$$

with $G_1(u)\in C^0(H^1_x,L^1_x)$. On the other hand, f satisfies also (5.1), which is of the form

$$\dot f=J(-\Delta+\omega_1+A')f+G_2(u),$$

with $G_2(u)\in C^0(H^1_x,L^1_x)$. Then we have

$$\kappa(u)f=G_1(u)-G_2(u) \text{ with } \kappa(u):=\omega_1-\dot\vartheta+\dot{\mathcal{R}}^{0,2}+A'. \tag{5.14}$$

We have $\kappa\in C^0(H^1_x,\mathbb{R})$ and we claim that $\kappa=0$. Indeed, if $\kappa(u(t_0))\neq 0$ for a given solution, we can find solutions for which $u_n(t,\cdot)\in\mathcal{S}(\mathbb{R}^2)$, $u_n(t_0,\cdot)\to u(t_0,\cdot)$ in $H^1(\mathbb{R}^2)$, $\|u_n(t_0)\|_{L^1(\mathbb{R}^2)}\to\infty$, $G_j(u_n(t_0))\to G_j(u(t_0))$ and $\kappa(u_n(t_0))\to\kappa(u(t_0))$. This yields a contradiction because on one hand $\|u_n(t_0)\|_{L^1(\mathbb{R}^2)}\to\infty$ implies for the corresponding f coordinates $\|f_n(t_0)\|_{L^1(\mathbb{R}^2)}\to\infty$, on the other hand (5.14) is telling us $\|f_n(t_0)\|_{L^1(\mathbb{R}^2)}\sim|\kappa(u(t_0))|^{-1}\|G_1(u(t_0))-G_2(u(t_0))\|_{L^1(\mathbb{R}^2)}$ and so $\|f_n(t_0)\|_{L^1(\mathbb{R}^2)}\not\to\infty$.

Integrating $\omega_1 - \dot{\vartheta} + \dot{\mathcal{R}}^{0,2} + A' = 0$ and by $\lim_{t\to+\infty} \mathcal{R}^{0,2}_{k,m} = 0$ we get for a fixed $\vartheta_0 \in \mathbb{R}$

$$\lim_{t\to+\infty} (\omega_1 t + \int_0^t A'(s)ds - \vartheta(t)) = \vartheta_0.$$

Then (5.12) for $h_+ := e^{J\vartheta_0} f_+$ becomes

$$\lim_{t\to+\infty} \|e^{-J\vartheta(t)} f(t) - e^{-Jt\Delta} h_+\|_{H^1} = 0.$$

Using this in formula (5.13) we obtain

$$\lim_{t\to+\infty} \|u(t) - e^{i\vartheta(t)}\phi_{\omega(t)} - e^{\mathrm{i}t\Delta} h_+\|_{H^1} = 0$$

which yields (2.12). Combining (3.13) and (5.13) wee have

$$q(\omega(t)) + Q(f) + \mathcal{R}^{0,2}_{k,m} = q(\omega_0).$$

Then

$$\lim_{t\to+\infty} q(\omega(t)) = q(\omega_0) - Q(h_+).$$

Then (H5) implies that there must be a ω_+ s.t. $\lim_{t\to+\infty} \omega(t) = \omega_+$. The last sentence of Proposition 2.5 follows from (5.13) and the estimates in Proposition 5.1.

□

By a standard continuity argument, Proposition 5.1 is a consequence of the following proposition.

Proposition 5.2. *There exists $\epsilon_0 >$ and a constant $c_0 > 0$ such that if $T > 0$ and if $u(t)$ is a solution of the NLS which satisfies $\sup_{t\in I} \inf_{\vartheta\in\mathbb{R}} \|u(t) - e^{\mathrm{i}\vartheta}\phi_{\omega_1}\|_{H^1} < \epsilon < \epsilon_0$ where $I = [0,T]$ then, if the inequalities (5.7)–(5.8) hold for this I and for $C = C_0 \geq c_0$, they hold also $C = C_0/2$.*

6 Proof of Proposition 5.2

The proof of Proposition 5.2 is basically the same of Proposition 6.7 in [16]. We give a schematic description of the main steps. The first is the following, which follows from theory in [21] and whose proof we review in Appendix A.

Lemma 6.1. *Assume the hypotheses of Prop. 5.2. Then there is a fixed c and an s_0 such that for all admissible pairs (p,q) and all $s > s_0$*

$$\|f\|_{L^p_t([0,T],W^{1,q}_x)} + \|f\|_{L^2_t([0,T],H^{1,-s}_x)} \leq c_s\epsilon + c_s \sum_{\lambda\cdot\mu>\omega_1} |z^\mu|^2_{L^2_t(0,T)}, \tag{6.1}$$

where we sum only on multiindices such that $\lambda \cdot \mu - \lambda_j < \omega_1$ for any j such that for the j–th component of μ we have $\mu_j \neq 0$.

□

The notation is simpler if we change frame. For M defined below we have $M^{-1}\mathrm{i}JM = \sigma_3$, see (1.17), and so we have

$$\mathcal{K}_\omega := M^{-1}\mathrm{i}\mathcal{L}_\omega M = \sigma_3(-\Delta + V + \omega) + \sigma_3 M^{-1}\mathcal{V}_\omega M,$$

$$\text{where } M := \begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}, \quad M^{-1} = \frac{1}{2}\begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then we set $h = M^{-1}f$, which satisfies for $\mathbf{G}_{\mu\nu} := M^{-1}G_{\mu\nu}(Q,0)$ and $\mathbf{E} := M^{-1}\mathfrak{R}$,

$$\mathrm{i}\dot{h} = \mathcal{K}_{\omega_1}h + A\sigma_3 h + \sum_{\mathbf{e}\cdot(\mu-\nu)\in\sigma_e(\mathcal{L}_{\omega_1})} z^\mu \overline{z}^\nu \mathbf{G}_{\mu\nu} + \mathbf{E}. \tag{6.2}$$

with A defined in (5.3). The last summation in (6.1) originates from the $z^\mu\overline{z}^\nu\mathbf{G}_{\mu\nu}$ terms in (6.2) (or the corresponding ones in the 1st line of (5.4)). We cancel these terms by a normal forms argument For

$$g = h + Y\,, \quad Y := \sum_{|\lambda\cdot(\mu-\nu)|>\omega_1} z^\mu\overline{z}^\nu R^+_{\mathcal{K}_{\omega_1}}(\lambda\cdot(\nu-\mu))\mathbf{G}_{\mu\nu}, \tag{6.3}$$

we have

$$\mathrm{i}\dot{g} = \mathcal{K}_{\omega_1}g + A\sigma_3 g + [\mathrm{i}\dot{Y} - \mathcal{K}_{\omega_1}Y] + \sum_{\mathbf{e}\cdot(\mu-\nu)\in\sigma_e(\mathcal{L}_{\omega_1})} z^\mu\overline{z}^\nu\mathbf{G}_{\mu\nu} + A\sigma_3 Y + \mathbf{E}. \tag{6.4}$$

We then compute

$$\begin{aligned} &\mathrm{i}\dot{Y} = \sum_{|\lambda\cdot(\mu-\nu)|>\omega_1} \lambda\cdot(\nu-\mu) z^\mu\overline{z}^\nu R^+_{\mathcal{K}_{\omega_1}}(\lambda\cdot(\nu-\mu))\mathbf{G}_{\mu\nu} + \mathbf{T} \text{ where} \\ &\mathbf{T} := \sum_j \left[\partial_{z_j}Y(\mathrm{i}\dot{z}_j + \lambda_j z_j) + \partial_{\overline{z}_j}Y(\mathrm{i}\dot{\overline{z}}_j - \lambda_j\overline{z}_j)\right]. \end{aligned} \tag{6.5}$$

Inserting (6.5) in (6.4) we obtain

$$\mathrm{i}\dot{g} = \mathcal{K}_{\omega_1}g + A\sigma_3 g + A\sigma_3 Y + \mathbf{T} + \mathbf{E}. \tag{6.6}$$

So we have canceled the $z^\mu\overline{z}^\nu\mathbf{G}_{\mu\nu}$ terms. Notice that $\mathbf{T}$ contains terms of this type but by (5.5) they are smaller. So g is smaller than h. In fact the following is true, and is proved in Lemma 4.6 in [21] (the statement in Lemma 4.6 [21] has a systematic typo and $L^2_tL^{2,M}_x$ should be replaced by $L^2_tL^{2,-M}_x$, where M there is like our s here).

Lemma 6.2. *Assume the hypotheses of Prop. 5.2. Then for fixed $s>1$ there exist a fixed c such that if ε_0 is sufficiently small, for any preassigned and large $L>1$ we have $\|g\|_{L^2((0,T),L^{2,-s}_x)} \le c\epsilon$.*

□

For M^T the transpose of M, and using $M^T = 2M^{-1}$, $f = Mh$ and $\mathbf{G}_{\mu\nu} := M^{-1}G_{\mu\nu}(Q,0)$, by direct computation we have

$$\mathrm{i}\langle JG_{\mu\nu}(Q,0), f\rangle = \mathrm{i}\langle M^TJMM^{-1}G_{\mu\nu}(Q,0), h\rangle = 2\langle\sigma_1\sigma_3\mathbf{G}_{\mu\nu}, h\rangle.$$

Notice that $G_{\mu\nu}(Q,0) = -\overline{G}_{\nu\mu}(Q,0)$, see in Lemma 4.1, implies by $\overline{M}^{-1} = \sigma_1 M^{-1}$

$$\overline{\mathbf{G}}_{\nu\mu} = \overline{M}^{-1}\overline{G}_{\nu\mu}(Q,0) = -\sigma_1 M^{-1}G_{\mu\nu}(Q,0) = -\sigma_1\mathbf{G}_{\mu\nu}.$$

Then substituting (6.3) in (5.5) we obtain

$$\begin{aligned} &\mathrm{i}s_j\dot{z}_j = \partial_{\overline{z}_j}(H_2' + Z_0) + 2\sum_{|\lambda\cdot(\mu-\nu)|>\omega_1} \nu_j \frac{z^\mu\overline{z}^\nu}{\overline{z}_j}\langle g, \sigma_3\overline{\mathbf{G}}_{\nu\mu}\rangle + \partial_{\overline{z}_j}\mathcal{R} \\ &+ 2\sum_{\substack{|\lambda\cdot(\alpha-\beta)|>\omega_1 \\ |\lambda\cdot(\mu-\nu)|>\omega_1}} \nu_j \frac{z^{\mu+\alpha}\overline{z}^{\nu+\beta}}{\overline{z}_j}\langle R^+_{\mathcal{K}_{\omega_1}}(\lambda\cdot(\beta-\alpha))\mathbf{G}_{\alpha\beta}, \sigma_3\overline{\mathbf{G}}_{\nu\mu}\rangle, \end{aligned} \tag{6.7}$$

Let us consider the set of multi–indexes

$$\mathcal{M} := \{\alpha : \lambda\cdot\alpha > \omega_1 \text{ and } \lambda\cdot\alpha - \lambda_k < \omega_1 \,\forall\, k \text{ s.t. } \alpha_k \neq 0\}. \tag{6.8}$$

Set also $\Lambda := \{\lambda\cdot\alpha : \alpha\in\mathcal{M}\}$.
Like in [16], there is a new set of variables $\zeta = z + O(z^2)$ s.t. for a fixed C

$$\|\zeta - z\|_{L^2_t} \le CC_0\epsilon^2\,, \quad \|\zeta - z\|_{L^\infty_t} \le C\epsilon^3 \quad \text{ and } \tag{6.9}$$

$$\begin{aligned} s_j \mathrm{i}\dot{\zeta}_j &= \partial_{\overline{\zeta}_j} H_2'(\zeta,h) + \partial_{\overline{\zeta}_j} Z_0(\zeta,h) + \mathcal{D}_j \\ &\quad + 2 \sum_{\substack{\lambda\cdot\alpha=\lambda\cdot\nu\\ (\alpha,\nu)\in\mathcal{M}^2}} \nu_j \frac{\zeta^\alpha\overline{\zeta}^\nu}{\overline{\zeta}_j} \langle R^+_{\mathcal{K}_{\omega_1}}(-\lambda\cdot\alpha)\mathbf{G}_{\alpha 0}, \sigma_3\overline{\mathbf{G}}_{\nu 0}\rangle, \end{aligned} \tag{6.10}$$

where for a fixed constant c_0 we have

$$\sum_{j=1}^{\mathbf{n}} \|\mathcal{D}_j\overline{\zeta}_j\|_{L^1[0,T]} \le c_0(1+C_0)\epsilon^2. \tag{6.11}$$

Now we consider, like in [16],

$$\begin{aligned} \partial_t \sum_{j=1}^{\mathbf{n}} s_j\lambda_j|\zeta_j|^2 &= 2\sum_{j=1}^{\mathbf{n}} \lambda_j \operatorname{Im}\left(\mathcal{D}_j'\overline{\zeta}_j\right) - \\ &- 4 \sum_{\substack{\lambda\cdot\alpha=\lambda\cdot\nu\\ (\alpha,\nu)\in\mathcal{M}^2}} \lambda\cdot\nu \operatorname{Im}\left(\zeta^\alpha\overline{\zeta}^\nu \langle R^+_{\mathcal{K}_{\omega_1}}(-\lambda\cdot\alpha)\mathbf{G}_{\alpha 0}, \sigma_3\,\overline{\mathbf{G}}_{\nu 0}\rangle\right). \end{aligned} \tag{6.12}$$

In the second line of (6.12) we have a sum

$$\begin{aligned} &\Gamma(\zeta) := -4\sum_{L\in\Lambda} L \operatorname{Im}\left\langle R^+_{\mathcal{K}_{\omega_1}}(-L)\mathbf{G}(L,\zeta), \sigma_3\overline{\mathbf{G}(L,\zeta)}\right\rangle, \text{ for} \\ &\mathbf{G}(L,\zeta) := \sum_{\substack{\lambda\cdot\alpha=L\\ \alpha\in\mathcal{M}}} \zeta^\alpha\mathbf{G}_{\alpha 0}. \end{aligned} \tag{6.13}$$

For $W = \lim_{t\to\infty} e^{-\mathrm{i}t\mathcal{K}_{\omega_1}} e^{\mathrm{i}t\sigma_3(-\Delta+\omega_1)}$, there exist $F^{(L,\zeta)} \in W^{k,p}(\mathbb{R}^2,\mathbb{C}^2)$ for all $k\in\mathbb{R}$ and $p\in(1,\infty)$ with $2\mathbf{G}(L,\zeta) = WF^{(L,\zeta)}$, see [21]. Then for ${}^tF^{(L,\zeta)} = (F_1^{(L,\zeta)}, F_2^{(L,\zeta)})$

$$\begin{aligned} \Gamma(\zeta) &= -4\sum_{L\in\Lambda} \lim_{\varepsilon\searrow 0} \operatorname{Im}[\langle R_{-\Delta}(-L-\omega_1+\mathrm{i}\varepsilon)F_1^{(L,\zeta)}, \overline{F}_1^{(L,\zeta)}\rangle - \langle R_\Delta(\omega_1 - L + \mathrm{i}\varepsilon)F_2^{(L,\zeta)}, \overline{F}_2^{(L,\zeta)}\rangle] \\ &= 4\sum_{L\in\Lambda} \lim_{\varepsilon\searrow 0} \operatorname{Im}\langle R_\Delta(\omega_1 - L + \mathrm{i}\varepsilon)F_2^{(L,\zeta)}, \overline{F}_2^{(L,\zeta)}\rangle \\ &= -4\sum_{L\in\Lambda} \lim_{\varepsilon\searrow 0} \int_{\mathbb{R}^2} \frac{\varepsilon}{(x^2-(\Lambda-\omega_1))^2+\varepsilon^2} |\widehat{F}_2^{(L,\zeta)}(x)|^2 dx \le 0. \end{aligned} \tag{6.14}$$

Now we assume:

(H13) for some fixed constant $\Gamma > 0$ and for all $\zeta\in\mathbb{C}^{\mathbf{n}}$ we have

$$\Gamma(\zeta) < -\Gamma \sum_{\alpha\in\mathcal{M}} |\zeta^\alpha|^2. \tag{6.15}$$

Then integrating and exploiting (6.9) we get for $t \in [0,T]$

$$\Gamma \sum_{\alpha \text{ as in (H14)}} \|z^\alpha\|^2_{L^2(0,t)} \le cC_0\epsilon^2 - \sum_j s_j\lambda_j|z_j(0)|^2 + \sum_j s_j\lambda_j|z_j(t)|^2. \tag{6.16}$$

We want to conclude for some other fixed c'

$$\text{l.h.s. of (6.16)} \le 3c'C_0\epsilon^2. \tag{6.17}$$

Since $\sup_{0\le t\le T}\inf_{\vartheta\in\mathbb{R}} \|u(t) - e^{\mathrm{i}\vartheta}\phi_{\omega_1}\|_{H^1} < \epsilon$ by hypothesis, we can conclude that $\sum_j \lambda_j|z_j(t)|^2 \le c'\epsilon^2$ for any t. Then (6.16) implies (6.17) since here we can assume $C_0 > 1$ we get (6.17).

We conclude that for $\epsilon_0 > 0$ sufficiently small and any $T > 0$, (5.7)–(5.8) in $I = [0,T]$ and with $C = C_0$ implies (5.7)–(5.8) in $I = [0,T]$ with $C = c(1+\sqrt{C_0})$ for c. This yields Proposition 5.2.

□

Remark 6.3. Notice that in the proof of asymptotic stability in [13], where $s_j = -1$ for all j, orbital stability is a consequence of (6.17) rather than the other way around. Indeed in that case we have

$$\sum_j \lambda_j|z_j(t)|^2 + \Gamma \sum_{\alpha \text{ as in (H14)}} \|z^\alpha\|^2_{L^2(0,t)} \le cC_0\epsilon^2 + \sum_j \lambda_j|z_j(0)|^2, \tag{6.18}$$

and taking the initial datum u_0 very close to ϕ_{ω_1} we can assume $\sum_j \lambda_j|z_j(0)|^2 \le c'\epsilon^2$. Then each term in the l.h.s. of (6.18) is small for all $t > 0$. Furthermore this and and the fact that from (5.5) we derive that the time derivatives $\dot z_j(t)$ remain small, we conclude that $z_j(t) \overset{t\to\infty}{\to} 0$ for all j.

7 Theorem 1.4 and cubic quintic equations

We have not carried out numerical experiments to check examples to which Theorem 1.4 applies. In this section we combine numerical results in [63] with number of assumptions to propose some possible applications of Theorem 1.4. We discuss mainly hypothesis (H14).

We consider the cubic–quintic NLS (1.5). For each $m \ge 0$ one can find a family of vortices $e^{\mathrm{i}m\theta}\psi_\omega(r)$ for $\omega \in \mathcal{O} = (0,\omega_*)$ with $\psi_\omega \ge 0$ [36]. Here, the upper bound ω_* of $\mathcal{O}$ is given by $\omega_* := \sup\{\omega > 0 \mid \exists s > 0,\ \frac{\omega}{2}s^2 - \frac14 s^4 + \frac16 s^6 < 0\}$ which is in this case $\frac{3}{16}$. Notice that if $\frac{\omega}{2}s^2 - \frac14 s^4 + \frac16 s^6 \ge 0$ for all $s > 0$, by Pohozaev identity we can show that there are no nontrivial bound states in the energy space.

As we have already discussed in Section 1, for each $m = 1,2,3,4,5$ [63] shows that there is a critical value ω_{cr} such that for $\omega \ge \omega_{cr}$ the vortices are spectrally stable, while for $\omega < \omega_{cr}$ they are spectrally unstable. For $m = 1$ the value is $\omega_{cr} \approx 0.1487$, see Section 5.5 [63]. In all these cases spectral instability is generated as follows. As ω approaches ω_{cr}^+ from above, two distinct eigenvalues on the imaginary axis $\mathrm{i}\lambda^{(1)}(\omega)$ and $\mathrm{i}\lambda^{(2)}(\omega)$ coalesce at $\omega = \omega_{cr}$ at a point $\mathrm{i}\lambda_{cr}$ ($\mathrm{i}\lambda_{cr} \approx \mathrm{i}0.0478$ for $m = 1$). As ω decreases further, two eigenvalues bifurcate from $\mathrm{i}\lambda_{cr}$ out of the imaginary axis. In [63] it is not stated explicitly whether or not the eigenvalues $\mathrm{i}\lambda^{(j)}(\omega)$ for $j = 1,2$ are simple and whether their algebraic and geometric dimensions coincide. The fact that only eigenvalues with different signatures can generate by collapse eigenvalues outside $\mathrm{i}\mathbb{R}$ is well known, and we formalize it as follows.

In the following, $\mathcal{L}_\omega$ will denote the linearized operator of the vortices $e^{\mathrm{i}\omega t}e^{\mathrm{i}m\theta}\psi_\omega(r)$ of the cubic-quintic NLS (1.5).

Lemma 7.1. *Consider equation* (1.5)*, an* $m = 1, 2, 3, 4, 5$ *, a corresponding vortex and the operators* $\mathcal{L}_\omega$*. Suppose that there exists* $\varepsilon_0 > 0$ *s.t. for* $\omega \in (\omega_{cr}, \omega_{cr} + \varepsilon_0)$ *the eigenvalues* $\mathrm{i}\lambda^{(j)}(\omega)$ *for* $j = 1, 2$ *are simple and their algebraic and geometric dimensions coincide (and so are 1). Then one of them satisfies (H14).*

Proof. Recall, preliminarily, that if z is an eigenvalue of $\mathcal{L}_\omega$ with $z \neq -\overline{z}$, that is if $z \notin \mathrm{i}\mathbb{R}$, then $\Omega(v, \overline{v}) = 0$ for any $v \in \ker(\mathcal{L}_\omega - z)$. This follows from

$$z\langle J^{-1}v, \overline{v}\rangle = \langle J^{-1}\mathcal{L}_\omega v, \overline{v}\rangle = -\langle v, \overline{J^{-1}\mathcal{L}_\omega v}\rangle = -\overline{z}\langle J^{-1}v, \overline{v}\rangle, \tag{7.1}$$

which uses $J^{-1}\mathcal{L}_\omega = -\mathcal{L}_\omega^* J^{-1}$. If $z = \mathrm{i}\lambda \in \mathbb{R}$ we have

$$\mathrm{i}\lambda\langle J^{-1}v, \overline{v}\rangle = \overline{\mathrm{i}\lambda\langle J^{-1}v, \overline{v}\rangle},$$

so that $\langle J^{-1}\mathcal{L}_\omega v, \overline{v}\rangle \in \mathbb{R}$ for any $v \in \ker(\mathcal{L}_\omega - \mathrm{i}\lambda)$. When $\mathrm{i}\lambda \in \mathbb{R}$ is simple we have $\langle J^{-1}\mathcal{L}_\omega v, \overline{v}\rangle \neq 0$ for any $v \in \ker(\mathcal{L}_\omega - \mathrm{i}\lambda)\backslash\{0\}$ and the sign is called the *Krein signature*. Since

$$\langle J^{-1}\mathcal{L}_\omega v, \overline{v}\rangle = \mathrm{i}\lambda\langle J^{-1}v, \overline{v}\rangle,$$

it is clear that the Krein signature of $\mathrm{i}\lambda$ is positive exactly if $\langle J^{-1}v, \overline{v}\rangle = \mathrm{i}\varsigma$ with $\varsigma < 0$ (this explains why in (H14) the hypothesis $s_j = 1$ implies *negative* Krein signature and that positive Krein signature corresponds to $s_j = -1$). Notice that for $\mathrm{i}\lambda$ to have a well defined Krein signature it is not necessary that be simple, and it is sufficient that $\langle J^{-1}\mathcal{L}_\omega v, \overline{v}\rangle$ have constant sign as $v \neq 0$ varies in $\ker(\mathcal{L}_\omega - \mathrm{i}\lambda)$.

Now let us proceed with the proof of Lemma 7.1. By hypothesis the $\mathrm{i}\lambda^{(j)}(\omega)$ for $\omega > \omega_{cr}$ are simple and hence they have a well defined Krein signature, which is constant in $\omega > \omega_{cr}$ (see [45]) and can be written in the form $s^{(j)} := -\mathrm{i}\Omega(\xi_\omega^{(j)}, \overline{\xi}_\omega^{(j)})$ for appropriate generators $\xi_\omega^{(j)} \in \ker(\mathcal{L}_\omega - \mathrm{i}\lambda^{(j)}(\omega))$. We have $s^{(j)} \in \{-1, 1\}$. For $\omega < \omega_{cr}$ in [63] it is proved that there are two eigenvalues $z^{(1)}(\omega)$ and $z^{(1)}(\omega)$ with $\operatorname{Re} z^{(j)}(\omega) \neq 0$, which for reasons of symmetry satisfy $z^{(2)}(\omega) = -\overline{z}^{(1)}(\omega)$. These two eigenvalues exit from $\mathrm{i}\lambda^{(j)}(\omega_{cr})$ as ω decreases. With an argument by contradiction we suppose now that $s^{(1)} = s^{(2)}$ for $\omega > \omega_{cr}$. Then, by Section 6.1 in [45], this continues to be true for $\omega < \omega_{cr}$ in the sense that the quadratic form $\langle J^{-1}\mathcal{L}_\omega \cdot, \bar{\cdot}\rangle$, which is definite in $\ker(\mathcal{L}_\omega - \mathrm{i}\lambda^{(1)}(\omega)) \oplus \ker(\mathcal{L}_\omega - \mathrm{i}\lambda^{(2)}(\omega))$ for $\omega > \omega_{cr}$ must continue to be definite also in $\ker(\mathcal{L}_\omega - z^{(1)}(\omega)) \oplus \ker(\mathcal{L}_\omega - z^{(2)}(\omega))$ for $\omega < \omega_{cr}$. But from what we saw in (7.1) this is not possible. This gets us to a contradiction. So $s^{(1)} \neq s^{(2)}$ for $\omega > \omega_{cr}$ and one of the two must be equal to 1. $\square$

Since (1.5) is translation invariant it is beyond the scope of our theory. In order to find examples of equations not translation invariant which satisfy (H14) it is natural to add to (1.5) a small potential $\varepsilon V(|x|)$ with a point of relative minimum in 0. We will show that the perturbed equation has vortices. As $\varepsilon \to 0$ they converge to vortices of (1.5) in any space $\Sigma_k(\mathbb{R}^2, \mathbb{C})$.

Then the spectrum and the eigenfunctions of the linearizations $\mathcal{L}_\omega^{(\varepsilon)}$ converge to spectrum and eigenfunctions of $\mathcal{L}_\omega$. In particular, assuming our mixed rigorous and numerical proof that (1.5) satisfies (H14) for $\omega > \omega_{cr}$, then we will have obtained this result also for the operators $\mathcal{L}_\omega^\varepsilon$ with $\varepsilon \neq 0$.

Let $\phi_\omega(e^{\mathrm{i}\theta}r) = e^{\mathrm{i}m\theta}\psi_\omega(r)$ be the vortex of [63] with $\psi_\omega \geq 0$. Under the assumption that for (1.5) the kernel of $\mathcal{L}_\omega$ restricted in $L_m^2 := \{u \in L^2 \mid e^{-\mathrm{i}m\theta}u \text{ is radially symmetric}\}$ is $\operatorname{Span}\{J\phi_\omega\}$ (in [63] it is shown that numerically this appears to be generically true) we show that for small ε there exists $\phi_{\omega,\varepsilon}(e^{\mathrm{i}\theta}r) = e^{\mathrm{i}m\theta}\psi_{\omega,\varepsilon}(r)$ which is a solution of

$$0 = -\Delta\phi_{\omega,\varepsilon} + \omega\phi_{\omega,\varepsilon} + \varepsilon V\phi_{\omega,\varepsilon} - |\phi_{\omega,\varepsilon}|^2\phi_{\omega,\varepsilon} + |\phi_{\omega,\varepsilon}|^4\phi_{\omega,\varepsilon}. \tag{7.2}$$

More precisely, we have the following proposition.

Proposition 7.2. *Assume* $\ker \mathcal{L}_{\omega_0}|_{L^2_m} = \{J\phi_{\omega_0}\}$. *Then there exist* $\delta_0 > 0$ *and* $\varepsilon_0 > 0$ *s.t. for* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ *and* $\omega \in (\omega_0 - \delta_0, \omega_0 + \delta_0)$ *there exists* $\phi_{\omega,\varepsilon} \in \cap_{k\geq 0}\Sigma_k(\mathbb{R}^2, \mathbb{C})$ *which satisfies* (7.2). *Furthermore, the map* $(\omega, \varepsilon) \to \phi_{\omega,\varepsilon}$ *is in* $C^1((\omega_0 - \delta_0, \omega_0 + \delta_0) \times (-\varepsilon_0, \varepsilon_0), \Sigma_k)$ *for arbitrary* $k \geq 0$.

To prove Proposition 7.2 we consider a preparatory and standard lemma.

Lemma 7.3. *Assume* $\ker \mathcal{L}_\omega|_{L^2_m} = \{J\phi_\omega\}$. *Then* $A_\omega = -\Delta_r + \frac{m^2}{r^2} + \omega - 3\psi_\omega^2 + 5\psi_\omega^4$ *is invertible in* $L^2_{rad}(\mathbb{R}^2, \mathbb{R})$ *and* $\|e^{\mathrm{i}m\theta} A_\omega^{-1} u\|_{H^2} \lesssim \|u\|_{L^2}$.

Proof. Let $v \in L^2_{rad}(\mathbb{R}^2, \mathbb{R})$ satisfy $A_\omega v = 0$. Then, multiplying by $e^{\mathrm{i}m\theta}$ we have

$$(-\Delta + \omega - 2|\phi_\omega|^2 + 3|\phi_\omega|^4)u + \left(-\phi_\omega^2 + 2\phi_\omega^2|\phi_\omega|^2\right)\bar{u} = 0,$$

where $u = e^{\mathrm{i}m\theta}v$. Therefore, using the natural identification between $\mathbb{C}$ and $\mathbb{R}^2$, we have

$$\mathcal{L}_\omega u = 0.$$

By the assumption, we have $u = aJ\phi_\omega$ and thus $v = aJ\psi_\omega$. However, v has values in $\mathbb{R} \times \{0\}$ while $J\psi_\omega$ has values in $\{0\} \times \mathbb{R}$. So $a = 0$ and $\ker A_\omega = \{0\}$. Therefore, A_ω is invertible. Finally suppose $A_\omega v = u$. Then, first we have $\|e^{\mathrm{i}m\theta}v\|_{L^2} = \|v\|_{L^2} \lesssim \|u\|_{L^2}$. Next, multiplying by $e^{\mathrm{i}m\theta}$, we have

$$(-\Delta + \omega)e^{\mathrm{i}m\theta}v = \left(3\psi_\omega^2 - 5\psi_\omega^4\right)e^{\mathrm{i}m\theta}v + e^{\mathrm{i}m\theta}u.$$

Taking the L^2 norm of both sides, we have the conclusion. $\square$

Proof of Proposition 7.2. Set $\delta = \omega - \omega_0$ and consider $\phi_{\omega_0+\delta,\varepsilon} = e^{\mathrm{i}m\theta}(\psi_{\omega_0+\delta} + v_{\delta,\varepsilon})$, where $v_{\delta,\varepsilon}$ is radially symmetric and real valued. Then, substituting this into (7.2) and for $v = v_{\delta,\varepsilon}$, we have

$$A_{\omega_0}v = G(\delta, \varepsilon, v) \text{ where } G(\delta, \varepsilon, v) := -\varepsilon V(\psi_{\omega_0+\delta} + v) + \left[\left(3\psi_\tau^2 - 5\psi_\tau^4\right)\right]_{\omega_0}^{\omega_0+\delta} v - \delta v + N_{\omega_0+\delta}(v), \tag{7.3}$$

with $N_{\omega_0+\delta}(v)$ nonlinear in v and the convention $[f(\tau)]_a^b = f(b) - f(a)$. We can rephrase (7.3) as

$$F(\delta, \varepsilon, v) = 0 \text{ where } F(\delta, \varepsilon, v) := v - A_{\omega_0}^{-1}G(\delta, \varepsilon, v). \tag{7.4}$$

The function F is in $C^1((\omega_0 - \delta_0, \omega_0 + \delta_0) \times (-\varepsilon_0, \varepsilon_0) \times H^2, H^2)$. An elementary application of the implicit function theorem yields a function $v_{\delta,\varepsilon}$ in $C^1((\omega_0 - \delta_0, \omega_0 + \delta_0) \times (-\varepsilon_0, \varepsilon_0), H^2)$. By a standard bootstrapping argument, H^2 can be replaced by Σ_k for arbitrary k. $\square$

In [63] it is checked numerically that for a generic vortex of the (1.5)

$$N_g(\mathcal{L}_\omega) = \mathrm{Span}\{J\phi_\omega, \partial_{x_1}\phi_\omega, \partial_{x_2}\phi_\omega, \partial_\omega\phi_\omega, Jx_1\phi_\omega, Jx_2\phi_\omega\}. \tag{7.5}$$

We have chosen $V(|x|)$ with a relative minimum at 0. The following fact is well known, see Theorem 4.1 [24] and Theorem 3.0.2 [25].

Lemma 7.4. *Consider an* $m = 1$ *vortex of the*(1.5). *Assume* $\frac{d}{d\omega}\|\phi_\omega\|_2 \neq 0$ *and* (7.5) *for the linearized operator* $\mathcal{L}_\omega$. *Then for* $\varepsilon > 0$ *sufficiently small we have*

$$\ker \mathcal{L}_\omega^{(\varepsilon)} = \mathit{Span}\{J\phi_{\omega,\varepsilon}\} \text{ and } N_g(\mathcal{L}_\omega^{(\varepsilon)}) = \mathit{Span}\{J\phi_{\omega,\varepsilon}, \partial_\omega\phi_{\omega,\varepsilon}\}. \tag{7.6}$$

Furthermore, $\mathcal{L}_\omega^{(\varepsilon)}$ *has an eigenvalue of algebraic and geometric multiplicity 2 which is of the form* $\mathrm{i}\mu(\varepsilon) = \mathrm{i}\varepsilon\sqrt{2e} + o(\varepsilon)$, *with* e *the eigenvalue of the Hessian matrix of the potential in 0, and with eigenfunctions*

$$\Psi_j^{(\varepsilon)} = \sqrt{2}\partial_{x_j}\phi_\omega + \mathrm{i}\sqrt{e}\varepsilon Jx_j\phi_\omega + o(\epsilon), \tag{7.7}$$

with $\|o(\varepsilon)\|_{\Sigma_k} = o(\varepsilon)$ *for any* k.

Proof. Everything is proved in Theorem 3.0.2 [25]. We only remark that there is only one small eigenvalue $\mathrm{i}\mu(\varepsilon) \in \mathrm{i}\mathbb{R}_+$ which has to be of multiplicity 2. Indeed, $\mathcal{L}_\omega^{(\varepsilon)} \Psi_1^{(\varepsilon)} = \mathrm{i}\mu(\varepsilon)\Psi_1^{(\varepsilon)}$ implies by the symmetry $J^{-1}\mathcal{V}_\omega(x_1, x_2)J = \mathcal{V}_\omega(x_2, -x_1)$ also $\mathcal{L}_\omega^{(\varepsilon)} J\Psi_1^{(\varepsilon)}(x_2, -x_1) = \mathrm{i}\mu(\varepsilon)J\Psi_1^{(\varepsilon)}(x_2, -x_1)$. By $J\phi_\omega(x_1, x_2) = -\phi_\omega(x_2, -x_1)$ we have

$$J\Psi_1^{(\varepsilon)}(x_2, -x_1) = J\sqrt{2}(\partial_{x_1}\phi_\omega)(x_2, -x_1) - \mathrm{i}\sqrt{e}\varepsilon x_2\phi_\omega(x_2, -x_1) + o(\epsilon) =$$
$$\sqrt{2}\partial_{x_2}[J\phi_\omega(x_2, -x_1)] + \mathrm{i}\sqrt{e}\varepsilon Jx_2\phi_\omega(x_1, x_2) + o(\epsilon) = \sqrt{2}\partial_{x_2}\phi_\omega(x_1, x_2) + \mathrm{i}\sqrt{e}\varepsilon Jx_2\phi_\omega(x_1, x_2) + o(\epsilon)$$

and hence necessarily $J\Psi_1^{(\varepsilon)}(x_2, -x_1) = \Psi_2^{(\varepsilon)}(x_1, x_2)$. This implies that $\mathrm{i}\mu(\varepsilon)$ has multiplicity 2. □

In the following remarks we discuss whether the perturbations of equation (1.5) satisfy the hypotheses of Theorem 1.4.

Remark 7.5. (H1)–(H3) are trivially satisfied.

Remark 7.6. It is known that equation (1.5) satisfies (H4) and the same holds for the perturbations.

Remark 7.7. According to the numerical experiments in [63] hypothesis (H5) is true generically. The same will be true for the perturbations.

Remark 7.8. (H6) is proved numerically for $\omega \geq \omega_{cr}$ in [63] and for the perturbations is a consequence of Proposition 7.2 and Lemma 7.4.

Remark 7.9. Assuming the numerical results in [63] which claim that (7.5) is true for generic ω, then for the perturbations (H7) is a consequence of Lemma 7.4.

Remark 7.10. We don't know the status of hypotheses (H8)–(H12) for (1.5), but if they hold, they hold also for the perturbations.
As we have already mentioned, failure of (H8) would yield some Jordan block of dimension 2 or higher forcing $e^{t\mathcal{L}_\omega}$ to grow algebraically in the invariant space $N_g^\perp(\mathcal{L}_\omega^*)$ in (2.10) producing essentially a linear instability, which would yield an easy to detect nonlinear instability. Since this is not what the numerical experiments show, we conclude that probably [63, 77] confirm (H8).
In [63] embedded eigenvalues and singularities at the edges are not discussed explicitly but hypotheses (H9)–(H10) seem to be confirmed. In [63] isolated eigenvalues are obtained as zeros of an Evans function. It is observed, see the discussion on pp. 371–372, that sometimes the Evans function is small near the continuous spectrum. This smallness is attributed not to eigenvalues sitting in the continuous spectrum, but rather to *resonances* on the other side of the continuous spectrum (in essence, to zeros of an analytic continuation of the Evans function beyond the continuous spectrum).
Hypotheses (H11)-(H12) state that the eigenvalues between 0 and $\mathrm{i}\omega$ are positioned in a generic way. This is plausible to expect and probably can be proved for V generic. We did not attempt the proof.

Remark 7.11. We have discussed at length Hypothesis (H13) in Section 1. It ought to be checked directly for perturbations of (1.5) or for equation (1.5). Notice that, since (1.5) is translation invariant, the search of an effective Hamiltonian is somewhat more involved, see [12, 16].

Remark 7.12. We have already discussed (H14) which is true, assuming that each of the eigenvalues $\mathrm{i}\lambda^{(j)}(\omega)$ has some fixed Krein signature. Indeed this is what happens if each eigenvalue is simple, as we assumed in Lemma 7.1. Even if there is an eigenvalue of higher multiplicity, it is enough to ask for $\mathrm{i}\Omega(v, \overline{v})$ to have fixed sign for any eigenvector. This property on the signatures continue to hold for the perturbations.

A Appendix: proof of Lemma 6.1

First of all, it is equivalent to consider equation 6.2 for h. Let $X_c = M^{-1}L_c^2(\omega_1)$ and by an abuse of notation let us set $\widetilde{P}_c = M^{-1}P_c(\omega_1)M$, where P_c is introduced under Lemma 2.3. Set also $\mathcal{K} = \mathcal{K}_{\omega_1}$ The following three lemmas are Lemma 3.1–3.3 in [21].

Lemma A.1 (Strichartz estimate)**.** *There exists a positive number C such that for any $k \in [0,2]$:*

(a) for any $h = \widetilde{P}_c h$ and any admissible all pair (p,q),

$$\|e^{-\mathrm{i}t\mathcal{K}}h\|_{L_t^p W_x^{k,q}} \le C\|h\|_{H^k};$$

(b) for any $g(t,x) \in S(\mathbb{R}^2)$ and any couple of admissible pairs (p_1,q_1) (p_2,q_2) we have

$$\|\int_0^t e^{-\mathrm{i}(t-s)\mathcal{K}}\widetilde{P}_c g(s,\cdot)ds\|_{L_t^{p_1}W_x^{k,q_1}} \le C\|g\|_{L_t^{p_2'}W_x^{k,q_2'}}.$$

Lemma A.2. *Let $s > 1$. $\exists\ C = C$ such that:*

(a) for any $f \in S(\mathbb{R}^2)$,

$$\|e^{-\mathrm{i}t\mathcal{K}}\widetilde{P}_c f\|_{L_t^2 L_x^{2,-s}} \le C\|f\|_{L^2};$$

(b) for any $g(t,x) \in S(\mathbb{R}^2)$

$$\left\|\int_{\mathbb{R}} e^{\mathrm{i}t\mathcal{K}}\widetilde{P}_c g(t,\cdot)dt\right\|_{L_x^2} \le C\|g\|_{L_t^2 L_x^{2,s}}.$$

Lemma A.3. *Let $s > 1$. $\exists\ C$ such that $\forall\ g(t,x) \in S(\mathbb{R}^2)$ and $t \in \mathbb{R}$:*

$$\left\|\int_0^t e^{-\mathrm{i}(t-s)\mathcal{K}}\widetilde{P}_c g(s,\cdot)ds\right\|_{L_t^2 L_x^{2,-s}} \le C\|g\|_{L_t^2 L_x^{2,s}}.$$

Lemma A.4. *Let (p,q) be an admissible pair and let $s > 1$. $\exists$ a constant $C > 0$ such that $\forall$ $g(t,x) \in S(\mathbb{R}^2)$ and $t \in \mathbb{R}$:*

$$\left\|\int_0^t e^{-\mathrm{i}(t-s)\mathcal{K}}\widetilde{P}_c g(s,\cdot)ds\right\|_{L_t^p L_x^q} \le C\|g\|_{L_t^2 L_x^{2,s}}.$$

The following is Proposition 1.2 in [21].

Lemma A.5. *The following limits are well defined isomorphism, inverse of each other:*

$$Wu = \lim_{t\to+\infty} e^{\mathrm{i}t\mathcal{K}}e^{\mathrm{i}t\sigma_3(\Delta-\omega_1)}u \text{ for any } u \in L^2$$
$$Zu = \lim_{t\to+\infty} e^{\mathrm{i}t(-\Delta+\omega_1)}e^{-\mathrm{i}t\mathcal{K}} \text{ for any } u = \widetilde{P}_c u.$$

For any $p \in (1,\infty)$ and any k the restrictions of W and Z to $L^2 \cap W^{k,p}$ extend into operators such that for for a constant C we have

$$\|W\|_{W^{k,p}(\mathbb{R}^2),W_c^{k,p}} + \|Z\|_{W_c^{k,p},W^{k,p}(\mathbb{R}^2)} < C$$

with $W_c^{k,p}$ the closure in $W^{k,p}(\mathbb{R}^2)$ of $W^{k,p}(\mathbb{R}^2) \cap \widetilde{P}_c L_c^2$.

The following is Lemma 3.5 [21].

Lemma A.6. *Consider the diagonal matrices $E_+ = diag(1,0)$ $E_- = diag(0,1)$. Set $P_\pm = ZE_\pm W$ with Z and W the wave operators associated to $\mathcal{K}$. Then we have for $u = \widetilde{P}_c u$*

$$P_+ u = \lim_{\epsilon\to 0^+} \frac{1}{2\pi \mathrm{i}} \lim_{M\to+\infty} \int_\omega^M \left[R_{\mathcal{K}}(\lambda + \mathrm{i}\epsilon) - R_{\mathcal{K}}(\lambda - \mathrm{i}\epsilon)\right] u d\lambda$$
$$P_- u = \lim_{\epsilon\to 0^+} \frac{1}{2\pi \mathrm{i}} \lim_{M\to+\infty} \int_{-M}^{-\omega} \left[R_{\mathcal{K}}(\lambda + \mathrm{i}\epsilon) - R_{\mathcal{K}}(\lambda - \mathrm{i}\epsilon)\right] u d\lambda$$

and for any s_1 and s_2 and for $C = C(s_1, s_2)$ we have

$$\|(P_+ - P_- - \widetilde{P}_c \sigma_3) f\|_{L^{2,s_1}} \le C \|f\|_{L^{2,s_2}}.$$

Now we look at the term $\mathbf{E}$ in (6.2).

Lemma A.7. *For any preassigned s and for $\epsilon_0 > 0$ small enough we have*

$$\mathbf{E} = R_1 + R_2 \text{ with } \|R_1\|_{L^1_t([0,T],H^1_x)} + \|R_2\|_{L^2_t([0,T],H^{1,-s}_x)} \le C(s, C_0)\epsilon^2. \tag{A.1}$$

Furthermore for a fixed constant c we have

$$\|A\|_{L^\infty((0,T),\mathbb{R})} \le c C_0^2 \epsilon^2. \tag{A.2}$$

Proof. The estimate on $A = A' + A''$ follows from the definitions of A' in (5.2) and of A'' in (5.3). $\mathbf{E}$ is a sum of various terms. For example we have

$$\begin{aligned}
&\|z^\mu \overline{z}^\nu M^{-1}[G_{\mu\nu}(Q,0) - G_{\mu\nu}(Q, Q(f))]\|_{L^2_t([0,T],H^{1,s}_x)} \\
&\le \|z^\mu \overline{z}^\nu\|_{L^2_t[0,T]} \|G_{\mu\nu}(Q,0) - G_{\mu\nu}(Q,Q(f))\|_{L^\infty_t([0,T],H^{1,s}_x)} \lesssim C_0^3 \epsilon^3.
\end{aligned}$$

So this term can be absorbed in R_2. Another example is $\beta(|f|^2)f = \chi_{|f|\le 1}\beta(|f|^2)f + \chi_{|f|\ge 1}\beta(|f|^2)f$. The 1st term can be bounded, schematically, by

$$\||f|^2 f\|_{L^1_t([0,T],H^1_x)} \lesssim \left\| \|f\|_{W^{1,6}_x} \|f\|^2_{L^6_x} \right\|_{L^1_t[0,T]} \le \|f\|^3_{L^3_t([0,T],W^{1,6}_x)} \lesssim C_0^3 \epsilon^3 \tag{A.3}$$

while the 2nd term can be bounded by

$$\|f^L\|_{L^1_t H^1_x} \lesssim \left\| \|f\|_{W^{1,2L}_x} \|f\|^{L-1}_{L^{2L}_x} \right\|_{L^1_t} \le \|f\|_{L_t^{\frac{2L}{L-1}} W^{1,2L}_x} \|f\|^{L-1}_{L_t^{2L\frac{L-1}{L+1}} W^{1,2L}_x} \lesssim C_0^L \epsilon^L, \tag{A.4}$$

where in the last step we use $\|f\|_{L_t^{2L\frac{L-1}{L+1}} W^{1,2L}_x} \lesssim \|f\|^\alpha_{L_t^{\frac{2L}{L-1}} L^{2L}_x} \|f\|^{1-\alpha}_{L^\infty_t H^1_x}$ for some $0 < \alpha < 1$ by $L > 3$ (which we can always assume), interpolation and Sobolev embedding.
Notice that by $\nabla_f \mathcal{R}^{1,2}_{k,m}(Q, \varrho, f)_{|\varrho = Q(f)} = S^{1,1}_{k,m-1}(Q, Q(f), f)$ we have by (3.16)

$$\begin{aligned}
&\|\nabla_f \mathcal{R}^{1,2}_{k,m}(Q, \varrho, f)_{|\varrho=Q(f)}\|_{L^2_t H^{1,s}_x} \le \| \|f\|_{L^{2,-\sigma}_x} \|_{L^2_t} (\|f\|_{L^2} + |z| + |Q(f)|)\|_{L^2_t} \\
&\le \|f\|_{L^2_t L^{2,-\sigma}_x} (\|z\|_{L^\infty_t} + \|f\|_{L^\infty_t L^2}) \le 2C_0^2 \epsilon^2.
\end{aligned}$$

Consider for example the contribution of

$$\begin{aligned}
&\nabla_f \int_{\mathbb{R}^2} B_L(x, f(x), Q, z, \varrho, f) f^L(x) dx_{|\varrho = Q(f)} \sim B_L(x, f(x), Q, z, Q(f), f) f^{L-1}(x) \\
&+ \int_{\mathbb{R}^2} \partial_6 B_L(x, f(x), Q, z, Q(f), f) f^L(x) dx + \partial_2 B_L(x, f(x), Q, z, Q(f), f) f^L(x).
\end{aligned} \tag{A.5}$$

The last term can be treated like f^L above, since $\|B_L(x, f(x), Q, z, Q(f), f)\|_{L^\infty_{tx}} \le C$ by (3.20). We can use (A.3) or (A.4) for the 1st term of the r.h.s., since $L-1 \ge 3$. Finally let us consider the 2nd term in the r.h.side. If we take $g \in L^\infty_t H^{-1}_x$ we need to bound

$$\begin{aligned}
&\int_0^T dt \int_{\mathbb{R}^2} |\langle g, \partial_6 B_L(x, f(x), Q, z, Q(f), f)\rangle_{L^2_{x'}} f^L(x) dx| \\
&\le \int_0^T dt \| |\langle g, \partial_6 B_L(x, f(x), Q, z, Q(f), f)\rangle_{L^2_{x'}} \|_{L^2_x} \|f^L\|_{L^2_x} \\
&\le \| |\langle g, \partial_6 B_L(x, f(x), Q, z, Q(f), f)\rangle_{L^2_{x'}} \|_{L^\infty_t L^2_x} \|f^L\|_{L^1_t L^2_x}
\end{aligned}$$

and we bound the last factor by (A.4). We have for fixed t

$$\| |\langle g, \partial_6 B_L(x, f(x), Q, z, Q(f), f)\rangle_{L^2_{x'}} \|_{L^2_x} \le \|\partial_6 B_L\|_{B(\Sigma_{-k}, \Sigma_k)} \|g\|_{\Sigma_{-k}}$$

so that by (3.20), or by its analogue for the B_L in Lemma 4.1, we have that the last quantity is bounded by $C\|g\|_{H^{-1}}$. This yields a bound $\|$1st term 2nd line (A.5)$\|_{L^1_t L^2_x} \lesssim C_0^L \epsilon^L$.

Proof of Lemma 6.1 We rewrite (6.2) as

$$\mathrm{i}\dot{h} = [\mathcal{K}h + A(P_+ - P_-)]h + A[\widetilde{P}_c \sigma_3 - P_+ + P_-]\sigma_3 h + \sum_{\mathbf{e}\cdot(\mu-\nu)\in\sigma_e(\mathcal{L}_{\omega_1})} z^\mu \overline{z}^\nu \mathbf{G}_{\mu\nu} + \widetilde{P}_c \mathbf{E}.$$

Then we have

$$\begin{aligned}
h(t) = \mathcal{U}(t,0) e^{\mathrm{i}t\mathcal{K}} h(0) + \int_0^t \mathcal{U}(t,s) e^{\mathrm{i}(t-s)\mathcal{K}} \Big[& A[\widetilde{P}_c \sigma_3 - P_+ + P_-]\sigma_3 h \\
& + \sum_{\mathbf{e}\cdot(\mu-\nu)\in\sigma_e(\mathcal{L}_{\omega_1})} z^\mu \overline{z}^\nu \mathbf{G}_{\mu\nu} + \widetilde{P}_c \mathbf{E} \Big] ds,
\end{aligned} \tag{A.6}$$

where the following operator commutes with $\mathcal{K}$:

$$\mathcal{U}(t,s) =^{\mathrm{i}\int_s^t A(s')ds'(P_+ - P_-)}.$$

Then

$$\begin{aligned}
\|h\|_{L^p_t W^{1,q}_x \cap L^2_t H^{1,-s}_x} \lesssim & \|h(0)\|_{H^1} + \sum_{\mu\nu} \|z^\mu \overline{z}^\nu\|_{L^2_t} \|\mathbf{G}_{\mu\nu}\|_{L^\infty_t H^{1,s}_x} \\
& + \|A\|_{L^\infty_t} \|h\|_{L^2_t H^{1,-s}_x} + \|R_1\|_{L^1_t H^1_x} + \|R_2\|_{L^2_t H^{1,s}_x}.
\end{aligned}$$

The terms on the second line are $O(\epsilon^2)$ and the r.h.s. is bounded by the r.h.s. of (6.1), proving Lemma 6.1.

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