# Isospectral flows for the inhomogeneous string density problem 

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#### Abstract

We derive isospectral flows of the mass density in the string boundary value problem corresponding to general boundary conditions. In particular, we show that certain class of rational flows produces in a suitable limit all flows generated by polynomials in negative powers of the spectral parameter. We illustrate the theory with concrete examples of isospectral flows of discrete mass densities which we prove to be Hamiltonian and for which we provide explicit solutions of equations of motion in terms of Stieltjes continued fractions and Hankel determinants.


## 1 Introduction

The 1-D wave equation $\frac{1}{c^{2}} u_{\tau \tau}-u_{x x}=0$ is a basic classical equation describing propagation of waves, in particular vibrations of a string. The coefficient $c^{2}$ has the physical dimension of velocity squared and is usually assumed to be independent of position. In general $\frac{1}{c^{2}}$ is proportional to the lineal mass density $\rho$ with the inverse of the proportionality constant being the string tension which for convenience we will set equal to 1 . Thus we can write the inhomogeneous string wave equation as:

$$
\begin{equation*}
\rho(x) u_{\tau \tau}-u_{x x}=0 . \tag{1.1}
\end{equation*}
$$

The normal modes $v$ are derived by substituting $u=v(x) \cos \omega t$ into (1.1), resulting in

$$
\begin{equation*}
v_{x x}=-\omega^{2} \rho(x) v \stackrel{\text { def }}{=}-z \rho(x) v . \tag{1.2}
\end{equation*}
$$

Furthermore, one needs to specify the length of the string; in this paper we will choose $0 \leq x \leq 1$ so that the string will have a unit length, as well as we impose some boundary conditions reflecting how the string is tied at the endpoints. The most typical boundary conditions take the form $v_{x}(0)-h v(0)=0$ and $v_{x}(1)+H \nu(1)=0$ where both $h$ and $H$ are non-negative or infinity. In the latter case, taking the left boundary condition as an example, the interpretation of $h=\infty$ is simply $\nu(0)=0$,

[^0]in other words $h=\infty$ corresponds to the Dirichlet condition at $x=0$. Thus in this paper the inhomogeneous string equation will mean the boundary value problem
\[

$$
\begin{gather*}
v_{x x}=-z \rho(x) v, \quad 0<x<1  \tag{1.3a}\\
v_{x}(0)-h v(0)=0, \quad v_{x}(1)+H v(1)=0 \tag{1.3b}
\end{gather*}
$$
\]

where $0 \leq h, H \leq \infty$. Physically those $z$ for which the boundary value problem has a solution represent the squares of frequencies and, clearly, they are expected to be positive, or zero. The literature on the inhomogeneous string problem includes $[3,15,17,13,10,18]$ and recent contributions [1, 2] dealing with a perturbative approach to computations of densities close to the homogenous one. Models leading, directly or indirectly, to inhomogeneous strings appear in many areas of science, from geophysics (see e.g. [16]) and fluid dynamics (see e.g. [4]) to particle physics (see e.g. [12]).

## 2 Isospectral deformations

The question of finding isospectral, that is leaving the spectrum invariant, deformations of the mass density $\rho$ appearing in the string equation (1.3a) was proposed in a sequel of interesting papers by P. Sabatier in [21,22,23]. In simple terms one is looking for evolution equations on $\rho$ with respect to the deformation parameter $t$ (not to be confused with physical time) for which the spectrum of the boundary value problem (1.3a) remains the same. Historically, this problem goes back at least to G. Borg [6] who was perhaps the first to systematically study the question of determining the potential in the Sturm-Liouville problem from the knowledge of eigenvalues; in its canonical form, called the Liouville normal form, the underlying equation is equivalent to the 1-D Schrödinger equation

$$
-\psi_{x x}+u(x) \psi=E \psi
$$

and the main message of [6] was that two spectra are needed to determine uniquely $u$. In other words, by knowing a single spectrum, we can only hope to determine a family of potentials, and that family will be isospectral relative to that chosen spectrum. The situation for the string equation (1.3a) is analogous, even though the string boundary value problem cannot be put in the canonical form mentioned above for general mass densities $\rho$. We will refer to the resulting family as isospectral strings. An additional motivation for studying this problem comes from the theory of completely integrable nonlinear partial differential equations, especially the Camassa-Holm (CH) equation [7]

$$
\begin{equation*}
u_{t}-u_{x x t}+3 u u_{x}-2 u_{x} u_{x x}-u u_{x x x}=0 \tag{2.1}
\end{equation*}
$$

modelling wave propagation in shallow water. The rationale for this connection comes from the result in [4] that the CH can be viewed as an isospectral deformation of the string equation with Dirichlet boundary conditions (see also reviews [5, 14]).

### 2.1 Summary of the main results of [10]

This section contains a summary of the approach to isospectral strings taken in [10] based on ideas originating in the theory of integrable systems. The starting point is
to postulate the "time" deformation of (1.3a) to be of the form

$$
\begin{equation*}
v_{t}=a v+b v_{x} \tag{2.2}
\end{equation*}
$$

Then a simple computation that amounts to checking $v_{x x t}=v_{t x x}$ yields

$$
\begin{array}{r}
z \rho_{t}=\frac{1}{2} b_{x x x}+z \mathscr{L}_{\rho} b, \\
a=-\frac{1}{2} b_{x}+\beta \tag{2.3b}
\end{array}
$$

where $\mathscr{L}_{\rho}=\rho D_{x}+D_{x} \rho$ and $\beta$ is a constant in $x$, which in principle depends on $t$ and also $z$. If, for example, $b$ is regular at $z=\infty$ and we write $b=\sum_{0 \leq j} \frac{b_{-j}}{z^{j}}$ then (2.3a) reads

$$
\begin{align*}
\rho_{t} & =\mathscr{L}_{\rho} b_{0},  \tag{2.4a}\\
0 & =\frac{1}{2} b_{-j, x x x}+\mathscr{L}_{\rho} b_{-(j+1)}, \quad 0 \leq j . \tag{2.4b}
\end{align*}
$$

If we assume the shape of $b$ to be

$$
\begin{equation*}
b=b_{0}+\frac{b_{-1}}{z+\varepsilon}, \quad \varepsilon>0 \tag{2.5}
\end{equation*}
$$

then the deformation equation (2.3a) reads

$$
\begin{gather*}
\rho_{t}=\mathscr{L}_{\rho} b_{0},  \tag{2.6a}\\
b_{0, x x x}+\frac{b_{-1, x x x}}{\varepsilon}=0, \quad \frac{1}{2} b_{-1, x x x}-\varepsilon \mathscr{L}_{\rho} b_{-1}=0 . \tag{2.6b}
\end{gather*}
$$

We note that equations (2.4b) and (2.6b) do not involve any derivatives with respect to $t$ and thus can be viewed as representing constraints. Resolving these constraints, that is finding a manageable parametrization of $b_{j}$ in terms of $\rho$, is one of the essential intermediate steps in solving (2.4a)(or (2.6a)). This problem was completely solved in [10] for the rational model given by (2.5).

Theorem 2.1 ([10]). Let $G_{\varepsilon}(x, y)$ be the Green's function for the boundary value problem

$$
\begin{equation*}
D_{x}^{2} f=\varepsilon \rho f, \quad f^{\prime}(0)-h f(0)=0, \quad f^{\prime}(1)+H f(1)=0, \quad h \geq 0, H \geq 0 \tag{2.7}
\end{equation*}
$$

Suppose that $(h, H) \neq(0,0)$ then $b_{-1}$ and $b_{0}$ defined by setting

$$
\begin{equation*}
b_{-1}(x)=G_{\varepsilon}(x, x), \quad b_{0}=\frac{\left[G_{\varepsilon=0}(x, x)-G_{\varepsilon}(x, x)\right]}{\varepsilon} \tag{2.8}
\end{equation*}
$$

satisfy (2.6b) and the resulting deformation given by (2.6a) is isospectral.
Remark 2.2. Observe that all $b_{j}$ are a priori defined only up to a quadratic polynomial in $x$ (see (2.6b)). Moreover, to satisfy the boundary conditions, the quadratic polynomial has to be proportional to the diagonal part, obtained by setting $x=y$, of the Green's function $G_{\varepsilon=0}(x, y)$.

It is not difficult to see that the limit $\varepsilon \rightarrow 0^{+}$exists and one obtains the following counterpart of the previous theorem, again proven in its entirety in [10].

Theorem 2.3 ([10] ). Let $G_{0}(x, y)=G_{\varepsilon=0}(x, y)$ be be the Green's function of the operator $D_{x}^{2}$ satisfying

$$
\begin{equation*}
G_{0, x}(0, y)-h G_{0}(0, y)=0, \quad G_{0, x}(1, y)+H G_{0}(1, y)=0, \quad h \geq 0, H \geq 0 \tag{2.9}
\end{equation*}
$$

Suppose $(h, H) \neq(0,0), 0 \leq j$, and define $b_{-j}$ by setting

$$
\begin{equation*}
b_{-j}=0, \text { for } 1<j, \quad b_{-1}(x)=G_{0}(x, x), \quad b_{0}=-G_{1}(x, x) \stackrel{\text { def }}{=}-\left.\frac{d G_{\varepsilon}(x, x)}{d \varepsilon}\right|_{\varepsilon=0} . \tag{2.10}
\end{equation*}
$$

Then the $b_{j}$ satisfy (2.4b) and the resulting deformation given by (2.4a) is isospectral.

## 3 Deformations with finitely many fields

The most natural types of flows involve only finitely many fields $b_{j}$; one can formally obtain them by truncating the infinite tower of constraints at certain level $j=k$ by requiring that $b_{-j}=0, k<j$. Thus the constraints take the form

$$
\begin{align*}
& 0=\frac{1}{2} b_{-j, x x x}+\mathscr{L}_{\rho} b_{-(j+1)}, \quad 0 \leq j \leq k-1, \\
& 0=\frac{1}{2} b_{-k, x x x}, \text { and } b_{-j}=0, \quad k<j . \tag{3.1}
\end{align*}
$$

The other finite type, generalizing the rational case $k=1$ above, can be taken to be

$$
\begin{equation*}
b=b_{0}+\frac{b_{-1}}{z+\varepsilon}+\frac{b_{-2}}{(z+\varepsilon)^{2}}+\cdots+\frac{b_{-k}}{(z+\varepsilon)^{k}} . \tag{3.2}
\end{equation*}
$$

The main objective of this paper is to show that, firstly, there exists a natural parametrization of the case (3.2) in terms of the same Green's function $G_{\varepsilon}$ used for $k=1$ and, secondly, that this parametrization has a nontrivial limit, parametrizing the truncated case (3.1).

To begin with, by direct computation, we get the evolution equation and the constraints for $b$ given by (3.2) to be

$$
\begin{align*}
\rho_{t} & =\mathscr{L}_{\rho} b_{0}, \\
0 & =\frac{1}{2} b_{0, x x x}+\mathscr{L}_{\rho} b_{-1}, \\
0 & =\frac{1}{2} b_{-j, x x x}+\mathscr{L}_{\rho} b_{-(j+1)}-\varepsilon \mathscr{L}_{\rho} b_{-j}, \quad 1 \leq j \leq k-1  \tag{3.3}\\
0 & =\frac{1}{2} b_{-k, x x x}-\varepsilon \mathscr{L}_{\rho} b_{-k} .
\end{align*}
$$

We will now argue that there exists a parametrization of these equations in terms of the Green's function $G_{\varepsilon}$ and its $\varepsilon$ derivatives $G_{\varepsilon}^{(j)} \stackrel{d e f}{=} \frac{d^{j} G_{\varepsilon}}{d \varepsilon^{j}}$. The proposed parametrization is given by the formulas

$$
\begin{align*}
b_{0} & =\frac{\left[G_{0}(x, x)-\sum_{j=0}^{k-1} \frac{G_{\varepsilon}^{(j)}(x, x)}{j!}(-\varepsilon)^{j}\right]}{\varepsilon^{k}},  \tag{3.4}\\
b_{-j} & =\frac{(-1)^{(k-j)} G_{\varepsilon}^{(k-j)}(x, x)}{(k-j)!}, \quad 1 \leq j \leq k .
\end{align*}
$$

The main ingredient of the proof rests on the observation that the $\varepsilon$ derivative of the last equation in (3.3) is, up to a correct choice of the sign, the previous equation on the list. Indeed, differentiating once we obtain

$$
\begin{equation*}
0=\frac{1}{2} b_{-k, x x x}^{(1)}-\mathscr{L}_{\rho} b_{-k}-\varepsilon \mathscr{L}_{\rho} b_{-k}^{(1)}, \tag{3.5}
\end{equation*}
$$

and by iterating we arrive at the general formula (3.4). The formula for $b_{0}$ follows by successive elimination of terms $\mathscr{L}_{\rho} b_{j}$ in terms of third spacial derivatives. The intermediate formula

$$
\mathscr{L}_{\rho} b_{-(k-j)}=\frac{1}{2} \frac{\left[b_{-k}+\varepsilon b_{-(k-1)}+\cdots+\varepsilon^{j} b_{-(k-j)}\right]_{x x x}}{\varepsilon^{j+1}}
$$

is then substituted into the first equation of (3.3) with an important proviso that the term $G_{0}$, which we recall is at most quadratic in $x$, is added to ensure the existence of the limit $\varepsilon \rightarrow 0^{+}$. Finally, the reason why the limit exists is that $G_{\varepsilon}$ is an analytic function of $\varepsilon$ for $\varepsilon$ small enough (thanks to the assumption $(h, H) \neq(0,0)$ ), while the limit for $b_{0}$ is ensured by observing that the numerator in $\frac{\sum_{j=0}^{k-1} \frac{G_{\varepsilon}^{(j)}(x, x)}{j!}(-\varepsilon)^{j}}{\varepsilon^{k}}$ is the Taylor expansion of $G_{0=\varepsilon-\varepsilon}$ about $\varepsilon$ and thus it is equal to $G_{0}$ with an error term $\mathscr{O}\left(\varepsilon^{k}\right)$. In the limit $\varepsilon \rightarrow 0^{+}$

$$
b_{0}=\left.(-1)^{k} \frac{G_{\varepsilon}^{(k)}}{k!}\right|_{\varepsilon=0}
$$

Before we discuss the formulas for $\varepsilon=0$ we will simplify our notation to facilitate the display of formulas. We will write

$$
\begin{equation*}
G_{\varepsilon}=G_{0}+G_{1} \varepsilon+G_{2} \varepsilon^{2}+\cdots \tag{3.6}
\end{equation*}
$$

Since $G_{\varepsilon}$ is the Green's function of $D_{x}^{2}-\varepsilon \rho$ the terms $G_{j}$ satisfy

$$
\begin{equation*}
D_{x}^{2} G_{0}(x, y)=\delta(x-y), \quad D_{x}^{2} G_{j+1}(x, y)=\rho(x) G_{j}(x, y), \quad 0 \leq j \tag{3.7}
\end{equation*}
$$

all $G_{j}$ subject to the boundary conditions $G_{j, x}(x=0, y)-h G_{j}(x=0, y)=0$, $G_{j, x}(x=1, y)+H G_{j}(x=1, y)=0$. This allows one to write an explicit formula for $G_{j}(x, y)$, namely,

$$
\begin{equation*}
G_{j}(x, y)=\int_{[0,1]^{j}} G_{0}\left(x, \xi_{j}\right) \rho\left(\xi_{j}\right) G_{0}\left(\xi_{j}, \xi_{j-1}\right) \rho\left(\xi_{j-1}\right) \cdots \rho\left(\xi_{1}\right) G_{0}\left(\xi_{1}, y\right) d \xi_{j} \cdots d \xi_{1} \tag{3.8}
\end{equation*}
$$

Remark 3.1. It is easy to check that $G_{0}(x, y)<0$ on $[0,1]$. Since in (3.8) there are $j+1$ factors involving $G_{0}$ and the remaining factors are positive we get that $(-1)^{j+1} G_{j}(x, y)>$ 0.

With this notation in place we conclude that the parametrization

$$
\begin{equation*}
b_{-j}(x)=(-1)^{k-j} G_{k-j}(x, x), \quad 0 \leq j \leq k, \tag{3.9}
\end{equation*}
$$

resolves the constraints (3.1), and in addition the $b_{-j}$ s so defined satisfy the correct boundary conditions ensuring isospectrality [10]. These formulae have a natural diagrammatic representation as illustrated in Figure 1, where we present a natural interpretation of $b_{0}(x)$ for the case $k=2$.

We are especially interested in discrete strings, also called degenerate strings in [23], which are finite discrete measures, i.e. $\rho=\sum_{j=1}^{n} m_{j} \delta_{x_{j}}, 0<x_{1}<x_{2}<\cdots<x_{n}<$ 1. The above considerations carry over to this case with one important exception; as explained in [10] in this case the action of $\mathscr{L}_{\rho}$ on continuous, piecewise smooth, functions $f$ is given by

$$
\begin{equation*}
\mathscr{L}_{\rho} f=D_{x}(f(x) \rho)+\left\langle f_{x}\right\rangle(x) \rho, \tag{3.10}
\end{equation*}
$$



Figure 1. A diagram illustrating the flow generated by $b_{0}(x)$ in the case $k=2$ with $K(x, y)=-G_{0}(x, y)$. The flow is generated by a formal sum of weights along all broken lines of this type. Each edge $\left(\xi_{i}, \xi_{i-1}\right)$ has a positive weight $K_{0}\left(\xi_{i}, \xi_{i-1}\right)$, each vertex $\xi_{i}$ has a positive weight $\rho\left(\xi_{i}\right)$ and the weight of the path is the product of weights. When $\rho$ is a discrete measure (a discrete string) the formal sum is a finite sum over all admissible broken lines.
where now $D_{x}$ means the distributional derivative, $\left\langle f_{x}\right\rangle(x) \delta_{x_{j}}=\left\langle f_{x}\right\rangle\left(x_{j}\right) \delta_{x_{j}}$, and $\langle h\rangle\left(x_{j}\right)$ denotes the arithmetic average of $h$ at $x_{j}$. The time derivative of $\rho$ is easily computed to be

$$
\begin{equation*}
\dot{\rho}=\sum_{j=1}\left(\dot{m}_{j} \delta_{x_{j}}-\dot{x}_{j} m_{j} \delta_{j}^{(1)}\right) \tag{3.11}
\end{equation*}
$$

and the evolution equation (2.4a) becomes a system of ODEs which for the flows truncated at level $k$ with $b_{-j}$ parametrized by (3.9) takes the following simple form

$$
\begin{align*}
\dot{x}_{j} & =(-1)^{k+1} G_{k}\left(x_{j}, x_{j}\right)  \tag{3.12a}\\
\dot{m}_{j} & =(-1)^{k} m_{j}\left\langle G_{k, x}(x, x)\right\rangle\left(x=x_{j}\right) \tag{3.12b}
\end{align*}
$$

Theorem 3.2. Equations (3.12) are Hamiltonian with respect to the canonical Poisson bracket

$$
\begin{equation*}
\left\{x_{i}, x_{j}\right\}=\left\{m_{i}, m_{j}\right\}=0, \quad\left\{x_{i}, m_{j}\right\}=\delta_{i, j} \tag{3.13}
\end{equation*}
$$

and the Hamiltonian

$$
\begin{equation*}
H^{(k)}=\frac{(-1)^{k+1}}{k+1} \int_{0}^{1} G_{k}(x, x) \rho(x) d x=\frac{(-1)^{k+1}}{k+1} \sum_{i=1}^{n} m_{i} G_{k}\left(x_{i}, x_{i}\right) . \tag{3.14}
\end{equation*}
$$

Proof. The computation of the $\left\{x_{j}, H^{(k)}\right\}$ bracket is straightforward:
$\left\{x_{j}, H^{(k)}\right\}=\frac{\partial H^{(k)}}{\partial m_{j}}=$
$\frac{(-1)^{k+1}}{k+1} \frac{\partial}{\partial m_{j}} \int_{[0,1]^{k+1}} \rho\left(\xi_{k+1}\right) G_{0}\left(\xi_{k+1}, \xi_{k}\right) \rho\left(\xi_{k}\right) \cdots \rho\left(\xi_{1}\right) G_{0}\left(\xi_{1}, \xi_{k+1}\right) d \xi_{k+1} d \xi_{k} \cdots d \xi_{1}=$ $\frac{(-1)^{k+1}}{k+1}\left((k+1) G_{k}\left(x_{j}, x_{j}\right)\right)=(-1)^{k+1} G_{k}\left(x_{j}, x_{j}\right)$,
where we used $(k+1)$ times that $\frac{\partial \rho}{\partial m_{j}}=\delta_{x_{j}}$. This proves (3.12a). The proof of (3.12b) can be broken into two steps. First we observe that, based on (3.8),

$$
\begin{align*}
& \left\langle G_{k, x}(x, x)\right\rangle\left(x=x_{j}\right)= \\
& \int_{[0,1]^{j}}\left\langle G_{0, x}\left(x, \xi_{j}\right)\right\rangle\left(x=x_{j}\right) \rho\left(\xi_{j}\right) G_{0}\left(\xi_{j}, \xi_{j-1}\right) \rho\left(\xi_{j-1}\right) \cdots \rho\left(\xi_{1}\right) G_{0}\left(\xi_{1}, y\right) d \xi_{j} \cdots d \xi_{1}+ \\
& \int_{[0,1]^{j}} G_{0}\left(x, \xi_{j}\right) \rho\left(\xi_{j}\right) G_{0}\left(\xi_{j}, \xi_{j-1}\right) \rho\left(\xi_{j-1}\right) \cdots \rho\left(\xi_{1}\right)\left\langle G_{0, x}\left(\xi_{1}, x\right)\right\rangle\left(x=x_{j}\right) d \xi_{j} \cdots d \xi_{1} . \tag{3.15}
\end{align*}
$$

The second observation hinges on the fact that, since $x_{i} \neq x_{j}, i \neq j$, we have that

$$
G_{0, x_{i}}\left(x_{i}, x_{j}\right)=\left\{\begin{array}{l}
\left\langle G_{0, x}\left(x, x_{j}\right)\right\rangle\left(x=x_{i}\right), \quad i \neq j  \tag{3.16}\\
2\left\langle G_{0, x}\left(x, x_{i}\right)\right\rangle\left(x=x_{i}\right), \quad i=j
\end{array}\right.
$$

which follows immediately from the shape of the Green's function, namely,

$$
G_{0}\left(x_{i}, x_{j}\right)= \begin{cases}c\left(x_{i}\right) \hat{c}\left(x_{j}\right), & i<j \\ c\left(x_{i}\right) \hat{c}\left(x_{i}\right), & i=j \\ c\left(x_{j}\right) \hat{c}\left(x_{i}\right), & j<i\end{cases}
$$

where $c(x), \hat{c}(x)$ are linear functions of $x$, satisfying appropriate boundary conditions at $x=0, x=1$, respectively. We need to compute, for simplicity expanding integrals in terms of sums,

$$
\begin{aligned}
& \left\{m_{j}, H^{(k)}\right\}=-\frac{\partial H^{(k)}}{\partial x_{j}}= \\
& \frac{(-1)^{k}}{k+1} \frac{\partial}{\partial x_{j}} \sum_{i_{k+1}, i_{k}, \cdots i_{1}} m_{i_{k+1}} m_{i_{k}} \cdots m_{i_{1}} G_{0}\left(x_{i_{k+1}}, x_{i_{k}}\right) G_{0}\left(x_{i_{k}}, x_{i_{k-1}}\right) \cdots G_{0}\left(x_{i_{1}}, x_{i_{k+1}}\right)= \\
& (-1)^{k} \sum_{i_{k+1}, i_{k}, \cdots, i_{1}} m_{i_{k+1}} m_{i_{k}} \cdots m_{i_{1}} \frac{\partial}{\partial x_{j}}\left(G_{0}\left(x_{i_{k+1}}, x_{i_{k}}\right)\right) G_{0}\left(x_{i_{k}}, x_{i_{k-1}}\right) \cdots G_{0}\left(x_{i_{1}}, x_{i_{k+1}}\right) \text {, }
\end{aligned}
$$

where in the second step we used that the expression under the sum is symmetric. Moreover,

$$
G_{0, x_{j}}\left(x_{i_{k+1}}, x_{i_{k}}\right)=\delta_{i_{k+1}, j} G_{0, x_{j}}\left(x_{j}, x_{i_{k}}\right)+\left(1-\delta_{i_{k+1}, j}\right) \delta_{i_{k}, j} G_{0, x_{j}}\left(x_{i_{k+1}}, x_{j}\right)
$$

and after substituting back into the above formula and making use of (3.16) we obtain

$$
\begin{aligned}
& \left\{m_{j}, H^{(k)}\right\}=(-1)^{k} m_{j}\left(\sum_{i_{k}, i_{i}, \cdots, i_{1}} m_{i_{k}} \cdots m_{i_{1}}\left\langle G_{0, x}\left(x, x_{i_{k}}\right)\right\rangle\left(x=x_{j}\right) G_{0}\left(x_{i_{k}}, x_{i_{k-1}}\right) \cdots G_{0}\left(x_{i_{1}}, x_{j}\right)+\right. \\
& \left.\sum_{i_{k}, i_{i}, \cdots, i_{1}} m_{i_{k}} \cdots m_{i_{1}} G_{0}\left(x_{i_{k}}, x_{i_{k-1}}\right) \cdots\left\langle G_{0, y}\left(x_{i_{1}}, y\right)\right\rangle\left(y=x_{j}\right)\right) \stackrel{(3.15)}{=} \\
& (-1)^{k} m_{j}\left\langle G_{k, x}(x, x)\right\rangle\left(x=x_{j}\right),
\end{aligned}
$$

which confirms (3.12b).

Remark 3.3. In view of (3.1) we see that Hamiltonians given by (3.14) are non-negative. Moreover, they can be shown to be simply related to spectral invariants (see i.e. [10])

$$
\begin{align*}
& I_{0}=\int_{[0,1]} \rho\left(\xi_{1}\right) G_{0}(\xi, \xi) d \xi  \tag{3.17a}\\
& I_{j}=\int_{0<\xi_{1}<\xi_{2}<\cdots<\xi_{j}<1} \rho\left(\xi_{j}\right)\left(\xi_{j}-\xi_{j-1}\right) \cdots \rho\left(\xi_{2}\right)\left(\xi_{2}-\xi_{1}\right) \rho\left(\xi_{1}\right) G_{0}\left(\xi_{1}, \xi_{j}\right) d \xi \cdots d \xi_{j} . \tag{3.17b}
\end{align*}
$$

For example, if $k=1$ then for $H^{(1)}$ computed from (3.14) we get

$$
H^{(1)}=\frac{1}{2} I_{1}^{2}+I_{2} .
$$

### 3.1 Evolution of spectral data

We will briefly discuss how one can solve equations (3.12a), (3.12b), or alternatively $\rho_{t}=\mathscr{L}_{\rho} b_{0}$ with $b_{0}(x)=(-1)^{k} G_{k}(x, x)$ and

$$
\begin{equation*}
\rho=\sum_{j=1}^{n} m_{j} \delta_{x_{j}}, \quad 0<x_{1}<x_{2}<\cdots<x_{n}<1 . \tag{3.18}
\end{equation*}
$$

Let $\phi$ satisfy (1.3a) and the left initial condition $\phi_{x}(0)-h \phi(0)=0$. Then for $0 \leq x<x_{1}$ (the position of the first mass) we can take

$$
\begin{equation*}
\phi(x)=h x+1 . \tag{3.19}
\end{equation*}
$$

Since $\phi$ changes with the deformation time according to (2.2) we have that for $0<$ $x<x_{1}$

$$
\begin{equation*}
0=\left(-\frac{1}{2} b_{x}+\beta\right)(h x+1)+h b \tag{3.20}
\end{equation*}
$$

which implies

$$
\begin{equation*}
\beta=\frac{1}{2} b_{x}(0)-h b(0) . \tag{3.21}
\end{equation*}
$$

In fact, for $b=b_{0}+\frac{b_{-1}}{z}+\cdots+\frac{b_{-k}}{z^{k}}$ and $b_{-j}$ given by (3.9), $\beta$ takes the following simple form

$$
\begin{equation*}
\beta(z)=\frac{1}{2 z^{k}} . \tag{3.22}
\end{equation*}
$$

The spectrum is given by the zeros of the function $D(z)=\phi_{x}(1 ; z)+H \phi(1 ; z)$ for $0 \leq$ $H<\infty$ and $D(z)=\phi(1 ; z)$ for $H=\infty$, and we have the following linearization result

Theorem 3.4. Let $0<H<\infty$, and let

$$
\begin{equation*}
N(z ; t) \stackrel{\text { def }}{=} \phi_{x}(1 ; z)-H \phi(1 ; z) . \tag{3.23}
\end{equation*}
$$

Then for every point $z_{i}$ of the spectrum the time derivative of $N\left(z_{i} ; t\right)$ satisfies

$$
\begin{equation*}
\dot{N}\left(z_{i} ; t\right)=\frac{1}{z_{i}^{k}} N\left(z_{i} ; t\right) . \tag{3.24}
\end{equation*}
$$

Proof. We start by computing $\dot{N}\left(z_{i} ; t\right)=\phi_{t, x}\left(1 ; z_{i}\right)-H \phi_{t}\left(1 ; z_{i}\right)$ with the help of (2.2), (2.3b) and employing one intermediate result proven in [10] stating that one of the necessary conditions for isospectrality can be written

$$
\frac{1}{2} b_{x x}(1)+H b_{x}(1)+H^{2} b(1)=0 .
$$

Then, by straightforward computation, and the fact that on the spectrum $D=0$, we obtain:

$$
\dot{N}\left(z_{i} ; t\right)=\left(\beta\left(z_{i}\right)-\frac{b_{x}\left(1 ; z_{i}\right)+2 H b\left(1 ; z_{i}\right)}{2}\right) N\left(z_{i} ; t\right) .
$$

For the flows $b=b_{0}+\frac{b_{-1}}{z}+\cdots+\frac{b_{-k}}{z^{k}}$ and $b_{-j}$ given by (3.9) the second term is zero, when evaluated at $x=1$, except for the last term $\frac{b_{-k}}{z_{i}^{k}}$ for which one obtains $\frac{-1}{z_{i}^{k}}$. The proof of this claim follows from $G_{0, x}(x ; y)(x=1, y \neq 1)+H G_{0}(x ; y)(x=1, y \neq 1)=0$ and $G_{0, x}(x ; x)(x=1)+H G_{0}(x ; x)(x=1)=-1$. Finally, using (3.22), we obtain the required result.

It is now easy to cover the remaining two special cases $H=0, H=\infty$. We modify the definition of $N$; we set $N=-\phi(1 ; z)$ and $N(z ; t)=\phi_{x}(1 ; z)$ respectively.

Theorem 3.5. Let $H=\infty$ or $H=0,(h, H) \neq 0$. Then for every point $z_{i}$ of the spectrum

$$
\begin{equation*}
\dot{N}\left(z_{i} ; t\right)=\frac{1}{z_{i}^{k}} N\left(z_{i} ; t\right) \tag{3.25}
\end{equation*}
$$

Proof. We will give the proof in the case $H=\infty$, leaving the case $H=0$ for interested readers. Following the same steps as above we obtain:

$$
\dot{N}\left(z_{i} ; t\right)=\left(\beta\left(z_{i}\right)+\frac{1}{2} b_{x}(1)\right) N\left(z_{i} ; t\right),
$$

and subsequently observe that the only change now is that $G_{0}(x ; y)(x=1, y \neq 1)=$ 0 and $G_{0, x}(x ; x)(x=1)=-1$, which in conjuncture with (3.22) imply the required result.

Remark 3.6. The reason why the case $(h, H)=(0,0)$ is excluded is because the formula for the Green's function using iterations (3.6) does not hold since $z=0$ is now in the spectrum.

As is well known the Weyl function (Weyl-Titchmarch function) is a convenient way of storing information about the boundary value problem (1.3a). We can define it as

$$
\begin{equation*}
W(z)=\frac{N(z)}{D(z)} \tag{3.26}
\end{equation*}
$$

$W(-z)$ is an example of a Herglotz function (i.e. p. 17 in [11]), in particular $-W(-z)$ admits an integral Stieltjes representation

$$
\begin{equation*}
-W(-z)=\gamma+\int \frac{d \mu(\zeta)}{z+\zeta}, \quad \gamma \in \mathbf{R} \tag{3.27}
\end{equation*}
$$

where $d \mu$ is a positive measure supported on positive reals $\mathbf{R}_{+}$. In the case of interest, when dealing with discrete strings, $d \mu=\sum_{j=1}^{n} \mu_{j} \delta_{z_{j}}, \mu_{j}>0$, where $z_{j}$ are the eigenvalues of the boundary value problem. The dependence on the deformation parameter, in view of (3.4) and 3.5, is given by $\mu_{j}(t)=\mu_{j}(0) e^{\frac{t}{z_{j}^{k}}}$ for the flow given by (3.1).

Example 3.7. We will illustrate the developed formalism on the case of the string boundary value problem (1.3a) corresponding to the mass density (3.18), with boundary conditions of the type $0<h<\infty$ and $H=0$. The (bare) Green's function in this case is

$$
G_{0}(x, y)= \begin{cases}\frac{h x+1}{-h}, & x<y \\ \frac{h y+1}{-h}, & y<x\end{cases}
$$

With the help of (3.8), (3.9) we can now write all coefficients in (3.1), in particular obtaining

$$
\begin{equation*}
b_{0}(x)=(-1)^{k} \sum_{j_{1}, j_{2}, \cdots, j_{k}} G_{0}\left(x, x_{j_{k}}\right) m_{j_{k}} G_{0}\left(x_{j_{k}}, x_{j_{k-1}}\right) m_{j_{k-1}} \cdots m_{j_{1}} G_{0}\left(x_{j_{1}}, x\right) . \tag{3.28}
\end{equation*}
$$

We observe that the first equation of motion, (3.12a), can be written in a compact way if one defines matrices $M=\operatorname{diag}\left(m_{1}, \ldots, m_{n}\right)$ and $K=\left[K_{i, j}\right]=\left[-G_{0}\left(x_{i}, x_{j}\right)\right]$. Then

$$
\begin{equation*}
\dot{x}_{j}=(K M K M \ldots M K)_{j, j} . \tag{3.29}
\end{equation*}
$$

In particular, if $k=1$, we get $\dot{x}_{j}=(K M K)_{j, j}$. Likewise, the second equation (3.12b) can be written in an analogous way by introducing a matrix $J=\left[J_{i, j}\right]=2\left\langle G_{0, x}\left(x, x_{j}\right)(x=\right.$ $\left.\left.x_{i}\right)\right\rangle$, resulting in

$$
\begin{equation*}
\dot{m}_{j}=(M J M K M K \cdots M K)_{j, j} . \tag{3.30}
\end{equation*}
$$

We now turn to outlining the integration process for the mass density (3.18).
Let

$$
\left.\phi\right|_{I_{j}}=\phi_{j}=p_{j}\left(x-x_{j}\right)+q_{j}, \quad \text { where } I_{j}=\left(x_{j}, x_{j+1}\right), \text { and } x_{0}=1, x_{n+1}=1,
$$

denote the solution to the initial value problem $-\phi_{x x}=z \rho \phi, \phi_{x}(0)-h \phi(0)=0$ whose construction proceeds as follows. We start off with $p_{0}=h$ and $q_{0}=1$ to satisfy the initial condition at $x=0$. Then letting $l_{j}$ denote the length of the interval $I_{j}$ and imposing the continuity of $\phi$ at $x=x_{j+1}$ one obtains that $p_{j}=\frac{q_{j+1}-q_{j}}{l_{j}}$. The jump in the derivative of $\phi$ at $x=x_{j+1}$ results in $p_{j+1}-p_{j}=-z m_{j+1} q_{j+1}$. On the last interval $I_{n}, \phi_{n}=p_{n}\left(x-x_{n}\right)+q_{n}$. Let us define the Weyl function for this problem

$$
W(z) \stackrel{\text { def }}{=}-\frac{\phi(1 ; z)}{\varphi_{x}(1 ; z)} .
$$

From the construction $-W(-z)=\frac{p_{n}(-z) l_{n}+q_{n}(-z)}{p_{n}(-z)}=l_{n}+\frac{q_{n}(-z)}{p_{n}(-z)}$. Iterating with the help of continuity and jump conditions we obtain

$$
-W(-z)=l_{n}+\frac{1}{z m_{n}+\frac{1}{l_{n-1}+\frac{1}{\ddots+\frac{1}{l_{0}+\frac{1}{h}}}}}
$$

and, upon comparing with (3.27), we obtain

$$
\begin{equation*}
\gamma=l_{n}, \quad \int \frac{e^{\frac{t}{\zeta^{k}}} d \mu(\zeta ; 0)}{z+\zeta}=\frac{1}{z m_{n}+\frac{1}{l_{n-1}+\frac{1}{\ddots \cdot+\frac{1}{l_{0}+\frac{1}{h}}}}}, \tag{3.31}
\end{equation*}
$$

where we denote $d \mu(\zeta ; 0)=d \mu(\zeta ; t=0)$, which shows that using Stieltjes' inversion formulas (see [25, 4]) we can recover $m_{n}, \cdots, m_{1}$ and $l_{n-1}, \cdots, l_{0}$ in terms of the Hankel determinants of the moments of the measure $e^{\frac{t}{\zeta^{k}}} d \mu(\zeta ; 0)$. We will briefly review
the relevant part of Stieltjes' theory. In [25] T. Stieltjes studied the continued fraction

$$
\begin{equation*}
f(z)=\frac{1}{a_{1} z+\frac{1}{a_{2}+\frac{1}{\ddots+\frac{1}{a_{2 j-1} z+\frac{1}{a_{2 j}+\ldots} \cdot}}}} \tag{3.32}
\end{equation*}
$$

under the assumption $a_{j}>0$, viewed as a function of the complex variable $z$. Then he considered a formal Laurent expansion at $z=\infty$ of the continued fraction, written as

$$
\frac{c_{0}}{z}-\frac{c_{1}}{z^{2}}+\frac{c_{3}}{z^{3}}+\cdots
$$

Then the main thrust of Stieltjes's theory went towards establishing the existence of a measure (Stieltjes measure) $d \alpha$ supported on $\mathbf{R}_{+}=[0, \infty)$ such that $c_{j}=\int_{\mathbf{R}_{+}} \zeta^{j} d \alpha(\zeta)$. In the case of interest this measure is unique (because the Laurent series converges) and is simply given by

$$
\begin{equation*}
d \alpha(\zeta)=e^{\frac{t}{\zeta^{k}}} d \mu(\zeta ; 0)=\sum_{j=1}^{n} \mu_{j}(0) e^{\frac{t}{z_{j}^{k}}} \delta_{z_{j}} \tag{3.33}
\end{equation*}
$$

Then one defines the Hankel matrix of moments

$$
H=\left(\begin{array}{cccccc}
c_{-1} & c_{0} & c_{1} & \cdots & c_{j} & \cdots  \tag{3.34}\\
c_{0} & c_{1} & c_{2} & \cdots & c_{j+1} & \cdots \\
c_{1} & c_{2} & c_{3} & \cdots & c_{j+2} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \\
c_{l-1} & \cdots & \cdots & \cdots & c_{j+l} & \cdots
\end{array}\right),
$$

where we included $c_{-1}$ because in our case the measure $d \alpha$ is supported away from 0 . With the help of $H$ one introduces certain minors, called Hankel determinants, denoted $\Delta_{k}^{l}$, which are defined as the determinants of $k \times k$ submatrices whose $(i, j)$ entries are $c_{i+j+l-2}$, while $\Delta_{0}^{l}=1$ by convention. Finally, one can express the coefficients $a_{j}$ in (3.32) in terms of these determinants using the formulae

$$
\begin{equation*}
a_{2 j}=\frac{\left(\Delta_{j}^{0}\right)^{2}}{\Delta_{j}^{1} \Delta_{j-1}^{1}}, \quad a_{2 j+1}=\frac{\left(\Delta_{j}^{1}\right)^{2}}{\Delta_{j}^{0} \Delta_{j+1}^{2}} \tag{3.35}
\end{equation*}
$$

For simplicity let us set $j^{\prime}=n-j$, then an immediate application of these formulae to our case yields:

$$
\begin{align*}
m_{j} & =\frac{\left(\Delta_{j^{\prime}}^{1}\right)^{2}}{\Delta_{j^{\prime}}^{0} \Delta_{(j-1)^{\prime}}^{2}}, \quad 1 \leq j \leq n  \tag{3.36a}\\
l_{j} & =\frac{\left(\Delta_{\left.j^{\prime}\right)^{2}}^{0}\right.}{\Delta_{j^{\prime}}^{1} \Delta_{(j+1)^{\prime}}^{1}}, \quad 1 \leq j \leq n-1  \tag{3.36b}\\
l_{0}+\frac{1}{h} & =\frac{\left(\Delta_{0^{\prime}}^{0}\right)^{2}}{\Delta_{0^{\prime}}^{1} \Delta_{1^{\prime}}^{1}}, \tag{3.36c}
\end{align*}
$$

In the final step, one can recover $l_{n}$ by observing that

$$
-W(0)=\frac{h+1}{h}=1+\frac{1}{h}=l_{n}+\int \frac{e^{\frac{t}{\zeta^{k}}}}{\zeta} d \mu(\zeta ; 0)=l_{n}+\Delta_{1}^{-1}
$$

and solve for $l_{n}$, or compute $l_{n}$ from the formula for the total length of the string: $l_{n}=1-\sum_{i=0}^{n-1} l_{i}$. We conclude this example by noting that the case $h=\infty$, which means the Dirichlet condition on the left end and the Neumann condition on the right, can be handled by taking the limit $h \rightarrow 0$. Interestingly, the Dirichlet-Neumann case appeared, somewhat unexpectedly, in the recent work on the modified CamassaHolm equation [9].

Example 3.8. The case of the CH equation (2.1) is not the main focus of this paper. However, one might be tempted to compare the known formulas with what one gets if the formalism is applied to that case. The literature on the CH equation is so vast that one can not do justice to many important contributions to the subject. We will only refer to papers the results of which overlap in scope with the presented formalism. One way of looking at the CH theory is to start from the spectral problem

$$
\begin{equation*}
-v_{x x}+\frac{1}{4} v=z m v, \quad-\infty<x<\infty \tag{3.37}
\end{equation*}
$$

with vanishing boundary conditions $v \rightarrow 0$ as $|x| \rightarrow \infty$ [7]. We note that this spectral problem also appears in [21]. In (3.37) $m$ is a measure, similar to the string mass density $\rho$ in (1.3a). For convenience we will assume that $m$ has a compact support as this is sufficient for our purposes. We deform $m$ in exactly the same way as in (2.2), i.e.

$$
v_{t}=a v+b v_{x}
$$

and from the condition $v_{t x x}=v_{x x t}$ we get

$$
\begin{align*}
z m_{t} & =\frac{1}{2} b_{x x x}-\frac{1}{2} b_{x}+z \mathscr{L}_{m} b,  \tag{3.38a}\\
a & =-\frac{1}{2} b_{x}+\beta . \tag{3.38b}
\end{align*}
$$

For $b$ regular at $z=\infty$ the evolution equation is the same as for the string, namely,

$$
m_{t}=\mathscr{L}_{m} b_{0}
$$

where $b_{0}$ is the limit of $b$ at $z=\infty$. However, the constraints are different. Let us analyze the constraints for the rational model specified in (3.2). The resulting constraints can be simply obtained by changing $D_{x}^{3}$ to $D_{x}^{3}-D_{x}$ in (3.3). Let us denote by $G_{\varepsilon}(x, y)$, the Green's function for $D_{x}^{2}-\frac{1}{4}-\varepsilon m$ vanishing as $|x| \rightarrow \infty$. For these boundary conditions

$$
G_{\varepsilon}=G_{0}+\varepsilon G_{1}+\varepsilon^{2} G_{2}+\cdots
$$

where

$$
\begin{equation*}
G_{0}(x, y)=-e^{\frac{|x-y|}{2}} \tag{3.39}
\end{equation*}
$$

The perturbative expansion produces essentially the same formula as in (3.8), namely

$$
\begin{equation*}
G_{j}(x, y)=\int_{\mathbf{R}^{j}} G_{0}\left(x, \xi_{j}\right) m\left(\xi_{j}\right) G_{0}\left(\xi_{j}, \xi_{j-1}\right) m\left(\xi_{j-1}\right) \cdots m\left(\xi_{1}\right) G_{0}\left(\xi_{1}, y\right) d \xi_{j} \cdots d \xi_{1} \tag{3.40}
\end{equation*}
$$

Now we only check that the diagonal of the Green's function, $G_{\varepsilon}(x, x)$, satisfies

$$
\begin{equation*}
G_{\varepsilon, x x x}(x, x)-G_{\varepsilon, x}(x, x)=\varepsilon \mathscr{L}_{m} G_{\varepsilon}(x, x), \tag{3.41}
\end{equation*}
$$

the proof of which is essentially identical to the one in Lemma 5.1 in [10]. This result, in conjuncture with the fact that, after changing $D_{x}^{3}$ to $D_{x}^{3}-D_{x}$, the last equation in (3.3) is satisfied by $G_{\varepsilon}(x, x)$ and the iterative procedure employed in the analysis of rational flows for the string goes through, resulting in the validity of the final formulae (3.4). In particular, the limit $\varepsilon \rightarrow 0^{+}$can be carried out yielding an explicit parametrization of the flow $b=b_{0}+\frac{b_{-1}}{z}+\cdots+\frac{b_{-k}}{z^{k}}$, structurally the same as (3.9), namely $b_{-j}(x)=(-1)^{k-j} G_{k-j}(x, x)$. The most relevant physically is the case of the discrete measure $m=\sum_{j}^{n} m_{j} \delta_{x_{j}}$ (the peakon sector for $k=1$ [7]). The equations of motion for $x_{j}$ and $m_{j}$ are given by equations (3.29) and (3.30) with $K=\left[e^{-\frac{\left|x_{i}-x_{j}\right|}{2}}\right]$ and $J=\left[\operatorname{sgn}\left(x_{j}-x_{i}\right) e^{-\frac{\left|x_{j}-x_{i}\right|}{2}}\right]$. For example, for $k=1$ we obtain

$$
\begin{aligned}
\dot{x}_{j} & =(K M K)_{j, j}=\sum_{i} m_{i} e^{-\left|x_{j}-x_{i}\right|}, \\
\dot{m}_{j} & =(M J M K)_{j, j}=m_{j} \sum_{i} \operatorname{sgn}\left(x_{j}-x_{i}\right) m_{i} e^{-\left|x_{j}-x_{i}\right|},
\end{aligned}
$$

well known from the CH theory [8].
For $k=2$ we obtain

$$
\begin{aligned}
& \dot{x}_{j}=(\text { KMKMK })_{j, j}=\sum_{i, l} e^{-\frac{\left|x_{j}-x_{i}\right|}{2}} m_{i} e^{-\frac{\left|x_{i}-x_{l}\right|}{2}} m_{l} e^{-\frac{\left|x_{l}-x_{j}\right|}{2}}, \\
& \dot{m}_{j}=(M J M K M K)_{j, j}=m_{j} \sum_{i, l} \operatorname{sgn}\left(x_{j}-x_{i}\right) e^{-\frac{\left|x_{j}-x_{i}\right|}{2}} m_{i} e^{-\frac{\left|x_{i}-x_{l}\right|}{2}} m_{l} e^{-\frac{\left|x_{l}-x_{j}\right|}{2}},
\end{aligned}
$$

clearly indicating a general pattern for arbitrary $k$. Although the higher order flows in the CH theory have been investigated $[19,24,20$ ] we are not aware of any work on the peakon sector of the CH hierarchy. In particular we conclude that explicit integration of higher-order peakon flows will be possible using Stieltjes continued fractions generalizing the results of [4]. We leave this however for future work.

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