On the regularity of solutions to the 2D Boussinesq equations satisfying Type I conditions

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Abstract

We prove continuation in time of the local smooth solutions satisfying various Type I conditions for the 2D inviscid Boussinesq equations.

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1 Introduction

We consider the Boussinesq equations in the space time cylinder $\mathbb{R}^2 \times (-1, 0)$

$$
(1.1) \qquad \begin{cases} \partial_t v + (v \cdot \nabla) v = e_1 \theta - \nabla p, \qquad \nabla \cdot u = 0, \\ \partial_t \theta + (v \cdot \nabla) \theta = 0, \end{cases}
$$

where $v = (v_1(x,t), v_2(x,t))$, $(x,t) \in \mathbb{R}^2 \times (-\infty, 0)$. This is an important equation modelling the dynamics of the heat convection in the atmospheric science(see e.g. [10]). Moreover, it has essentially same structure as the axisymmetric 3D Euler equations off the axis([9]). Therefore, the study of the system (1.1) could provide us with information useful to understand the Euler equations. In [2] the first author of this paper proved the local well-posedness in standard Soblev space setting $H^m(\mathbb{R}^2)$, $m > 2$, and also the following Beale-Kato-Majda[1] type (non)blow-up criterion is deduced; for any $m > 2$

$$
(1.2) \quad \limsup_{t \to 0^-} (\|v(t)\|_{H^m} + \|\theta(t)\|_{H^m}) < +\infty \quad \text{if and only if} \quad \int_{-1}^0 \|\nabla \theta(t)\|_{L^\infty} dt < +\infty.
$$

See also [11, 7] for the other forms of criterion, using different functional setting, while a special type of scenario of singularity is excluded in [5]. We note the following scaling property of the system (1.1); it is invariant under the transform

$$
(v(x,t), \theta(x,t)) \mapsto \big(\lambda^\alpha v(\lambda x, \lambda^{\alpha+1} t), \lambda^{2\alpha+1} \theta(\lambda x, \lambda^{\alpha+1} t)\big) \quad \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R}
$$

for all $\lambda > 1$, $\alpha > -1$. This leads us to the following natural definition.

Definition 1.1. Let $(v, \theta) \in C([-1, 0); W^{2,p}(\mathbb{R}^2))$, $p > 2$, be a local in time classical solution of (1.1), which blows up at $t = 0$. We say it is of *Type I with respect to* v, if

$$\sup_{-1<t<0} (-t)\|\nabla v(t)\|_{L^\infty} < +\infty,$$

while we say it is of *Type I with respect to* θ, if

$$\sup_{-1<t<0} (-t)^2\|\nabla \theta(t)\|_{L^\infty} < +\infty.$$

The aim of the present paper is to exclude a possible Type I blow-up at time $t = 0$. For the definition of Besov space $\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)$, used in the theorem below, see Section 2.

Theorem 1.2. *Let (v, θ) be a solution of (1.1) in $\mathbb{R}^2 \times (-1, 0)$, which is regular in $[-1, 0)$. Furthermore, we assume that $v(-1), \theta(-1) \in W^{2,p}(\mathbb{R}^2)$ for $p > 2$, and at least one of the following conditions,*

(i)

$$\limsup_{t\to 0^-} (-t)\|\nabla v(t)\|_{L^\infty} < 2.$$

(ii)

$$\int_{-1}^{0} (-t)\|\nabla \theta(t)\|_{L^\infty} dt < +\infty.$$

(iii)

$$\int_{-1}^{0} \|ø(t)\|_{\dot{B}^0_{\infty,\infty}} dt + \int_{-1}^{0} (-t)\|\nabla \theta(t)\|_{\dot{B}^0_{\infty,\infty}} dt < +\infty.$$

(iv)

$$\int_{-1}^{0} \|ø(t)\|_{\dot{B}^0_{\infty,\infty}} dt + \sup_{-1<t<0} (-t)^2\|\nabla \theta(t)\|_{L^\infty} < +\infty.$$

Then both v and θ belong to $L^\infty(-1, 0; W^{2,p}(\mathbb{R}^2))$.

Remark 1.3. (a) In [2] (see also [8]) it is proved that if a solution to the 3D Euler equations on $\mathbb{R}^3 \times [-1, 0)$ satisfies $v(-1) \in W^{2,p}(\mathbb{R}^2)$ and

$$\limsup_{t\to 0^-} (-t)\|\nabla v(t)\|_{L^\infty} < 1. \tag{1.3}$$

Then, $v \in L^\infty(-1, 0; W^{2,p}(\mathbb{R}^2))$. Note that the condition (i) is a relaxed version of (1.3). It is also interesting to notice that in a recent paper[6] Elgindi and Jeong constructed explicitly blowing up solution, which has linear growth at spatial infinity, and is defined in a domain $D \subset \mathbb{R}^2$ with a corner. The solution satisfies

$$\lim_{t\to 0^-} \left\{(-t)\|\nabla v(t)\|_{L^\infty(D)} + (-t)^2\|\nabla \theta(t)\|_{L^\infty(D)}\right\} < +\infty,$$

and the blow-up happens at every point in D.

(b) The main novelty in the conditions (ii) and (iii) is the extra factor $(-t)$ in the integral of the norms of $|\nabla\theta(t)|$. This factor makes the integral $\int_{-1}^0(-t)\|\nabla\theta(t)\|_{L^\infty}dt$ scaling invariant quantity, while the stronger integral $\int_{-1}^0\|\nabla\theta(t)\|_{L^\infty}dt$ is not. Similar remark holds for $\int_{-1}^0\|\nabla\theta(t)\|_X dt$ with $X = BMO$ or $\dot{B}^0_{\infty,\infty}$.

(c) As far as the authors know it is still an open question if the regularity of the system (1.1) is guaranteed only by the vorticity integral condition, say

$$\int_{-1}^0\|\omega(t)\|_{L^\infty}dt < +\infty. \tag{1.4}$$

The above theorem with the condition (iv) says that if (1.4) holds, then any singularity, which is of Type I with respect to θ is excluded.

At this moment we could not omit the vorticity integral in the condition (iv) above, but if we modify Type I condition on $\nabla\theta$ logaritmmically as well as imposing the smallness, then this is possible as follows.

Theorem 1.4. *Let (v,θ) be a solution of (1.1) in $\mathbb{R}^2\times(-1,0)$ which is regular in $[-1,0)$. Furthermore, we assume that $v(-1),\theta(-1)\in W^{2,p_0}(\mathbb{R}^2)$ for some $p_0>2$. There exists $\varepsilon>0$ depending only on p_0, such that if*

$$\limsup_{t\to 0^-}(-t)^2\log(-1/t)\|\nabla\theta(t)\|_{L^\infty}\le\varepsilon, \tag{1.5}$$

then both v and θ belong to $L^\infty(-1,0;W^{2,p_0}(\mathbb{R}^2))$.

2 Proof of the Main Theorems

We introduce the space $\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)$ below. Let $\varphi\in\mathcal{S}$, where $\mathcal{S}$ is the the Schwartz class of rapidly decreasing functions, and let $\hat\varphi$ be its Fourier transform, defined by $\hat\varphi(\xi)=\int_{\mathbb{R}^n}e^{-2\pi ix\cdot\xi}\varphi(x)dx$. Then, we consider φ satisfying the following conditions

$$Supp\,\hat\varphi\subset\{\xi\in\mathbb{R}^n\,|\,\frac{1}{2}\le|\xi|\le 2\},\quad \hat\varphi\ge c>0\quad\text{if}\quad\frac{2}{3}<|\xi|<\frac{3}{2},\quad\text{and}\quad\sum_{j\in\mathbb{Z}}\hat\varphi_j(\xi)=1,$$

where we defined $\hat\varphi_j=\hat\varphi(2^{-j}\xi)$. Construction of the sequence $\{\varphi_j\}_{j\in\mathbb{Z}}$ is well-known(see e.g. [4]). Then, we say $f\in\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)$ if and only if $\sup_{j\in\mathbb{R}^n}\|\varphi_j * f\|_{L^\infty}:=\|f\|_{\dot{B}^0_{\infty,\infty}}<+\infty$. The basic properties of $\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)$ useful for us are the followings.

(i) Embedding properties:

$$L^\infty(\mathbb{R}^n)\hookrightarrow BMO(\mathbb{R}^n)\hookrightarrow\dot{B}^0_{\infty,\infty}(\mathbb{R}^n), \tag{2.1}$$

(ii) The logarithmic Sobolev inequality,

$$\|f\|_{L^\infty} \le c(1+\|f\|_{\dot{B}^0_{\infty,\infty}} \log(e+\|f\|_{W^{s,p}})), \quad s>n/p. \tag{2.2}$$

where the constant $c =: c_{ls}$ depends on s and p.

(iii) Boundedness on the Calderon-Zygmund operators, in particular applying to the Bio-Savart formular one has

$$\|\nabla v\|_{\dot{B}^0_{\infty,\infty}} \le c\|\omega\|_{\dot{B}^0_{\infty,\infty}}, \tag{2.3}$$

where (v,ω) satisfies $\nabla\cdot v = 0, \nabla\times v = \omega$.

Proof of Theorem1.2 :
Proof for (iii): Let $q>2$. We apply the operator ∂_i to the vorticity equation, multiplying the resultant equation by $\partial_i\omega|\nabla\omega|^{q-1}$, and integrating it over $\mathbb{R}^2$. Then, after the integration by part and using the Hölder inequality, we are led to

$$\begin{aligned}\frac{d}{dt}\|\nabla\omega\|_{L^q} &\le \|\nabla v\|_{L^\infty}\|\nabla\omega\|_{L^q} + \|\nabla^2\theta\|_{L^q}\\ &= \|\nabla v\|_{L^\infty}\|\nabla\omega\|_{L^q} + (-t)^{-1}(-t)\|\nabla^2\theta\|_{L^q}.\end{aligned} \tag{2.4}$$

Next, we apply the operator $\partial_i\partial_j$ to both sides of the θ equation, multiply both sides the by $\partial_i\partial_j\theta|\nabla^2\theta|^{q-2}$, and sum over $i,j=1,2,3$, and the integrate it over $\mathbb{R}^2$. This, applying the integration by part and the Hölder inequality, yields the following inequality

$$\frac{d}{dt}\|\nabla^2\theta\|_{L^q} \le 2\|\nabla v\|_{L^\infty}\|\nabla^2\theta\|_{L^q} + \|\nabla\theta\|_{L^\infty}\|\nabla^2 v\|_{L^q}. \tag{2.5}$$

Multiplying both sides of (2.5) by $(-t)$, we see that

$$\begin{aligned}&\frac{d}{dt}(-t)\|\nabla^2\theta\|_{L^q} + \|\nabla^2\theta\|_{L^q}\\ &\quad\le 2\|\nabla v\|_{L^\infty}(-t)\|\nabla^2\theta\|_{L^q} + (-t)\|\nabla\theta\|_{L^\infty}\|\nabla^2 v\|_{L^q}\\ &\quad\le 2\|\nabla v\|_{L^\infty}(-t)\|\nabla^2\theta\|_{L^q} + c_{cz}(-t)\|\nabla\theta\|_{L^\infty}\|\nabla\omega\|_{L^q}\end{aligned} \tag{2.6}$$

Now define

$$\Psi(t) := \|\nabla\omega\|_{L^q} + (-t)\|\nabla^2\theta\|_{L^q}, \quad t\in(-1,0).$$

Adding the last two inequalities (2.4) and (2.6), we are led to

$$\Psi' \le \Big(2\|\nabla v(t)\|_{L^\infty} + (-t)^{-1} + c_{cz}(-t)\|\nabla\theta(t)\|_{L^\infty}\Big)\Psi. \tag{2.7}$$

By means of the logarithmic Sobolev embedding, we find

$$
\begin{aligned}
\|\nabla v(t)\|_{L^\infty} &\le c\left\{1+\|\nabla v(t)\|_{\dot B^0_{\infty,\infty}}\log(e+\|\nabla^2 v(t)\|_{L^q})\right\}\\
&\le c\left\{1+\|\omega(t)\|_{\dot B^0_{\infty,\infty}}\log(e+\|\nabla \omega(t)\|_{L^q})\right\}\\
&\le c\left\{1+\|\omega(t)\|_{\dot B^0_{\infty,\infty}}\log(e+\Psi(t))\right\}. \qquad (2.8)
\end{aligned}
$$

Similarly,

$$
\|\nabla\theta\|_{L^\infty} \le c\left\{1+\|\nabla\theta(t)\|_{\dot B^0_{\infty,\infty}}\log(e+\Psi(t))\right\}. \qquad (2.9)
$$

Inserting (2.8) and (2.9) into (2.7), it follows

$$
\Psi' \le \left\{c\left[1+(\|\omega(t)\|_{\dot B^0_{\infty,\infty}}+(-t)\|\nabla\theta(t)\|_{\dot B^0_{\infty,\infty}})\log(e+\Psi(t))\right]+(-t)^{-1}\right\}\Psi(t). \qquad (2.10)
$$

Setting $y(t)=\log(e+\Psi(t))$, we infer from (2.10) the differential inequality

$$
y' \le ca(t)y + c(-t)^{-1}, \qquad a(t)=\|\omega(t)\|_{\dot B^0_{\infty,\infty}}+(-t)\|\nabla\theta(t)\|_{\dot B^0_{\infty,\infty}} \qquad (2.11)
$$

which can be solved as

$$
\begin{aligned}
y(t) &= \log(e+\Psi(t))\\
&\le y(t_0)e^{c\int_{t_0}^t a(s)ds} + c\int_{t_0}^{t}(-s)^{-1}e^{c\int_s^t a(\tau)d\tau}ds \qquad (2.12)
\end{aligned}
$$

We now choose t_0 so that $e^{c\int_{t_0}^0 a(s)ds}<2$. Then, (2.12) implies

$$
\log(e+\Psi(t)) \le c\log(e+\Psi(t_0)) + c\log(-1/t) \qquad \forall t\in(t_0,0), \qquad (2.13)
$$

where $c>2$ is another constant. From θ-equation we have immediately

$$
\frac{\partial}{\partial t}|\nabla\theta| + (v\cdot\nabla)|\nabla\theta| \le |\nabla v||\nabla\theta|. \qquad (2.14)
$$

Let $t\in(-1,0)$ be arbitrarily chosen but fixed. Let $x_0\in\mathbb{R}^2$. By $X(x_0,t)$ we denote the trajectory of the particle which is located at x_0 at time $t=t_0$, defined by the following ODE

$$
\frac{dX(x_0,t)}{dt} = v(X(x_0,t),t) \quad \text{in} \quad [-1,0), \quad X(x_0,t_0)=x_0. \qquad (2.15)
$$

The Lipschitz continuity of $v(s)$ in $\mathbb{R}^2$ for all $s\in(-1,0)$ ensures the existence and uniqueness a solution to (2.15) in $[-1,0)$. Then, (2.14) can be written as

$$
\frac{\partial}{\partial t}|\nabla\theta(X(x_0,t),t)| \le |\nabla v(X(x_0,t),t)||\nabla\theta(X(x_0,t),t)|, \qquad (2.16)
$$

which can be integrated along the trajectories as

$$|\nabla\theta(X(x_0,t),t)| \le |\nabla\theta(x_0)| \exp\left(\int_{t_0}^t |\nabla v(X(x_0,s),s)|ds\right).$$

Therefore, we estimate, using (2.13) as

$$\begin{aligned}
\|\nabla\theta(t)\|_{L^\infty} &\le \|\nabla\theta(t_0)\|_{L^\infty} \exp\left(\int_{t_0}^t \|\nabla v\|_{L^\infty} ds\right)\\
&\le \|\nabla\theta(t_0)\|_{L^\infty} \exp\left(c\int_{t_0}^t \left\{\|\omega(s)\|_{\dot B^0_{\infty,\infty}} [\log(e+\Psi(t_0)) + \log(-1/s)] + 1\right\} ds\right)\\
&\le \|\nabla\theta(t_0)\|_{L^\infty} \exp\left(c\left\{\log(e+\Psi(t_0)) + \log(-1/t)\right\}\int_{t_0}^t \|\omega(s)\|_{\dot B^0_{\infty,\infty}} ds + c(t-t_0)\right) \qquad (2.17)
\end{aligned}$$

Choosing $t_0 \in (-1,0)$ so that

$$c\int_{t_0}^0 \|\omega(s)\|_{\dot B^0_{\infty,\infty}} ds < \frac{1}{2},$$

we deduce from (2.17) that

$$\|\nabla\theta(t)\|_{L^\infty} \le \|\nabla\theta(t_0)\|_{L^\infty} (e+\Psi(t_0))^c e^c (-t)^{-\frac{1}{2}} \quad \forall t \in (t_0, 0).$$

Therefore, $\int_{-1}^0 \|\nabla\theta\|_{L^\infty} dt < +\infty$. Applying the well-known blow-up criterion in [3], we obtain the desired result.

Proof for (iv) : Under the hypothesis of (iv) (2.7) is replaced by

$$\Psi' \le \left(2\|\nabla v(t)\|_{L^\infty} + c(-t)^{-1}\right)\Psi, \qquad (2.18)$$

and the remaing part of the proof is the same as in (iii).

Proof for (ii): Applying curl to the velocity equation in (1.1), we obtain

$$\partial_t\omega + v\cdot\nabla\omega = -\partial_2\theta \quad \text{in} \quad \mathbb{R}^2 \times [-1,0), \qquad (2.19)$$

where $\omega = \partial_1 v_2 - \partial_2 v_1$.

Using the particle trajectories(with $X(x_0,-1) = x_0$) as the above, we have from (2.19)

$$\frac{d}{dt}|\omega(X(x_0,t),t)| \le |\partial_2\theta(X(x_0,t),t)| \quad \text{in} \quad [-1,0), \qquad (2.20)$$

which implies that

$$\|\omega(s)\|_{L^\infty} \le \|\omega(-1)\|_{L^\infty} + \int_{-1}^s \|\partial_2\theta(\tau)\|_{L^\infty} d\tau. \qquad (2.21)$$

Integrating both sides of (2.21) over $[-1,t)$, $t\in(-1,0)$ with respect to s, and applying integration by parts, we get

$$\begin{aligned}\int_{-1}^t \|\omega(s)\|_{L^\infty}ds &\le (1+t)\|\omega(-1)\|_{L^\infty}+\int_{-1}^t\int_{-1}^s \|\partial_2\theta(\tau)\|_{L^\infty}d\tau ds\\ &=(1+t)\|\omega(-1)\|_{L^\infty}+\int_{-1}^t\left\{\frac{d}{ds}(s)\int_{-1}^s\|\partial_2\theta(\tau)\|_{L^\infty}d\tau\right\}ds\\ &=(1+t)\|\omega(-1)\|_{L^\infty}+\int_{-1}^t(-s)\|\partial_2\theta(s)\|_{L^\infty}ds+t\int_{-1}^t\|\partial_2\theta(s)\|_{L^\infty}ds\\ &\le\|\omega(-1)\|_{L^\infty}+\int_{-1}^t(-s)\|\partial_2\theta(s)\|_{L^\infty}ds.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{-1}^{t}\|\omega(s)\|_{L^\infty}ds+\int_{-1}^{t}(-s)\|\nabla\theta(s)\|_{L^\infty}ds\\ &\le\|\omega(-1)\|_{L^\infty}+2\int_{-1}^{0}(-s)\|\nabla\theta(s)\|_{L^\infty}ds<+\infty. \qquad (2.22)\end{aligned}$$

Therefore, from the embedding (2.1) the condition (iii) is satisfied.

Proof for (i) : By hypothesis (i) there exists $t_0\in(-1,0)$ and $\delta>0$ such that

$$\sup_{t_0<t<0}(-t)\|\nabla v(t)\|_{L^\infty}\le 2-\delta.$$

Multiplying (2.14) by $-\tau$, we have

$$\frac{\partial}{\partial\tau}((-\tau)|\nabla\theta|)+|\nabla\theta|+(v\cdot\nabla)(-\tau|\nabla\theta|)\le(-\tau)|\nabla v||\nabla\theta|\le(2-\delta)|\nabla\theta|,$$

which after integration over (t_0,s) along the trajectory gives

$$(-s)|\nabla\theta(X(x_0,s),s)|\le(-t_0)|\nabla\theta(x_0,t_0)|+(1-\delta)\int_{t_0}^s\|\nabla\theta(\tau)\|_{L^\infty}d\tau.$$

Let $t\in(t_0,0)$. Then, for all $s\in(t_0,t)$ we have

$$(-s)\|\nabla\theta(s)\|_{L^\infty}\le(-t_0)\|\nabla\theta(t_0)\|_{L^\infty}+(1-\delta)\int_{t_0}^s\|\nabla\theta(\tau)\|_{L^\infty}d\tau.$$

Integrating the both sides of the above over (t_0,t), and integrating by part, we get

$$\begin{aligned}&\int_{t_0}^t(-s)\|\nabla\theta(s)\|_{L^\infty}ds\le(-t_0)(t-t_0)\|\nabla\theta(t_0)\|_{L^\infty}+(1-\delta)\int_{t_0}^s\|\nabla\theta(\tau)\|_{L^\infty}d\tau\\ &\le(-t_0)^2\|\nabla\theta(t_0)\|_{L^\infty}+(1-\delta)\left\{(s\|\nabla\theta(s)\|_{L^\infty}-t_0\|\nabla\theta(t_0)\|_{L^\infty})-\int_{t_0}^s\tau\|\nabla\theta(\tau)\|_{L^\infty}d\tau\right\}\\ &\le(-t_0)^2\|\nabla\theta(t_0)\|_{L^\infty}+(1-\delta)(-t_0)\|\nabla\theta(t_0)\|_{L^\infty}+(1-\delta)\int_{t_0}^t(-\tau)\|\nabla\theta(\tau)\|_{L^\infty}d\tau.\end{aligned}$$

which implies

$$\delta\int_{t_0}^{t}(-s)\|\nabla\theta(s)\|_{L^\infty}ds \le (-t_0)^2\|\nabla\theta(t_0)\|_{L^\infty} + (1-\delta)(-t_0)\|\nabla\theta(t_0)\|_{L^\infty},$$

Passing $t\to 0^-$, we obtain finally

$$\delta\int_{t_0}^{0}(-s)\|\nabla\theta(s)\|_{L^\infty}ds \le (-t_0)^2\|\nabla\theta(t_0)\|_{L^\infty} + (1-\delta)(-t_0)\|\nabla\theta(t_0)\|_{L^\infty} < +\infty$$

and the condition (ii) is satisfied. ■

Proof of Theorem 1.4 : From (1.5), we find $t_0 \in (-e^{-2}, 0)$ such that

$$\|\nabla\theta(s)\|_{L^\infty} \le \frac{\varepsilon}{(-s)^2\log(-1/s)} \quad \forall\, s\in[t_0,0). \tag{2.23}$$

The inequality (2.20), following the argument of the proof for (ii), and combined with (2.23) yields

$$\begin{aligned}
\|\omega(t)\|_{L^\infty} &\le \|\omega(t_0)\|_{L^\infty} + \int_{t_0}^{t}\|\nabla\theta(s)\|_{L^\infty}ds\\
&\le \|\omega(t_0)\|_{L^\infty} + \varepsilon\int_{t_0}^{t}\frac{1}{(-s)^2\log(-1/s)}ds\\
&\le \|\omega(t_0)\|_{L^\infty} + 2\varepsilon\int_{t_0}^{t}\frac{\log(-1/s)-1}{((-s)\log(-1/s))^2}ds\\
&= \|\omega(t_0)\|_{L^\infty} + \frac{2\varepsilon}{(-t)\log(-1/t)} - \frac{2\varepsilon}{(-t_0)\log(-1/t_0)}\\
&\le \|\omega(t_0)\|_{L^\infty} + \frac{2\varepsilon}{(-t)\log(-1/t)}, \qquad (2.24)
\end{aligned}$$

where we used the fact that $\log(-1/s) \le 2\log(-1/s) - 2$ for all $s\in(-e^{-2},0)$ in the third inequality. We now define $\varepsilon > 0$ as follows

$$\varepsilon := \frac{1}{4\max\{c_{ls}, c_{cz}\}}. \tag{2.25}$$

Then from (2.7) combined with (2.8) together with (2.23) and (2.24) we find

$$\begin{aligned}
y' &\le c_{ls}\|\omega\|_\infty y + (-t)^{-1} + c_{cz}(-t)\|\nabla\theta(t)\|_{L^\infty}\\
&\le \Big(\frac{2\varepsilon c_{ls}}{(-t)\log(-1/t)} + c_{ls}\|\omega(t_0)\|_{L^\infty}\Big)y + (-t)^{-1} + \varepsilon c_{cz}(-t)^{-1}\log(-1/t)\\
&\le \Big(\frac{1}{2(-t)\log(-1/t)} + c_{ls}\|\omega(t_0)\|_{L^\infty}\Big)y + \frac{5}{4}(-t)^{-1} \quad \text{in} \quad (t_0,0), \qquad (2.26)
\end{aligned}$$

where

$$y(t) = \log(e+\Psi(t)), \quad \Psi(t) := \|\nabla\omega(t)\|_{L^{p_0}} + (-t)\|\nabla^2\theta(t)\|_{L^{p_0}}, \quad t \in (-1,0).$$

Integrating (2.26), we obtain

$$y(t) \le y(t_0)e^{\int_{t_0}^{t} a(s)ds} + \frac{5}{4}\int_{t_0}^{t} (-s)^{-1} e^{\int_{s}^{t} a(\tau)d\tau} ds, \tag{2.27}$$

where we set

$$a(t) = \frac{1}{2(-t)\log(-1/t)} + c_{ls}\|\omega(t_0)\|_{L^\infty}.$$

Applying integration by parts, we infer

$$\begin{aligned}
&\int_{t_0}^{t} (-s)^{-1} e^{\int_{s}^{t} a(\tau)d\tau} ds \\
&\quad = \int_{t_0}^{t} \frac{d}{ds}\log(-1/s) e^{\int_{s}^{t} a(\tau)d\tau} ds \\
&\quad = \log(-1/t) - \log(-1/t_0)e^{\int_{t_0}^{t} a(\tau)d\tau} + \int_{t_0}^{t} \log(-1/s)a(s)e^{\int_{s}^{t} a(\tau)d\tau} ds \\
&\quad \le \log(-1/t) + \frac{1}{2}\int_{t_0}^{t} (-s)^{-1} e^{\int_{s}^{t} a(\tau)d\tau} ds + c_{ls}\|\omega(t_0)\|_{L^\infty}\int_{t_0}^{t} \log(-1/s) e^{\int_{s}^{t} a(\tau)d\tau} ds.
\end{aligned}$$

Absorbing the second term on the right hand side into the left, one has

$$\begin{aligned}
&\int_{t_0}^{t} (-s)^{-1} e^{\int_{s}^{t} a(\tau)d\tau} ds \\
&\quad \le 2\log(-1/t) + 2c_{ls}\|\omega(t_0)\|_{L^\infty}\int_{t_0}^{t} \log(-1/s) e^{\int_{s}^{t} a(\tau)d\tau} ds.
\end{aligned}$$

Calculating

$$e^{\int_{s}^{t} a(\tau)d\tau} = e^{c_{ls}\|\omega(t_0)\|_{L^\infty}} e^{\frac{1}{2}\int_{s}^{t} \frac{1}{(-\tau)\log(-1/\tau)}d\tau} \le e^{c_{ls}\|\omega(t_0)\|_{L^\infty}} \{\log(-1/t)\}^{\frac{1}{2}} \tag{2.28}$$

for all $s \in [t_0, 0)$, we obtain from the above inequality

$$
\begin{aligned}
&\int_{t_0}^{t} (-s)^{-1} e^{\int_s^t a(\tau)d\tau} ds \\
&\quad \le 2\log(-1/t) + 2c_{ls}\|\omega(t_0)\|_{L^\infty} e^{c_{ls}\|\omega(t_0)\|_{L^\infty}} \{\log(-1/t)\}^{\frac{1}{2}} \int_{-1}^{0} \log(-1/s) ds \\
&\quad \le 2\log(-1/t) + c\{\log(-1/t)\}^{\frac{1}{2}},
\end{aligned}
$$

where $c =$ const is independent on t. Estimating the second term in (2.27) by the estimate we have just obtained and the first term by (2.28) for $s = t_0$, we arrive at

$$
y(t) \le \frac{5}{2}\log(-1/t) + c\{\log(-1/t)\}^{\frac{1}{2}} \quad \forall\, t \in [t_0, 0),
$$

for some constant independent of t. Accordingly, there exists $t_1 \in (t_0, 0)$ such that

$$
y(t) \le 3\log(-1/t) \quad \forall\, t \in [t_1, 0), \tag{2.29}
$$

By the aid of the logarithmic Sobolev embedding inequality, and observing (2.24) together with (2.29) and (2.25), we see that for all $t \in [t_1, 0)$

$$
\begin{aligned}
\|\nabla v(t)\|_{L^\infty} &\le c_{ls}\|\omega(t)\|_{L^\infty} y(t) + c_{ls} \le 3c_{ls}\|\omega(t_0)\|_{L^\infty}\log(-1/t) + 6\varepsilon c_{ls}(-t)^{-1} + c_{ls} \\
&\le 3c_{ls}\|\omega(t_0)\|_{L^\infty}\log(-1/t) + \frac{3}{2}(-t)^{-1} + c_{ls}.
\end{aligned} \tag{2.30}
$$

Thus,

$$
\limsup_{t\to 0^-} (-t)\|\nabla v(t)\|_{L^\infty} \le \frac{3}{2} < 2. \tag{2.31}
$$

Applying Theorem 1.2 (i), we get the assertion of the theorem. ■

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