Steady-state bifurcation of a non-parallel flow involving energy dissipation over a Hartmann boundary layer

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Abstract

A plane non-parallel vortex flow in a square fluid domain is examined. The energy dissipation of the flow is dominated by viscosity and linear friction effect of a Hartmann layer. This is a traditional Navier-Stokes flow when the linear friction effect is not involved, whereas it is a magnetohydrodynamic flow when the energy dissipation is fundamentally dominated by the friction. It is proved that linear critical values of a spectral problem are nonlinear thresholds leading to the onset of secondary steady-state flows, the nonlinear phenomenon observed in laboratory experiments.

Keywords: Non-parallel flows, Navier-Stokes equations, bifurcation, vortex flows, linear friction effect of Hartmann layer

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1. Introduction

To study the inverse energy cascade towards large scales [9] of plane flows, Sommeria and Verron [13, 14, 16] presented magnetohydrodynamic experiments by using electronically driven flows in a closed box, containing a thin horizontal layer of liquid metal. The box is bottomed with electromagnets producing a uniform vertical magnetic field. The flow velocity is small so that the upper free surface is negligible. The three-dimensional motion reduces a two-dimensional one as the vertical movement in the thin horizontal layer fluid can be ignored. The energy dissipation of the fluid motion counts for viscosity and the Hartmann layer friction applied on the bottom of liquid metal.

The non-dimensional governing equations of the two-dimensional approximation motion for the velocity v and pressure p in the domain $(0,1) \times (0,1)$ are [13, 14, 16]

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p - \frac{1}{Re} \Delta \mathbf{v} + \frac{\mathbf{v}}{Rh} = \mathbf{f}, \ \nabla \cdot \mathbf{v} = 0. \tag{1}$$

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Here Re is the Reynolds number, Rh is the Rayleigh number measuring the Hartmann bottom friction and f is the Lorentz driving force defined by electric currents so that

$$\nabla \times \boldsymbol{f} = \frac{\pi^2}{2} \sin(2\pi x) \sin(2\pi y)$$
 and $\int_0^1 \int_0^1 |\nabla \times \boldsymbol{f}| dx dy = 2$.

The stream function ψ and the vorticity of the fluid motion are defined as

$$\left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right) = v, \ \omega = \nabla \times v = -\Delta \psi.$$

The vorticity formulation of (1) is

$$-\frac{\partial \Delta \psi}{\partial t} + J(\psi, \Delta \psi) - \frac{\Delta \psi}{Rh} + \frac{\Delta^2 \psi}{Re} = \frac{\pi^2}{2} \sin(2\pi x) \sin(2\pi y), \tag{2}$$

where the nonlinear convective term is written as the Jacobian

$$J(\psi, \Delta \psi) = \partial_x \psi \, \partial_y \Delta \psi - \partial_y \psi \partial_x \Delta \psi.$$

The basic flow of (2) is dependant on the parameters Re and Rh. It is convenient to use the modified system [15] of (2) expressed through

$$-\frac{\partial \Delta \psi}{\partial t} + J(\psi, \Delta \psi) + (-\mu \Delta + \nu \Delta^2)(\psi - \sin x \sin y) = 0.$$
 (3)

In reference to [13, 14, 16, 15], the stream function is assumed to satisfy the free slip boundary condition

$$\psi|_{\partial\Omega} = \Delta\psi|_{\partial\Omega} = 0 \tag{4}$$

for the modified fluid domain $\Omega=(0,2\pi)\times(0,2\pi)$, and is demonstrated in the Fourier expansion

$$\psi = \sum_{n,m\geq 1} a_{n,m} \sin \frac{nx}{2} \sin \frac{my}{2}.$$
 (5)

The parameters ν and μ are defined by the transformations [4]

$$Rh = \frac{2\sqrt{\mu + 2\nu}}{\mu\pi} \text{ and } Re = \frac{8\pi\sqrt{\mu + 2\nu}}{\nu}.$$
 (6)

With this modification, we have the basic steady-state flow

$$\psi_0 = \sin x \sin y$$
.

The basic flow exhibits four vortices in Figure 1. The experiments [13, 14, 16] show transitions of the basic flow in a scenario of inverse energy cascade towards to large scales. The principal transition amongst them is the steady-state bifurcation of ψ_0 into a secondary flow, which is sketched in Figure 1 (b).

To the understanding of the transition, Thess [15] demonstrated critical stability parameters (v_c, μ_c) of a spectral problem linearized from (3)-(4) so that linear stable and unstable domains are defined. The author [4] provided nonlinear stability analysis of a vortex flow and employed a numerical spectral scheme to study all possible linear spectral solutions together with secondary flows as a result of nonlinear saturation of primary linear instability. The vortex instability with respect to two vortex merging phenomena was also discussed by Meunier *et al.* [10] and Cerretelli and Williamson[2]. The experimental studies [13, 14] of the non-parallel flow $\sin x \sin y$ are developed from the magnetohydrodynamic experiment of Bondarenko *et al.* [1] on the steady-state bifurcation of the parallel Kolmogorov flow $\sin x$. The existence of secondary steady-state flows and secondary temporal periodic flows bifurcating from the Kolmogorov flow has been studied extensively [5, 6, 8].

However, the basic flow ψ_0 is non-parallel and rigorous instability analysis for the secondary flow existence is missing. In the study of the linear spectral problem, Thess [15] emphasized that linear stability theory is not able to predict the structure of flows above the instability threshold, but it is a matter of bifurcation theory to decide whether stationary secondary solutions exist at all. He also suggested the linear spectral study to be continued in the following two directions: (i) the formation of a secondary flow as a result of nonlinear saturation of the primary instability and (ii) linear stability analysis of the secondary flow (see Orszag and Patera [11] on this stability problem for a parallel flow). In the present study, we solve problem (i) by showing the formation of a secondary steady-state flow resulting from nonlinear saturation of the linear instability. For problem (ii), no existing rigorous analysis is available. The linear stability analysis of secondary flow [11] on the parallel Poiseuille flow problem is not applicable to (ii). To sketch stability of the secondary steady-state flow, we use numerical computation via a finite difference scheme. Selected numerical results show that the secondary flow, close to its threshold and observed in the laboratory experiments [13, 14, 16], can be obtained by taking an initial state in its vicinity. The stability of the secondary flow and the positive viscosity v > 0 ensure the convergence of the numerical scheme so that the secondary flow attracts the non-stationary flow starting from the initial state.

It is the principal purpose of present paper to show the existence of the secondary flows, which are to be contained in the Hilbert space

$$H^{4} = \left\{ \psi = \sum_{n,m \ge 1} a_{n,m} \sin \frac{nx}{2} \sin \frac{my}{2} \right|$$

$$\|\psi\|_{H^{4}} = \left(\sum_{n,m \ge 1} \left(1 + \left(\frac{n^{2} + m^{2}}{4} \right)^{2} \right)^{2} |a_{n,m}|^{2} \right)^{\frac{1}{2}} < \infty \right\}.$$

The secondary flows will be constructed by nonlinear perturbation of the basic flow ψ_0 by using the eigenfunctions of the spectral problem

$$\lambda \Delta \psi = -\mu \Delta \psi + v \Delta^2 \psi + J(\psi_0, (2+\Delta)\psi), \ 0 \neq \psi = \sum_{n,m \geq 1} a_{n,m} \sin \frac{nx}{2} \sin \frac{my}{2}, \tag{7}$$

which is linearized from (3). The eigenfunctions will be studied in the following three

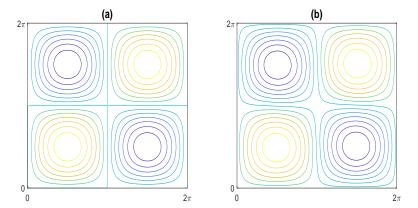


Figure 1: (a) The basic steady-state flow ψ_0 ; (b) the contour lines of the function $\psi_0 - 0.1 \sin \frac{x}{2} \sin \frac{y}{2}$, representing a profile of the secondary flow in the magnetohydrodynamic experiment [14].

linear orthogonal subspaces of H^4 :

$$E_1 = \left\{ \psi \in H^4 | \psi = \sum_{n,m \ge 1; n,m \text{ odd}} a_{n,m} \sin \frac{nx}{2} \sin \frac{my}{2} \right\}, \tag{8}$$

$$E_2 = \left\{ \psi \in H^4 | \psi = \sum_{n,m \ge 1; n \text{ odd}; m \text{ even}} a_{n,m} \sin \frac{nx}{2} \sin \frac{my}{2} \right\}, \tag{9}$$

$$E_3 = \left\{ \psi \in H^4 | \psi = \sum_{n,m \ge 1; n \text{ even}; m \text{ odd}} a_{n,m} \sin \frac{nx}{2} \sin \frac{my}{2} \right\}. \tag{10}$$

The algebraic form of spectral equation (7) displayed in next section shows that the mode $\sin \frac{nx}{2} \sin \frac{my}{2}$ is influenced by the four modes $\sin \frac{(n\pm 2)x}{2} \sin \frac{(m\pm 2)y}{2}$ in x and y directions. The introduction of these orthogonal subspaces implies the validity of the invariance property of (7) in the following sense:

$$\sin\frac{(n\pm 2)x}{2}\sin\frac{(m\pm 2)y}{2} \in E_i \text{ whenever } \sin\frac{nx}{2}\sin\frac{my}{2} \in E_i \ i=1,2,3.$$

The main result of the present paper reads as:

Theorem 1.1. (i). Let v > 0, $\mu \ge 0$ and $\lambda + \mu + \frac{1}{2}v > 0$. Then spectral problem (7) has at most three linear independent eigenfunctions. These eigenfunctions are contained in the set $E_1 \cup E_2 \cup E_3$.

(ii) Assume that spectral problem (7) admits a critical solution $(\lambda, \psi, \nu, \mu) = (0, \psi_c, \nu_c, \mu_c)$ for $\nu_c > 0$, $\mu_c \ge 0$ and $\psi_c \in E_i$ for an integer $1 \le i \le 3$. Then there exist a function $\psi_i \in H^4$ and a real δ so that system (3)-(4) has a steady-state solution (ψ, ν, μ) branching off the bifurcation point (ψ_0, ν_c, μ_c) in the direction of ψ_c :

$$\psi = \psi_0 + \varepsilon \psi_c + \varepsilon^2 \psi_i, \quad v = v_c + \varepsilon \delta v_c, \quad \mu = \mu_c + \varepsilon \delta \mu_c, \tag{11}$$

provided that $\varepsilon > 0$ is sufficiently small. Here σ and ψ_i are uniformity bounded functions of ε for small ε , and ψ_i is in the orthogonal complement of the eigenfunction space span{ ψ_c } or $\psi_i \in H^4$ /span{ ψ_c }.

Remark 1.1. This theorem shows the secondary flow bifurcating in the direction of ψ_c . If ψ_c is replaced by the eigenfunction $-\psi_c$, we have another secondary flow bifurcating in the direction of $-\psi_c$.

This paper is structured as follows. The spectral analysis for the theoretical base of Theorem 1.1 is established in Section 2, which contains the proof of Theorem 1.1 (i). The second assertion of this theorem is proven in Section 3 by developing a bifurcation technique of Rabinowitz [12] on a Bénard problem. Theorem 1.1 requires the existence of a linear critical spectral solution (v_c, μ_c, ψ_c) . Section 4 contains a discussion of the existence and connection of Theorem 1.1 with Crandall- Rabinowitz bifurcation theorem. To enrich the theoretical result, we display numerical spectral solutions and use a finite difference scheme to locate a secondary flow in accordance with the experimental observation of [14, 16] in Section 5. This numerical study aids the stability detection of the secondary flow. Moreover, in addition to the initial stage of inverse energy cascade profiled in Figure 1, a larger scale topological transition via three vortices into two are presented in Section 5.

2. Linear spectral analysis

We begin with the spectral assertion of Theorem 1.1.

2.1. Proof of Theorem 1.1 (i)

Proof. Let (\cdot,\cdot) denote the inner product of real L_2 as

$$(\varphi, \phi) = \frac{1}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} \varphi \phi dx dy.$$

Taking the L_2 inner product of the spectral equation (7) with $-(\Delta+2)\psi$ and employing integration by parts, we have

$$0 = (-\lambda \Delta - \mu \Delta \psi + \nu \Delta^2 \psi + J(\psi_0, (\Delta + 2)\psi), (-\Delta - 2)\psi)$$

= $(-\lambda \Delta - \mu \Delta \psi + \nu \Delta^2 \psi, (-\Delta - 2)\psi).$ (12)

This together with (7) becomes

$$0 = \sum_{n,m\geq 1} \beta_{n,m} (\lambda + \mu + \nu \beta_{n,m}) (\beta_{n,m} - 2) |a_{n,m}|^2 \text{ for } \beta_{n,m} = \frac{1}{4} (n^2 + m^2),$$
 (13)

or

$$\sum_{n,m\geq 1;\beta_{n,m}>2} \beta_{n,m}(\lambda + \mu + \nu \beta_{n,m})(\beta_{n,m} - 2)|a_{n,m}|^{2}$$

$$= \beta_{1,1}(\lambda + \mu + \nu \beta_{1,1})(2 - \beta_{1,1})|a_{1,1}|^{2} + \beta_{1,2}(\lambda + \mu + \nu \beta_{1,2})(2 - \beta_{1,2})|a_{1,2}|^{2} + \beta_{2,1}(\lambda + \mu + \nu \beta_{2,1})(2 - \beta_{2,1})|a_{2,1}|^{2}.$$
(14)

By the observation $2 > \beta_{1,2} = \beta_{2,1} > \beta_{1,1} = \frac{1}{2}$ and the condition $\lambda + \mu + \frac{1}{2}\nu > 0$, we see that

$$\lambda + \mu + \nu \beta_{n,m} \ge \lambda + \mu + \frac{1}{2}\nu > 0 \text{ for } n, m \ge 1$$

and the both sides of (14) are non-negative. Hence the right-hand side of (14) is positive due to the non-zero eigenfunction property $\psi \neq 0$. Therefore we may firstly assume the term involving $a_{1,1}$ on the right-hand side of (14) being positive or $a_{1,1} \neq 0$,

On the other hand, let $a_{n,m} = 0$ whenever $n \le 0$ or $m \le 0$. Spectral problem (7) is formulated as [4]

$$\sum_{n,m\geq 1} \beta_{n,m}(\lambda + \mu + \nu \beta_{n,m}) a_{n,m} \sin \frac{nx}{2} \sin \frac{my}{2}$$

$$= -\sum_{n,m\geq -2}^{\infty} \left\{ \frac{n-m}{8} [(\beta_{n-2,m-2} - 2) a_{n-2,m-2} - (\beta_{n+2,m+2} - 2) a_{n+2,m+2}] + \frac{n+m}{8} [(\beta_{n-2,m+2} - 2) a_{n-2,m+2} - (\beta_{n+2,m-2} - 2) a_{n+2,m-2}] \right\} \sin \frac{nx}{2} \sin \frac{my}{2}.$$
(15)

This implies that the non-zero coefficient $a_{1,1}$ produces the coefficients $a_{n,m}$ for odd integers $n, m \ge 1$. That is, the eigenfunction $\psi \in E_1$ is generated by the mode $\sin \frac{x}{2} \sin \frac{y}{2}$. Moreover, the derivation of (14) implies that

$$\beta_{1,1}(\lambda + \mu + \nu \beta_{1,1})(2 - \beta_{1,1})|a_{1,1}|^{2} = \sum_{n,m \geq 1: n,m \text{ odd: } \beta_{n,m} \geq 2} \beta_{n,m}(\lambda + \mu + \nu \beta_{n,m})(\beta_{n,m} - 2)|a_{n,m}|^{2}.$$
(16)

Similarly, we may suppose $\psi \in E_2$ when $a_{1,2} \neq 0$ and $\psi \in E_3$ when $a_{2,1} \neq 0$. Additionally, the corresponding coefficients are subject to the equations

$$\beta_{1,2}(\lambda + \mu + \nu \beta_{1,2})(2 - \beta_{1,2})|a_{1,2}|^{2} = \sum_{n,m \ge 1; n \text{ odd}; m \text{ even}; \beta_{n,m} > 2} \beta_{n,m}(\lambda + \mu + \nu \beta_{n,m})(\beta_{n,m} - 2)|a_{n,m}|^{2}$$
(17)

for $a_{1,2} \neq 0$, and

$$\beta_{2,1}(\lambda + \mu + \nu \beta_{2,1})(2 - \beta_{2,1})|a_{2,1}|^{2} = \sum_{n,m \ge 1; n \text{ even}; m \text{ odd}; \beta_{n,m} > 2} \beta_{n,m}(\lambda + \mu + \nu \beta_{n,m})(\beta_{n,m} - 2)|a_{n,m}|^{2}$$
(18)

for $a_{2,1} \neq 0$. The proof of Assertion (i) is complete.

2.2. Spectral simplicity property

To construct the secondary flows, we have to use the eigenfunction simplicity property shown in the following result.

Theorem 2.1. Let v > 0, $\mu \ge 0$ and $\lambda + \mu + \frac{1}{2}v > 0$. Then we have eigenfunction space dimension estimate:

$$\dim\left\{\psi\in E_i|\ \lambda\Delta\psi=-\mu\Delta\psi+\nu\Delta^2\psi+J(\psi_0,(\Delta+2)\psi)\right\}\leq 1,\ i=1,2,3.$$
 (19)

Moreover, if $(\lambda, \psi, \nu, \mu)$ *is a spectral solution of* (7) *for* $\psi \in E_1 \cup E_2 \cup E_3$, *then we have*

$$((-\lambda \Delta - \mu \Delta + \nu \Delta^2)\psi, \psi^*) \neq 0.$$
(20)

Here ψ^* is the conjugate eigenfunction of ψ subject to the conjugate equation of (7):

$$\lambda \Delta \psi^* = -\mu \Delta \psi^* + v \Delta^2 \psi^* + (-\Delta - 2) J(\psi_0, \psi^*),$$

$$\psi^* = \sum_{n,m>1}^{\infty} a_{n,m}^* \sin \frac{nx}{2} \sin \frac{my}{2} \neq 0,$$
(21)

produced by employing the L_2 pairing (\cdot,\cdot) .

Proof. To show the validity of (19), we suppose that there is a spectral solution $(\lambda, \psi, \nu, \mu)$ with the eigenfunction $\psi \in E_1$. To the contrary, if (19) with i=1 is not true, there exists an additional spectral solution $(\lambda, \hat{\psi}, \nu, \mu)$ with the eigenfunction $\hat{\psi} \in E_1$ linearly independent of ψ and involving expansion coefficients $\hat{a}_{n,m}$. It follows from (16) that $\hat{a}_{1,1} \neq 0$. Therefore, we have the additional spectral solution $(\lambda, \psi - \frac{a_{1,1}}{\hat{a}_{1,1}}\hat{\psi}, \nu, \mu)$. Using the eigenfunction $\psi - \frac{a_{1,1}}{\hat{a}_{1,1}}\hat{\psi}$ instead of ψ in (16), we have

$$0 = \sum_{n,m \text{ odd: } \beta_{n,m} > 2} \beta_{n,m} (\lambda + \mu + \nu \beta_{n,m}) (\beta_{n,m} - 2) |a_{n,m} - \frac{a_{1,1}}{\hat{a}_{1,1}} \hat{a}_{n,m}|^2,$$
 (22)

which together with the condition $\lambda + \mu + v^{\frac{1}{2}} > 0$ gives

$$a_{n,m} = \frac{a_{1,1}}{\hat{a}_{1,1}} \hat{a}_{n,m}$$
 or $\psi = \frac{a_{1,1}}{\hat{a}_{1,1}} \hat{\psi}_{1,1}$.

Hence $\hat{\psi}$ and ψ are linearly dependent. This leads to a contraction and thus (19) holds true for i = 1.

Arguing in the same way, we obtain (19) for i = 2 and 3.

To verify (20), we first assume the eigenfunction $\psi \in E_1$. Consider the conjugate spectral problem (21), which can be formulated in the algebraic equation

$$0 = \sum_{n,m \ge -2; n,m \text{ odd}} \left\{ \beta_{n,m} (\lambda + \mu + \nu \beta_{n,m}) a_{n,m}^* - (\beta_{n,m} - 2) \left\{ \frac{n-m}{8} (a_{n-2,m-2}^* - a_{n+2,m+2}^*) + \frac{n+m}{8} (a_{n-2,m+2}^* - a_{n+2,m-2}^*) \right\} \right\} \sin \frac{nx}{2} \sin \frac{my}{2}.$$

Here $a_{n,m}^* = 0$ whenever $n \le 0$ or $m \le 0$. The previous equation is rewritten as

$$0 = \sum_{n,m \ge -2; n,m \text{ odd}} \left\{ [(\lambda + \mu)\beta_{n,m} + \nu \beta_{n,m}^2] \frac{a_{n,m}^*}{\beta_{n,m} - 2} - \frac{n-m}{8} (a_{n-2,m-2}^* - a_{n+2,m+2}^*) + \frac{n+m}{8} (a_{n-2,m+2}^* - a_{n+2,m-2}^*) \right\} \sin \frac{nx}{2} \sin \frac{my}{2}.$$
(23)

Moreover, for $a'_{m,n} = \frac{a^*_{n,m}}{\beta_{n,m-2}}$, equation (23) becomes

$$0 = \sum_{n,m \ge -2; n,m \text{ odd}} \left\{ [(\lambda + \mu)\beta_{m,n} + \nu \beta_{m,n}^2] a'_{m,n} + \frac{n-m}{8} [(\beta_{m-2,n-2} - 2)a'_{m-2,n-2} - (\beta_{m+2,n+2} - 2)a'_{m+2,n+2}] + \frac{n+m}{8} [(\beta_{m+2,n-2} - 2)a'_{m+2,n-2} - (\beta_{m-2,n+2} - 2)a'_{m-2,n+2}] \right\} \sin \frac{nx}{2} \sin \frac{my}{2}.$$

Replacing the indices n, m by \tilde{m} , \tilde{n} respectively, and then omitting the index superscript tildes, we have

$$0 = \sum_{n,m \ge -2; n,m \text{ odd}} \left\{ [(\lambda + \mu)\beta_{n,m} + \nu \beta_{n,m}^{2}] a'_{n,m} + \frac{n-m}{8} [(\beta_{n-2,m-2} - 2)a'_{n-2,m-2} - (\beta_{n+2,m+2} - 2)a'_{n+2,m+2}] + \frac{n+m}{8} [(\beta_{n-2,m+2} - 2)a'_{n-2,m+2} - (\beta_{n+2,m-2} - 2)a'_{n+2,m-2}] \right\} \sin \frac{mx}{2} \sin \frac{ny}{2} (24)$$

The algebraic equation system for the coefficients of (24) is identical to that for the coefficients of (15), when the supercritical primes are omitted. Thus by (19), we have the relationship between the expansion coefficients of the eigenfunction ψ and those of its conjugate counterpart ψ^* :

$$a_{n,m}^* = (\beta_{m,n} - 2)a_{m,n}.$$

Hence, we have

$$(\Delta(-\lambda - \mu + \nu \Delta)\psi, \psi^*) = \sum_{n,m \ge 1; n,m \text{ odd}} \beta_{n,m}(\lambda + \mu + \nu \beta_{n,m})(\beta_{n,m} - 2)a_{n,m}a_{m,n}.$$
(25)

Note that $\psi \neq 0$ implies $a_{1,1} \neq 0$ due to (16). Moreover, from the algebraic equation defined by the first component of (15) with respect to the mode $\sin \frac{x}{2} \sin \frac{y}{2}$, it follows that the coefficient $a_{1,1}$ is proportional to $a_{1,3} - a_{3,1}$. Hence $a_{3,1} \neq a_{1,3}$ due to $a_{1,1} \neq 0$. Therefore the Cauchy inequality

$$2a_{1,3}a_{3,1} < a_{1,3}^2 + a_{3,1}^2 (26)$$

holds true. It follows from (25), (26) and the Cauchy inequality $2a_{n,m}a_{m,n} \le a_{n,m}^2 + a_{m,n}^2$ that

$$\begin{array}{lll} (\Delta(-\lambda-\mu+\nu\Delta)\psi,\psi^*) & = & -\beta_{1,1}(\lambda+\mu+\nu\beta_{1,1})(2-\beta_{1,1})a_{1,1}^2 \\ & + \sum\limits_{n,m\geq 1;\,n,m\; \text{odd};\,\beta_{n,m}>2} \beta_{n,m}(\lambda+\mu+\nu\beta_{n,m})(\beta_{n,m}-2)a_{n,m}a_{m,n} \\ & < & -\beta_{1,1}(\lambda+\mu+\nu\beta_{1,1})(2-\beta_{1,1})a_{1,1}^2 \\ & + \sum\limits_{n,m\geq 1;\,n,m\; \text{odd};\,\beta_{n,m}>2} \beta_{n,m}(\lambda+\mu+\nu\beta_{n,m})(\beta_{n,m}-2)a_{n,m}^2, \end{array}$$

which equals zero due to (16). This gives the validity of (20) when $\psi \in E_1$. Moreover, if (7) has a spectral solution $(\lambda, \psi, \nu, \mu)$ with the eigenfunction

$$\psi = \sum_{n,m \ge 1; n \text{ odd}; m \text{ even}} a_{n,m} \sin \frac{nx}{2} \sin \frac{my}{2} \in E_2, \tag{27}$$

the algebraic equation (15) for the coefficients $a_{n,m}$ of ψ reduces to

$$0 = \sum_{n,m \ge -2; n \text{ odd}; m \text{ even}} \left\{ \beta_{n,m} (\lambda + \mu + \nu \beta_{n,m}) a_{n,m} + \frac{n-m}{8} [(\beta_{n-2,m-2} - 2) a_{n-2,m-2} - (\beta_{n+2,m+2} - 2) a_{n+2,m+2}] + \frac{n+m}{8} [(\beta_{n-2,m+2} - 2) a_{n-2,m+2} - (\beta_{n+2,m-2} - 2) a_{n+2,m-2}] \right\} \sin \frac{nx}{2} \sin \frac{my}{2}. (28)$$

Therefore, the conjugate spectral problem (21) has a solution $(\lambda, \psi^*, \nu, \mu)$, subject to the algebraic equation

$$0 = \sum_{n,m \ge -2; n \text{ odd}; m \text{ even}} \left\{ [(\lambda + \mu)\beta_{n,m} + \nu \beta_{n,m}^2] \frac{a_{n,m}^*}{\beta_{n,m} - 2} - \frac{n - m}{8} (a_{n-2,m-2}^* - a_{n+2,m+2}^*) + \frac{n + m}{8} (a_{n-2,m+2}^* - a_{n+2,m-2}^*) \right\} \sin \frac{nx}{2} \sin \frac{my}{2}.$$
(29)

Here the assumption $a_{n,m}^* = 0$ whenever $n \le 0$ or $m \le 0$ is used. For $a'_{m,n} = \frac{a'_{n,m}}{\beta_{n,m}-2}$, equation (29) becomes

$$0 = \sum_{n,m \ge -2; n \text{ even}; m \text{ odd}} \left\{ [(\lambda + \mu)\beta_{n,m} + \nu \beta_{n,m}^{2}] a'_{n,m} + \frac{n-m}{8} [(\beta_{n-2,m-2} - 2)a'_{n-2,m-2} - (\beta_{n+2,m+2} - 2)a'_{n+2,m+2}] + \frac{n+m}{8} [(\beta_{n-2,m+2} - 2)a'_{n-2,m+2} - (\beta_{n+2,m-2} - 2)a'_{n+2,m-2}] \right\} \sin \frac{mx}{2} \sin \frac{ny}{2}.$$
 (30)

This together with (19) implies that spectral problem (7) has a spectral solution $(\lambda, \tilde{\psi}, v, \mu)$ with

$$\tilde{\psi} = \sum_{n,m \ge 1; n \text{ even}; m \text{ odd}} a_{n,m} \sin \frac{nx}{2} \sin \frac{my}{2} \in E_3, \tag{31}$$

subject to the equation

$$0 = \sum_{n,m \ge -2; n \text{ even}; m \text{ odd}} \left\{ [(\lambda + \mu)\beta_{n,m} + \nu \beta_{n,m}^{2}] a_{n,m} + \frac{n-m}{8} [(\beta_{n-2,m-2} - 2)a_{n-2,m-2} - (\beta_{n+2,m+2} - 2)a_{n+2,m+2}] + \frac{n+m}{8} [(\beta_{n-2,m+2} - 2)a_{n-2,m+2} - (\beta_{n+2,m-2} - 2)a_{n+2,m-2}] \right\} \sin \frac{mx}{2} \sin \frac{ny}{2}.$$
 (32)

Therefore, we have

$$a_{n\,m}^* = (\beta_{n,m} - 2)a_{m,n}. (33)$$

We show that $a_{n,m} \not\equiv a_{m,n}$. Otherwise, if $a_{n,m} \equiv a_{m,n}$, equation (32) becomes

$$0 = \sum_{n,m \ge -2; n \text{ odd}; m \text{ even}} \left\{ [(\lambda + \mu)\beta_{n,m} + \nu \beta_{n,m}^{2}] a_{n,m} - \frac{n-m}{8} [(\beta_{n-2,m-2} - 2)a_{n-2,m-2} - (\beta_{n+2,m+2} - 2)a_{n+2,m+2}] - \frac{n+m}{8} [(\beta_{n-2,m+2} - 2)a_{n-2,m+2} - (\beta_{n+2,m-2} - 2)a_{n+2,m-2}] \right\} \sin \frac{mx}{2} \sin \frac{ny}{2}.$$
 (34)

Adding (34) to (28), we have

$$\sum_{n,m\geq 1; n \text{ odd}; m \text{ even}} [(\lambda+\mu)\beta_{n,m} + \nu\beta_{n,m}^2] a_{n,m} \sin\frac{mx}{2} \sin\frac{ny}{2} = 0$$

or $a_{n,m} \equiv 0$. This leads to a contradiction. Hence

$$a_{n,m} \not\equiv a_{m,n}. \tag{35}$$

By (33), we have

$$(\Delta(-\lambda - \mu + \nu \Delta)\psi, \psi^*) = \sum_{n,m \ge 1; n \text{ odd}; m \text{ even}} \beta_{n,m}(\lambda + \mu + \nu \beta_{n,m})(\beta_{n,m} - 2)a_{n,m}a_{m,n}$$
 (36)

If $a_{1,2} = a_{2,1}$, we use (12), (35) and Cauchy inequality to obtain from (36) that

$$\begin{split} &(\Delta(-\lambda-\mu+\nu\Delta)\psi,\psi^*)\\ &< \quad \beta_{1,2}(\lambda+\mu+\nu\beta_{1,2})(\beta_{1,2}-2)a_{1,2}^2\\ &+ \sum_{n,m\geq 1; n \text{ odd}; m \text{ even}; \beta_{n,m}>2} \beta_{n,m}(\lambda+\mu+\nu\beta_{n,m})(\beta_{n,m}-2)\frac{a_{n,m}^2+a_{m,n}^2}{2}\\ &= \quad \beta_{1,2}(\lambda+\mu+\nu\beta_{1,2})(\beta_{1,2}-2)\frac{a_{1,2}^2+a_{2,1}^2}{2}\\ &+ \frac{1}{2} \sum_{n,m\geq 1; n \text{ odd}; m \text{ even}; \beta_{n,m}>2} \beta_{n,m}(\lambda+\mu+\nu\beta_{n,m})(\beta_{n,m}-2)a_{n,m}^2\\ &+ \frac{1}{2} \sum_{n,m\geq 1; n \text{ even}; m \text{ odd}; \beta_{n,m}>2} \beta_{n,m}(\lambda+\mu+\nu\beta_{n,m})(\beta_{n,m}-2)a_{n,m}^2 \end{split}$$

which equals zero due to (17) and (18). This gives (20) under the condition $a_{1,2} = a_{2,1}$. Actually, we can assume $a_{1,2} = a_{2,1} = 1$, since the spectral problem is linear. This gives (20) for the eigenfunction $\psi \in E_2$. This derivation also implies the validity of (20) when the eigenfunction $\psi \in E_3$. The proof of Theorem 2.1 is completed.

3. Proof of Theorem 1.1 (ii)

Proof. Firstly, we introduce a flow invariant space so that Fredholm alternative theory can be applied for the critical eigenfunction $\psi_c \in E_i$. Theorem 2.1 shows that $\psi_c \in E_i$ is a simple eigenfunction. That is,

$$\dim \left\{ \psi \in E_i \, \middle| \, 0 = -\mu_c \Delta \psi + \nu_c \Delta^2 \psi + J(\psi_0, (\Delta + 2)\psi) \right\} = 1$$

and (20) holds true. E_i is an invariant space of linear spectral problem (7) but is not flow invariant for nonlinear problem (3)-(4). The linear perturbation part of the bifurcating solution is expected to be in the eigenfunction space span $\{\psi_c\} \subset E_i$. Therefore, we need to consider the bifurcation in a nonlinear flow invariant space of (3)-(4) and the space is generated nonlinearly from the linear space E_i .

To do so, we use the summation notation

$$\sum_{1} = \sum_{1 \leq n, m \text{ odd}} + \sum_{2 \leq n, m \text{ even}},$$

$$\sum_{2} = \sum_{1 \leq n \text{ odd}; 2 \leq m \text{ even}} + \sum_{2 \leq n, m \text{ even}},$$

$$\sum_{3} = \sum_{2 \leq n \text{ even}; 1 \leq m \text{ odd}} + \sum_{2 \leq n, m \text{ even}}.$$

We define the H^4 subspaces

$$H_i^4 = \left\{ \psi \in H^4 \mid \psi = \sum_i a_{n,m} \sin \frac{nx}{2} \sin \frac{my}{2} \right\} \text{ for } i = 1, 2, 3,$$

and the L_2 subspaces

$$H_i = \left\{ \psi = \sum_{i} a_{n,m} \sin \frac{nx}{2} \sin \frac{my}{2} \left| \|\psi\|_{L_2} = \left(\sum_{i} a_{n,m}^2\right)^{\frac{1}{2}} < \infty \right\} \text{ for } i = 1, 2, 3.$$

This definition ensures $H_i^4 \supset E_i$ for i = 1, 2, 3 and H_i^4 is orthogonal to E_j if $i \neq j$. Hence the assertion of Theorem 2.1 remains valid when E_i is replaced by H_i^4 . That is, the eigenfunction simplicity property holds true in H_i^4 . The nonlinear flow invariant property of H_i^4 is valid in the following sense

$$\Delta^{-2}J(\varphi,\Delta\phi) \in H_i^4 \text{ whenever } \varphi, \phi \in H_i^4.$$
 (37)

This invariance property is confirmed by the estimate

$$\begin{array}{lcl} \|\Delta^{-2}J(\varphi,\Delta\phi)\|_{H^4} & \leq & \|J(\varphi,\Delta\phi)\|_{L_2} \\ & \leq & \|\nabla\varphi\|_{L_4}\|\nabla\Delta\phi\|_{L_4} \leq C\|\varphi\|_{H^4}\|\phi\|_{H^4}, \end{array}$$

due to Hölder inequality and Sobolev imbedding, and the multiplication computation

$$J(\varphi, \Delta \phi) = \left(\sum_{i} a_{n,m} \frac{n}{2} \cos \frac{nx}{2} \sin \frac{my}{2}\right) \left(\sum_{i} b_{n,m} \frac{m}{2} \beta_{n,m} \sin \frac{nx}{2} \cos \frac{my}{2}\right)$$
$$-\left(\sum_{i} a_{n,m} \frac{m}{2} \sin \frac{nx}{2} \cos \frac{my}{2}\right) \left(\sum_{i} b_{n,m} \frac{n}{2} \beta_{n,m} \cos \frac{nx}{2} \sin \frac{my}{2}\right)$$
$$= \sum_{i} c_{n,m} \sin \frac{nx}{2} \sin \frac{my}{2}$$

for coefficients $c_{n,m}$ rearranged from $a_{n,m}$ and $b_{n,m}$. Here we use the functions

$$\varphi = \sum_i a_{n,m} \sin \frac{nx}{2} \sin \frac{my}{2} \in H_i^4 \text{ and } \phi = \sum_i b_{n,m} \sin \frac{nx}{2} \sin \frac{my}{2} \in H_i^4.$$

Now we rewrite the critical spectral problem as

$$\mathcal{L}\psi_c = 0$$
 for $\mathcal{L}\psi = -\mu_c \Delta \psi + \nu_c \Delta^2 \psi + J(\psi_0, (\Delta + 2)\psi)$.

We see that \mathcal{L} maps H_i^4 into H_i . To employ the Fredholm theory, we define the range of \mathcal{L} as

$$\operatorname{Ran}(\mathscr{L}) = \left\{ \varphi \in H_i \middle| \text{ there exists } \phi \in H_i^4 \text{ so that } \mathscr{L}\phi = \varphi \right\}.$$

It readily seen that $Ran(\mathcal{L})$ is the space orthogonal to ψ_c^* , the conjugate eigenfunction of ψ_c , in the following sense:

$$\operatorname{Ran}(\mathcal{L}) = \left\{ \psi \in H_i \middle| (\psi, \psi_i^*) = 0 \right\}.$$

By the Fredholm alternative theory of Laplacian operators, $\mathscr L$ has an inverse operator

$$\mathcal{L}^{-1}: \operatorname{Ran}(\mathcal{L}) \mapsto H_i^4 \tag{38}$$

so that

$$\|\mathscr{L}^{-1}\psi\|_{H_i^4} \le C_1 \|\psi\|_{L_2}, \ \psi \in \operatorname{Ran}(\mathscr{L})$$
(39)

for a constant C_1 .

Secondly, following Rabinowitz [12] on a Bénard problem, we seek the secondary steady-state solution (ψ, ν, μ) branching from the bifurcation point (ψ_0, ν_c, μ_c) in the direction of ψ_c as

$$\psi = \psi_0 + \varepsilon \psi_c + \varepsilon^2 \psi_i, \ \nu = \nu_c + \varepsilon \sigma \nu_c, \ \mu = \mu_c + \varepsilon \sigma \mu_c \tag{40}$$

for a function $\psi_i \in H_i^4$ and a real σ , provided that $\varepsilon > 0$ is sufficiently small.

Substitution of the predicted solution (40) into the stationary form of (3), or the equation

$$\begin{split} 0 = & (-\mu\Delta + \nu\Delta^2)(\psi - \psi_0) + J(\psi_0, (2+\Delta)(\psi - \psi_0)) + J(\psi - \psi_0, \Delta(\psi - \psi_0)) \\ = & (-(\mu - \mu_c)\Delta + (\nu - \nu_c)\Delta^2)(\psi - \psi_0) + \mathcal{L}(\psi - \psi_0) + J(\psi - \psi_0, \Delta(\psi - \psi_0)), \end{split}$$

produces the equation

$$0 = (-\varepsilon \sigma \mu_c \Delta + \varepsilon \sigma \nu_c \Delta^2)(\varepsilon \psi_c + \varepsilon^2 \psi_i) + \mathcal{L}(\varepsilon \psi_c + \varepsilon^2 \psi_i) + J(\varepsilon \psi_c + \varepsilon^2 \psi_i, \Delta(\varepsilon \psi_c + \varepsilon^2 \psi_i)).$$

Since $\mathcal{L}\psi_c = 0$, the previous equation can be rewritten as

$$\sigma(-\mu_c \Delta + \nu_c \Delta^2) \psi_c + \mathcal{L} \psi_i = F_{\varepsilon}(\sigma, \psi_i)$$
(41)

with

$$F_{\varepsilon}(\sigma, \psi_i) = -\varepsilon \sigma(-\mu_c \Delta + \nu_c \Delta^2) \psi_i - J(\psi_c + \varepsilon \psi_i, \Delta(\psi_c + \varepsilon \psi_i)).$$

To show the existence of the unknowns ψ_i and σ , we take L_2 inner product of (41) with ψ_c^* to obtain

$$\sigma((-\mu_c \Delta + \nu_c \Delta^2) \psi_c, \psi_c^*) + (\mathcal{L} \psi_i, \psi_c^*) = (F_{\varepsilon}(\sigma, \psi_i), \psi_c^*). \tag{42}$$

Applying Theorem 2.1 with E_i replaced by H_i^4 , the invariance property (37) and the identity

$$(\mathcal{L}\psi_i, \psi_c^*) = (\psi_i, \mathcal{L}^*\psi_c^*) = 0,$$

we may rewrite (42) as

$$\sigma = \frac{(F_{\varepsilon}(\sigma, \psi_i), \psi_c^*)}{((-\mu_c \Delta + \nu_c \Delta^2)\psi_c, \psi_c^*)}.$$
(43)

The combination of (41) and (43) yields

$$\mathcal{L}\psi_i = F_{\varepsilon}(\sigma, \psi_i) - \frac{(-\mu_c \Delta + \nu_c \Delta^2)\psi_c(F_{\varepsilon}(\sigma, \psi_i), \psi_c^*)}{((-\mu_c \Delta + \nu_c \Delta^2)\psi_c, \psi_c^*)}.$$
(44)

The nonlinear invariance property (37) implies $F_{\varepsilon}(\sigma, \psi_i) \in H_i$. It is readily seen that the right-hand side of (44) is in $Ran(\mathcal{L})$. Therefore, we may use the inverse of \mathcal{L} to produce

$$\psi_i = \mathcal{L}^{-1} \left(F_{\varepsilon}(\sigma, \psi_i) - \frac{(-\mu_c \Delta + \nu_c \Delta^2) \psi_c(F_{\varepsilon}(\sigma, \psi_i), \psi_c^*)}{((-\mu_c \Delta + \nu_c \Delta^2) \psi_c, \psi_c^*)} \right). \tag{45}$$

For simplicity of notation, we rewrite the equations (43) and (45) in the following form

$$(\sigma, \psi_i) = G_{\varepsilon}(\sigma, \psi_i), \tag{46}$$

where the two components of the operator $G_{\varepsilon}(\sigma, \psi_i)$ represent respectively the right-hand sides of (43) and (45). Thus, to seek the solution (ψ, ν, μ) in (40) becomes to confirm the existence of the fixed point for the operator G_{ε} .

Finally, it remains to prove that G_{ε} is a contraction operator mapping a complete metric space into itself. The complete matric space is defined as

$$X = \left\{ (\sigma, \psi) \in (-\infty, \infty) \times H_i^4 \middle| \ \|(\sigma, \psi)\|_X = |\sigma| + \|\psi\|_{H_i^4} \le C \right\}.$$

Here C > 0 is a constant to be defined afterward.

To show the contraction property, we use the boundedness of \mathcal{L}^{-1} in (39), the expressions (43) and (45), and Hölder inequality to produce

$$||G_{\varepsilon}(\sigma, \psi_{i})||_{X} \leq \left(\frac{||\psi_{c}^{*}||_{L_{2}}}{|((-\mu_{c}\Delta + \nu_{c}\Delta^{2})\psi_{c}, \psi_{c}^{*})|} + C_{1}\left(1 + \frac{||(-\mu_{c}\Delta + \nu_{c}\Delta^{2})\psi_{c}||_{L_{2}}||\psi_{c}^{*}||_{L_{2}}}{|((-\mu_{c}\Delta + \nu_{c}\Delta^{2})\psi_{c}, \psi_{c}^{*})|}\right)\right)||F_{\varepsilon}(\sigma, \psi_{i})||_{L_{2}}.$$

This yields, by renaming the constant bounded by the large brackets in the right-hand side of the previous equation as C_2 ,

$$||G_{\varepsilon}(\sigma, \psi_i)||_X \leq C_2 ||F_{\varepsilon}(\sigma, \psi_i)||_{L_2}.$$

Hence, by Hölder inequality and Sobolev imbedding inequality, we have

$$\begin{split} \|G_{\varepsilon}(\sigma, \psi_{i})\|_{X} & \leq C_{2} \|J(\psi_{c} + \varepsilon\psi_{i}, \Delta(\psi_{c} + \varepsilon\psi_{i})) + \varepsilon\sigma(-\mu_{c}\Delta + \nu_{c}\Delta^{2})\psi_{i}\|_{L_{2}} \\ & \leq C_{2} \Big(\|\nabla(\psi_{c} + \varepsilon\psi_{i})\|_{L_{4}} \|\nabla\Delta(\psi_{c} + \varepsilon\psi_{i})\|_{L_{4}} \\ & + \varepsilon\sigma\|(-\mu_{c}\Delta + \nu_{c}\Delta^{2})\psi_{i}\|_{L_{2}} \Big) \\ & \leq C_{3} \Big(\|\psi_{c}\|_{H_{i}^{4}}^{2} + 2\varepsilon\|\psi_{c}\|_{H_{i}^{4}} \|\psi_{i}\|_{H_{i}^{4}} + \varepsilon^{2} \|\psi_{i}\|_{H_{i}^{4}}^{2} + \varepsilon\sigma\|\psi_{i}\|_{H_{i}^{4}} \Big) \\ & \leq C_{4} \Big(1 + \varepsilon C + \varepsilon^{2}C^{2} + \varepsilon C^{2} \Big) \end{split}$$

for the constants C_k independent of $(\sigma, \psi_i) \in X$ and $\varepsilon > 0$. Therefore, we obtain

$$||G_{\varepsilon}(\sigma, \psi_i)||_X \le C \text{ for } (\sigma, \psi_i) \in X,$$
 (47)

provided that

$$\frac{C}{2} = C_4$$

and

$$C_4(\varepsilon C + \varepsilon^2 C^2 + \varepsilon C^2) = C_4(\varepsilon + 2\varepsilon^2 C_4 + 2\varepsilon C_4)C \le \frac{1}{2}C,$$

by taking $\varepsilon > 0$ sufficiently small. The property (47) implies the injection property $G_{\varepsilon}: X \mapsto X$.

Arguing in the same manner, we have the contraction property:

$$\|G_{\varepsilon}(\sigma,\psi_i) - G_{\varepsilon}(\sigma',\psi_i')\|_X \leq \frac{1}{2} \|(\sigma,\psi_i) - (\sigma',\psi_i')\|_X$$

for (σ, ψ_i) , $(\sigma', \psi'_i) \in X$, provided that $\varepsilon > 0$ sufficiently small. Therefore, by the Banach contraction mapping principle, the operator G_{ε} with small $\varepsilon > 0$ admits a unique fixed point $(\sigma, \psi_i) \in X$. This confirms the existence of the steady-state solution (ψ, μ, ν) of (3) and (4) in the form of (40).

The uniform boundedness of the σ and ψ_i with respect to ε is given by (47). The property $\psi_i \in H_i^4/\text{span}\{\psi_c\}$ is implied from (45) due to the Fredholm operator property $\mathcal{L}^{-1}: \text{Ran}(\mathcal{L}) \mapsto H_i^4/\text{span}\{\psi_c\}$. The proof of Theorem 1.1 is completed.

4. Discussion on Theorem 1.1

4.1. Discussion on the existence of critical spectral solution

From viewpoint of numerical computation, a critical spectral solution (v_c, μ_c, ψ_c) of spectral problem (7) with $\lambda=0$ can be calculated by using algebraic equation (15). However, from viewpoint of rigorous analysis, its existence remains unsolved. In the unidirectional Kolmogorov flow problem, its linear spectral equation can be transformed into a continuous fraction equation, from which the existence of critical value is obtained. The coefficients of the corresponding eigenfunction can expressed as multiplications of continuous fractions (see [5, 6, 8]) for both real and non-real eigenfunctions. However, this continuous fraction approach is no longer applicable to the present multi-directional flow problem. To facilitate the understanding of the difficulty, we discuss the problem with the aid of a truncated form.

For simplicity, we only consider eigenfunction in E_1 . Linear spectral problem (7) or (15) with $\lambda = 0$ can be rewritten in the form

$$0 = \sum_{-2 \le n, m; n+m \le k; n, m \text{ odd}} \left\{ \alpha_{n,m} b_{n,m} + (n-m) [b_{n-2,m-2} - b_{n+2,m+2}] + (n+m) [b_{n-2,m+2} - b_{n+2,m-2}] \right\} \sin \frac{nx}{2} \sin \frac{my}{2}$$

$$(48)$$

as $k \to \infty$. Here

$$\alpha_{n,m} = 8 \frac{\beta_{n,m}(\mu + \nu \beta_{n,m})}{\beta_{n,m} - 2}, \ b_{n,m} = (\beta_{n,m} - 2)a_{n,m} \text{ for } n,m \ge 1, \text{ otherwise } b_{n,m} = 0.$$

If A_k represents the square matrix of influence coefficients and B_k the truncated eigenvector consists of $b_{n,m}$, then (48) becomes

$$A_k B_k = 0$$

For example, for the approximating spectral problem for k = 8, we have the 10-dimensional truncation equation

$$\begin{aligned} 0&=\alpha_{1,1}b_{1,1}-2b_{1,3}+2b_{3,1}\\ 0&=\alpha_{1,3}b_{1,3}-4b_{1,5}+2b_{3,5}+2b_{1,1}-4b_{3,1}\\ 0&=\alpha_{3,1}b_{3,1}+4b_{5,1}-2b_{5,3}-2b_{1,1}+4b_{1,3}\\ 0&=\alpha_{1,5}b_{1,5}-6b_{1,7}+4b_{1,3}-6b_{3,3}\\ 0&=\alpha_{5,1}b_{5,1}+6b_{7,1}-4b_{3,1}+6b_{3,3}\\ 0&=\alpha_{3,3}b_{3,3}+6b_{1,5}-6b_{5,1}\\ 0&=\alpha_{1,7}b_{1,7}+6b_{1,5}-8b_{3,5}\\ 0&=\alpha_{7,1}b_{7,1}-6b_{5,1}+8b_{5,3}\\ 0&=\alpha_{3,5}b_{3,5}-2b_{1,3}+8b_{1,7}-8b_{5,3}\\ 0&=\alpha_{5,3}b_{5,3}+2b_{3,1}+8b_{3,5}-8b_{7,1} \end{aligned}$$

or

$$0 = A_8 B_8 = \begin{pmatrix} \alpha_{1,1} & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & \alpha_{1,3} & -4 & -4 & 0 & 0 & 0 & 0 & 2 & 0 \\ -2 & 4 & \alpha_{3,1} & 0 & 4 & 0 & 0 & 0 & 0 & -2 \\ 0 & 4 & 0 & \alpha_{1,5} & 0 & -6 & -6 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & \alpha_{5,1} & 6 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 6 & -6 & \alpha_{3,3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & \alpha_{1,7} & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 & -6 & 0 & 0 & \alpha_{7,1} & 0 & 8 \\ 0 & -2 & 0 & 0 & 0 & 0 & 8 & 0 & \alpha_{3,5} & -8 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & -8 & 8 & \alpha_{5,3} \end{pmatrix} \begin{pmatrix} b_{1,1} \\ b_{1,3} \\ b_{3,1} \\ b_{3,1} \\ b_{3,3} \\ b_{1,7} \\ b_{7,1} \\ b_{3,5} \\ b_{5,3} \end{pmatrix}. (49)$$

The truncation seems quit harsh, but the solution to this equation is reasonably close to that of the original one. The existence of the corresponding spectral critical solution becomes the existence of the critical value $(v, \mu) = (v_c, \mu_c) \neq (0, 0)$ satisfying the determinant equation

$$\det(A_8(v, \mu)) = 0.$$

Since $\alpha_{1,1} < 0$ and $\alpha_{n,m} > 0$ for $(n,m) \neq (1,1)$, we see that $\lim_{v + \mu \to \infty} \det(A_8(v,\mu)) \to -\infty$. Therefore, the existence of the root (v_c, μ_c) can be confirmed if one finds some (v,μ) so that $\det(A_8(v,\mu)) > 0$. However, it is laborious to prove this positivity property although A_k is always skew-symmetric when $v = \mu = 0$. To show the positivity property and then the root existence, we use the selected computation result in Figure 2, which shows the roots $(v_c, \mu_c) \approx (0.0.236)$ from (a) and $(v_c, \mu_c) \approx (0.237, 0)$ from (b)-(c). This result is comparable with that computed by Thess [15].

Similar behaviour exists for the polynomial $det(A_k)$ as k increases. Actually, the behavior of the polynomial $det(A_8)$ such as transversal crossing the horizontal zero line is comparable to that of the third degree polynomial (see Figure 2 (d))

$$\det(A_4) = \alpha_{1,1}\alpha_{1,3}\alpha_{3,1} + 16\alpha_{1,1} + 4\alpha_{1,3} + 4\alpha_{3,1},$$

although the deviation with respect to the root increases. Here A_4 is the 3×3 matrix in the top-left corner of A_8 .

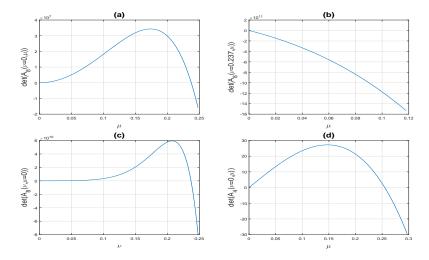


Figure 2: Selected computation results for the polynomials $\det(A_8(v,\mu))$ and $\det(A_4(v,\mu))$ crossing horizontal zero lines.

4.2. Connection to Crandall-Rabinowitz bifurcation theorem

The bifurcation result can be obtained by Crandall-Rabinowitz bifurcation theorem [7], although the secondary flow shown in Theorem 1.1 is more informative due to the construction by Banach fixed point theorem. As an alternative way, we would like to show the existence of the secondary flow as a consequence of the following.

Theorem 4.1. (Crandall and Rabinowitz [7, Theorem 1.7]) Let X, Y be Banach spaces, V a neighborhood of 0 in X and

$$F: (-1,1) \times V \mapsto Y \tag{50}$$

have the properties

- (a) $F(\tau,0) = 0$ for $|\tau| < 1$,
- **(b)** The partial derivatives F_{τ} , F_{ψ} and $F_{\tau\psi}$ exist and are continuous,
- (c) The kernel space $N(F_{\psi}(0,0))$ and the orthogonal compliment $Y/\text{Ran}(F_{\psi}(0,0))$ are one-dimensional,
- (d) $F_{\tau\psi}(0,0)\psi_c \notin \operatorname{Ran}(F_{\psi}(0,0))$, where

$$N(F_{\mathbf{\Psi}}(0,0)) = \operatorname{span}\{\psi_c\}.$$

Then there is a neighborhood U of (0,0) in $R \times X$, an interval (-a,a), and continuous functions

$$\kappa: (-a,a) \mapsto (-\infty,\infty), \quad \Psi: (-a,a) \mapsto X/N(F_{\Psi}(0,0))$$

such that $\kappa(0) = 0$, $\Psi(0) = 0$ and

$$F^{-1}(0) \cap U = \left\{ \left(\kappa(\varepsilon), \varepsilon \psi_c + \varepsilon \Psi(\varepsilon) \right| \ |\varepsilon| < a \right\} \cup \left\{ (\tau, 0) \, \middle| \, (\tau, 0) \in U \right\}.$$

Theorem 4.2. Assume that the spectral problem (7) admits a critical solution $(\lambda, \psi, \nu, \mu) = (0, \psi_c, \nu_c, \mu_c)$ for $\nu_c > 0$, $\mu_c \ge 0$. Then there exist continuous functions

$$\kappa: (-a,a) \mapsto (-\infty,\infty), \ \Psi: (-a,a) \mapsto H^4/span\{\psi_c\} \ with \ \kappa(0) = 0, \ \Psi(0) = 0,$$

for a small constant a > 0, so that system (3)-(4) has the bifurcating steady-state solution (ψ, ν, μ) expressed as

$$\Psi = \Psi_0 + \varepsilon \Psi_c + \varepsilon \Psi(\varepsilon), \ v = v_c + v_c \kappa(\varepsilon), \ \mu = \mu_c + \mu_c \kappa(\varepsilon), \ for \ \varepsilon \in (-a, \omega 1)$$

It should be noted that Theorem 4.2 is the same as Theorem 1.1, after we use the setting

$$\kappa(\varepsilon) = \varepsilon \delta(\varepsilon, \psi_c), \ \Psi(\varepsilon) = \varepsilon \psi_i(\varepsilon, \psi_c) \ \text{for } 0 \le \varepsilon \le a,$$

$$\kappa(\varepsilon) = -\varepsilon \delta(-\varepsilon, -\psi_c), \ \Psi(\varepsilon) = \varepsilon \psi_i(-\varepsilon, -\psi_c) \ \text{for} \ -a \le \varepsilon < 0.$$

Proof. For employing Theorem 4.1, we formulate the fluid motion problem into the functional framework of Theorem 4.1 and then show the validity of properties (a)-(d) and (50). Indeed, using the perturbation

$$\psi = \psi_0 + \psi', \ \nu = \nu_c + \tau \nu_c, \ \mu = \mu_c + \tau \mu_c,$$
(52)

we may rewrite (3) as

$$F(\tau, \psi) = 0, \tag{53}$$

for

$$F(\tau, \psi) = -(\tau + 1)\mu_c \Delta \psi + (\tau + 1)\nu_c \Delta^2 \psi + J(\psi_0, (2 + \Delta)\psi) + J(\psi, \Delta\psi), \tag{54}$$

after omitting the superscript prime. Thus (τ, ψ) being a steady-state flow means $(\tau, \psi) \in F^{-1}(0)$ and we are seeking steady-state solutions branching off the basic flow $(\tau, \psi) = (0, 0)$. It follows from Theorem 1.1(i) that

$$\psi_c \in E_i \text{ for some integer } 1 \le i \le 3.$$
 (55)

Let $X = H_i^4$ and $Y = H_i$. The injection property (50) is valid due to (37), and then the definition given by (54) implies the validity of properties (a)-(b).

Moreover, note that $v_c > 0$ and

$$F_{\psi}(0,0)\psi = -\mu_c \Delta \psi + \nu_c \Delta^2 \psi + J(\psi_0, (2+\Delta)\psi).$$

The linear operator $F_W(0,0): X \mapsto Y$ is Fredholm index zero. Hence the complement

$$Y/\operatorname{Ran}(F_{\psi}(0,0)) = \operatorname{span}\{\psi_c^*\}$$

for ψ_c^* the conjugate counterpart of ψ_c , since

$$N(F_{\psi}(0,0)) = \operatorname{span}\{\psi_c\}$$

due to (19) and (55). We thus have property (c). Additionally, upon the observation

$$F_{\tau\psi}(0,0)\psi = -\mu_c\Delta\psi + \nu_c\Delta^2\psi,$$

equation (20) becomes

$$(F_{\tau\psi}(0,0)\psi_c,\psi_c^*) \neq 0 \text{ or } F_{\tau\psi}(0,0)\psi_c \notin \text{Ran}(F_{\phi}(0,0)),$$
 (56)

and hence property (d) holds true. Here we have used the property

$$\operatorname{Ran}(F_{\psi}(0,0)) = \left\{ \phi \in Y \middle| (\phi, \psi_c^*) = 0 \right\}.$$

Therefore, by Theorem 4.1, we have the desired functions κ and Ψ so that the equation $F(\tau, \psi') = 0$ has solutions

$$\psi' = \varepsilon \psi_c + \varepsilon \Psi(\varepsilon), \ \tau = \kappa(\varepsilon)$$

This together with (52) implies Theorem 4.2. The proof is completed. \Box

5. Numerical computation

Firstly, we compute critical spectral solutions. Let **S** be the set of all spectral solutions (ψ_c, ν_c, μ_c) with $\nu_c, \mu_c \ge 0$ and $(\nu_c, \mu_c) \ne (0, 0)$ satisfying the spectral problem

$$0 = -\mu_c \Delta \psi_c + \nu_c \Delta^2 \psi_c + J(\psi_0, (2+\Delta)\psi_c),$$

$$0 \neq \psi_c = \sum_{n,m \geq 1} a_{n,m} \sin \frac{nx}{2} \sin \frac{my}{2} \in H^4.$$
(57)

By Theorem 2.1, the set of all critical values $\{(v_c, \mu_c)\}$ is the union of the following three subsets

$$S_i = \{ (\mu_c, \nu_c) | (\psi_c, \nu_c, \mu_c) \in \mathbf{S} \text{ and } \psi_c \in E_i \}, i = 1, 2, 3.$$
 (58)

Numerical spectral solutions are obtained by using the MATLAB eig function. S_1 is the same as the set of numerical data in [15, Table I] and is the curve joining the points (0.2371,0) and (0,0.2310) in Figure 3 (a), which shows that S_1 is the neutral line separating the linear stable and linear unstable domains. However, for the eigenfunction $\psi_c \in E_2$, the proof of Theorem 2.1 shows the coexistence of two critical orthogonal eigenfunctions. The other one is given by (31). The numerical simulation of the two eigenfunctions sharing with the same critical vector value (v_c, μ_c) was given in [4]. In fact, we have $S_2 = S_3$, since horizontal coordinate is symmetric with the vertical coordinate in the spectral problem (57). The subset $S_2 = S_3$ forms the line touching the points (0.0415,0) and (0,0.01515) inside the linear unstable domain displayed in Figure 3(a). For displaying purpose, following three classes of approximating critical eigenfunctions

$$\psi_c \approx \sum_{n,m \text{ odd: } 1 \le n,m \le 11} a_{n,m} \sin \frac{nx}{2} \sin \frac{mx}{2} \in E_1, \tag{59}$$

$$\psi_c \approx \sum_{n \text{ odd}; m \text{ even}; 1 \le n, m \le 11} a_{n,m} \sin \frac{nx}{2} \sin \frac{mx}{2} \in E_2,$$
(60)

$$\psi_c \approx \sum_{n \text{ even}; m \text{ odd}; 1 \le n, m \le 11} a_{n,m} \sin \frac{nx}{2} \sin \frac{mx}{2} \in E_3,$$
(61)

at some typical critical values in $S_1 \cup S_2 \cup S_3$, are exhibited respectively in Figure 3(b)-(d).

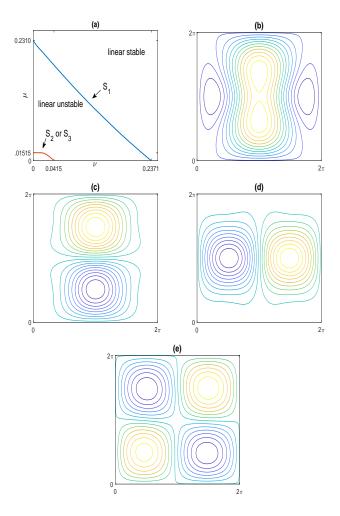


Figure 3: (a) All critical values $(v_c, \mu_c) \in S_1 \cup S_2 \cup S_3$; (b) critical eigenfunction $\psi_c \in E_1$ for $(v_c, \mu_c) = (0.00054, 0.2315)$ or $(Re_c, Rh_c) = (22446, 1.326)$; (c) critical eigenfunction $\psi_c \in E_2$ for $(v_c, \mu_c) = (0.00054, 0.0177)$ or $(Re_c, Rh_c) = (6378, 4.929)$; (d) critical eigenfunction $\psi_c \in E_3$ for $(v_c, \mu_c) = (0.00054, 0.0177)$ or $(Re_c, Rh_c) = (6378, 4.929)$; (e) numerical presentation of the nonlinear secondary flow at $(v, \mu) = (0.0005, 0.23)$.

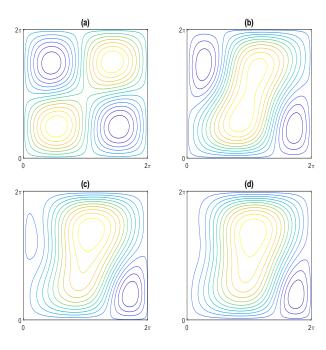


Figure 4: The steady-state flows when (a) (Re,Rh) = (700,1.55), (b) (Re,Rh) = (700,3), (c) (Re,Rh) = (700,5), and (d) (Re,Rh) = (700,7).

Secondly, we compute the bifurcating steady-state flow shown in Theorem 1.1. To understand the experimental magnetohydrodynamic flows, we follow [13, 14, 16] to consider almost inviscid flows so that their energy dissipation is essentially controlled by the Hartmann layer friction μ or the Rayleigh number Rh. When the critical eigenfunction $\psi_c \in E_2 \cup E_3$, the secondary flows branching from the corresponding critical vector values in the linear unstable domain is unobservable in laboratory experiments, although they are contributed to the complexity of flow dynamic behaviour towards to turbulence. We only consider the flows related to (ψ_c, v_c, μ_c) with $\psi_c \in E_1$.

If a secondary flow bifurcating in the direction $\psi_c \in E_1$ is stable, it attracts flows initially from the states in its vicinity and thus can be reached by numerical computation. Indeed, the desired stable secondary flow is obtained by employing a finite difference scheme with a 80×80 gridding mesh of the fluid domain Ω . For example, a numerical secondary solution is obtained for (v,μ) close to the critical condition $(v_c,\mu_c)=(0.00054,0.2306)$ or $(Re_c,Rh_c)=(22402,1.329)$. In Figure 3(e), we present nonlinear secondary steady-state flow at $(v,\mu)=(0.0005,0.23)$ or (Re,Rh)=(24158,1.33), which represents the secondary flow bifurcating from ψ_0 at $(v_c,\mu_c)=(0.00054,0.2306)$. This secondary flow is actually the limit of the flow initially from the state (see figure 1 (b))

$$\psi_0 - 0.1 \sin \frac{x}{2} \sin \frac{y}{2}. \tag{62}$$

Therefore, the steady-state flow is obtained by computing the non-stationary flow in

the numerical computation.

The secondary flow in Figure 3(e) shows the topological transition for the merging of two vortices, observed by Sommaria and Verron [14, 16]. Their experimental threshold for the onset of secondary flow is $Rh_c = 1.52$, which is close to but higher than the present numeric one $Rh_c = 1.329$. This is due to the neglect of the energy dissipation inside the lateral boundary layers of the original three-dimensional fluid motion problem (see [15]).

When (v, μ) is close to the threshold (v_c, μ_c) , the nonlinear secondary flow in Figure 3(e) is comparable with the initial form (62) expressed in Figure 1 (b). This is owing to the principal mode $\sin \frac{x}{2} \sin \frac{y}{2}$ generating the eigenfunction ψ_c . By numerical computation and (16), the principal coefficient $a_{1,1}$ of the principal mode is significantly larger than other coefficients $a_{n,m}$.

Finally, we show an additional nonlinear topological bifurcation of the flow motion by considering another bifurcating flow with respect to the initial state

$$\psi_0 + 0.1 \sin \frac{x}{2} \sin \frac{y}{2}. \tag{63}$$

By choosing a smaller Reynolds number value, the initial state leads to the occurrence of the four steady-state flows shown respectively in Figure 4 (a)-(d) in four different (Re,Rh) values. Their corresponding (v,μ) values are (0.0173,0.1982), (0.0102,0.0603), (0.0072,0.0254) and (0.0058,0.0148), respectively. Figure 4 shows the inverse energy cascade towards large scales by merging four vortices into three and then combining three vortices into two.

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