

# Quasistatic hypoplasticity at large strains Eulerian

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**Abstract.** The isothermal quasistatic (i.e. acceleration neglected) hardening-free plasticity at large strains is considered, based on the standard multiplicative decomposition of the total strain and the isochoric plastic distortion. The Eulerian velocity-strain formulation is used. The mass density evolves too, but acts only via the force term with a given external acceleration. This rather standard model is then re-formulated in terms of rates (so-called hypoplasticity) and the plastic distortion is completely eliminated, although it can be a-posteriori re-constructed. Involving gradient theories for dissipation, existence and regularity of weak solutions is proved rather constructively by a suitable regularization combined with a Galerkin approximation. The local non-interpenetration through a blowup of stored energy when elastic-strain determinant approaches zero is enforced and exploited. The plasticity is considered rate dependent and, as a special case, also creep in Jeffreys' viscoelastic rheology in the shear is covered while the volumetric response obeys the Kelvin-Voigt rheology.

*Keywords:* Finitely-strained plasticity, creep in Jeffreys' rheology, multiplicative decomposition, rate formulation, quasistatic, Galerkin approximation, weak solutions.

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## 1 Introduction

Many materials undergo large inelastic process, specifically plastification or creep. Typically, beside metals (some of whose exhibit so-called “superplasticity”), it concerns polymers and particularly geomaterials as rocks, soils, and ice which can exhibit very large inelastic strains on long time scales. Mechanically, *large-strain* (sometimes called finite-strain) *plasticity* or *creep* models have been developed during decades, see the monographs [6, 7, 23–25, 27, 28, 37, 41] and references therein. Following a general idea to express strain-stress responses rather in terms of rates, like hypo-elasticity being an alternative description to hyperelasticity, a rate formulation (sometimes called *hypoplasticity*) has been developed as an alternative to the classical theory of elasto-plasticity, cf. [13, 29, 33], although the label “hypoplasticity” has rather free meaning and is used in various ways, even not entirely identical as here, cf. also [60]. One attribute is that this formulation works without any explicit decomposition of the strain rate tensor to a reversible and an irreversible parts, although it is implicitly based on it.

Let us summarize the main ingredients, which are actually quite standard and generally accepted, and which will be employed:

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- deformation in actual configuration (i.e. *Eulerian approach*) and corresponding evolution of the deformation gradient, cf. (2.1) below,
- corresponding transport of mass (i.e. *mass conservation*), cf. (2.3),
- Lie-Liu-Kröner *multiplicative decomposition* of the deformation gradient to the elastic and the inelastic (plastic) strains, cf. (2.4), with the plastic distortion being *isochoric*, i.e. having determinant equal 1,
- a stored energy dependent on elastic strain, expected generally *nonconvex, frame indifferent, and singular* when respecting local non-interpenetration by a blow-up within infinite compression, i.e. if determinant of the elastic strain goes to zero, cf. (4.2b),
- a dissipation potential acting on the symmetric velocity gradient and on the plastic distortion rate,
- the evolution based on the mentioned conservation of mass and evolution of the deformation gradient, in addition on the momentum equilibrium and on the flow rule of plastic distortion through the plastic distortion rate, and
- gradient theories, here applied on the dissipative potential.

Ultimately, we focus to an energetics of the models, which will make a solid base for a rigorous analysis.

Any reference configuration (i.e. the Lagrangian approach as in [14, 32, 38, 44, 45, 59]) is thus eliminated from the formulation of the problem. This is very natural especially for materials where such a reference configuration cannot be identified naturally, as e.g. in geological materials (rocks, soils, ice, etc.) which are permanently evolving on long time scales and which do not possess any “original” stress-free configuration, cf. e.g. [53]. Rather, they have a continuously evolving natural configuration, sometimes presented under the name of “multiple natural configurations” [54, 56]. This is one of conventional approaches to inelasticity, dated back to C. Eckart [15], including both creep and plasticity. Rather for explanatory lucidity, we will present it in detail in Section 2 first a classical way including inelastic (plastic) distortion and the multiplicative decomposition of the deformation gradient. The plastic isochoricity is build in through the dissipation potential.

Then, in Section 3 we re-formulate the problem in rates and eliminate thus the plastic distortion, casting thus a hypoplastic model. Such a rate formulation based on velocity and elastic strain together with the plastic distortion rate is sometimes used in engineering, although without any rigorous analysis. A conceptual benefit from avoiding the plastic distortion is elimination of discussions about an intermediate stress free configuration arising from it, which is felt as a fictitious and physically meaningless. The eliminated plastic strain can be “reconstructed” a-posteriori. At this point, we involve higher gradients in the dissipative potential, which allows for a rigorous mathematical analysis, together with rigorous control of invertibility of elastic strain by the stored energy.

The mentioned energy dissipation balance is used eventually in Section 4 to perform the analysis of the hypoplastic model by a discretization in space (Galerkin method) together with some regularization. In this way, existence of weak solutions is proved by a constructive method, giving also some conceptual numerical algorithm.

To highlight the main attributes of the model and its mathematical treatment, we present

it as quasistatic with the acceleration (and related inertial forces) neglected and (still non-constant and evolving) mass density thus occurring only in the bulk-load term. A lot of nontrivial analytical technicalities, now well understood from compressible fluid dynamics [18, 36], are thus avoided. For the same reason, we present the model isothermally. The second simplifying assumption (but most frequently adopted in literature) is nonpenetrability of the boundary (i.e. normal velocity zero), which allows also for considering fixed boundary even for the Eulerian description. In applied sciences, a rough approach to live with this nonpenetrable boundary is considering time-varying domains embedded into a fictitious fixed domain and let the material being inhomogeneous, composed from the viscoelastoplastic solid and a very soft one. In geodynamical modelling, this trick is sometimes called the sticky-air approach. Eventually, we will exploit suitable gradient theories to facilitate the proof of existence of weak solutions to the hypoplastic model. There seems to be a general agreement that large-strain models ultimately needs some higher gradients to cope with geometrical nonlinearities. In engineering models, various gradients are used to control internal length-scales. Two principle options are usage of higher gradients in the conservative way (i.e. enhancing the stored energy) or in the dissipative way (i.e. enhancing the dissipation potential). In Section 3, we will accept the latter option. Of course, various combinations of both options can be considered, too. Sometimes, even a diffusion is added into the evolution rule of the deformation gradient [4], which seems only artificial if not used in a modified form [58, Remark 3] where it might have an interpretation of Brenner's stress diffusion [8]; in creep (fluid) models cf. e.g. [9] or for an incompressible case also [1, 10, 16, 39], although even this is considered disputable.

The main notation used in this paper is summarized in the following table:

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| <p><math>\mathbf{v}</math> velocity (in m/s),<br/> <math>\varrho</math> mass density (in kg/m<sup>3</sup>),<br/> <math>\mathbf{F}</math> deformation gradient,<br/> <math>\mathbf{F}_e</math> elastic strain,<br/> <math>\mathbf{F}_p</math> inelastic (plastic) distortion,<br/> <math>\mathbf{T}</math> Cauchy stress (symmetric - in Pa),<br/> <math>\mathbf{S} = \varphi(\mathbf{F}_e)</math> Piola stress (in Pa),<br/> <math>\mathfrak{H}</math> hyperstress (in Pa m),<br/> <math>\mathbb{R}_{\text{dev}}^{d \times d} = \{A \in \mathbb{R}^{d \times d}; \text{tr } A = 0\}</math>,</p> | <p><math>\varphi = \varphi(\mathbf{F}_e)</math> stored energy (in J/m<sup>3</sup>=Pa),<br/> <math>\mathbf{e}(\mathbf{v}) = \frac{1}{2} \nabla \mathbf{v}^\top + \frac{1}{2} \nabla \mathbf{v}</math> small strain rate (in s<sup>-1</sup>),<br/> <math>\xi = \xi(\mathbf{e}(\mathbf{v}))</math> viscous dissipation potential,<br/> <math>\zeta = \zeta(\mathbf{L}_p)</math> plastic dissipation potential,<br/> <math>\mathbf{L}_p = \dot{\mathbf{F}}_p \mathbf{F}_p^{-1}</math> plastic distortion rate (in s<sup>-1</sup>),<br/> <math>(\bullet)^\cdot = \frac{\partial}{\partial t} \bullet + (\mathbf{v} \cdot \nabla) \bullet</math> convective time derivative,<br/> <math>\mathbf{g}</math> external bulk load (gravity acceleration in m/s<sup>2</sup>),<br/> <math>\mathbf{f}</math> traction load,<br/> <math>\mathbb{R}_{\text{sym}}^{d \times d} = \{A \in \mathbb{R}^{d \times d}; A^\top = A\}</math>.</p> |
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## 2 Plasticity at large strains classically

In large-strain continuum mechanics, the basic geometrical concept is the time-evolving deformation  $\mathbf{y} : \Omega \rightarrow \mathbb{R}^d$  as a mapping from a reference configuration  $\Omega \subset \mathbb{R}^d$  into a physical space  $\mathbb{R}^d$ . The ‘‘Lagrangian’’ space variable in the reference configuration will be

denoted as  $\mathbf{X} \in \Omega$  while in the ‘‘Eulerian’’ physical-space variable by  $\mathbf{x} \in \mathbb{R}^d$ . The basic geometrical object is the deformation gradient  $\mathbf{F} = \nabla_{\mathbf{X}} \mathbf{y}$ .

We will be interested in deformations  $\mathbf{x} = \mathbf{y}(t, \mathbf{X})$  evolving in time, which are sometimes called ‘‘motions’’. The important quantity is the Eulerian velocity  $\mathbf{v} = \dot{\mathbf{y}} = \frac{\partial}{\partial t} \mathbf{y} + (\mathbf{v} \cdot \nabla) \mathbf{y}$ . Here and thorough the whole article, we use the dot-notation  $(\cdot)'$  for the *convective time derivative* applied to scalars or, component-wise, to vectors or tensors.

Then the velocity gradient  $\nabla \mathbf{v} = \nabla_{\mathbf{X}} \mathbf{v} \nabla_{\mathbf{x}} \mathbf{X} = \dot{\mathbf{F}} \mathbf{F}^{-1}$ , where we used the chain-rule calculus and  $\mathbf{F}^{-1} = (\nabla_{\mathbf{X}} \mathbf{x})^{-1} = \nabla_{\mathbf{x}} \mathbf{X}$ . This gives the *transport equation-and-evolution for the deformation gradient* as

$$\dot{\mathbf{F}} = (\nabla \mathbf{v}) \mathbf{F}. \quad (2.1)$$

From this, we also obtain the transport equation for the determinant  $\det \mathbf{F}$  as

$$\overline{\dot{\det \mathbf{F}}} = (\det \mathbf{F})(\operatorname{div} \mathbf{v}). \quad (2.2)$$

The understanding of (2.1) and (2.2) is a bit delicate because it mixes the Eulerian  $\mathbf{x}$  and the Lagrangian  $\mathbf{X}$ ; note that  $\nabla \mathbf{v} = \nabla_{\mathbf{x}} \mathbf{v}(\mathbf{x})$  while standardly  $\mathbf{F} = \nabla_{\mathbf{X}} \mathbf{y} = \mathbf{F}(\mathbf{X})$ . In fact, we consider  $\mathbf{F} \circ \boldsymbol{\xi}$  where  $\boldsymbol{\xi} : \mathbf{x} \mapsto \mathbf{y}^{-1}(t, \mathbf{X})$  is the so-called *return* (sometimes called also a *reference*) *mapping*. Thus  $\mathbf{F}$  depends on  $\mathbf{x}$  and (2.1) and (2.2) are equalities for a.a.  $\mathbf{x}$ . The reference mapping  $\boldsymbol{\xi}$ , which is well defined through its transport equation  $\dot{\boldsymbol{\xi}} = \mathbf{0}$ , actually does not explicitly occur in the formulation of the problem. Here we will benefit from the boundary condition  $\mathbf{v} \cdot \mathbf{n} = 0$  below, which causes that the actual domain  $\Omega$  does not evolve in time. The same concerns  $\mathbf{T}$  in (2.5b) below, which will make the problem indeed fully Eulerian, as announced in the title itself. Cf. the continuum-mechanics textbooks as e.g. [23, 40].

The mass density (in  $\text{kg}/\text{m}^3$ ) is an extensive variable, and its transport (expressing that the conservation of mass) writes as the *continuity equation*  $\frac{\partial}{\partial t} \rho + \operatorname{div}(\rho \mathbf{v}) = 0$ , or, equivalently, the *mass transport equation*

$$\dot{\rho} = -\rho \operatorname{div} \mathbf{v}. \quad (2.3)$$

Introducing a (generally non-symmetric) *plastic distortion* tensor  $\mathbf{F}_p$ , a conventional large-strain plasticity is based on Kröner-Lie-Liu [31, 34] *multiplicative decomposition*

$$\mathbf{F} = \mathbf{F}_e \mathbf{F}_p. \quad (2.4)$$

The interpretation of  $\mathbf{F}_p$  is a transformation of the reference configuration into an intermediate stress-free configuration, and then the *elastic strain*  $\mathbf{F}_e$  transforms this intermediate configuration into the current actual configuration.

The main ingredients of the model are the (volumetric) *stored energy* and the *dissipation potential*, i.e. the physical unit of the stored energy is  $\text{Pa} = \text{J}/\text{m}^3$  and of the dissipation potential is  $\text{Pa}/\text{s}$ . The stored energy  $\widehat{\varphi}(\mathbf{F}, \mathbf{F}_p)$  depends naturally on the elastic strain  $\mathbf{F}_e =$

$\mathbf{F}\mathbf{F}_p^{-1}$  and possibly also on  $\mathbf{F}_p$  itself if an isotropical hardening were considered, but not directly on  $\mathbf{F}$ . In this section we will consider  $\widehat{\varphi}(\mathbf{F}, \mathbf{F}_p) = \varphi(\mathbf{F}\mathbf{F}_p^{-1})$ . The other ingredient is the dissipation potential depending on the symmetric velocity gradient  $\mathbf{e}(\mathbf{v}) = \frac{1}{2}\nabla\mathbf{v}^\top + \frac{1}{2}\nabla\mathbf{v}$  and on the *plastic distortion rate*  $\dot{\mathbf{F}}_p\mathbf{F}_p^{-1}$ . We will consider this dissipation potential as  $\xi(\mathbf{e}(\mathbf{v})) + \widehat{\zeta}(\mathbf{F}_p, \dot{\mathbf{F}}_p)$  with the plastic dissipation potential  $\widehat{\zeta}$  depending on the plastic distortion rate, i.e.  $\widehat{\zeta}(\mathbf{F}_p, \dot{\mathbf{F}}_p) = \zeta(\dot{\mathbf{F}}_p\mathbf{F}_p^{-1})$  for some potential  $\zeta$ . If quadratic, these two parts of the dissipation potential involve linear Kelvin-Voigt-type and Maxwell-type viscosities into the model, and altogether with the elastic part determined by the stored energy, we obtain the *Jeffreys viscoelastic rheological model* in the shear while the volumetric response obeys the *Kelvin-Voigt rheology* if  $\mathbf{F}_p$  is purely isochoric, as in Sections 3 and 4 below. A quadratic  $\zeta(\cdot)$  thus describes *creep*. Yet,  $\zeta$  may be non-quadratic and even non-smooth at the rate zero, which models an “activated creep” as in ice or *plasticity*, or even out of zero rate as in the Tresca plasticity. This nonsmoothness makes the convex subdifferential  $\partial\zeta$  set-valued and thus why we wrote an inclusion “ $\ni$ ” in (2.5d).

The quasistatic evolution system then consists from the mass transport equation, momentum equilibrium, the deformation gradient transport (2.1), and a flow rule for the plastic distortion  $\mathbf{F}_p$ . Specifically, in terms of  $\widehat{\varphi}$  and  $\widehat{\zeta}$  the system for  $(\varrho, \mathbf{v}, \mathbf{F}, \mathbf{F}_p)$  reads as:

$$\dot{\varrho} = -\varrho \operatorname{div} \mathbf{v}, \quad (2.5a)$$

$$\operatorname{div} \mathbf{T} + \varrho \mathbf{g} = \mathbf{0} \quad \text{with} \quad \mathbf{T} = \widehat{\varphi}'_{\mathbf{F}}(\mathbf{F}, \mathbf{F}_p) \mathbf{F}^\top + \widehat{\varphi}(\mathbf{F}, \mathbf{F}_p) \mathbb{I} + \xi'(\mathbf{e}(\mathbf{v})) \quad (2.5b)$$

$$\dot{\mathbf{F}} = (\nabla \mathbf{v}) \mathbf{F}, \quad (2.5c)$$

$$\partial_{\dot{\mathbf{F}}_p} \widehat{\zeta}(\mathbf{F}_p, \dot{\mathbf{F}}_p) \ni -\widehat{\varphi}'_{\mathbf{F}_p}(\mathbf{F}, \mathbf{F}_p), \quad (2.5d)$$

where  $\widehat{\varphi}'_{\mathbf{F}}(\mathbf{F}, \mathbf{F}_p)$  is the so-called *Piola stress* and while (2.5d) has the standard structure of the so-called Biot equation.

In (2.5b),  $\mathbf{g}$  means a given acceleration (typically the *gravity acceleration*) while we neglected the inertial force  $\varrho \dot{\mathbf{v}}$ . This last point substantially simplifies the analytical arguments below while keeping the main phenomena under our focus in the game, although the absence of the kinetic energy makes estimation of the bulk force quite technical, cf. (4.3) below. In particular, although (2.5b) neglects the acceleration  $\dot{\mathbf{v}}$  and thus the mentioned inertial force  $\varrho \dot{\mathbf{v}}$ , the mass  $\varrho$  and its transport (2.5a) are still involved. Such models are called *quasistatic* (or, in geophysics, sometimes also *quasidynamic*).

It should be noted that the system (2.5) is truly standard, and can often be found in literature, at least in its parts. Its structure is, to a large extent, dictated by pursuing a consistent energetics and the gradient doubly-nonlinear structure. The evolution of  $\mathbf{F}_e$  and the multiplicative decomposition (2.5c) is indeed most often considered as a model for large-strain elastoplasticity and does not need any comments here, as well as the mass transport (2.5a) and the momentum equilibrium with the Kelvin-Voigt type Cauchy stress. The conservative (elastic) part of the Cauchy stress  $\widehat{\varphi}'_{\mathbf{F}}(\mathbf{F}, \mathbf{F}_p) \mathbf{F}^\top + \widehat{\varphi}(\mathbf{F}, \mathbf{F}_p) \mathbb{I}$  involves also a pressure contribution since the free energy  $\varphi$  is here considered per actual volume (and not per the referential volume or mass), cf. [58, Rem. 2]; the notation  $\mathbb{I}$  here and in what

follows stands for the identity matrix. The symmetry of such Cauchy stress is a standard consequence of the frame indifference of  $\varphi$  which is to be assumed, although we will not explicitly use it. The form of the rate  $\dot{\mathbf{F}}_p \mathbf{F}_p^{-1}$  which occurs in the dissipation potential, is used most often in a position of an inelastic distortion rate, cf. [7, 11, 12, 22, 23, 35, 44, 54, 64, 65], and is also compatible with the so-called plastic indifference, cf. e.g. [43]. The plastic flow rule (2.5d) is exactly as in [59], cf. also [32, Sec.9.4]. Sometimes, however, the plastic flow rule is formulated in the rate  $\mathbf{F}_e \dot{\mathbf{F}}_p \mathbf{F}_p^{-1} \mathbf{F}_e^{-1}$ , cf. Remark 3.2 below or, in Lagrangian setting, as  $(\mathbf{F}_p^\top \mathbf{F}_p)^\cdot = \dot{\mathbf{F}}_p^\top \mathbf{F}_p + \mathbf{F}_p^\top \dot{\mathbf{F}}_p$  in [21], too.

To reveal the mentioned energy dissipation balance behind the system (2.5), we complete it by suitable boundary conditions. An important aspect is to impose impenetrability of the boundary, which allows also for working on a fix domain even in the Eulerian description. This also simplifies many analytical arguments and is most often used in literature, too. Moreover, in our quasistatic case where inertial forces are neglected, we need to fix the body at a part of the boundary at least viscously. Thus we consider a combination of a homogeneous Dirichlet combination in the normal direction and the Newton (or Navier) condition in the tangential direction:

$$\mathbf{v} \cdot \mathbf{n} = 0 \quad \text{and} \quad [\mathbf{T}\mathbf{n}]_T + \kappa \mathbf{v}_T = \mathbf{f}, \quad (2.6)$$

where  $(\cdot)_T$  denotes the tangential component of a vector on the boundary  $\Gamma$  and  $\mathbf{n}$  is the unit outward normal to  $\Gamma$ . The first condition in (2.6) simplifies considerably the situation and allows for working on a fixed domain  $\Omega$ . Then, formally, we obtain the energetics by testing (2.5b) by  $\mathbf{v}$  and using (2.5c) tested by  $\mathbf{S}$  and by testing (2.5d) by  $\dot{\mathbf{F}}_p \mathbf{F}_p^{-1}$ , while (2.5a) does not directly contribute to the energetics because the inertial term  $\rho \dot{\mathbf{v}}$  has been neglected. The former test gives

$$\begin{aligned} \int_{\Omega} \operatorname{div} \mathbf{T} \cdot \mathbf{v} \, dx &= \int_{\Gamma} (\mathbf{T}\mathbf{n}) \cdot \mathbf{v} \, dS - \int_{\Omega} \mathbf{T} : \mathbf{e}(\mathbf{v}) \, dx \\ &= \int_{\Gamma} (\mathbf{T}\mathbf{n}) \cdot \mathbf{v} \, dS - \int_{\Omega} \left( \tilde{\varphi}'_{\mathbf{F}}(\mathbf{F}, \mathbf{F}_p) \mathbf{F}^\top + \hat{\varphi}(\mathbf{F}, \mathbf{F}_p) \mathbb{I} + \xi'(\mathbf{e}(\mathbf{v})) \right) : \mathbf{e}(\mathbf{v}) \, dx \\ &= \int_{\Gamma} (\mathbf{T}\mathbf{n}) \cdot \mathbf{v} \, dS - \int_{\Omega} \left( \tilde{\varphi}'_{\mathbf{F}}(\mathbf{F}, \mathbf{F}_p) : (\nabla \mathbf{v}) \mathbf{F} + \hat{\varphi}(\mathbf{F}, \mathbf{F}_p) \operatorname{div} \mathbf{v} + \xi'(\mathbf{e}(\mathbf{v})) : \mathbf{e}(\mathbf{v}) \right) : \mathbf{e}(\mathbf{v}) \, dx \\ &= \int_{\Gamma} (\mathbf{T}\mathbf{n}) \cdot \mathbf{v} \, dS - \int_{\Omega} \tilde{\varphi}'_{\mathbf{F}}(\mathbf{F}, \mathbf{F}_p) : \dot{\mathbf{F}} + \hat{\varphi}(\mathbf{F}, \mathbf{F}_p) \operatorname{div} \mathbf{v} + \xi'(\mathbf{e}(\mathbf{v})) : \mathbf{e}(\mathbf{v}) \, dx \\ &= \int_{\Gamma} (\mathbf{T}\mathbf{n}) \cdot \mathbf{v} \, dS - \frac{d}{dt} \int_{\Omega} \hat{\varphi}(\mathbf{F}, \mathbf{F}_p) \, dx \\ &\quad - \int_{\Omega} \tilde{\varphi}'_{\mathbf{F}}(\mathbf{F}, \mathbf{F}_p) : (\mathbf{v} \cdot \nabla) \mathbf{F} + \hat{\varphi}(\mathbf{F}, \mathbf{F}_p) \operatorname{div} \mathbf{v} + \xi'(\mathbf{e}(\mathbf{v})) : \mathbf{e}(\mathbf{v}) - \tilde{\varphi}'_{\mathbf{F}_p}(\mathbf{F}, \mathbf{F}_p) : \frac{\partial \mathbf{F}_p}{\partial t} \, dx \\ &= \int_{\Gamma} (\mathbf{T}\mathbf{n}) \cdot \mathbf{v} \, dS - \frac{d}{dt} \int_{\Omega} \hat{\varphi}(\mathbf{F}, \mathbf{F}_p) \, dx - \int_{\Omega} \left( \tilde{\varphi}'_{\mathbf{F}}(\mathbf{F}, \mathbf{F}_p) : (\mathbf{v} \cdot \nabla) \mathbf{F} + \hat{\varphi}(\mathbf{F}, \mathbf{F}_p) \operatorname{div} \mathbf{v} \right. \\ &\quad \left. + \xi'(\mathbf{e}(\mathbf{v})) : \mathbf{e}(\mathbf{v}) - \tilde{\varphi}'_{\mathbf{F}_p}(\mathbf{F}, \mathbf{F}_p) : \dot{\mathbf{F}}_p + \tilde{\varphi}'_{\mathbf{F}_p}(\mathbf{F}, \mathbf{F}_p) : (\mathbf{v} \cdot \nabla) \mathbf{F}_p \right) \, dx \\ &= \int_{\Gamma} \mathbf{f} \cdot \mathbf{v} - \kappa |\mathbf{v}|^2 \, dS - \frac{d}{dt} \int_{\Omega} \hat{\varphi}(\mathbf{F}, \mathbf{F}_p) \, dx - \int_{\Omega} \xi'(\mathbf{e}(\mathbf{v})) : \mathbf{e}(\mathbf{v}) + \partial_{\dot{\mathbf{F}}_p} \hat{\zeta}(\mathbf{F}_p, \dot{\mathbf{F}}_p) : \dot{\mathbf{F}}_p \, dx, \quad (2.7) \end{aligned}$$

where the last equality results when using the inclusion (2.5d) tested by  $\dot{\mathbf{F}}_p$ ; actually, such test generally gives only an inequality but we implicitly rely on certain smoothness of  $\zeta$  out of 0 as assumed later in (4.2d). Here we have used several times the matrix algebra

$$A : (BC) = (B^\top A) : C = (AC^\top) : B \quad (2.8)$$

for any three square matrices  $A$ ,  $B$ , and  $C$ . For the last equality in (2.7), we also used the calculus

$$\begin{aligned} & \int_{\Omega} \widehat{\varphi}'_{\mathbf{F}}(\mathbf{F}, \mathbf{F}_p) : (\mathbf{v} \cdot \nabla) \mathbf{F} + \widehat{\varphi}'_{\mathbf{F}_p}(\mathbf{F}, \mathbf{F}_p) : (\mathbf{v} \cdot \nabla) \mathbf{F}_p + \widehat{\varphi}(\mathbf{F}, \mathbf{F}_p) \operatorname{div} \mathbf{v} \, dx \\ &= \int_{\Omega} \nabla \widehat{\varphi}(\mathbf{F}, \mathbf{F}_p) \cdot \mathbf{v} + \widehat{\varphi}(\mathbf{F}, \mathbf{F}_p) \operatorname{div} \mathbf{v} \, dx = \int_{\Gamma} \widehat{\varphi}(\mathbf{F}, \mathbf{F}_p) (\mathbf{v} \cdot \mathbf{n}) \, dS = 0, \end{aligned} \quad (2.9)$$

where we employed the Green formula and the boundary conditions  $\mathbf{v} \cdot \mathbf{n} = 0$ . Altogether, we obtain (at least formally) the expected energy dissipation balance

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \underbrace{\widehat{\varphi}(\mathbf{F}, \mathbf{F}_p)}_{\text{stored energy}} \, dx + \int_{\Omega} \underbrace{\xi'(\mathbf{e}(\mathbf{v})) : \mathbf{e}(\mathbf{v}) + \partial_{\dot{\mathbf{F}}_p} \widehat{\zeta}(\mathbf{F}_p, \dot{\mathbf{F}}_p) : \dot{\mathbf{F}}_p}_{\text{bulk dissipation rate}} \, dx \\ + \int_{\Gamma} \underbrace{\kappa |\mathbf{v}|^2}_{\text{boundary dissipation rate}} \, dS = \int_{\Omega} \underbrace{\rho \mathbf{g} \cdot \mathbf{v}}_{\text{power of external load}} \, dx + \int_{\Gamma} \underbrace{\mathbf{f} \cdot \mathbf{v}}_{\text{power of traction load}} \, dS. \end{aligned} \quad (2.10)$$

Actually, a usual assumption is that the inelastic deformation  $\mathbf{F}_p$  only concerns shear and does not affect volume variations. We call it *isochoric* and it means not only that that  $\det \mathbf{F}_p$  is positive (to make  $\mathbf{F}_p$  invertible) but even that  $\det \mathbf{F}_p = 1$ . Yet, the constraint  $\det \mathbf{F}_p = 1$  is not affine and, if it would be ensured by the conservative part (i.e. through the stored energy  $\phi$ ) and thus explicitly involved into (2.5) together with a corresponding Lagrange multiplier, the analytical treatment of such a differential-algebraic-type system would become extremely difficult and likely impossible. One modelling option is to consider this isochoric constraint only approximately by casting a hardening-like term acting on  $\det \mathbf{F}_p$  to ensure that  $\det \mathbf{F}_p$  is positive and close to 1; cf. [14, 32, 45, 59] for a Lagrangian formulation. Another option is, instead of the control of  $\det \mathbf{F}_p$  in the stored energy (implementing thus the isochoricity only approximately), to implement the isochoricity exactly in the dissipative part relying on the calculus

$$\overline{\det \dot{\mathbf{F}}_p} = \operatorname{Cof} \mathbf{F}_p : \dot{\mathbf{F}}_p = (\det \mathbf{F}_p) \mathbf{F}_p^{-\top} : \dot{\mathbf{F}}_p = (\det \mathbf{F}_p) \operatorname{tr}(\dot{\mathbf{F}}_p \mathbf{F}_p^{-1}) \quad (2.11)$$

and by considering

$$\zeta : \mathbb{R}^{d \times d} \rightarrow [0, +\infty] \text{ is convex, } \zeta(0) = 0, \quad \zeta(\mathbb{R}^{d \times d} \setminus \mathbb{R}_{\text{dev}}^{d \times d}) = +\infty \quad (2.12)$$

together with prescribing the constraint  $\det \mathbf{F}_p = 1$  on the initial condition; cf. also [23, Sect. 91.3]. Then (2.5d) ensures  $\operatorname{tr}(\dot{\mathbf{F}}_p \mathbf{F}_p^{-1}) = 0$  and, by (2.11),  $\det \mathbf{F}_p = 1$  provided the

initial inelastic deformation is isochoric. It is important that the trace-free constraint in (2.12) is linear, in contrast to the non-affine constraint  $\det \mathbf{F}_p = 1$ .

Let us however emphasize that the rigorous analysis of the system (2.5) would need still gradients in the dissipation potential, which we will use in the following sections, cf. (3.7) below, but which we intentionally ignored in (2.5) in order to explain the main structure of the model without unnecessary technicalities.

**Remark 2.1** (*A gradient structure of  $(\mathbf{v}, \mathbf{F})$* ). Implicitly, we have in mind the situation when  $(\mathbf{v}, \mathbf{F})$  is a gradient in the sense that  $(\mathbf{v}, \mathbf{F}) = [(\cdot), \nabla] \mathbf{y}$  of some deformation  $\mathbf{y}$  which, however, does not explicitly occur in (2.5). Indeed, an existence of some  $\mathbf{y}$  so that  $\mathbf{F} = \nabla_{\mathbf{x}} \mathbf{y}$  and  $\mathbf{v} = \dot{\mathbf{y}}$  is not automatic even if  $\mathbf{F}$  is a gradient of some deformation at an initial time. Rather, we can always construct the return mapping  $\boldsymbol{\xi}$  mentioned above by solving the simple transport equation  $\dot{\boldsymbol{\xi}} = \mathbf{0}$  with the initial condition  $\boldsymbol{\xi}(0) = \text{identity}$ . Then  $\mathbf{F} = (\nabla_{\mathbf{x}} \boldsymbol{\xi})^{-1}$  and, if  $\boldsymbol{\xi}(t) : \Omega \rightarrow \Omega$  is injective, the underlying deformation is  $\mathbf{y}(t) = \boldsymbol{\xi}^{-1}(t)$ . This global injectivity seems not automatic, however; cf. also [58, Rem. 7].

### 3 Rate form of plasticity – hypoplasticity

We will now express the original model (2.5) in terms of the energies  $\varphi$  and  $\zeta$  instead of  $\widehat{\varphi}$  and  $\widehat{\zeta}$ . By this way, the plastic distortion  $\mathbf{F}_p$  will be eliminated from the model, although it will be possible to reconstruct it if the initial condition is known. The plasticity evolution will be formulated exclusively in terms of plastic distortion rate  $\mathbf{L}_p$ , cf. (3.5) below, called here *hypoplasticity* in parallel how a rate formulated hyperelasticity is called hypo-elasticity [62]. Actually, formulating the model in terms of  $\mathbf{F}_e$  and  $\mathbf{L}_p$  instead of the multiplicative decomposition, was explicitly advocated in [7, p.249], emphasizing that  $\mathbf{F}_p$  bears no physical relevance.

Moreover, we will eliminate the equation for the mass transport (2.5a), although it will stay implicitly contained in the model. Relying on (2.2), one can determine the density  $\varrho$  instead of the transport equation for mass density (2.3) from the algebraic relation

$$\varrho = \frac{\varrho_0}{\det \mathbf{F}} \quad (3.1)$$

where  $\varrho_0$  is the mass density in the reference configuration. Indeed, one has the calculus

$$\frac{\dot{\varrho}}{\varrho} = \left( \varrho_0 \left( \frac{\dot{1}}{\det \mathbf{F}} \right) + \frac{\dot{\varrho}_0}{\det \mathbf{F}} \right) \frac{\det \mathbf{F}}{\varrho_0} = - \frac{\dot{\det \mathbf{F}}}{\det \mathbf{F}} = -\text{div } \mathbf{v} \quad (3.2)$$

because  $\dot{\varrho}_0 = 0$  and because, analogously to (2.11), we have

$$\begin{aligned} \overline{\dot{\det \mathbf{F}}} &= \text{Cof } \mathbf{F} : \dot{\mathbf{F}} = (\det \mathbf{F}) \mathbf{F}^{-\top} : \dot{\mathbf{F}} \\ &= (\det \mathbf{F}) \mathbb{I} : \dot{\mathbf{F}} \mathbf{F}^{-1} = (\det \mathbf{F}) \mathbb{I} : \nabla \mathbf{v} = (\det \mathbf{F}) \text{div } \mathbf{v}; \end{aligned} \quad (3.3)$$

here we used also (2.1) and the matrix algebra (2.8). Thus, the last equality in (3.2) is the transport equation (2.2) while (3.2) itself is just the continuity equation (2.3). This would allow (and is actually often used) for elimination of the continuity equation (2.5a) in Section 2.

Here, assuming again (2.12) and isochoricity of the initial plastic distortion and, thus, having  $\det \mathbf{F}_p = 1$  during the whole evolution, we have  $\det \mathbf{F} = \det(\mathbf{F}_e \mathbf{F}_p) = \det \mathbf{F}_e \det \mathbf{F}_p = \det \mathbf{F}_e$  and (3.1) can be written as

$$\varrho = \frac{\varrho_0}{\det \mathbf{F}_e}. \quad (3.4)$$

Moreover, applying the material derivative on (2.4) and using (2.5c), we obtain  $(\nabla \mathbf{v})\mathbf{F} = \dot{\mathbf{F}} = \dot{\mathbf{F}}_e \mathbf{F}_p + \mathbf{F}_e \dot{\mathbf{F}}_p$  and, multiplying it by  $\mathbf{F}^{-1} = \mathbf{F}_p^{-1} \mathbf{F}_e^{-1}$ , eventually we obtain

$$\nabla \mathbf{v} = \underbrace{\dot{\mathbf{F}}_e \mathbf{F}_e^{-1}}_{\substack{\text{elastic} \\ \text{distortion} \\ \text{rate}}} + \mathbf{F}_e \underbrace{\dot{\mathbf{F}}_p \mathbf{F}_p^{-1}}_{\substack{\text{plastic} \\ \text{distortion} \\ \text{rate} =: \mathbf{L}_p}} \mathbf{F}_e^{-1}, \quad (3.5)$$

cf. e.g. [5,7,12,22–25,27,28,35,42,54,55,64,65]; the terms “distortion rates” are due to [22,23] while sometimes  $\mathbf{L}_p$  is called a “plastic dissipation tensor” [7] or “velocity gradient of purely plastic deformation” in [35], etc.

Interestingly, in terms of plastic distortion rate  $\mathbf{L}_p$ , we will not see explicitly the plastic distortion  $\mathbf{F}_p$  and, multiplying (3.5) by  $\mathbf{F}_e$ , we obtain an evolution rule for  $\mathbf{F}_e$  even without any explicit occurrence of  $\mathbf{F}_e^{-1}$ , namely

$$\dot{\mathbf{F}}_e = (\nabla \mathbf{v})\mathbf{F}_e - \mathbf{F}_e \mathbf{L}_p. \quad (3.6)$$

Mainly for analytical reasons, we will enhance the dissipation potential from Sect. 2 by generally non-quadratic gradient terms as

$$\xi(\mathbf{e}(\mathbf{v})) + \zeta(\mathbf{L}_p) + \frac{\nu}{p} |\nabla \mathbf{e}(\mathbf{v})|^p + \frac{\mu}{q} |\nabla \mathbf{L}_p|^q \quad (3.7)$$

with some (presumably small) coefficients  $\nu, \mu > 0$ , cf. Remark 3.1 below. In the next Section 4, we will need both the gradient-term exponents sufficiently big, namely  $p > d$  and  $q > d$ .

The stress  $\hat{\varphi}'_{\mathbf{F}}(\mathbf{F}, \mathbf{F}_p) \mathbf{F}_p^\top$  in (2.5b) is to be written in terms of  $\varphi$  by the calculus

$$\hat{\varphi}'_{\mathbf{F}}(\mathbf{F}, \mathbf{F}_p) \mathbf{F}^\top = [\varphi(\mathbf{F} \mathbf{F}_p^{-1})]'_{\mathbf{F}} \mathbf{F}^\top = \varphi'(\mathbf{F}_e) \mathbf{F}_p^{-\top} (\mathbf{F}_e \mathbf{F}_p)^\top = \varphi'(\mathbf{F}_e) \mathbf{F}_e^\top.$$

When writing the plastic flow rule (2.5d) in terms of  $\mathbf{L}_p$  as a purely “algebraic” relation without any explicit reference to the relation  $\mathbf{L}_p = \dot{\mathbf{F}}_p \mathbf{F}_p^{-1}$  and using (3.1) together with the isochoric-inelasticity concept, we obtain the hypo-elastoplastic system as

$$\operatorname{div} \boldsymbol{\Sigma} + \varrho \mathbf{g} = \mathbf{0} \quad \text{with} \quad \varrho = \frac{\varrho_0}{\det \mathbf{F}_e}, \quad \boldsymbol{\Sigma} = \mathbf{T} - \operatorname{div} \mathfrak{H},$$

$$\text{and } \mathbf{T} = \mathbf{S}\mathbf{F}_e^\top + \varphi(\mathbf{F}_e)\mathbb{I} + \xi'(\mathbf{e}(\mathbf{v}))$$

$$\text{where } \mathbf{S} = \varphi'(\mathbf{F}_e) \quad \text{and} \quad \mathfrak{H} = \nu|\nabla\mathbf{e}(\mathbf{v})|^{p-2}\nabla\mathbf{e}(\mathbf{v}), \quad (3.8a)$$

$$\dot{\mathbf{F}}_e = (\nabla\mathbf{v})\mathbf{F}_e - \mathbf{F}_e\mathbf{L}_p, \quad (3.8b)$$

$$\partial\zeta(\mathbf{L}_p) - \text{div}(\mu|\nabla\mathbf{L}_p|^{q-2}\nabla\mathbf{L}_p) \ni \mathbf{F}_e^\top\mathbf{S}. \quad (3.8c)$$

The right-hand side in (3.8c), being a driving stress for the plastification process, can be identified as the *Eshelby stress* [5, 11, 17, 42]; actually, the Eshelby stress standardly contains also a pressure part like does also the elastic Cauchy stress  $\varphi'(\mathbf{F}_e)\mathbf{F}_e^\top + \varphi(\mathbf{F}_e)\mathbb{I}$  but such a pressure would not affect the isochoric plastic evolution. Again, the form of this stress is dictated essentially in order to achieve the desired energy dissipation balance (2.10), i.e. now (3.14) below.

The system (3.8) is to be completed by suitable boundary conditions counting also the gradient terms arising from the enhanced dissipation potential (3.7), say

$$\mathbf{v}\cdot\mathbf{n} = 0, \quad [\boldsymbol{\Sigma}\mathbf{n} - \text{div}_s(\mathfrak{H}\mathbf{n})]_{\Gamma} + \kappa\mathbf{v}_{\Gamma} = \mathbf{f}, \quad \nabla\mathbf{e}(\mathbf{v}):(\mathbf{n}\otimes\mathbf{n}) = \mathbf{0}, \quad \text{and} \quad \nabla\mathbf{L}_p\cdot\mathbf{n} = \mathbf{0}, \quad (3.9)$$

where the  $(d-1)$ -dimensional surface divergence is defined as

$$\text{div}_s = \text{tr}(\nabla_s) \quad \text{with} \quad \nabla_s v = \nabla v - \frac{\partial v}{\partial \mathbf{n}}\mathbf{n}, \quad (3.10)$$

where  $\text{tr}(\cdot)$  is the trace of a  $(d-1)\times(d-1)$ -matrix and  $\nabla_s v$  is the surface gradient of  $v$ .

The energetics behind the model (3.8) can be revealed by testing (3.8a) by  $\mathbf{v}$ , and (3.8b) by  $\mathbf{S}$  (or, more precisely, testing (3.5) by  $\mathbf{S}\mathbf{F}_e^\top$ ), and (3.8c) by  $\mathbf{L}_p$ . Using the Green formula and (3.5) tested by  $\mathbf{S}\mathbf{F}_e^\top$ , we obtain from the Cauchy stress  $\mathbf{T}$ :

$$\begin{aligned} \int_{\Omega} \text{div } \mathbf{T}\cdot\mathbf{v} \, dx &= \int_{\Gamma} \mathbf{v}\cdot\mathbf{T}\mathbf{n} \, dS - \int_{\Omega} (\mathbf{S}\mathbf{F}_e^\top):\nabla\mathbf{v} + \varphi(\mathbf{F}_e)\text{div } \mathbf{v} + \xi'(\mathbf{e}(\mathbf{v})):\mathbf{e}(\mathbf{v}) \, dx \\ &= \int_{\Gamma} \mathbf{v}\cdot\mathbf{T}\mathbf{n} \, dS - \int_{\Omega} \varphi'(\mathbf{F}_e)\mathbf{F}_e^\top:(\dot{\mathbf{F}}_e + \mathbf{F}_e\mathbf{L}_p)\mathbf{F}_e^{-1} + \varphi(\mathbf{F}_e)\text{div } \mathbf{v} + \xi'(\mathbf{e}(\mathbf{v})):\mathbf{e}(\mathbf{v}) \, dx \\ &= \int_{\Gamma} \mathbf{v}\cdot\mathbf{T}\mathbf{n} \, dS - \int_{\Omega} \varphi'(\mathbf{F}_e):\dot{\mathbf{F}}_e + \mathbf{F}_e^\top\varphi'(\mathbf{F}_e):\mathbf{L}_p + \varphi(\mathbf{F}_e)\text{div } \mathbf{v} + \xi'(\mathbf{e}(\mathbf{v})):\mathbf{e}(\mathbf{v}) \, dx \\ &= \int_{\Gamma} \mathbf{v}\cdot\mathbf{T}\mathbf{n} \, dS - \frac{d}{dt} \int_{\Omega} \varphi(\mathbf{F}_e) \, dt - \int_{\Omega} \partial\zeta(\mathbf{L}_p):\mathbf{L}_p + \mu|\nabla\mathbf{L}_p|^q + \xi'(\mathbf{e}(\mathbf{v})):\mathbf{e}(\mathbf{v}) \, dx, \end{aligned} \quad (3.11)$$

where we also used the matrix algebra (2.8) for  $\varphi'(\mathbf{F}_e)\mathbf{F}_e^\top:(\dot{\mathbf{F}}_e\mathbf{F}_e^{-1}) = \varphi'(\mathbf{F}_e)\mathbf{F}_e^\top\mathbf{F}_e^{-\top}:\dot{\mathbf{F}}_e = \varphi'(\mathbf{F}_e):\dot{\mathbf{F}}_e$  and for  $\varphi'(\mathbf{F}_e)\mathbf{F}_e^\top:(\mathbf{F}_e\mathbf{L}_p\mathbf{F}_e^{-1}) = \varphi'(\mathbf{F}_e):(\mathbf{F}_e\mathbf{L}_p) = \mathbf{F}_e^\top\varphi'(\mathbf{F}_e):\mathbf{L}_p$ . In comparison with Sect. (2), note that  $\partial_{\dot{\mathbf{F}}_p}\widehat{\zeta}(\mathbf{F}_p, \dot{\mathbf{F}}_p):\dot{\mathbf{F}}_p = \partial\zeta(\mathbf{L}_p):\mathbf{L}_p$ . The pressure term is to be treated similarly as in (2.9) by the calculus

$$\begin{aligned} &\int_{\Omega} \varphi'(\mathbf{F}_e) \left( \frac{\partial\mathbf{F}_e}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{F}_e \right) + \varphi(\mathbf{F}_e)\text{div } \mathbf{v} \, dx \\ &= \int_{\Omega} \frac{\partial}{\partial t} \varphi(\mathbf{F}_e) + \nabla\varphi(\mathbf{F}_e)\cdot\mathbf{v} + \varphi(\mathbf{F}_e)\text{div } \mathbf{v} \, dx = \frac{d}{dt} \int_{\Omega} \varphi(\mathbf{F}_e) \, dx + \int_{\Gamma} \underbrace{\varphi(\mathbf{F}_e)(\mathbf{v}\cdot\mathbf{n})}_{=0} \, dS. \end{aligned} \quad (3.12)$$

The further contribution from the hyperstress gives, using Green formula over  $\Omega$  twice and the surface Green formula over  $\Gamma$ , that

$$\begin{aligned}
\int_{\Omega} \operatorname{div}^2 \mathfrak{H} \cdot \mathbf{v} \, dx &= \int_{\Gamma} \mathbf{v} \cdot \operatorname{div} \mathfrak{H} \mathbf{n} \, dS - \int_{\Omega} \operatorname{div} \mathfrak{H} : \nabla \mathbf{v} \, dx \\
&= \int_{\Omega} \mathfrak{H} : \nabla^2 \mathbf{v} \, dx + \int_{\Gamma} \mathbf{n} \cdot \mathfrak{H} : \nabla \mathbf{v} - \mathbf{v} \cdot \operatorname{div} \mathfrak{H} \mathbf{n} \, dS \\
&= \int_{\Omega} \mathfrak{H} : \nabla^2 \mathbf{v} \, dx + \int_{\Gamma} \mathfrak{H} : (\mathbf{n} \otimes \mathbf{n}) + \mathbf{n} \cdot \mathfrak{H} : \nabla_s \mathbf{v} - \mathbf{v} \cdot \operatorname{div} \mathfrak{H} \mathbf{n} \, dS \\
&= \int_{\Omega} \nu |\nabla e(\mathbf{v})|^p \, dx + \int_{\Gamma} \mathfrak{H} : (\mathbf{n} \otimes \mathbf{n}) - (\operatorname{div}_s(\mathbf{n} \cdot \mathfrak{H}) + \operatorname{div} \mathfrak{H} \mathbf{n}) \cdot \mathbf{v} \, dS, \quad (3.13)
\end{aligned}$$

where we used the decomposition of  $\nabla \mathbf{v}$  into its normal and tangential parts, i.e. written componentwise  $\nabla \mathbf{v}_i = (\mathbf{n} \cdot \nabla \mathbf{v}_i) \mathbf{n} + \nabla_s \mathbf{v}_i$ .

Merging the boundary integrals in (3.11) and in (3.13) and using the boundary condition  $(\boldsymbol{\Sigma} \mathbf{n} - \operatorname{div}_s(\mathfrak{H} \mathbf{n}))_{\Gamma} + \kappa \mathbf{v}_{\Gamma} = \mathbf{f}$ , we thus obtain (at least formally) the energy dissipation balance

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega} \underbrace{\varphi(\mathbf{F}_e)}_{\text{stored energy}} \, dx + \int_{\Omega} \underbrace{\xi'(e(\mathbf{v})) : e(\mathbf{v}) + \partial \zeta(\mathbf{L}_p) : \mathbf{L}_p + \nu |\nabla e(\mathbf{v})|^p + \mu |\nabla \mathbf{L}_p|^q}_{\text{dissipation rate}} \, dx \\
+ \int_{\Gamma} \underbrace{\kappa |\mathbf{v}|^2}_{\text{boundary dissipation rate}} \, dS = \int_{\Omega} \underbrace{\rho \mathbf{g} \cdot \mathbf{v}}_{\text{power of external load}} \, dx + \int_{\Gamma} \underbrace{\mathbf{f} \cdot \mathbf{v}}_{\text{power of traction load}} \, dS. \quad (3.14)
\end{aligned}$$

If one is interested in an “a posteriori” reconstruction of the plastic distortion  $\mathbf{F}_p$ , one should prescribe also an initial condition  $\mathbf{F}_p|_{t=0} = \mathbf{F}_{p,0}$  and, by re-arranging (3.5), to use the *plastic-strain evolution* rule

$$\dot{\mathbf{F}}_p = \mathbf{L}_p \mathbf{F}_p. \quad (3.15)$$

Only at this “a posteriori” point, one should consider the assumption  $\det \mathbf{F}_{p,0} = 1$  on which the system (3.8) relied when arising from (2.5).

**Remark 3.1** (*Gradient theories in rates*). So-called gradient theories in continuum mechanical models are very standard, determining some internal length scales and often facilitating mathematical analysis. They can be applied to the stored energy or to the dissipative potential, i.e. they contribute to the conservative or to the dissipative parts of the model. Here we used the latter option in (3.7). The first gradient term leads to the hyperstress  $\mathfrak{H}$  in the momentum equation (3.8a) and is compliant with the so-called *2nd-grade non-simple fluid* concept devised by E. Fried and M. Gurtin [20] and, earlier and even more generally and nonlinearly, as *multipolar fluids* by J. Nečas et al. [2, 3, 49–52], following ideas of R.A. Toupin [61] and R.D. Mindlin [47] for elastic solids. The further gradient term in (3.7) gives rise to  $\operatorname{div}(\mu |\nabla \mathbf{L}_p|^{q-2} \nabla \mathbf{L}_p)$  in the plastic-rate evolution (3.8c). This causes a certain “dynamical” hardening involves a certain length scale to the plastic distortion but,

does not make spurious hardening effects during long lasting plastification or creep, unlike the conventional kinematic or isotropic hardening in the conservative part. Similarly, in Lagrangian formulation, [14] used the plastic distortion rate in  $\Delta \frac{\partial}{\partial t} \mathbf{F}_p$ .

**Remark 3.2** (*An alternative model*). There is not a general agreement on an interpretation of the additive split of the rate in (3.5). One can also work with the plastic rate as  $\mathbf{L}_p = \mathbf{F}_e \dot{\mathbf{F}}_p \mathbf{F}_p^{-1} \mathbf{F}_e^{-1}$ , cf. [6, Sect.10.4], [7, Formula (7.1.4)], [30, Formulae (2.5)–(2.7)], or [63, Formulae (95)–(96)]. This gives the elastic-strain evolution rule  $\dot{\mathbf{F}}_e = (\nabla \mathbf{v} - \mathbf{L}_p) \mathbf{F}_e$  instead of (3.6), cf. [7, Sect.7.1], and the right-hand side of the plastic flow rule (3.8c) is  $-\mathbf{S} \mathbf{F}_e^\top$  instead of the Eshelby stress  $-\mathbf{F}_e^\top \mathbf{S}$ . The dissipation potential should act on this alternative  $\mathbf{L}_p$ . The plastic distortion can be reconstructed, instead of (3.15), by  $\dot{\mathbf{F}}_p = \mathbf{F}_e^{-1} \mathbf{L}_p \mathbf{F}_e \mathbf{F}_p$ . This alternative model can capture the zero plastic spin, provided the plastic spin is understood as the skew-symmetric part of  $\mathbf{L}_p$  because the driving (Cauchy) stress  $\mathbf{S} \mathbf{F}_e^\top$  is symmetric. Sometimes, the plastic spin is however understood as the skew-symmetric part of  $\mathbf{L}_p$  when the splitting (3.5) is used, cf. [23, Sect.91.3]. For the discussion according both variant see [7, Sect.7.1] or [48, Sect.4]. Actually, for small elastic deformations where  $\mathbf{F}_e \sim \mathbb{I}$ , both variants do not differ much from each other.

## 4 Analysis – weak solutions of (3.8)

We will provide a proof of existence and certain regularity of weak solutions. It should be emphasized that, even with the nonlinear dissipative gradient terms which have regularizing effects, it is quite nontrivial.

We will use the standard notation concerning the Lebesgue and the Sobolev spaces, namely  $L^p(\Omega; \mathbb{R}^n)$  for Lebesgue measurable functions  $\Omega \rightarrow \mathbb{R}^n$  whose Euclidean norm is integrable with  $p$ -power, and  $W^{k,p}(\Omega; \mathbb{R}^n)$  for functions from  $L^p(\Omega; \mathbb{R}^n)$  whose all derivative up to the order  $k$  have their Euclidean norm integrable with  $p$ -power. We also write briefly  $H^k = W^{k,2}$ . The notation  $p^*$  will denote the exponent from the embedding  $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ , i.e.  $p^* = dp/(d-p)$  for  $p < d$  while  $p^* \geq 1$  arbitrary for  $p = d$  or  $p^* = +\infty$  for  $p > d$ . Moreover, for a Banach space  $X$  and for  $I = [0, T]$ , we will use the notation  $L^p(I; X)$  for the Bochner space of Bochner measurable functions  $I \rightarrow X$  whose norm is in  $L^p(I)$  while  $W^{1,p}(I; X)$  denotes for functions  $I \rightarrow X$  whose distributional derivative is in  $L^p(I; X)$ . Also,  $C(\cdot)$  and  $C^1(\cdot)$  will denote spaces of continuous and continuously differentiable functions.

A highly applicable assertion was originally devised for situations when  $\mathbf{F} = \nabla \mathbf{y}$  with  $\mathbf{y} \in W^{2,p}(\Omega; \mathbb{R}^d)$  but actually it holds in more general situations, as used also in [32,45,46,59]:

**Lemma 4.1** (T.J. Healey and S. Krömer [26]). *Let  $\varkappa > rd/(r-d)$  for some  $r > d$ . Then, for any  $C < +\infty$ , there is  $\epsilon > 0$  such that, for any  $\mathbf{F} \in W^{1,r}(\Omega; \mathbb{R}^{d \times d})$  with  $\det \mathbf{F} > 0$  a.e. on  $\Omega$ , it holds*

$$\|\mathbf{F}\|_{W^{1,r}(\Omega; \mathbb{R}^{d \times d})} + \int_{\Omega} \frac{1}{(\det \mathbf{F})^\varkappa} dx \leq C \quad \Rightarrow \quad \det \mathbf{F} \geq \epsilon \quad \text{on } \bar{\Omega}.$$

To devise a weak formulation of the initial-boundary-value problem for the system (3.8), we use also by-part integration in time and the Green formula also for  $\dot{\mathbf{F}}_e$  in the evolution rule (3.8b) tested by a smooth  $\tilde{\mathbf{S}}$  with  $\tilde{\mathbf{S}}(T) = 0$  together  $\mathbf{v} \cdot \mathbf{n} = 0$ , we obtain

$$\begin{aligned} \int_0^T \int_{\Omega} \dot{\mathbf{F}}_e : \tilde{\mathbf{S}} \, dx dt &= \int_0^T \int_{\Omega} \left( \frac{\partial \mathbf{F}_e}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{F}_e \right) : \tilde{\mathbf{S}} \, dx dt = \int_0^T \int_{\Gamma} (\mathbf{v} \cdot \mathbf{n}) (\mathbf{F}_e : \tilde{\mathbf{S}}) \, dS dt \\ &\quad - \int_0^T \int_{\Omega} \mathbf{F}_e : \frac{\partial \tilde{\mathbf{S}}}{\partial t} + (\operatorname{div} \mathbf{v}) \mathbf{F}_e : \tilde{\mathbf{S}} + \mathbf{F}_e : ((\mathbf{v} \cdot \nabla) \tilde{\mathbf{S}}) \, dx dt - \int_{\Omega} \mathbf{F}_{e,0} : \tilde{\mathbf{S}}(0) \, dx. \end{aligned}$$

Actually,  $\mathbf{F}_e : \frac{\partial}{\partial t} \tilde{\mathbf{S}} + \mathbf{F}_e : ((\mathbf{v} \cdot \nabla) \tilde{\mathbf{S}})$  can be written “elegantly” as  $\dot{\tilde{\mathbf{S}}}$  but it combines the testing  $\tilde{\mathbf{S}}$  with the solution  $\mathbf{v}$  and we will better not use such a “too compact” form.

**Definition 4.2** (Weak solutions to (3.8)). *A triple  $(\mathbf{v}, \mathbf{F}_e, \mathbf{L}_p) \in L^2(I; W^{2,p}(\Omega; \mathbb{R}^d)) \times L^\infty(I; L^r(\Omega; \mathbb{R}^{d \times d})) \times L^\infty(I; W^{1,q}(\Omega; \mathbb{R}_{\text{dev}}^{d \times d}))$  will be called a weak solution to the system (3.8) with the boundary conditions (3.9) and the initial condition  $\mathbf{F}_e|_{t=0} = \mathbf{F}_{e,0}$  if  $\mathbf{v} \cdot \mathbf{n} = 0$ ,  $\det \mathbf{F}_e > 0$  a.e. such that  $\varrho = \varrho_0 / \det \mathbf{F}_e \in L^\infty(I \times \Omega)$ , and the integral identities*

$$\begin{aligned} \int_0^T \int_{\Omega} (\varphi'(\mathbf{F}_e) \mathbf{F}_e^\top + \xi'(e(\mathbf{v}))) : \nabla \tilde{\mathbf{v}} - \varphi(\mathbf{F}_e) (\operatorname{div} \tilde{\mathbf{v}}) + \nu |\nabla e(\mathbf{v})|^{p-2} \nabla e(\mathbf{v}) : \nabla e(\tilde{\mathbf{v}}) \, dx dt \\ + \int_0^T \int_{\Gamma} \kappa \mathbf{v} \cdot \tilde{\mathbf{v}} \, dS dt = \int_0^T \int_{\Omega} \varrho \mathbf{g} \cdot \tilde{\mathbf{v}} \, dx dt + \int_0^T \int_{\Gamma} \mathbf{f} \cdot \tilde{\mathbf{v}} \, dS dt \end{aligned} \quad (4.1a)$$

and

$$\begin{aligned} \int_0^T \int_{\Omega} \left( \mathbf{F}_e : \frac{\partial \tilde{\mathbf{S}}}{\partial t} + \left( (\operatorname{div} \mathbf{v}) \mathbf{F}_e + (\nabla \mathbf{v}) \mathbf{F}_e - \mathbf{F}_e \mathbf{L}_p \right) : \tilde{\mathbf{S}} \right. \\ \left. + \mathbf{F}_e : ((\mathbf{v} \cdot \nabla) \tilde{\mathbf{S}}) \right) \, dx dt = - \int_{\Omega} \mathbf{F}_{e,0} : \tilde{\mathbf{S}}(0) \, dx \end{aligned} \quad (4.1b)$$

hold for any  $\tilde{\mathbf{v}}$  and  $\tilde{\mathbf{S}}$  smooth with  $\tilde{\mathbf{v}} \cdot \mathbf{n} = 0$ ,  $\tilde{\mathbf{v}}(T) = 0$ , and  $\tilde{\mathbf{S}}(T) = \mathbf{0}$ , and also the variational inequality

$$\int_0^T \int_{\Omega} \zeta(\tilde{\mathbf{L}}_p) + \frac{\mu}{q} |\nabla \tilde{\mathbf{L}}_p|^q - \mathbf{F}_e^\top \mathbf{S} : (\tilde{\mathbf{L}}_p - \mathbf{L}_p) \, dx dt \geq \int_0^T \int_{\Omega} \zeta(\mathbf{L}_p) + \frac{\mu}{q} |\nabla \mathbf{L}_p|^q \, dx dt \quad (4.1c)$$

holds for any  $\tilde{\mathbf{L}}_p \in L^\infty(I; W^{1,q}(\Omega; \mathbb{R}_{\text{dev}}^{d \times d}))$ .

Before stating the main analytical result, let us summarize the data qualification (for some  $\epsilon > 0$ ):

$$\Omega \text{ a smooth bounded domain of } \mathbb{R}^d, \quad d = 2, 3 \quad (4.2a)$$

$$\varphi \in C^1(\operatorname{GL}^+(d)), \quad \varphi(F) \geq \epsilon / (\det F)^\varkappa, \quad (4.2b)$$

$$\xi \in C^1(\mathbb{R}_{\text{sym}}^{d \times d}) \text{ convex, } \xi(0) = 0, \quad \sup_{\mathbb{R}_{\text{sym}}^{d \times d}} |\xi'(\cdot)| / (1 + |\cdot|^{p-1}) < \infty, \quad (4.2c)$$

$$\zeta : \mathbb{R}^{d \times d} \rightarrow [0, +\infty] \text{ satisfy (2.12), } \forall L \in \mathbb{R}_{\text{dev}}^{d \times d} : \lambda \mapsto \zeta(\lambda L) \text{ is differentiable at } \lambda = 1,$$

$$\exists q_0 \geq 1 : \inf_{\mathbb{R}_{\text{dev}}^{d \times d} \setminus \{0\}} \zeta(\cdot) / |\cdot|^{q_0} > 0 \quad \text{and} \quad \sup_{\mathbb{R}_{\text{dev}}^{d \times d}} |\partial \xi(\cdot)| / (1 + |\cdot|^{q-1}) < \infty, \quad (4.2d)$$

$$\kappa \in L^\infty(\Gamma), \quad \text{ess inf } \kappa > 0, \quad \nu > 0, \quad \mu > 0, \quad (4.2e)$$

$$\mathbf{g} \in L^{2\kappa/(\kappa-2)}(I; L^{\kappa'}(\Omega; \mathbb{R}^d)), \quad \mathbf{f} \in L^2(I \times \Gamma; \mathbb{R}^d), \quad (4.2f)$$

$$\mathbf{F}_{e,0} \in W^{1,r}(\Omega; \mathbb{R}^{d \times d}) \quad \text{with} \quad \text{ess inf}_\Omega \det \mathbf{F}_{e,0} > 0, \quad (4.2g)$$

$$\varrho_0 \in L^\infty(\Omega) \cap W^{1,1}(\Omega) \quad \text{with} \quad \text{ess inf}_\Omega \varrho_0 > 0. \quad (4.2h)$$

where  $\text{GL}^+(d) = \{F \in \mathbb{R}^{d \times d}; \det F > 0\}$  denotes the orientation-preserving general linear group. As we admit  $q_0 = 1$  in (4.2d), the ‘‘essential part’’ of the dissipation potential (3.7) can be degree-1 homogeneous, which would lead to a rate-independent plasticity like e.g. [21, 38, 43, 45] although all other dissipative mechanisms stay rate dependent.

**Proposition 4.3** (Existence and regularity of weak solutions). *Let  $\min(p, q) > d$  and the assumptions (4.2) hold for  $\kappa > rd/(r-d)$  with some  $r > d$ . Then:*

- (i) *there exist a weak solution according Definition 4.2 such that also  $\frac{\partial}{\partial t} \mathbf{F}_e \in L^2(I; L^r(\Omega; \mathbb{R}^{d \times d}))$  and  $\nabla \mathbf{F}_e \in L^\infty(I; L^r(\Omega; \mathbb{R}^{d \times d \times d}))$ . Moreover, it conserves energy in the sense that the energy dissipation balance (3.14) integrated over time interval  $[0, t]$  with the initial condition  $\mathbf{F}_e|_{t=0} = \mathbf{F}_{e,0}$  holds.*
- (ii) *If also  $\mathbf{F}_{p,0} \in L^s(\Omega; \mathbb{R}^{d \times d})$  with some  $s > 1$  and  $\det \mathbf{F}_{p,0} = 1$  a.e. on  $\Omega$ , then the corresponding plastic distortion  $\mathbf{F}_p$  reconstructed as a unique weak solution to (3.15) belongs to  $L^\infty(I; L^s(\Omega; \mathbb{R}^{d \times d}))$  and  $\det \mathbf{F}_p = 1$  a.e. on  $I \times \Omega$ .*
- (iii) *If also  $\mathbf{F}_{p,0} \in W^{1,s}(\Omega; \mathbb{R}_{\text{dev}}^{d \times d})$  with some  $s > 1$ , then the plastic distortion  $\mathbf{F}_p$  belongs to  $L^\infty(I; W^{1,s}(\Omega; \mathbb{R}_{\text{dev}}^{d \times d})) \cap W^{1,p}(I; L^s(\Omega; \mathbb{R}_{\text{dev}}^{d \times d}))$  and the deformation gradient  $\mathbf{F} = \mathbf{F}_e \mathbf{F}_p \in L^\infty(I; W^{1, \min(s, s^*r/(s^*+r))}(\Omega; \mathbb{R}^{d \times d}))$  with  $\frac{\partial}{\partial t} \mathbf{F} \in L^2(I; L^{s^*r/(s^*+r)}(\Omega; \mathbb{R}^{d \times d})) + L^p(I; L^s(\Omega; \mathbb{R}^{d \times d}))$ , and  $\text{ess inf}_{I \times \Omega} \det \mathbf{F} > 0$ .*

*Proof.* For clarity, we will divide the proof into nine steps.

*Step 1: a regularization and discretization.* Let us first make formally the a priori estimates which follow from the energetics (3.14) when use the assumptions (4.2) for  $\kappa > rd/(r-d)$  with some  $r > d$  and  $\min(p, q) > d$  and the Healey-Krömer Lemma.

The only difficult term is  $\varrho \mathbf{g} \cdot \mathbf{v}$  on the right-hand side of (3.14), which can be estimated by Hölder’s and Young’s inequalities as

$$\begin{aligned} \int_\Omega \varrho \mathbf{g} \cdot \mathbf{v} \, dx &= \int_\Omega \frac{\varrho_0}{\det \mathbf{F}_e} \mathbf{g} \cdot \mathbf{v} \, dx \leq \left\| \frac{\varrho_0}{\det \mathbf{F}_e} \right\|_{L^\kappa(\Omega)} \|\mathbf{v}\|_{L^\infty(\Omega; \mathbb{R}^d)} \|\mathbf{g}\|_{L^{\kappa'}(\Omega; \mathbb{R}^d)} \\ &\leq C_\epsilon \left( 1 + \left\| \frac{\varrho_0}{\det \mathbf{F}_e} \right\|_{L^\kappa(\Omega)}^\kappa + \|\mathbf{g}\|_{L^{\kappa'}(\Omega; \mathbb{R}^d)}^{2\kappa/(\kappa-2)} \right) + \epsilon \|\nabla \mathbf{e}(\mathbf{v})\|_{L^p(\Omega; \mathbb{R}^{d \times d \times d})}^p + \epsilon \|\mathbf{v}|_\Gamma\|_{L^2(\Gamma; \mathbb{R}^d)}^2, \end{aligned} \quad (4.3)$$

where we used  $\|\mathbf{v}\|_{L^\infty(\Omega; \mathbb{R}^d)} \leq C(\|\nabla \mathbf{e}(\mathbf{v})\|_{L^p(\Omega; \mathbb{R}^{d \times d \times d})} + \|\mathbf{v}|_\Gamma\|_{L^2(\Gamma; \mathbb{R}^d)})$ ; here a Korn-Poincaré inequality with the Navier boundary condition for  $\mathbf{v}$  is exploited; for even a stronger variant exploiting only an  $L^2$ -norm of the deviatoric part of  $\mathbf{e}(\mathbf{v})$  with  $L^1$ -norm of the instead of the  $L^p$ -norm of  $\nabla \mathbf{e}(\mathbf{v})$  cf. [19, Theorems 10.16–10.17]. The term  $\|\varrho_0/\det \mathbf{F}_e\|_{L^\kappa(\Omega)}^\kappa$  in (4.3) can thus be treated by the Gronwall inequality relying on the blow-up assumption  $\varphi(F) \geq \epsilon/(\det F)^\kappa$  in (4.2b). Of course, we choose  $\epsilon > 0$  sufficiently small so that the

last and the penultimate terms in (4.3) can be absorbed in the left-hand side of the energy balance. From (3.14), we thus obtain

$$\|\mathbf{v}\|_{L^2(I;W^{2,p}(\Omega;\mathbb{R}^d))} \leq C \quad \text{with} \quad \|\nabla \mathbf{e}(\mathbf{v})\|_{L^p(I \times \Omega; \mathbb{R}^{d \times d \times d})} \leq C, \quad (4.4a)$$

$$\|\mathbf{L}_p\|_{L^{q_0}(I \times \Omega; \mathbb{R}_{\text{dev}}^{d \times d})} \leq C \quad \text{with} \quad \|\nabla \mathbf{L}_p\|_{L^q(I \times \Omega; \mathbb{R}^{d \times d \times d})} \leq C. \quad (4.4b)$$

and  $\int_{\Omega} \varphi(\mathbf{F}_e(t)) \, dx$  bounded uniformly in time. The exponent 2 in the  $L^2$ -estimate (4.4a) is due to the linearity of the Navier boundary condition and thus due to the only quadratic growth of the term  $\kappa|\mathbf{v}|^2$  in (3.14). The former estimate in (4.4a) together with the qualification (4.2g) of the initial condition  $\mathbf{F}_{e,0}$  can then be exploited for the estimation as (4.11) and (4.18) below to obtain  $W^{1,r}$ -regularity of  $\mathbf{F}_e$ . The usage of Lemma 4.1 gives  $1/\det \mathbf{F}_e(t)$  bounded in  $L^\infty(\Omega)$ . Altogether, for some  $\varepsilon > 0$ , we obtain

$$\|\mathbf{F}_e\|_{L^\infty(I;W^{1,r}(\Omega;\mathbb{R}^{d \times d}))} \leq C \quad \text{with} \quad \text{essinf}_{I \times \Omega} \det \mathbf{F}_e > \varepsilon. \quad (4.4c)$$

As  $r > d$ , (4.4c) implies also

$$|\mathbf{F}_e| < \frac{1}{\varepsilon}, \quad \varphi(\mathbf{F}_e) < \frac{1}{\varepsilon}, \quad \text{and} \quad |\varphi'(\mathbf{F}_e)| < \frac{1}{\varepsilon} \quad \text{a.e. on } I \times \Omega; \quad (4.4d)$$

without loss of generality, we may take  $\varepsilon > 0$  small enough so that both (4.4c) and (4.4d) hold with the same  $\varepsilon$ . Thus,  $|\varphi'(\mathbf{F}_e)\mathbf{F}_e^\top| \leq 1/\varepsilon^2$  and  $|\mathbf{F}_e^\top \varphi'(\mathbf{F}_e)| \leq 1/\varepsilon^2$ . Moreover, from (3.8c) by comparison realizing that the Eshelby stress on the right-hand side is bounded on  $I \times \Omega$ , we can even improve the time integrability of (4.4b) as

$$\|\mathbf{L}_p\|_{L^\infty(I;W^{1,q}(\Omega;\mathbb{R}_{\text{dev}}^{d \times d}))} \leq C. \quad (4.4e)$$

Then, taking this  $\varepsilon > 0$  from (4.4c,d), we make a regularization of the right-hand side of the momentum equation and the Cauchy and the Eshelby stresses and, for  $k \in \mathbb{N}$ , a parabolic regularization of the evolution equation (3.8b) for  $\mathbf{F}_e$ . Altogether, we devise the regularized system as

$$\operatorname{div} \boldsymbol{\Sigma} + \varrho \mathbf{g} = \mathbf{0} \quad \text{with} \quad \varrho = \frac{\varrho_0}{\max(\det \mathbf{F}_e, \varepsilon)} \quad \text{and} \quad \boldsymbol{\Sigma} = \mathbf{T} - \operatorname{div} \boldsymbol{\mathfrak{H}},$$

$$\text{where } \mathbf{T} = \frac{\mathbf{S}\mathbf{F}_e^\top}{1 + (|\mathbf{S}\mathbf{F}_e^\top| - 1/\varepsilon^2)^+} + \frac{\varphi(\mathbf{F}_e)\mathbb{I}}{1 + (\varphi(\mathbf{F}_e) - 1/\varepsilon)^+} + \xi'(\mathbf{e}(\mathbf{v}))$$

$$\text{with } \mathbf{S} = \varphi'(\mathbf{F}_e) \quad \text{and} \quad \boldsymbol{\mathfrak{H}} = \nu |\nabla \mathbf{e}(\mathbf{v})|^{p-2} \nabla \mathbf{e}(\mathbf{v}), \quad (4.5a)$$

$$\dot{\mathbf{F}}_e = (\nabla \mathbf{v})\mathbf{F}_e - \mathbf{F}_e \mathbf{L}_p + k^{-1} \operatorname{div}(|\nabla \mathbf{F}_e|^{r-2} \nabla \mathbf{F}_e), \quad (4.5b)$$

$$\partial \zeta(\mathbf{L}_p) - \operatorname{div}(\mu |\nabla \mathbf{L}_p|^{q-2} \nabla \mathbf{L}_p) \ni \frac{\mathbf{F}_e^\top \mathbf{S}}{1 + (|\mathbf{F}_e^\top \mathbf{S}| - 1/\varepsilon^2)^+} \quad (4.5c)$$

with  $(\cdot)^+ = \max(\cdot, 0)$ . The boundary conditions (3.9) must now be complemented by some boundary condition for the regularizing term in (4.5b), say  $\mathbf{n} \cdot \nabla \mathbf{F}_e = \mathbf{0}$ . Of course, we will be interested in weak solutions to (4.5) with these boundary conditions and the initial condition

for  $\mathbf{F}_e$ . The corresponding weak formulation a'la Definition 4.2 is quite straightforward and we will not explicitly write it, also because it is obvious from its Galerkin version (4.6) below. The philosophy of the regularization (4.5) is that the estimation of (4.5a) and (4.5b,c) decouples and simultaneously the a priori estimates are the same as the formal estimates (4.4) and, when taking  $\varepsilon > 0$  small to comply with (4.4c,d), the  $\varepsilon$ -regularization becomes eventually inactive, cf. Step 7 below. Moreover, the parabolic regularization of the flow rule (4.5b) can be suppressed, cf. Step 6.

Then we make a conformal Galerkin approximation of (4.5a) by using a nested finite-dimensional subspaces  $\{V_k\}_{k \in \mathbb{N}}$  whose union is dense in  $W^{2,p}(\Omega; \mathbb{R}^d)$ ; note that they are indexed by the same  $k \in \mathbb{N}$  as used in (4.5b). Separately, we make a Galerkin approximation of (4.5b) and (4.5c) by using other nested finite-dimensional subspaces  $\{W_l\}_{l \in \mathbb{N}}$  whose union is dense in  $W^{1,\max(q,r)}(\Omega; \mathbb{R}^{d \times d})$ , using another index  $l \in \mathbb{N}$ . Also, the trace-free functions from  $\{W_l\}_{l \in \mathbb{N}}$  are dense in the space  $\{\mathbf{L} \in W^{1,q}(\Omega; \mathbb{R}^{d \times d}); \text{tr}(\mathbf{L}) = 0\}$ . Without loss of generality, we may assume  $\mathbf{v} \in V_1$  and  $\mathbf{F}_{e,0} \in W_1$ .

The approximate solution of the regularized system (4.5) will be denoted by  $(\mathbf{v}_{kl}, \mathbf{F}_{e,kl}, \mathbf{L}_{p,kl}) : I \rightarrow V_k \times W_l \times W_l$ . Specifically, such a triple should satisfy the following integral identities

$$\begin{aligned} \int_0^T \int_\Omega \left( \frac{\varphi'(\mathbf{F}_{e,kl}) \mathbf{F}_{e,kl}^\top}{1 + (|\varphi'(\mathbf{F}_{e,kl}) \mathbf{F}_{e,kl}^\top| - 1/\varepsilon^2)^+} + \xi'(\mathbf{e}(\mathbf{v}_{kl})) \right) : \nabla \tilde{\mathbf{v}} + \frac{\varphi(\mathbf{F}_{e,kl}) \text{div} \tilde{\mathbf{v}}}{1 + (\varphi(\mathbf{F}_{e,kl}) - 1/\varepsilon)^+} \\ + \nu |\nabla \mathbf{e}(\mathbf{v}_{kl})|^{p-2} \nabla \mathbf{e}(\mathbf{v}_{kl}) : \nabla \mathbf{e}(\tilde{\mathbf{v}}) \, dx dt + \int_0^T \int_\Gamma \kappa \mathbf{v}_{kl} \cdot \tilde{\mathbf{v}} \, dS dt \\ = \int_0^T \int_\Omega \frac{\varrho_0 \mathbf{g}}{\max(\det \mathbf{F}_{e,kl}, \varepsilon)} \cdot \tilde{\mathbf{v}} \, dx dt + \int_0^T \int_\Gamma \mathbf{f} \cdot \tilde{\mathbf{v}} \, dS dt, \end{aligned} \quad (4.6a)$$

and

$$\begin{aligned} \int_0^T \int_\Omega \left( \mathbf{F}_{e,kl} : \frac{\partial \tilde{\mathbf{S}}}{\partial t} + \left( (\text{div} \mathbf{v}_{kl}) \mathbf{F}_{e,kl} + (\nabla \mathbf{v}_{kl}) \mathbf{F}_{e,kl} - \mathbf{F}_{e,kl} \mathbf{L}_{p,kl} \right) : \tilde{\mathbf{S}} \right. \\ \left. + \mathbf{F}_{e,kl} : ((\mathbf{v}_{kl} \cdot \nabla) \tilde{\mathbf{S}}) - \frac{1}{k} |\nabla \mathbf{F}_{e,kl}|^{r-2} \nabla \mathbf{F}_{e,kl} : \nabla \tilde{\mathbf{S}} \right) \, dx dt = - \int_\Omega \mathbf{F}_{e,0} : \tilde{\mathbf{S}}(0) \, dx \end{aligned} \quad (4.6b)$$

for any  $\tilde{\mathbf{v}} \in L^\infty(I; V_k)$  and  $\tilde{\mathbf{S}} \in L^\infty(I; W_l)$  with  $\tilde{\mathbf{v}} \cdot \mathbf{n} = 0$ ,  $\tilde{\mathbf{v}}(T) = 0$ , and  $\tilde{\mathbf{S}}(T) = \mathbf{0}$ , and also the variational inequality

$$\begin{aligned} \int_0^T \int_\Omega \zeta(\tilde{\mathbf{L}}_p) + \frac{\mu}{q} |\nabla \tilde{\mathbf{L}}_p|^q - \frac{\mathbf{F}_{e,kl}^\top \varphi'(\mathbf{F}_{e,kl})}{1 + (|\mathbf{F}_{e,kl}^\top \varphi'(\mathbf{F}_{e,kl})| - 1/\varepsilon^2)^+} : (\tilde{\mathbf{L}}_p - \mathbf{L}_{p,kl}) \, dx dt \\ \geq \int_0^T \int_\Omega \zeta(\mathbf{L}_{p,kl}) + \frac{\mu}{q} |\nabla \mathbf{L}_{p,kl}|^q \, dx dt \end{aligned} \quad (4.6c)$$

should hold for any  $\tilde{\mathbf{L}}_p \in L^\infty(I; W_l)$ ,  $\text{tr}(\tilde{\mathbf{L}}_p) = 0$  a.e. on  $I \times \Omega$ .

Existence of this solution is based on the theory of systems of ordinary differential equations first locally in time, and then by successive prolongation on the whole time interval based on the  $L^\infty$ -estimates below.

*Step 2: first a priori estimates.* The basic test of the Galerkin approximation of (4.5) can be done by  $(\mathbf{v}_{kl}, \mathbf{F}_{e,kl}, \mathbf{L}_{p,kl})$ . In particular, for (4.5b), we use the estimate

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} \frac{1}{2} |\mathbf{F}_{e,kl}|^2 dx + \frac{1}{k} \int_{\Omega} |\nabla \mathbf{F}_{e,kl}|^r dx \\
& \leq \int_{\Omega} \left( (\nabla \mathbf{v}_{kl}) \mathbf{F}_{e,kl} - (\mathbf{v}_{kl} \cdot \nabla) \mathbf{F}_{e,kl} - \mathbf{F}_{e,kl} \mathbf{L}_{p,kl} \right) : \mathbf{F}_{e,kl} dx \\
& = \int_{\Omega} (\nabla \mathbf{v}_{kl}) \mathbf{F}_{e,kl} : \mathbf{F}_{e,kl} + \frac{\operatorname{div} \mathbf{v}_{kl}}{2} |\mathbf{F}_{e,kl}|^2 - \mathbf{L}_{p,kl} : (\mathbf{F}_{e,kl}^{\top} \mathbf{F}_{e,kl}) dx \\
& \leq \left( \frac{3}{2} \|\nabla \mathbf{v}_{kl}\|_{L^{\infty}(\Omega; \mathbb{R}^{d \times d})} + \|\mathbf{L}_{p,kl}\|_{L^{\infty}(\Omega; \mathbb{R}^{d \times d})} \right) \|\mathbf{F}_{e,kl}\|_{L^2(\Omega; \mathbb{R}^{d \times d})}^2; \quad (4.7)
\end{aligned}$$

here we used also the calculus (for  $\mathbf{F} = \mathbf{F}_{e,kl}$  and  $\mathbf{v} = \mathbf{v}_{kl}$ )

$$\begin{aligned}
\int_{\Omega} (\mathbf{v} \cdot \nabla) \mathbf{F} : \mathbf{F} dx &= \int_{\Gamma} |\mathbf{F}|^2 (\mathbf{v} \cdot \mathbf{n}) dS \\
&\quad - \int_{\Omega} \mathbf{F} : (\mathbf{v} \cdot \nabla) \mathbf{F} + (\operatorname{div} \mathbf{v}) |\mathbf{F}|^2 dx = -\frac{1}{2} \int_{\Omega} (\operatorname{div} \mathbf{v}) |\mathbf{F}|^2 dx \quad (4.8)
\end{aligned}$$

together with the boundary condition  $\mathbf{v} \cdot \mathbf{n} = 0$ . By the Gronwall inequality exploiting the first left-hand-side term which does not contain the factor  $1/k$ , we obtain the estimate

$$\|\mathbf{F}_{e,kl}\|_{L^{\infty}(I; L^2(\Omega; \mathbb{R}^{d \times d}))} \leq \|\mathbf{F}_{e,0}\|_{L^2(\Omega; \mathbb{R}^{d \times d})} e^{\|\nabla \mathbf{v}_{kl}\|_{L^1(I; L^{\infty}(\Omega; \mathbb{R}^{d \times d}))} + \|\mathbf{L}_{p,kl}\|_{L^1(I; L^{\infty}(\Omega; \mathbb{R}^{d \times d}))}. \quad (4.9)$$

Thus, by this test, we obtain

$$\|\mathbf{v}_{kl}\|_{L^2(I; W^{2,p}(\Omega; \mathbb{R}^d))} \leq C, \quad (4.10a)$$

$$\|\mathbf{F}_{e,kl}\|_{L^{\infty}(I; L^2(\Omega; \mathbb{R}^{d \times d}))} \leq C \quad \text{with} \quad \|\nabla \mathbf{F}_{e,kl}\|_{L^r(I \times \Omega; \mathbb{R}^{d \times d \times d})} \leq C \sqrt[r]{k}, \quad (4.10b)$$

$$\|\mathbf{L}_{p,kl}\|_{L^{\infty}(I; W^{1,q}(\Omega; \mathbb{R}_{\text{dev}}^{d \times d}))} \leq C. \quad (4.10c)$$

Particularly, let us note that the regularized force  $\varrho_0 \mathbf{g} / \max(\det \mathbf{F}_{e,kl}, \varepsilon)$  is a priori bounded in  $L^{2\kappa/(\kappa-2)}(I; L^{\kappa'}(\Omega; \mathbb{R}^d))$ , cf. (4.2f), and that the constant  $C$  in (4.10a) depends on  $\varepsilon$  but not on  $k, l$ , because also the conservative (elastic) part of the regularized Cauchy stress in (4.5a) and also the Eshelby stress in (4.5c) are bounded independently of  $\mathbf{F}_e$ , and therefore each of the equations in (4.5) can be estimated separately.

*Step 3: second a priori estimates.* In Step 1, we could also estimate  $\frac{\partial}{\partial t} \mathbf{F}_{e,kl} + (\mathbf{v}_{kl} \cdot \mathbf{F}_{e,kl} - (\nabla \mathbf{v}_{kl}) \mathbf{F}_{e,kl} + \mathbf{F}_{e,kl} \mathbf{L}_{p,kl})$  by comparison from (4.10b) in  $L^{r'}(I; W^{1,r}(\Omega; \mathbb{R}^{d \times d})^*)$ , but it would not be enough for (4.17) below and thus for making the test in Step 5 legitimate. To get a better estimate, we can also test the Galerkin approximation of (4.5b) by  $\frac{\partial}{\partial t} \mathbf{F}_{e,kl}$ . By Hölder and Young inequalities, we can estimate

$$\begin{aligned}
& \int_{\Omega} \left| \frac{\partial \mathbf{F}_{e,kl}}{\partial t} \right|^2 dx + \frac{1}{rk} \frac{d}{dt} \int_{\Omega} |\nabla \mathbf{F}_{e,kl}|^r dx \\
& \leq \int_{\Omega} \left( (\nabla \mathbf{v}_{kl}) \mathbf{F}_{e,kl} - (\mathbf{v}_{kl} \cdot \nabla) \mathbf{F}_{e,kl} - \mathbf{F}_{e,kl} \mathbf{L}_{p,kl} \right) : \frac{\partial \mathbf{F}_{e,kl}}{\partial t} dx
\end{aligned}$$

$$\begin{aligned}
&\leq \left( \|\nabla \mathbf{v}_{kl}\|_{L^\infty(\Omega; \mathbb{R}^{d \times d})} + \|\mathbf{L}_{p,kl}\|_{L^\infty(\Omega; \mathbb{R}^{d \times d})} \right)^2 \|\mathbf{F}_{e,kl}\|_{L^2(\Omega; \mathbb{R}^{d \times d})}^2 \\
&\quad + C_r \|\mathbf{v}_{kl}\|_{L^\infty(\Omega; \mathbb{R}^d)}^2 \left( 1 + \|\nabla \mathbf{F}_{e,kl}\|_{L^r(\Omega; \mathbb{R}^{d \times d})}^r \right) + \frac{1}{2} \left\| \frac{\partial \mathbf{F}_{e,kl}}{\partial t} \right\|_{L^2(\Omega; \mathbb{R}^{d \times d})}^2
\end{aligned} \tag{4.11}$$

with some  $C_r \in \mathbb{R}$ ; here we used that surely  $r > 2$ . Using the already obtained estimates (4.10) and the Gronwall inequality, we obtain

$$\left\| \frac{\partial \mathbf{F}_{e,kl}}{\partial t} \right\|_{L^2(I \times \Omega; \mathbb{R}^{d \times d})} \leq C e^k / k \quad \text{and} \quad \|\nabla \mathbf{F}_{e,kl}\|_{L^\infty(I; L^r(\Omega; \mathbb{R}^{d \times d \times d}))} \leq C e^{k/r}. \tag{4.12}$$

Note that here the Gronwall inequality uses not the first but the second left-hand-side term which contains the factor  $1/k$  so that both estimates in (4.12) are  $k$ -dependent.

*Step 4: limit passage with  $l \rightarrow \infty$ .* Now, by the Banach selection principle, we extract a subsequence and some  $(\mathbf{v}_k, \mathbf{F}_{e,k}, \mathbf{L}_{p,k}) : I \rightarrow V_k \times W^{1,r}(\Omega; \mathbb{R}^{d \times d}) \times W^{1,q}(\Omega; \mathbb{R}_{\text{dev}}^{d \times d})$  such that

$$\mathbf{v}_{kl} \rightarrow \mathbf{v}_k \quad \text{weakly* in } L^2(I; W^{2,p}(\Omega; \mathbb{R}^d)), \tag{4.13a}$$

$$\mathbf{F}_{e,kl} \rightarrow \mathbf{F}_{e,k} \quad \text{weakly* in } L^\infty(I; W^{1,r}(\Omega; \mathbb{R}^{d \times d})) \cap H^1(I; L^2(\Omega; \mathbb{R}^{d \times d})), \tag{4.13b}$$

$$\mathbf{L}_{p,kl} \rightarrow \mathbf{L}_{p,k} \quad \text{weakly* in } L^\infty(I; W^{1,q}(\Omega; \mathbb{R}_{\text{dev}}^{d \times d})). \tag{4.13c}$$

By the Aubin-Lions theorem, we have also  $\mathbf{F}_{e,kl} \rightarrow \mathbf{F}_{e,k}$  strongly in  $L^c(I \times \Omega; \mathbb{R}^{d \times d})$  with any  $c < \infty$ ; recall that  $r > d$ . Thus, by the continuity of the corresponding Nemytskiĭ (or here simply superposition) mappings, also the regularized Eshelby stress converges

$$\frac{\mathbf{F}_{e,kl}^\top \varphi'(\mathbf{F}_{e,kl})}{1 + (|\mathbf{F}_{e,kl}^\top \varphi'(\mathbf{F}_{e,kl})| - 1/\varepsilon^2)^+} \rightarrow \frac{\mathbf{F}_{e,k}^\top \varphi'(\mathbf{F}_{e,k})}{1 + (|\mathbf{F}_{e,k}^\top \varphi'(\mathbf{F}_{e,k})| - 1/\varepsilon^2)^+} \quad \text{strongly in } L^c(I \times \Omega; \mathbb{R}^{d \times d}).$$

As the right-hand side of the discretized plasticity-rate inclusion (4.5c) converge strongly, by the uniform monotonicity of its left-hand side, we have also

$$\mathbf{L}_{p,kl} \rightarrow \mathbf{L}_{p,k} \quad \text{strongly in } L^c(I; W^{1,q}(\Omega; \mathbb{R}_{\text{dev}}^{d \times d})) \text{ with any } c < \infty. \tag{4.13d}$$

By the mentioned continuity of the corresponding Nemytskiĭ mappings, we have also

$$\varrho_{kl} = \frac{\varrho_0}{\max(\det \mathbf{F}_{e,kl}, \varepsilon)} \rightarrow \frac{\varrho_0}{\max(\det \mathbf{F}_{e,k}, \varepsilon)} = \varrho_k \quad \text{strongly in } L^c(I \times \Omega) \text{ with any } c < \infty$$

and similarly the regularized elastic part of the Cauchy stress in (4.5a) converges strongly:

$$\begin{aligned}
\mathbf{T}_{\varepsilon,kl} &= \frac{\varphi'(\mathbf{F}_{e,kl}) \mathbf{F}_{e,kl}}{1 + (|\varphi'(\mathbf{F}_{e,kl}) \mathbf{F}_{e,kl}| - 1/\varepsilon^2)^+} + \frac{\varphi(\mathbf{F}_{e,kl}) \mathbb{I}}{1 + (\varphi(\mathbf{F}_{e,kl}) - 1/\varepsilon)^+} \\
&\rightarrow \frac{\varphi'(\mathbf{F}_{e,k}) \mathbf{F}_{e,k}}{1 + (|\varphi'(\mathbf{F}_{e,k}) \mathbf{F}_{e,k}| - 1/\varepsilon^2)^+} + \frac{\varphi(\mathbf{F}_{e,k}) \mathbb{I}}{1 + (\varphi(\mathbf{F}_{e,k}) - 1/\varepsilon)^+} = \mathbf{T}_{\varepsilon,k} \quad \text{strongly in } L^c(I \times \Omega; \mathbb{R}^{d \times d}).
\end{aligned} \tag{4.14}$$

For the limit passage in the Galerkin approximation of (4.5a) for  $l \rightarrow \infty$  (which still will remain discretized as  $k$  is considered fixed in this step), we need the Minty trick or,

having here the strong monotonicity of the hyperstress term, we use just strong convergence of  $\nabla \mathbf{e}(\mathbf{v}_{kl})$ . In fact, as we do not consider any acceleration and thus do not have  $\frac{\partial}{\partial t} \mathbf{v}_{kl}$  under control in the quasistatic case, we will anyhow need strong convergence of  $\nabla \mathbf{v}_{kl}$  later in Step 6 for the convective term  $(\mathbf{v} \cdot \nabla) \mathbf{F}_e$ . So, using the Galerkin approximation of the momentum equation tested by  $\mathbf{v}_{kl} - \mathbf{v}_k$ , we can estimate

$$\begin{aligned} & \nu c_p \|\nabla \mathbf{e}(\mathbf{v}_{kl} - \mathbf{v}_k)\|_{L^p(I \times \Omega; \mathbb{R}^{d \times d \times d})}^p \leq \int_0^T \int_{\Omega} \left( (\xi'(\mathbf{e}(\mathbf{v}_{kl})) - \xi'(\mathbf{e}(\mathbf{v}_k))) : \mathbf{e}(\mathbf{v}_{kl} - \mathbf{v}_k) \right. \\ & \quad \left. + \nu (|\nabla \mathbf{e}(\mathbf{v}_{kl})|^{p-2} \nabla \mathbf{e}(\mathbf{v}_{kl}) - |\nabla \mathbf{e}(\mathbf{v}_k)|^{p-2} \nabla \mathbf{e}(\mathbf{v}_k)) : \nabla \mathbf{e}(\mathbf{v}_{kl} - \mathbf{v}_k) \right) dx dt \\ & = \left( \int_0^T \int_{\Omega} \left( \rho_{kl} \mathbf{g} \cdot (\mathbf{v}_{kl} - \mathbf{v}_k) - \nu (|\nabla \mathbf{e}(\mathbf{v}_k)|^{p-2} \nabla \mathbf{e}(\mathbf{v}_k)) : \nabla \mathbf{e}(\mathbf{v}_{kl} - \mathbf{v}_k) \right. \right. \\ & \quad \left. \left. - (\mathbf{T}_{\varepsilon, kl} + \xi'(\mathbf{e}(\mathbf{v}_k))) : \nabla (\mathbf{v}_{kl} - \mathbf{v}_k) \right) dx dt + \int_0^T \int_{\Gamma} \mathbf{f} \cdot (\mathbf{v}_{kl} - \mathbf{v}_k) dS dt \right) \rightarrow 0 \end{aligned} \quad (4.15)$$

with some  $c_p > 0$  related to the inequality  $c_p |G - \tilde{G}|^p \leq (|G|^{p-2} G - |\tilde{G}|^{p-2} \tilde{G}) : (G - \tilde{G})$  holding for  $p \geq 2$ . We also use (4.14) and that  $\nabla(\mathbf{v}_{kl} - \mathbf{v}_k) \rightarrow 0$  weakly in  $L^p(I \times \Omega; \mathbb{R}^{d \times d})$ , so that  $\int_0^T \int_{\Omega} \mathbf{T}_{\varepsilon, kl} : \nabla(\mathbf{v}_{kl} - \mathbf{v}_k) dx dt \rightarrow 0$ , and also the growth assumption (4.2c) which ensures  $\xi'(\mathbf{e}(\mathbf{v}_k)) \in L^p(I \times \Omega; \mathbb{R}^{d \times d})$ . Thus we obtain the desired strong convergence of  $\nabla \mathbf{e}(\mathbf{v}_{kl})$  in  $L^p(I \times \Omega; \mathbb{R}^{d \times d \times d})$ .

The limit passage in the quasilinear parabolic evolution equation (4.5b) in its Galerkin approximation (4.6b) is very standard when realizing that  $(\nabla \mathbf{v}_{kl}) \mathbf{F}_{e, kl} - \mathbf{F}_{e, kl} \mathbf{L}_{p, kl}$  converges to  $(\nabla \mathbf{v}_k) \mathbf{F}_{e, k} - \mathbf{F}_{e, k} \mathbf{L}_{p, k}$  strongly in  $L^p(I; L^{q^*}(\Omega; \mathbb{R}^{d \times d}))$  while  $(\mathbf{v}_{kl} \cdot \nabla) \mathbf{F}_{e, kl} \rightarrow (\mathbf{v}_k \cdot \nabla) \mathbf{F}_{e, k}$  only weakly\* in  $L^\infty(L^r(\Omega; \mathbb{R}^{d \times d}))$  but this is enough when tested by  $\tilde{\mathbf{F}}_{e, kl} - \mathbf{F}_{e, kl} \rightarrow 0$  strongly in  $L^1(I; L^{r'}(\Omega; \mathbb{R}^{d \times d}))$  with some approximation  $\tilde{\mathbf{F}}_{e, kl} \in L^1(I; W_l)$  strongly converging to  $\mathbf{F}_{e, k}$  for  $l \rightarrow \infty$ ; the strong convergence  $\mathbf{F}_{e, kl} \rightarrow \mathbf{F}_{e, k}$  is due to the Aubin-Lions theorem and the estimates (4.12); here we rely on that surely  $1/r + 1/r^* < 1$ .

The limit passage in the Galerkin approximation of (4.5c) written as the variational inequality (4.6c) can be made easily by a weak convergence and by the weak lower semicontinuity of the functional  $\mathbf{L}_p \mapsto \int_0^T \int_{\Omega} \zeta(\mathbf{L}_p) + \mu |\nabla \mathbf{L}_p|^q / q dx dt$ .

*Step 5: third a priori estimates.* Since now

$$\left\| \frac{\partial \mathbf{F}_{e, k}}{\partial t} + (\mathbf{v}_k \cdot \nabla) \mathbf{F}_{e, k} - (\nabla \mathbf{v}_k) \mathbf{F}_{e, k} + \mathbf{F}_{e, k} \mathbf{L}_{p, k} \right\|_{L^2(I \times \Omega; \mathbb{R}^{d \times d})} \leq C e^{k/r}, \quad (4.16)$$

by comparison we also obtain

$$\left\| \operatorname{div}(|\nabla \mathbf{F}_{e, k}|^{r-2} \nabla \mathbf{F}_{e, k}) \right\|_{L^2(I \times \Omega; \mathbb{R}^{d \times d})} \leq k C e^{k/r}. \quad (4.17)$$

Although this estimate blows up when  $k \rightarrow \infty$ , we have now at least the information that  $\operatorname{div}(|\nabla \mathbf{F}_{e, k}|^{r-2} \nabla \mathbf{F}_{e, k}) \in L^2(I \times \Omega; \mathbb{R}^{d \times d})$ . It is now important that we have (4.5b) continuous, i.e. non-discretized. Therefore, we can legitimately use  $\operatorname{div}(|\nabla \mathbf{F}_{e, k}|^{r-2} \nabla \mathbf{F}_{e, k})$  as a test. Since  $\min(p, q) > d$ , we have  $\min(p, q)^{-1} + (r^*)^{-1} + (r')^{-1} \leq 1$ , and thus by the Hölder and Young inequalities, we can estimate

$$\frac{d}{dt} \int_{\Omega} \frac{1}{r} |\nabla \mathbf{F}_{e, k}|^r dx + \frac{1}{k} \int_{\Omega} |\operatorname{div}(|\nabla \mathbf{F}_{e, k}|^{r-2} \nabla \mathbf{F}_{e, k})|^2 dx$$

$$\begin{aligned}
&= \int_{\Omega} \nabla \left( (\mathbf{v}_k \cdot \nabla) \mathbf{F}_{e,k} - (\nabla \mathbf{v}_k) \mathbf{F}_{e,k} - \mathbf{F}_{e,k} \mathbf{L}_{p,k} \right) : (|\nabla \mathbf{F}_{e,k}|^{r-2} \nabla \mathbf{F}_{e,k}) \, dx \\
&= \int_{\Omega} |\nabla \mathbf{F}_{e,k}|^{r-2} (\nabla \mathbf{F}_{e,k} \otimes \nabla \mathbf{F}_{e,k}) : \mathbf{e}(\mathbf{v}_k) - \frac{1}{r} |\nabla \mathbf{F}_{e,k}|^r \operatorname{div} \mathbf{v}_k \\
&\quad - \left( (\nabla \mathbf{v}_k) \nabla \mathbf{F}_{e,k} + (\nabla^2 \mathbf{v}_k) \mathbf{F}_{e,k} + \nabla \mathbf{F}_{e,k} \mathbf{L}_{p,k} + \mathbf{F}_{e,k} \nabla \mathbf{L}_{p,k} \right) : (|\nabla \mathbf{F}_{e,k}|^{r-2} \nabla \mathbf{F}_{e,k}) \, dx \\
&\leq C_r \left( \|\nabla \mathbf{v}_k\|_{L^\infty(\Omega; \mathbb{R}^{d \times d})} + \|\mathbf{L}_{p,k}\|_{L^\infty(\Omega; \mathbb{R}^{d \times d})} \right) \|\nabla \mathbf{F}_{e,k}\|_{L^r(\Omega; \mathbb{R}^{d \times d \times d})}^r \\
&\quad + C_r \left( \|\nabla^2 \mathbf{v}_k\|_{L^p(\Omega; \mathbb{R}^{d \times d \times d})} + \|\nabla \mathbf{L}_{p,k}\|_{L^q(\Omega; \mathbb{R}^{d \times d \times d})} \right) \|\mathbf{F}_{e,k}\|_{L^{r^*}(\Omega; \mathbb{R}^{d \times d})} \|\nabla \mathbf{F}_{e,k}\|_{L^r(\Omega; \mathbb{R}^{d \times d \times d})}^{r-1} \\
&\leq C_r \left( \|\nabla \mathbf{v}_k\|_{L^\infty(\Omega; \mathbb{R}^{d \times d})} + \|\mathbf{L}_{p,k}\|_{L^\infty(\Omega; \mathbb{R}^{d \times d})} \right) \|\nabla \mathbf{F}_{e,k}\|_{L^r(\Omega; \mathbb{R}^{d \times d \times d})}^r \\
&\quad + C_r N \left( \|\nabla^2 \mathbf{v}_k\|_{L^p(\Omega; \mathbb{R}^{d \times d \times d})} + \|\mathbf{L}_{p,k}\|_{L^2(\Omega; \mathbb{R}^{d \times d})} \right) \|\mathbf{F}_{e,k}\|_{L^2(\Omega; \mathbb{R}^{d \times d})} (1 + \|\nabla \mathbf{F}_{e,k}\|_{L^r(\Omega; \mathbb{R}^{d \times d \times d})}^r) \\
&\quad + C_r N \left( \|\nabla^2 \mathbf{v}_k\|_{L^p(\Omega; \mathbb{R}^{d \times d \times d})} + \|\nabla \mathbf{L}_{p,k}\|_{L^q(\Omega; \mathbb{R}^{d \times d \times d})} \right) \|\nabla \mathbf{F}_{e,k}\|_{L^r(\Omega; \mathbb{R}^{d \times d \times d})}^r, \tag{4.18}
\end{aligned}$$

where we used  $\min(p, q) > d$  also for the embedding of  $\nabla \mathbf{v}_k$  and  $\mathbf{L}_{p,k}$  into  $L^\infty(\Omega; \mathbb{R}^{d \times d})$  and where we further used the calculus (to be used for  $\mathbf{F} = \mathbf{F}_{e,k}$ )

$$\begin{aligned}
&\int_{\Omega} \nabla \left( (\mathbf{v} \cdot \nabla) \mathbf{F} \right) : |\nabla \mathbf{F}|^{r-2} \nabla \mathbf{F} \, dx \\
&= \int_{\Omega} |\nabla \mathbf{F}|^{r-2} (\nabla \mathbf{F} \otimes \nabla \mathbf{F}) : \mathbf{e}(\mathbf{v}) + (\mathbf{v} \cdot \nabla) \nabla \mathbf{F} : |\nabla \mathbf{F}|^{r-2} \nabla \mathbf{F} \, dx \\
&= \int_{\Gamma} |\nabla \mathbf{F}|^r \mathbf{v} \cdot \mathbf{n} \, dS + \int_{\Omega} \left( |\nabla \mathbf{F}|^{r-2} (\nabla \mathbf{F} \otimes \nabla \mathbf{F}) : \mathbf{e}(\mathbf{v}) \right. \\
&\quad \left. - (\operatorname{div} \mathbf{v}) |\nabla \mathbf{F}|^r - (r-1) |\nabla \mathbf{F}|^{r-2} \nabla \mathbf{F} : (\mathbf{v} \cdot \nabla) \nabla \mathbf{F} \right) \, dx \\
&= \int_{\Gamma} \frac{|\nabla \mathbf{F}|^r}{r} \mathbf{v} \cdot \mathbf{n} \, dS + \int_{\Omega} |\nabla \mathbf{F}|^{r-2} (\nabla \mathbf{F} \otimes \nabla \mathbf{F}) : \mathbf{e}(\mathbf{v}) - (\operatorname{div} \mathbf{v}) \frac{|\nabla \mathbf{F}|^r}{r} \, dx.
\end{aligned}$$

Here  $\nabla \mathbf{F} \otimes \nabla \mathbf{F}$  denoted the symmetric matrix  $[\nabla \mathbf{F} \otimes \nabla \mathbf{F}]_{ij} = \sum_{k,l=1}^d \frac{\partial}{\partial x_i} \mathbf{F}_{kl} \frac{\partial}{\partial x_j} \mathbf{F}_{kl}$ . Again, the boundary integral vanishes in (4.18) if  $\mathbf{v} \cdot \mathbf{n} = 0$ . For the last inequality in (4.18), we have used  $\|\mathbf{F}_{e,k}\|_{L^{r^*}(\Omega; \mathbb{R}^{d \times d})} \leq N(\|\mathbf{F}_{e,k}\|_{L^2(\Omega; \mathbb{R}^{d \times d})} + \|\nabla \mathbf{F}_{e,k}\|_{L^r(\Omega; \mathbb{R}^{d \times d \times d})})$  where  $N$  is the norm of the embedding  $W^{1,r}(\Omega) \subset L^{r^*}(\Omega)$  if  $W^{1,r}(\Omega)$  is endowed with the norm  $\|\cdot\|_{L^2(\Omega)} + \|\nabla \cdot\|_{L^r(\Omega; \mathbb{R}^d)}$ .

Thus one can apply the Gronwall inequality to (4.18) and the estimates (4.10b) and (4.12) can be strengthened. Specifically, using the already obtained estimates (4.9) and having assumed  $\mathbf{F}_{e,0} \in W^{1,r}(\Omega; \mathbb{R}^{d \times d})$ , one obtains the estimates

$$\|\nabla \mathbf{F}_{e,k}\|_{L^\infty(I; L^r(\Omega; \mathbb{R}^{d \times d \times d}))} \leq C \quad \text{and} \tag{4.19a}$$

$$\|\operatorname{div}(|\nabla \mathbf{F}_{e,k}|^{r-2} \nabla \mathbf{F}_{e,k})\|_{L^2(I \times \Omega; \mathbb{R}^{d \times d})} \leq C \sqrt{k}. \tag{4.19b}$$

Besides, although the former estimate in (4.12) on  $\frac{\partial}{\partial t} \mathbf{F}_{e,k}$  is not inherited also on the limit, we have by comparison from  $\frac{\partial}{\partial t} \mathbf{F}_{e,k} = (\nabla \mathbf{v}_k) \mathbf{F}_{e,k} - (\mathbf{v}_k \cdot \nabla) \mathbf{F}_{e,k} + k^{-1} \operatorname{div}(|\nabla \mathbf{F}_{e,k}|^{r-2} \nabla \mathbf{F}_{e,k})$  in its Galerkin approximation (4.6b) a weaker estimate

$$\left\| \frac{\partial \mathbf{F}_{e,k}}{\partial t} \right\|_{L^2(I; L^2(\Omega; \mathbb{R}^{d \times d}) + W_K^{1,r}(\Omega; \mathbb{R}^{d \times d})^*)} \leq C \quad \text{for } k \geq K, \tag{4.19c}$$

where  $W_K^{1,r}(\Omega; \mathbb{R}^{d \times d})^*$  is considered endowed with the seminorm

$$|\cdot|_K = \sup_{\|F\|_{W^{1,r}(\Omega; \mathbb{R}^{d \times d})} \leq 1, F \in W_K} \int_{\Omega} \nabla \cdot : \nabla F \, dx.$$

It is important that  $C$  in (4.19c) can be taken independent of  $K \in \mathbb{N}$ .

*Step 6: limit passage with  $k \rightarrow \infty$ .* We use the Banach selection principle as in Step 4 now also taking (4.19a) into account instead of the latter estimate in (4.12) which was not uniform in  $k$ . Thus, for a subsequence and some  $(\mathbf{v}, \mathbf{F}_e, \mathbf{L}_p)$ , we have

$$\mathbf{v}_k \rightarrow \mathbf{v} \quad \text{weakly}^* \text{ in } L^2(I; W^{2,p}(\Omega; \mathbb{R}^d)), \quad (4.20a)$$

$$\mathbf{F}_{e,k} \rightarrow \mathbf{F}_e \quad \text{weakly}^* \text{ in } L^\infty(I; W^{1,r}(\Omega; \mathbb{R}^{d \times d})), \quad (4.20b)$$

$$\mathbf{L}_{p,k} \rightarrow \mathbf{L}_p \quad \text{strongly in } L^c(I; W^{1,q}(\Omega; \mathbb{R}_{\text{dev}}^{d \times d})) \text{ with any } c < \infty. \quad (4.20c)$$

Moreover, exploiting (4.20b) together with the estimate (4.19c), by the Aubin-Lions theorem generalized for time derivatives controlled in Hausdorff locally convex spaces [57, Lemma 7.7] we obtain also  $\mathbf{F}_{e,k} \rightarrow \mathbf{F}_e$  strongly in  $L^c(I \times \Omega; \mathbb{R}^{d \times d})$  for any  $1 \leq c < +\infty$  to be used analogously as we did in Step 4.

The momentum equation (4.5a) (still regularized by  $\varepsilon$  and discretised) and the plastic-rate inclusion (4.5c) are to be treated like in Step 4; in fact, (4.15) is to be slightly modified by using some approximation  $\tilde{\mathbf{v}}_k$  of the limit  $\mathbf{v}$  valued in the Galerkin finite-dimensional space so that  $\mathbf{v}_k - \tilde{\mathbf{v}}_k$  is a legitimate test for the Galerkin approximation of the momentum equation (4.5a) and such that  $\nabla \mathbf{e}(\tilde{\mathbf{v}}_k) \rightarrow \nabla \mathbf{e}(\mathbf{v})$  strongly in  $L^p(I \times \Omega; \mathbb{R}^{d \times d \times d})$ . Due to (4.19b), we have  $k^{-1} \text{div}(|\nabla \mathbf{F}_k|^{r-2} \nabla \mathbf{F}_k) = \mathcal{O}(\sqrt{1/k}) \rightarrow 0$  in  $L^2(I \times \Omega; \mathbb{R}^{d \times d})$  and thus this regularizing term in the elastic-strain evolution equation (4.5b) disappears in the limit. The rest is a linear equation in terms of  $\mathbf{F}_e$ , while its coefficients  $\mathbf{v}_k$ ,  $\nabla \mathbf{v}_k$ , and  $\mathbf{L}_{p,k}$  converge strongly. Altogether, we showed that  $(\mathbf{v}, \mathbf{F}_e, \mathbf{L}_p)$  is a weak solution of a problem like (4.5) but regularized only by  $\varepsilon > 0$ , i.e. the last term in (4.5b) is omitted.

*Step 7: the original problem.* Let us note that the limit  $\mathbf{F}_e$  lives in  $L^\infty(I; W^{1,r}(\Omega)) \cap H^1(I; W^{1,r}(\Omega)^*)$  and this space is embedded into  $C(I \times \bar{\Omega})$  if  $r > d$ . Therefore  $\mathbf{F}_e$  and its determinant evolve continuously in time, being valued respectively in  $C(\bar{\Omega}; \mathbb{R}^{d \times d})$  and  $C(\bar{\Omega})$ . Let us recall that the initial condition  $\mathbf{F}_{e,0}$  complies with the bounds (4.4c,d) and we used this  $\mathbf{F}_{e,0}$  also for the  $\varepsilon$ -regularized system. Therefore  $\mathbf{F}_e$  satisfies these bounds not only at  $t = 0$  but also at least for small times. Yet, it means that the  $\varepsilon$ -regularization is nonactive and  $(\mathbf{v}, \mathbf{F}_e, \mathbf{L}_p)$  solves, at least for a small time, the original nonregularized system for which the a priori bounds (4.4) hold. Here we used Lemma 4.1. By the continuation argument, we may see that the  $\varepsilon$ -regularization remains therefore inactive within the whole evolution of  $(\mathbf{v}, \mathbf{F}_e, \mathbf{L}_p)$  on the whole time interval  $I$ .

*Step 8: energy balance.* It is now important that the tests and then all the subsequent calculations leading to (3.14) integrated over a current time interval  $[0, t]$  are really legitimate.

Since  $\nabla \mathbf{e}(\mathbf{v}) \in L^p(I \times \Omega; \mathbb{R}^{d \times d \times d})$ , we have  $\text{div}^2(\nu |\nabla \mathbf{e}(\mathbf{v})|^{p-2} \nabla \mathbf{e}(\mathbf{v})) \in L^{p'}(I; W^{2,p}(\Omega; \mathbb{R}^d)^*)$  in duality with  $\mathbf{v}$ . Also  $\text{div} \xi'(e(\mathbf{v})) \in L^{p'}(I; W^{1,p^*}(\Omega; \mathbb{R}^d)^*)$  is in duality with  $\mathbf{v}$  due to the

growth condition (4.2c) and  $\varrho_0 \mathbf{g} / \det \mathbf{F}_e + \operatorname{div}(\varphi'(\mathbf{F}_e) \mathbf{F}_e^\top + \varphi(\mathbf{F}_e) \mathbb{I})$  is even better. Further, by comparison,  $\frac{\partial}{\partial t} \mathbf{F}_e = (\nabla \mathbf{v}) \mathbf{F}_e - (\mathbf{v} \cdot \nabla) \mathbf{F}_e - \mathbf{F}_e \mathbf{L}_p \in L^p(I; L^r(\Omega; \mathbb{R}^{d \times d})) + L^q(I \times \Omega; \mathbb{R}^{d \times d}) \subset L^{\min(p,q)}(I; L^{\min(r,q)}(\Omega; \mathbb{R}^{d \times d}))$ . Therefore, it is surely in duality with the Piola stress  $\mathbf{S} = \varphi'(\mathbf{F}_e) \in L^\infty(I \times \Omega; \mathbb{R}^{d \times d})$ . Also, by comparison from (3.8c),  $\operatorname{div}(\mu |\nabla \mathbf{L}_p|^{q-2} \nabla \mathbf{L}_p) \in \partial \zeta(\mathbf{L}_p) + \mathbf{F}_e^\top \mathbf{S}$  is a bounded set in  $L^q(I; L^\infty(\Omega; \mathbb{R}_{\text{dev}}^{d \times d}))$ , cf. the growth condition in (4.2d). Therefore, it is in duality with  $\mathbf{L}_p \in L^q(I; W^{1,q}(\Omega; \mathbb{R}_{\text{dev}}^{d \times d}))$ . Here we note that, due to (4.2d),  $\partial \zeta$  is possibly multivalued but, due to (4.2d), the plastic dissipation rate  $\partial \zeta(\mathbf{L}_p) : \mathbf{L}_p$  is, in fact, always a single-valued function in  $L^1(I \times \Omega)$ .

Therefore, the calculations (3.11)–(3.13) are legitimate.

*Step 9: additional information – plastic distortion.* The corresponding plastic distortion  $\mathbf{F}_p$  satisfies the evolution rule  $\dot{\mathbf{F}}_p = \mathbf{L}_p \mathbf{F}_p$ , cf. (3.15). Then, for the  $W^{1,s}$ -estimate, it suffices to apply the same procedure as we did for (3.8b) modified and even simplified since there is no term like  $(\nabla \mathbf{v}) \mathbf{F}_p$ , i.e. we regularize as the linear transport-evolution equation  $\dot{\mathbf{F}}_p = \mathbf{L}_p \mathbf{F}_p + k \operatorname{div}(|\nabla \mathbf{F}_p|^{s-2} \nabla \mathbf{F}_p)$  and use the calculus like (4.7), (4.11), and (4.18). Actually, for the mere  $L^s$ -estimate with  $s = 2$ , it suffices to use only the first estimate (4.7) while, for  $L^s$ -estimate with  $s \neq 2$ , the estimate (4.7) is to be modified for a test by  $|\mathbf{F}_p|^{s-2} \mathbf{F}_p$  in the spirit of (4.18). Moreover,  $\mathbf{L}_p \mathbf{F}_p \in L^\infty(I; L^{s^*}(\Omega; \mathbb{R}^{d \times d}))$  and  $(\mathbf{v} \cdot \nabla) \mathbf{F}_p \in L^p(I; L^s(\Omega; \mathbb{R}^{d \times d}))$  so that we have  $\frac{\partial}{\partial t} \mathbf{F}_p = \mathbf{L}_p \mathbf{F}_p - (\mathbf{v} \cdot \nabla) \mathbf{F}_p \in L^p(I; L^s(\Omega; \mathbb{R}^{d \times d}))$ .

The same arguments can be applied to the evolution-and-transport equation of  $\det \mathbf{F}_p$ , cf. (2.11). Due to (2.12),  $\operatorname{tr} \mathbf{L}_p = 0$  so that (2.11) reduces to

$$\overline{\dot{\det \mathbf{F}_p}} = 0. \quad (4.21)$$

If  $\det \mathbf{F}_{p,0}$  is constant, then (4.21) reduces to  $\frac{\partial}{\partial t}(\det \mathbf{F}_p) = 0$ , so that  $\det \mathbf{F}_p$  stays equal to this constant during the whole evolution. In particular it holds for  $\det \mathbf{F}_{p,0} = 1$ .

Eventually, for the deformation gradient  $\mathbf{F} = \mathbf{F}_e \mathbf{F}_p \in L^\infty(I; L^{s^*}(\Omega; \mathbb{R}^{d \times d}))$ , we have also  $\nabla \mathbf{F} = \nabla \mathbf{F}_e \mathbf{F}_p + \mathbf{F}_e \nabla \mathbf{F}_p \in L^\infty(I; L^{\min(s, s^*r/(s^*+r))}(\Omega; \mathbb{R}^{d \times d \times d}))$  and  $\frac{\partial}{\partial t} \mathbf{F} = (\frac{\partial}{\partial t} \mathbf{F}_e) \mathbf{F}_p + \mathbf{F}_e \frac{\partial}{\partial t} \mathbf{F}_p \in L^2(I; L^{s^*r/(s^*+r)}(\Omega; \mathbb{R}^{d \times d})) + L^p(I; L^s(\Omega; \mathbb{R}^{d \times d}))$ . As  $\det \mathbf{F}_p = 1$  and  $\det \mathbf{F}_e$  stays away from 0, the same holds for  $\det \mathbf{F} = \det \mathbf{F}_e$ .  $\square$

**Remark 4.4** (*Classical solutions*). In fact, we proved that all the terms in the transport-evolution equations (3.8b) and, under the assumptions of Proposition 4.3(iii), also (3.15) are surely in  $L^1(I \times \Omega; \mathbb{R}^{d \times d})$ . Thus these equations are satisfied even a.e. on  $I \times \Omega$ . If  $\varphi$  is twice continuously differentiable,  $\operatorname{div}(\varphi'(\mathbf{F}_e) \mathbf{F}_e^\top + \varphi(\mathbf{F}_e) \mathbb{I}) \in L^\infty(I; L^r(\Omega; \mathbb{R}^d))$  due to the regularity of  $\nabla \mathbf{F}_e \in L^\infty(I; L^r(\Omega; \mathbb{R}^{d \times d}))$ . If also  $\zeta$  is twice continuously differentiable, then also  $\operatorname{div}(\zeta'(\mathbf{e}(\mathbf{v})) \in L^p(L^\infty(\Omega; \mathbb{R}^d))$ . Then, by comparison,  $\operatorname{div}^2(\nu |\nabla \mathbf{e}(\mathbf{v})|^{p-2} \nabla \mathbf{e}(\mathbf{v})) \in L^p(I; L^r(\Omega; \mathbb{R}^d))$  and therefore also the momentum equation (3.8a) holds a.e. on  $I \times \Omega$ . If the plastic-distortion-rate inclusion (3.8c) is understood on the linear subspace  $\operatorname{tr} \mathbf{L}_p = \mathbf{0}$ , then also  $\operatorname{div}(\mu |\nabla \mathbf{L}_p|^{q-2} \nabla \mathbf{L}_p) \in L^q(I \times \Omega; \mathbb{R}^{d \times d})$  and also the inclusion (3.8c) holds a.e. on  $I \times \Omega$ . This is more than the weak formulation (4.1). Recovery of the boundary conditions a.e. on  $I \times \Gamma$  would need still more regularity, however.

**Remark 4.5** (*Uniqueness*). For a given  $\mathbf{v}$  and  $\mathbf{L}_p$ , the weak solution of the transport-evolution equations (3.8b) is unique. The highest-order terms of the momentum equation (3.8a) and the plastic-distortion-rate inclusion (3.8c) are strictly monotone but, anyhow, the uniqueness of a weak solution to the whole system (3.8) seems problematic. The troublesome attribute is that the conservative part of the Cauchy stress  $\varphi'(\mathbf{F}_e)\mathbf{F}_e^\top + \varphi(\mathbf{F}_e)\mathbb{I}$  as well as the Eshelby stress  $\mathbf{F}_e^\top\varphi'(\mathbf{F}_e)$  are highly nonmonotone.

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