arXiv:2112.14426v1 [math.AP] 29 Dec 2021

LINEAR INSTABILITY OF BREATHERS FOR THE FOCUSING NONLINEAR SCHRÖDINGER EQUATION

MARIANA HARAGUS, DMITRY E. PELINOVSKY

Abstract. Relying upon tools from the theory of integrable systems, we discuss the linear instability of the Kuznetsov–Ma breathers and the Akhmediev breathers of the focusing nonlinear Schrödinger equation. We use the Darboux transformation to construct simultaneously the breathers and the exact solutions of the Lax system associated with the breathers. We obtain a full description of the Lax spectra for the two breathers, including multiplicities of eigenvalues. Solutions of the linearized NLS equations are then obtained from the eigenfunctions and generalized eigenfunctions of the Lax system. While we do not attempt to prove completeness of eigenfunctions, we aim to determine the entire set of solutions of the linearized NLS equations generated by the Lax system in appropriate function spaces.

1. Introduction

The focusing nonlinear Schrödinger (NLS) equation in the space of one dimension is a fundamentally important model which brings together nonlinearity and dispersion of modulated waves in many physical systems [17, 29]. It has been used as the main testbed for rogue waves in fluids and optics [22, 32], where the rogue waves appear from nowhere and disappear without any trace. One of the important properties of the focusing cubic NLS equation is its integrability, which allows to construct the basic solutions for the rogue waves in a closed analytical form. Although these solutions have been constructed long ago in the works of Akhmediev *et al.* [2], Kuznetsov [25], Ma [26], Peregrine [28], and Tajiri & Watanabe [31] they have been studied a lot in the past few years in physics literature [13, 16].

To explain the current state of art in the mathematical studies of these breather solutions, we set up the stage and take the NLS equation in the following dimensionless form:

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0, \tag{1.1}$$

where the unknown ψ is a complex-valued function depending on time $t \in \mathbb{R}$ and space $x \in \mathbb{R}$. The NLS equation (1.1) is invariant under the scaling transformation

$$\text{if } \psi(x,t) \text{ is a solution, so is } c\psi(cx, c^2t), \quad \text{for every } c \in \mathbb{R},$$

and under translations in x and t. Up to these symmetries, the NLS equation (1.1) admits the following exact solutions on the background of the constant-amplitude wave $\psi(x,t) = e^{it}$:

- Akhmediev breather (AB)

$$\psi(x,t) = \left[-1 + \frac{2(1-\lambda^2)\cosh(\lambda kt) + i\lambda k\sinh(\lambda kt)}{\cosh(\lambda kt) - \lambda\cos(kx)}\right] e^{it}, \tag{1.2}$$

  where $k = 2\sqrt{1-\lambda^2}$, and $\lambda \in (0,1)$ is the only free parameter.

Date: December 30, 2021.

- Kuznetsov–Ma breather (KMB)

$$\psi(x,t) = \left[-1 + \frac{2(\lambda^2-1)\cos(\lambda\beta t) + i\lambda\beta\sin(\lambda\beta t)}{\lambda\cosh(\beta x) - \cos(\lambda\beta t)} \right] e^{it}, \tag{1.3}$$

  where $\beta = 2\sqrt{\lambda^2-1}$, and $\lambda \in (1,\infty)$ is the only free parameter.
- Peregrine's rogue wave (PRW)

$$\psi(x,t) = \left[-1 + \frac{4(1+2it)}{1+4(x^2+t^2)} \right] e^{it}. \tag{1.4}$$

Note that PRW can be obtained in the limit $\lambda \to 1$ from either AB or KMB. Also note the formal transformation $k = i\beta$ between AB and KMB. The main goal of this work is to study the linear instability of AB and KMB.

Stability of breathers is a challeging question that has been extensively studied in the mathematics literature. A major difficulty comes from the nontrivial dependence of the breathers on time t. Therefore, many of the analytical methods developed for stability of stationary or traveling waves in nonlinear partial differential equations do not apply to breathers. For instance, spectral methods are out of reach for AB and PRW which are localized in time t. Since KMB is periodic in time t, Floquet theory can be used, at least formally, to compute stable and unstable modes of KMB. This has been done numerically in [14] after KMB were truncated on a spatially periodic domain in x. Further studies of KMB in discrete setting of the NLS equation can be found in [30].

Very recently, the authors of [35] set up a basis for a rigorous investigation of stability of breathers which are periodic in time t and localized in space x. Using tools from the theory of semigroups and Fredholm operators, they analyzed properties of the monodromy operator for the linearization of the cubic-quintic complex Ginzburg–Landau equation about such solutions, and computed its essential spectrum. These results being obtained in a dissipative setting do not directly apply to KMB due to the Hamiltonian nature of the NLS equation.

Most of the existing instability results for breathers of the NLS equation strongly rely upon the integrability properties of the NLS equation. Instability of AB was concluded by using the variational characterization of breathers in the energy space in [3]. Perturbations to the AB were considered in the periodic space $H^s_{\rm per}$ for $s > 1/2$. Similar techniques were applied to KMB and PRW in [4] (see also the review in [5]). It was shown that both KMB and PRW are unstable with respect to perturbations in $H^s(\mathbb{R})$ for $s > 1/2$.

Evolution of KMB and PRW under perturbations was studied in [18] and [7], where inverse scattering transform was applied to the NLS equation in the class of functions decaying to the nonzero boundary conditions. Instability of PRW was visualized numerically in [24] by using time-dependent simulations of the NLS equation. Linear instability of PRW was also studied numerically in [9]. By using perturbation theory for embedded eigenvalues of the Lax system, it was shown in [23] that the perturbed PRW is transformed to either KMB or two counter-propagating breathers, the latter solutions were later constructed explicitly in [33].

Our approach to linear instability is closely related to the recent works [6, 8, 19], where solutions of the linearized NLS equation are constructed from solutions of the associated Lax system. Eigenfunctions of the Lax system related to the Lax spectrum provide solutions of the linearized NLS equation relevant for the linear instability of breathers. The completeness of the resulting solution set is a particularly challenging question. In the class of spatially localized

functions, it was shown in [6, Section 3.4] how to obtain a complete set of solutions of the linearized NLS equation at PRW. Stability of AB under periodic perturbations of the same period was stated in [8] without the proof of completeness. It was recently discovered in [19] that the set of eigenfunctions constructed in [8] is incomplete and two unstable modes exist for AB. The spatially periodic unstable modes for AB were constructed in [19] by taking a suitable combination of unbounded solutions of the linearized NLS equation.

The purpose of this paper is twofold. Firstly, we give a full description of the Lax spectra for AB and KMB, including algebraic multiplicities of eigenvalues. Secondly, we obtain all solutions of the linearized NLS equations at AB and KMB generated by eigenfunctions and generalized eigenfunctions of the Lax systems. These solutions are spatially periodic for AB and spatially localized for KMB. The completeness question is outside the scope of this paper and will be the subject of subsequent studies.

Similar to [6, 8, 19], we use the Darboux transformation to obtain AB and KMB from the constant-amplitude wave and then to precisely determine the Lax spectra at AB and KMB from the Lax spectrum at the constant-amplitude wave. For AB we focus on solutions of the linearized NLS equation with the first three spatially periodic Fourier modes, whereas for KMB we focus on spatially localized solutions.

Aiming for a presentation accessible to readers who are not expert in integrable systems, we review some properties of the Lax system and the Darboux transformation in Section 2. In Section 3 we consider the constant-amplitude wave. We compute the Lax spectrum and establish the explicit relation between the solutions of the linearized NLS equation obtained by a standard Fourier analysis and the ones generated by the Lax system. We focus on spatially periodic and spatially localized solutions. Then using the Darboux transformation, we determine the Lax spectra and the resulting solutions of the linearized NLS equations for AB in Section 4 and for KMB in Section 5. The paper is concluded at Section 6 with a discussion of further directions.

Acknowledgments: M. Haragus was partially supported by the project Optimal (ANR-20-CE30-0004) and the EUR EIPHI program (ANR-17-EURE-0002). D. E. Pelinovsky was partially supported by the National Natural Science Foundation of China (No. 11971103).

2. Preliminaries

We recall the Lax system for the NLS equation (1.1), its connection with the linearized NLS equation, and the Darboux transformation for the NLS equation and its Lax system.

For our purpose, it is convenient to write $\psi(x,t) = u(x,t)e^{it}$, where u satisfies the normalized NLS equation

$$iu_t + \frac{1}{2}u_{xx} + (|u|^2 - 1)u = 0. \tag{2.1}$$

The constant-amplitude wave $\psi(x,t) = e^{it}$ of the NLS equation (1.1) becomes $u(x,t) = 1$ and the breathers (1.2), (1.3), and (1.4) provide exact solutions of the normalized equaiton (2.1) without the factor e^{it} in these formulas.

2.1. Lax system.

The normalized NLS equation (2.1) for $u = u(x,t)$ is a compatibility condition $\varphi_{xt} = \varphi_{tx}$ for a 2-vector $\varphi = \varphi(x,t)$ satisfying the Lax system

$$\varphi_x = U(u,\lambda)\varphi, \quad U(u,\lambda) = \begin{pmatrix} \lambda & u \\ -\bar{u} & -\lambda \end{pmatrix} \tag{2.2}$$

and

$$\varphi_t = V(u,\lambda)\varphi, \quad V(u,\lambda) = i \begin{pmatrix} \lambda^2 + \frac{1}{2}(|u|^2 - 1) & \lambda u + \frac{1}{2}u_x \\ -\lambda \bar{u} + \frac{1}{2}\bar{u}_x & -\lambda^2 - \frac{1}{2}(|u|^2 - 1) \end{pmatrix}, \tag{2.3}$$

where λ a complex number. The x-derivative equation (2.2) is the Zakharov–Shabat (ZS) spectral problem, which is a particular case of the AKNS spectral problem; see pioneering works [34] and [1]. The t-derivative equation (2.3) gives the time evolution of the solution $\varphi(x,t)$ of the ZS spectral problem (2.2).

Spatially bounded solutions of the Lax system are referred to as *eigenfunctions* and the corresponding values λ as *eigenvalues*. The set of eigenvalues λ form the *Lax spectrum* of the ZS spectral problem (2.2). Rigorously, this terminology corresponds to considering the ZS spectral problem in the space $C_b^0(\mathbb{R})$ of x-dependent functions which are bounded and continuous on $\mathbb{R}$. However, depending on the properties of the solution $u = u(x,t)$ to the NLS equation (2.1) other function spaces may be considered as, for instance, the space of L-periodic functions $L^2_{\rm per}(0,L)$, or the space of L-antiperiodic functions $L^2_{\rm antiper}(0,L)$, or the space of localized functions $L^2(\mathbb{R})$. The choice of the function space affects the nature of the Lax spectrum, as this is usual for spectra of differential operators. For the spaces mentioned above, the Lax spectrum is a purely point spectrum consisting of isolated eigenvalues for $L^2_{\rm per}(0,L)$, or $L^2_{\rm antiper}(0,L)$, whereas it is a purely continuous spectrum, up to possibly a finite number of eigenvalues for $L^2(\mathbb{R})$.

The ZS spectral problem (2.2) can be rewritten as a classical eigenvalue problem

$$(\mathcal{L} - \lambda I)\,\varphi = 0, \quad \mathcal{L} := \begin{pmatrix} \partial_x & -u \\ -\bar{u} & -\partial_x \end{pmatrix}. \tag{2.4}$$

In particular, this allows to define generalized eigenfunctions and algebraic multiplicities of eigenvalues in the usual way by the bounded solutions of $(\mathcal{L} - \lambda I)^k \varphi = 0$ for $k \in \mathbb{N}$. If λ is a double eigenvalue with the only eigenfunction φ satisfying (2.4), then there exists a generalized eigenfunction φ_g satisfying the nonhomogeneous linear equation

$$(\mathcal{L} - \lambda I)\,\varphi_g = \varphi. \tag{2.5}$$

In this case, λ has geometric multiplicity *one* and algebraic multiplicity *two*.

Remark 1. *Solutions of the Lax equations (2.2) and (2.3) satisfy the following symmetry. If $\varphi = (p,q)^T$ is a solution for λ, then $\varphi = (-\bar{q},\bar{p})^T$ is a solution for $-\bar{\lambda}$.*

Taking a solution $u = u(x,t)$ to the normalized NLS equation (2.1), solutions $v = v(x,t)$ of the corresponding linearized NLS equation

$$iv_t + \frac{1}{2}v_{xx} + (2|u|^2 - 1)v + u^2\bar{v} = 0, \tag{2.6}$$

can be constructed from solutions $\varphi = \varphi(x,t)$ of the Lax system (2.2)–(2.3). The following well-known property is a result of a straightforward calculation.

Proposition 1. *Assume u is a solution to the normalized NLS equation (2.1). If $\varphi = (\varphi_1,\varphi_2)^T$ is a solution to the Lax system (2.2)–(2.3) for some λ, then*

$$v = \varphi_1^2 - \bar{\varphi}_2^2, \quad \bar{v} = -\varphi_2^2 + \bar{\varphi}_1^2 \tag{2.7}$$

and

$$v = i(\varphi_1^2 + \bar{\varphi}_2^2), \quad \bar{v} = -i(\varphi_2^2 + \bar{\varphi}_1^2) \tag{2.8}$$

are solutions to the linearized NLS equation (2.6).

Proof. Due to the symmetry in Remark 1 and the linear superposition principle, it is sufficient to confirm the relations (2.7) and (2.8) by using $v = \varphi_1^2$ and $\bar{v} = -\varphi_2^2$. This is obtained directly:

$$\begin{aligned}
&iv_t + \frac{1}{2}v_{xx} + (2|u|^2 - 1)v + u^2\bar{v} \\
&= \varphi_1(2i\varphi_{1t} + \varphi_{1xx}) + (\varphi_{1x})^2 + (2|u|^2 - 1)\varphi_1^2 - u^2\varphi_2^2 \\
&= \varphi_1((1 - |u|^2)\varphi_1 - 2\lambda^2\varphi_1 - 2\lambda u\varphi_2 - u_x\varphi_2 + \lambda(\lambda\varphi_1 + u\varphi_2) + u_x\varphi_2 + u(-\bar{u}\varphi_1 - \lambda\varphi_2)) \\
&\quad + (\lambda\varphi_1 + u\varphi_2)^2 + (2|u|^2 - 1)\varphi_1^2 - u^2\varphi_2^2 \\
&= 0.
\end{aligned}$$

Extending the solution by using (2.7) and (2.8) ensures that $\bar{v}$ is a complex conjugate of v. □

Remark 2. *Solutions $\varphi = \varphi(x,t)$ to the Lax system (2.2)–(2.3) which are bounded functions in x generate bounded solutions $v = v(x,t)$ to the linearized NLS equation (2.6) by means of the transformations (2.7) and (2.8). On the other hand, solutions $\varphi = \varphi(x,t)$ which are unbounded functions in x generate unbounded solutions $v = v(x,t)$ but the linear superposition of unbounded solutions may become bounded* [19]*. This latter property must be taken into account when constructing solutions to the linearized NLS equation (2.6) either in $L^2_{\text{per}}(0,L)$ or in $L^2(\mathbb{R})$ by using Proposition 1.*

The result in Proposition 1 can be extended by taking two linearly independent solutions $\varphi = (\varphi_1, \varphi_2)^T$ and $\phi = (\phi_1, \phi_2)^T$ to the Lax system (2.2)–(2.3) for the same value of λ. Then from these two solutions we can construct the three pairs of solutions of the linearized NLS equation (2.6) given in Table 1. The symmetry of the Lax system in Remark 1 implies that the solutions of the Lax system for $-\bar{\lambda}$ lead, up to sign, to the same solutions of the linearized NLS equation (2.6).

Pair I	Pair II	Pair III
$v = \varphi_1^2 - \bar{\varphi}_2^2$	$v = \varphi_1\phi_1 - \bar{\varphi}_2\bar{\phi}_2$	$v = \phi_1^2 - \bar{\phi}_2^2$
$v = i\varphi_1^2 + i\bar{\varphi}_2^2$	$v = i\varphi_1\phi_1 + i\bar{\varphi}_2\bar{\phi}_2$	$v = i\phi_1^2 + i\bar{\phi}_2^2$

TABLE 1. Table of possible solutions of the linearized NLS equation (2.6) generated from two solutions $\varphi = (\varphi_1, \varphi_2)^T$ and $\phi = (\phi_1, \phi_2)^T$ to the Lax system (2.2)–(2.3) for the same value of λ.

Remark 3. *If λ is a double eigenvalue with the only eigenfunction $\varphi = (\varphi_1, \varphi_2)^T$ satisfying (2.4) and the generalized eigenfunction $\varphi_g = (\varphi_{g1}, \varphi_{g2})^T$ satisfying (2.5), then the linearized NLS equation (2.6) admits the solutions*

$$v = 2\varphi_1\varphi_{g1} - 2\bar{\varphi}_2\bar{\varphi}_{g2}, \quad v = 2i\varphi_1\varphi_{g1} + 2i\bar{\varphi}_2\bar{\varphi}_{g2}, \tag{2.9}$$

in addition to the two solutions in Pair I of Table 1.

2.2. **Darboux transformation.** For the construction of breathers, we use the following version of the one-fold Darboux transformation from [12, Propositions 2.2 and 3.1].

Proposition 2. *Assume that $u = u_0(x,t)$ is a solution to the normalized NLS equation (2.1) and pick $\lambda_0 \in \mathbb{C}$. If $\varphi = (p_0, q_0)^T$ is a particular solution of the Lax system (2.2)–(2.3) with $u = u_0$ and $\lambda = \lambda_0$, then*

$$\hat{u}_0 = u_0 + \frac{2(\lambda_0 + \bar{\lambda}_0) p_0 \bar{q}_0}{|p_0|^2 + |q_0|^2} \tag{2.10}$$

is a solution to the normalized NLS equation (2.1) and $\varphi = (\hat{p}_0, \hat{q}_0)^T$ with

$$\left[\begin{array}{c} \hat{p}_0 \\ \hat{q}_0 \end{array} \right] = \frac{\lambda_0 + \bar{\lambda}_0}{|p_0|^2 + |q_0|^2} \left[\begin{array}{c} -\bar{q}_0 \\ \bar{p}_0 \end{array} \right] \tag{2.11}$$

is a particular solution of the Lax system (2.2)–(2.3) with $u = \hat{u}_0$ and $\lambda = \lambda_0$. Furthermore, the following identity holds:

$$|\hat{u}_0|^2 = |u_0|^2 + \frac{\partial^2}{\partial x^2} \log(|p_0|^2 + |q_0|^2). \tag{2.12}$$

Remark 4. *By the symmetry in Remark 1, $\varphi = (-\bar{q}_0, \bar{p}_0)^T$ is a solution of the Lax system (2.2)–(2.3) with $u = u_0$ and $\lambda = -\bar{\lambda}_0$, whereas $\varphi = (-\bar{\hat{q}}_0, \bar{\hat{p}}_0)^T$ is a solution of the Lax system for $u = \hat{u}_0$ and $\lambda = -\bar{\lambda}_0$.*

Remark 5. *The result in Proposition 2 provides new solutions to the normalized NLS equation (2.1), and to the associated Lax system (2.2)–(2.3), when $\lambda_0 + \bar{\lambda}_0 \neq 0$, i.e., when λ_0 is not purely imaginary. When $\lambda_0 + \bar{\lambda}_0 = 0$, it gives the same solution $\hat{u}_0 = u_0$ to the normalized NLS equation (2.1) and the trivial solution $\varphi = (0,0)^T$ to the Lax system (2.2)–(2.3). Breathers are found by taking $u_0 = 1$ and positive values λ_0: $\lambda_0 \in (0,1)$ for AB, $\lambda_0 \in (1,\infty)$ for KMB, and $\lambda_0 = 1$ for PRW.*

In addition to the Darboux transformation $u_0 \mapsto \hat{u}_0$ in Proposition 2, we have a Darboux transformation $\Phi(\lambda) \mapsto \hat{\Phi}(\lambda)$ between solutions of the Lax system (2.2)–(2.3). More precisely, assuming that $\Phi(\lambda)$ is a 2×2 matrix solution to the Lax system with $u = u_0$, then

$$\hat{\Phi}(\lambda) = D(\lambda)\Phi(\lambda) \tag{2.13}$$

is a 2×2 matrix solution to the Lax system with $u = \hat{u}_0$ if $\lambda \neq \{\lambda_0, -\bar{\lambda}_0\}$, where the Darboux matrix $D(\lambda)$ is given by

$$D(\lambda) := I + \frac{1}{\lambda - \lambda_0} \left[\begin{array}{c} \hat{p}_0 \\ \hat{q}_0 \end{array} \right] [-q_0 \;\; p_0] \tag{2.14}$$

and I stands for the 2×2 identity matrix. Since

$$\det D(\lambda) = \frac{\lambda + \bar{\lambda}_0}{\lambda - \lambda_0},$$

the matrix $D(\lambda)$ is invertible, and the correspondence between the 2×2 matrix solutions $\Phi(\lambda)$ and $\hat{\Phi}(\lambda)$ is one-to-one, when $\lambda \neq \{\lambda_0, -\bar{\lambda}_0\}$.

3. Constant-amplitude background

Here we discuss the simple case of the constant solution $u = 1$. We determine the Lax spectrum and compare the set of solutions of the linearized NLS equation

$$iv_t + \frac{1}{2}v_{xx} + v + \bar{v} = 0 \tag{3.1}$$

obtained using standard tools of Fourier analysis with the one given by the solutions of the Lax system (2.2)–(2.3). This comparison will be useful in the study of linear instability of AB and KMB in Sections 4 and 5 respectively.

3.1. **Lax spectrum.** Since the solution $u = 1$ is constant, the Lax system (2.2)–(2.3) can be solved explicitly. Two linearly independent solutions exist for every λ since the Lax system (2.2)–(2.3) is of the second order. We only consider real and purely imaginary values of λ because the solutions found for the other complex values λ are unbounded.

For $\lambda \in \mathbb{R}_+$, two solutions to the Lax equations (2.2)–(2.3) are given by:

$$\varphi = \begin{bmatrix} \sqrt{\lambda - \frac{i}{2}k(\lambda)} \\ -\sqrt{\lambda + \frac{i}{2}k(\lambda)} \end{bmatrix} e^{-\frac{1}{2}ik(\lambda)(x+i\lambda t)}, \quad \phi = \begin{bmatrix} \sqrt{\lambda + \frac{i}{2}k(\lambda)} \\ -\sqrt{\lambda - \frac{i}{2}k(\lambda)} \end{bmatrix} e^{+\frac{1}{2}ik(\lambda)(x+i\lambda t)}, \tag{3.2}$$

where $k(\lambda) := 2\sqrt{1-\lambda^2}$. These solutions are bounded for $\lambda \in (0,1]$ and are linearly independent for $\lambda \neq 1$, that is, for $k(\lambda) \neq 0$. For $\lambda = 1$, two linearly independent solutions are given by

$$\lambda = 1: \quad \varphi = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \phi = \begin{bmatrix} x + it + 1 \\ -x - it \end{bmatrix}. \tag{3.3}$$

Solutions for $\lambda \in \mathbb{R}_-$, and in particular, for $\lambda \in [-1, 0)$, are found from the symmetry property of the Lax equations in Remark 1. This implies that any $\lambda \in (-1,0) \cup (0,1)$ is a geometrically double eigenvalue, whereas $\lambda = \pm 1$ are geometrically simple.

For $\lambda = i\gamma$ with $\gamma \in \mathbb{R}$, two solutions to the Lax equations (2.2)–(2.3) are given by:

$$\varphi = \begin{bmatrix} \sqrt{\frac{1}{2}k(\gamma) - \gamma} \\ -i\sqrt{\frac{1}{2}k(\gamma) + \gamma} \end{bmatrix} e^{-\frac{1}{2}ik(\gamma)(x-\gamma t)}, \quad \phi = \begin{bmatrix} \sqrt{\frac{1}{2}k(\gamma) + \gamma} \\ i\sqrt{\frac{1}{2}k(\gamma) - \gamma} \end{bmatrix} e^{+\frac{1}{2}ik(\gamma)(x-\gamma t)}, \tag{3.4}$$

where $k(\gamma) := 2\sqrt{1+\gamma^2}$. These solutions are bounded and linearly independent for every $\gamma \in \mathbb{R}_+$. Solutions for $\gamma \in \mathbb{R}_-$ are found from the symmetry property of the Lax equations in Remark 1. Consequently, any $\lambda = i\gamma$ with $\gamma \in \mathbb{R}\backslash\{0\}$ is a geometrically double eigenvalue.

For $\lambda = 0$ ($\gamma = 0$), there are two linearly independent solutions,

$$\lambda = 0: \quad \varphi = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{-ix}, \quad \phi = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{+ix}, \tag{3.5}$$

implying that $\lambda = 0$ is a geometrically double eigenvalue. In contrast to the eigenvalues above, the eigenvalue $\lambda = 0$ has algebraic multiplicity four because the bounded solutions of $\mathcal{L}^2\varphi = 0$ are spanned by (3.5) and two additional solutions

$$\lambda = 0: \quad \varphi_{\rm g} = \begin{bmatrix} t \\ -1 - it \end{bmatrix} e^{-ix}, \quad \phi_{\rm g} = \begin{bmatrix} -t \\ -1 - it \end{bmatrix} e^{+ix}. \tag{3.6}$$

These computations are summarized in the following lemma, where we have also checked algebraic multiplicities of all eigenvalues.

Lemma 1. *The Lax spectrum of the spectral problem (2.2) with $u = 1$ in the space $C_b^0(\mathbb{R})$ of bounded continuous functions is the set*

$$\Sigma_0 = i\mathbb{R} \cup [-1, 1], \tag{3.7}$$

and the following properties hold:

(1) $\lambda = \pm 1$ *are algebraically simple eigenvalues;*
(2) *each* $\lambda \in \Sigma_0 \backslash \{0, \pm 1\}$ *is a geometrically and algebraically double eigenvalue;*
(3) $\lambda = 0$ *is an eigenvalue with geometric multiplicity two and algebraic multiplicity four.*

Proof. Geometric multiplicity of all eigenvalues has been checked with direct computations resulting in (3.2), (3.3), (3.4), and (3.5). In order to check the algebraic multiplicity of eigenvalues, we note that for each eigenvalue λ, the bounded eigenfunctions φ and ϕ in $C_b^0(\mathbb{R})$ are periodic in x with some spatial period L. For the algebraic multiplicity of λ, we need to solve $(\mathcal{L} - \lambda I)\varphi_g = \varphi$ and $(\mathcal{L} - \lambda I)\phi_g = \phi$ in the space of periodic functions with the same period L. Consequently, we can check the Fredholm condition in $L^2(0, L)$ equipped with the standard inner product $\langle \cdot, \cdot \rangle$.

Let $\varphi = (\varphi_1, \varphi_2)$ be the bounded eigenfunction of the eigenvalue problem $(\mathcal{L} - \lambda I)\varphi = 0$. By the symmetry, the adjoint problem $\left(\mathcal{L}^* - \bar{\lambda} I\right) \varphi^* = 0$ admits the eigenfunction $\varphi^* = (\bar{\varphi}_2, \bar{\varphi}_1)^T$. If $\lambda \in \Sigma_0 \backslash \{+1, -1\}$, there exists another linearly independent eigenfunction $\phi = (\phi_1, \phi_2)^T$, for which we have similarly $\phi^* = (\bar{\phi}_2, \bar{\phi}_1)^T$. Since $\langle \psi^*, \varphi \rangle = \langle \varphi^*, \phi \rangle = 0$, the generalized eigenfunctions φ_g and ϕ_g exist if and only if $\langle \varphi^*, \varphi \rangle = 0$ and $\langle \phi^*, \phi \rangle = 0$.

For $\lambda \in (0, 1)$, we obtain

$$\langle \varphi^*, \varphi \rangle = -2\lambda L e^{\lambda k(\lambda) t}, \quad \langle \phi^*, \phi \rangle = -2\lambda L e^{-\lambda k(\lambda) t},$$

which are both nonzero for $\lambda \neq 0$. For $\lambda = 1$, only one linearly independent eigenfunction φ in (3.3) exists and we check that $\langle \varphi^*, \varphi \rangle = -2L \neq 0$. For $\lambda = i\gamma$ with $\gamma \in \mathbb{R}$, we obtain

$$\langle \varphi^*, \varphi \rangle = -2i\gamma L e^{i\gamma k(\gamma) t}, \quad \langle \phi^*, \phi \rangle = -2i\gamma L e^{-i\gamma k(\gamma) t},$$

which are both nonzero for $\gamma \neq 0$. Hence, the algebraic multiplicity of all nonzero eigenvalues is equal to their geometric multiplicity.

For the eigenvalue $\lambda = 0$ with the eigenfunctions (3.5), we obtain $\langle \varphi^*, \varphi \rangle = \langle \phi^*, \phi \rangle = 0$, in agreement with the existence of the generalized eigenfunctions (3.6). On the other hand, we also have

$$\langle \varphi^*, \varphi_g \rangle = -L, \quad \langle \phi^*, \phi \rangle = -L,$$

which implies that no new generalized eigenfunctions satisfying $\mathcal{L}^3 \varphi = 0$ exist. Hence, the zero eigenvalue has algebraic multiplicity equal to four. □

Replacing the space $C_b^0(\mathbb{R})$ by $L^2(\mathbb{R})$ in Lemma 1 the Lax spectrum does not change, the difference being that Σ_0 becomes a purely continuous spectrum in $L^2(\mathbb{R})$. In the space $L^2_{\rm per}(0, L)$ of L-periodic functions, the Lax spectrum only contains the eigenvalues $\lambda \in \Sigma_0$ with L-periodic associated eigenfunctions, hence the purely point spectrum is located at

$$\Sigma_0^{(P)} = \{\pm \lambda_m^{(P)}, \ \ m \in \{0, \mathbb{N}_{\rm even}\}\}, \quad \lambda_m^{(P)} := \sqrt{1 - \frac{\pi^2}{L^2} m^2}. \tag{3.8}$$

Similarly, in the space $L^2_{\text{antiper}}(0,L)$ of L-antiperiodic functions, the Lax spectrum only contains the eigenvalues $\lambda \in \Sigma_0$ with L-antiperiodic associated eigenfunctions, hence the purely point spectrum is located at

$$\Sigma_0^{(A)} = \{\pm\lambda_m^{(A)}, \quad m \in \mathbb{N}_{\text{odd}}\}, \quad \lambda_m^{(A)} := \sqrt{1 - \frac{\pi^2}{L^2}m^2}. \tag{3.9}$$

The algebraic and geometric multiplicities of these eigenvalues remain the same, as given by Lemma 1. Notice that $\lambda = 0$ is an eigenvalue only for particular periods $L \in \pi\mathbb{N}$.

Figure 1 illustrates these results. The left panel shows the purely continuous spectrum of Σ_0 in $L^2(\mathbb{R})$ given by (3.7). The right panel shows the union $\Sigma_0^{(P)} \cup \Sigma_0^{(A)}$ of the purely point spectra in $L^2_{\text{per}}(0,L)$ and $L^2_{\text{antiper}}(0,L)$ given by (3.8) and (3.9), respectively.

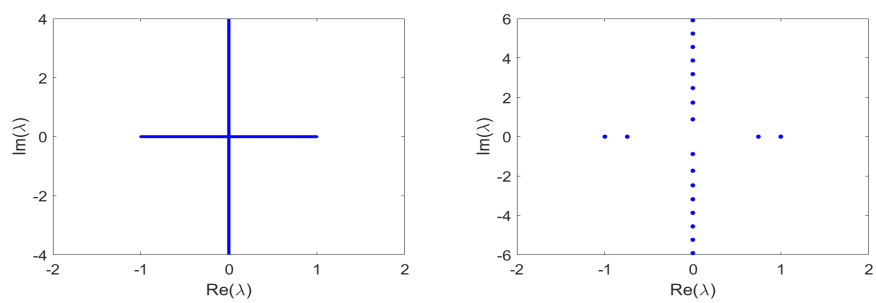

FIGURE 1. Left: The Lax spectrum Σ_0 in $L^2(\mathbb{R})$. Right: The union $\Sigma_0^{(P)} \cup \Sigma_0^{(A)}$ of the Lax spectra in $L^2_{\text{per}}(0,L)$ and $L^2_{\text{antiper}}(0,L)$ for some positive $L \notin \pi\mathbb{N}$.

3.2. **Localized solutions.** Since the linearized NLS equation (3.1) has constant coefficients, the Fourier transform provides a basis of bounded solutions in x which can be used to represent a general solution in $L^2(\mathbb{R})$. The following proposition gives the result.

Proposition 3. *For every $v_0 \in L^2(\mathbb{R})$, there exists a unique solution $v \in C^0(\mathbb{R}, L^2(\mathbb{R}))$ to the linearized NLS equation (3.1) satisfying $v(x,0) = v_0(x)$ in the form of a linear superposition*

$$v(x,t) = \int_0^\infty \left[c_k^+ \widetilde{v}_k^+(t) + c_k^- \widetilde{v}_k^-(t)\right] \cos(kx)dk + \int_0^\infty \left[d_k^+ \widetilde{v}_k^+(t) + d_k^- \widetilde{v}_k^-(t)\right] \sin(kx)dk, \tag{3.10}$$

where coefficients $c_k^\pm$ and $d_k^\pm$ are uniquely found from $v_0 \in L^2(\mathbb{R})$, and the functions $\widetilde{v}_k^\pm(t)$ for $k \geq 0$ are given as follows:

$$k = 0: \qquad \begin{cases} \widetilde{v}_0^+(t) = 2i, \\ \widetilde{v}_0^-(t) = 1 + 2it, \end{cases} \tag{3.11}$$

$$k \in (0,2): \qquad \begin{cases} \widetilde{v}_k^+(t) = (2i\lambda + k)e^{k\lambda t}, \\ \widetilde{v}_k^-(t) = (2i\lambda - k)e^{-k\lambda t}, \end{cases} \qquad \lambda = \lambda(k) = \frac{1}{2}\sqrt{4 - k^2}, \tag{3.12}$$

$$k = 2: \qquad \begin{cases} \widetilde{v}_2^+(t) = 2, \\ \widetilde{v}_2^-(t) = i + 2t, \end{cases} \tag{3.13}$$

$$k \in (2,\infty): \qquad \begin{cases} \widetilde{v}_k^+(t) = k\cos(k\gamma t) - 2i\gamma\sin(k\gamma t), \\ \widetilde{v}_k^-(t) = 2i\gamma\cos(k\gamma t) + k\sin(k\gamma t), \end{cases} \qquad \gamma = \gamma(k) = \frac{1}{2}\sqrt{k^2 - 4}. \tag{3.14}$$

Proof. The proof is based on separation of variables and straightforward computations. Indeed, substituting $v(t,x) = \widetilde{v}_k(t)e^{ikx}$ into (3.1) yields the linear differential equation

$$i\frac{d}{dt}\widetilde{v}_k + \left(1 - \frac{k^2}{2}\right)\widetilde{v}_k + \overline{\widetilde{v}}_k = 0,$$

with two linearly independent solutions $\widetilde{v}_k^+(t)$ and $\widetilde{v}_k^-(t)$ given by (3.11), (3.12), (3.13), and (3.14) for different values of $k \geq 0$. Completeness of the basis of bounded functions in $L^2(\mathbb{R})$ is given by the Fourier theory. □

Remark 6. *Since the points $k = 0$ and $k = 2$ are of measure zero in the integral (3.10) we actually do not need the solutions for $k = 0$ and $k = 2$. However these solutions play a role when the space $L^2(\mathbb{R})$ is replaced by the space $L^2_{\rm per}(0,L)$ of L-periodic functions.*

Remark 7. *From the Fourier decomposition* (3.10) *we can determine the spectrum of the linear operator from the linearized NLS equation (3.1), when acting in the space $L^2(\mathbb{R})$. We find that the purely continuous spectrum is located at $\{\pm k\lambda(k)\ ;\ k \geq 0\} = i\mathbb{R} \cup [-1,1]$. This implies the spectral instability of $u = 1$ in the linearized NLS equation (3.1).*

It follows from Proposition 1 that solutions of the linearized NLS equation (3.1) can be constructed from solutions of the Lax equations (2.2)–(2.3) with $u = 1$ and $\lambda \in \mathbb{C}$. We show below how to recover the Fourier basis in the decomposition (3.10) from the eigenfunctions associated to the Lax spectrum in Lemma 1. We use the three pairs of solutions given in Table 1.

Pair II of Table 1. Using φ and ϕ in either (3.2) or (3.4), we obtain the same constant solutions

$$v = 0, \quad \widetilde{v}_0^+(t) = 2i, \tag{3.15}$$

where $\widetilde{v}_0^+$ is the same as in (3.11). Using φ and ϕ from (3.3), we find the solutions

$$\widetilde{v}_0^-(t) = 1 + 2it, \quad v(x) = i(2x+1), \tag{3.16}$$

where $\widetilde{v}_0^-$ is the same as in (3.11). The two bounded solutions in the Fourier decomposition (3.10) with $k = 0$ are recovered.

Pairs I and III of Table 1. Using the eigenfunction φ from (3.3) associated to the simple eigenvalue $\lambda = 1$ we obtain the solutions from (3.15), again. By the symmetry of the Lax system in Remark 1, the solutions obtained for $\lambda = -1$ are, up to sign, the same.

Next, using φ and ϕ in (3.2) for $\lambda \in (0,1)$, we find the following four linearly independent bounded solutions:

$$v_\lambda^+(x,t) = -(2i\lambda + k)e^{\lambda kt}\sin(kx), \quad v_{-\lambda}^+(x,t) = (2i\lambda + k)e^{\lambda kt}\cos(kx), \tag{3.17}$$

and

$$v_\lambda^-(x,t) = (2i\lambda - k)e^{-\lambda kt}\sin(kx), \quad v_{-\lambda}^-(x,t) = (2i\lambda - k)e^{-\lambda kt}\cos(kx), \tag{3.18}$$

in which $k = k(\lambda) \in (0,2)$. These are, up to sign, equal to the four solutions in the Fourier decomposition (3.10) given by (3.12) so that we have a one-to-one correspondence between the solutions provided by the Lax system with $\lambda \in (0,1)$ and the solutions in (3.10) with $k \in (0,2)$ through the equalities $k = k(\lambda)$ and $\lambda = \lambda(k)$. By the symmetry of the Lax system in Remark 1, the solutions obtained for $\lambda = (-1,0)$ are, up to sign, the same.

Using φ and ϕ in (3.4) for $\lambda = i\gamma$ with $\gamma \in \mathbb{R}_+$, we only find two linearly independent solutions

$$\begin{aligned} v_\lambda^+(x,t) &= k\cos(kx - k\gamma t) + 2i\gamma\sin(kx - k\gamma t), \\ v_{-\lambda}^+(x,t) &= -2i\gamma\cos(kx - k\gamma t) + k\sin(kx - k\gamma t), \end{aligned} \tag{3.19}$$

in which $k = k(\gamma) \in (2,\infty)$. However, using φ and ϕ in (3.4) with $-\gamma$ instead of γ, we obtain other two linearly independent solutions,

$$\begin{aligned} v_\lambda^-(x,t) &= k\cos(kx + k\gamma t) - 2i\gamma\sin(kx + k\gamma t), \\ v_{-\lambda}^-(x,t) &= 2i\gamma\cos(kx + k\gamma t) + k\sin(kx + k\gamma t). \end{aligned} \tag{3.20}$$

Solutions (3.19) and (3.20) are linear combinations of the four solutions in the Fourier decomposition (3.10) with $k \in (2,\infty)$ given by (3.14), and we have a one-to-one correspondence between these solutions through the equalities $k = k(\gamma)$ and $\gamma = \gamma(k)$.

Finally, using φ and ϕ in (3.5) for $\lambda = 0$, we obtain two linearly independent solutions

$$v_0^+(x,t) = -2\sin(2x), \quad v_{-0}^+(x,t) = 2\cos(2x). \tag{3.21}$$

These recover the two solutions with $k = 2$ in the Fourier decomposition (3.10) corresponding to $\widetilde{v}_2^+$ in (3.13). In order to recover the two solutions given by $\widetilde{v}_2^-$ in (3.13), we use (2.9) with the eigenfunctions (3.5) and the generalized eigenfunctions (3.6) to obtain

$$v_0^-(x,t) = 2(i+2t)\cos(2x) - 2\sin(2x), \quad v_{-0}^-(x,t) = 2(i+2t)\sin(2x) + 2\cos(2x). \tag{3.22}$$

Using (2.9) with ϕ and ϕ_g produces the same solutions as (3.22) up to the change of signs. Solutions (3.21) and (3.22) for $\lambda = 0$ recover the four solutions in the Fourier decomposition (3.10) given by (3.13) for $k = 2$.

Summarizing, the set of eigenfunctions of the Lax equations (2.2)–(2.3) with $u = 1$ and $\lambda \in \Sigma_0$ allows us to recover the Fourier basis in the decomposition (3.10), except for the two functions $\widetilde{v}_2^-(t)\cos(2x)$ and $\widetilde{v}_2^-(t)\sin(2x)$ with $k = 2$. The entire basis is recovered when also using the generalized eigenfunctions (3.6) associated to the eigenvalue $\lambda = 0$. This leads to an alternative expansion for solutions $v \in C^0(\mathbb{R}, L^2(\mathbb{R}))$ to the linearized NLS equation (3.1),

$$v(x,t) = \int_0^\infty \left[c_k^+ v_{\lambda(k)}^+(x,t) + c_k^- v_{\lambda(k)}^-(x,t) + c_{-k}^+ v_{-\lambda(k)}^+(x,t) + c_{-k}^- v_{-\lambda(k)}^-(x,t) \right] dk, \tag{3.23}$$

where coefficients $c_{\pm k}^\pm$ are uniquely defined from the initial condition $v(\cdot,0) = v_0 \in L^2(\mathbb{R})$, and $v_{\pm\lambda(k)}^\pm(x,t)$ are given by (3.17)–(3.18) if $k \in (0,2)$ and by (3.19)–(3.20) if $k \in (2,\infty)$. Since the points $k = 0$ and $k = 2$ are of measure zero in the integral (3.23) we do not need solutions (3.15), (3.16), (3.21), and (3.22).

Remark 8. *Since the solutions for $k = 0$ and $k = 2$ are not used in the expansion (3.23), the solutions found from Pair II of Table 1 and from the eigenvalues $\lambda = 0$ and $\lambda = \pm 1$ play no role in the dynamics of localized perturbations on the background of $u = 1$. In particular, linearly growing in t solutions play no role in this dynamics. All relevant solutions are obtained using the eigenfunctions of the Lax system for $\lambda \in \Sigma_0 \setminus \{0, \pm 1\}$ in Pairs I and III of Table 1.*

3.3. **Periodic solutions.** Solutions of the linearized NLS equation (3.1) in the space $L^2_{\rm per}(0,L)$ of periodic functions with the fundamental period $L > 0$ are found by restricting the continuous Fourier decomposition (3.10) to the discrete values

$$k_m := \frac{2\pi m}{L}, \quad m \in \mathbb{N}_0 := \{0, \mathbb{N}\}. \tag{3.24}$$

This leads to a decomposition in Fourier series

$$v(x,t) = \sum_{m\in\mathbb{N}_0} \left[c^+_{k_m}\widetilde{v}^+_{k_m}(t) + c^-_{k_m}\widetilde{v}^-_{k_m}(t)\right]\cos(k_m x) + \sum_{m\in\mathbb{N}} \left[d^+_{k_m}\widetilde{v}^+_{k_m}(t) + d^-_{k_m}\widetilde{v}^-_{k_m}(t)\right]\sin(k_m x), \quad (3.25)$$

where coefficients $c^\pm_{k_m}$ and $d^\pm_{k_m}$ are uniquely found from the initial condition $v(\cdot,0) = v_0 \in L^2_{\rm per}(0,L)$, and the functions $\widetilde{v}^\pm_{k_m}(t)$ are given by (3.11)–(3.14).

We obtain an equivalent decomposition using the eigenfunctions of the Lax system. For the Lax system we have to consider both L-periodic and L-antiperiodic solutions, because the solutions of the linearized NLS equation (3.1) are constructed using squares of solutions of the Lax system.

The Lax spectra $\Sigma_0^{(P)}$ in $L^2_{\rm per}(0,L)$ and $\Sigma_0^{(A)}$ in $L^2_{\rm antiper}(0,L)$ are given in (3.8) and (3.9), respectively. For notational simplicity we set $\lambda(k_m) = \lambda_m^{(P)}$, if m is even, and $\lambda(k_m) = \lambda_m^{(A)}$, if m is odd, so that $\Sigma_0^{(P)} \cup \Sigma_0^{(A)} = \{\pm\lambda(k_m),\ m\in\mathbb{N}_0\}$. The arguments above show that all functions in the Fourier series (3.25) are recovered from the eigenfunctions of the Lax system associated to the eigenvalues $\lambda \in \Sigma_0^{(P)} \cup \Sigma_0^{(A)}$. Indeed, for $m=0$ we have the eigenvalues $\pm\lambda(0) = \pm 1 \in \Sigma_0^{(P)}$ leading to the solutions $\widetilde{v}_0^+$ and $\widetilde{v}_0^-$ given by (3.15) and (3.16), respectively, which are constant in x. If $0 < \pi m < L$, then $\lambda(k_m) \in (0,1)$, and we have the four linearly independent solutions in (3.17)–(3.18) with $\lambda = \lambda(k_m)$. If $\pi m > L$, then $\lambda(k_m) = i\gamma(k_m)$ is purely imaginary, and we have the four linearly independent solutions in (3.19)–(3.20) with $\gamma = \gamma(k_m)$. In the particular case $L = \pi m$, for some $m\in\mathbb{N}$, we have $\lambda(k_m) = 0$ and four linearly independent solutions are given in (3.21) and (3.22).

As a consequence, an arbitrary solution of the linearized NLS equation (3.1) in $L^2_{\rm per}(0,L)$ can be written in the series form:

$$\begin{aligned} v(x,t) &= c_0^+\widetilde{v}_0^+(t) + c_0^-\widetilde{v}_0^-(t) \\ &\quad + \sum_{m\in\mathbb{N}} \left[c_m^+ v^+_{\lambda(k_m)}(x,t) + c_m^- v^-_{\lambda(k_m)}(x,t) + c_{-m}^+ v^+_{-\lambda(k_m)}(x,t) + c_{-m}^- v^-_{-\lambda(k_m)}(x,t)\right], \quad (3.26) \end{aligned}$$

where coefficients $c^\pm_{\pm m}$ are uniquely defined from the initial condition $v(\cdot,0) = v_0 \in L^2_{\rm per}(0,L)$, and $v^\pm_{\pm\lambda(k_m)}(x,t)$ are given by (3.17)–(3.18) if $0 < \pi m < L$, by (3.21)–(3.22) if $\pi m = L$, and by (3.19)–(3.20) if $\pi m > L$.

Remark 9. *When $L \notin \pi\mathbb{N}$, the functions $v^\pm_{\pm\lambda(k_m)}(x,t)$ in the decomposition* (3.26) *are all obtained from the eigenfunctions associated to nonzero eigenvalues $\pm\lambda(k_m)$. When $L = \pi m$, the eigenvalues $\pm\lambda(k_m)$ vanish and the associated eigenfunctions only provide the two linearly independent solutions* (3.21)*. The generalized eigenfunctions associated to the eigenvalue $\lambda(k_m) = 0$ must be used in this case to obtain the other two solutions in* (3.22)*.*

4. Akhmediev breather (AB)

By using the Darboux transformation in Proposition 2, we obtain AB from the constant solution $u = 1$. We describe the associate Lax spectrum in Section 4.1 and construct periodic solutions of the linearized NLS equation in Section 4.2.

Let $\lambda_0 \in (0,1)$ and define the particular solution $\varphi = (p_0, q_0)^T$ of the Lax system (2.2)–(2.3) with $u = 1$ and $\lambda = \lambda_0$:

$$\left\{ \begin{array}{l} p_0(x,t) = \sqrt{\lambda_0 - \frac{i}{2}k_0}\, e^{\frac{1}{2}(-ik_0x+\sigma_0 t)} - \sqrt{\lambda_0 + \frac{i}{2}k_0}\, e^{\frac{1}{2}(ik_0x-\sigma_0 t)}, \\ q_0(x,t) = -\sqrt{\lambda_0 + \frac{i}{2}k_0}\, e^{\frac{1}{2}(-ik_0x+\sigma_0 t)} + \sqrt{\lambda_0 - \frac{i}{2}k_0}\, e^{\frac{1}{2}(ik_0x-\sigma_0 t)}, \end{array} \right. \tag{4.1}$$

where $k_0 = 2\sqrt{1-\lambda_0^2} \in (0,2)$ and $\sigma_0 = \lambda_0 k_0$. Elementary computations give

$$\begin{aligned} |p_0|^2 + |q_0|^2 &= 4\left[\cosh(\sigma_0 t) - \lambda_0 \cos(k_0 x)\right], \\ |p_0|^2 - |q_0|^2 &= 2k_0 \sin(k_0 x), \\ p_0 \bar{q}_0 &= 2\cos(k_0 x) - 2\lambda_0 \cosh(\sigma_0 t) + ik_0 \sinh(\sigma_0 t). \end{aligned}$$

and the one-fold Darboux transformation (2.10) yields the formula for AB:

$$\hat{u}_0(x,t) = -1 + \frac{2(1-\lambda_0^2)\cosh(\sigma_0 t) + i\sigma_0 \sinh(\sigma_0 t)}{\cosh(\sigma_0 t) - \lambda_0 \cos(k_0 x)}. \tag{4.2}$$

The AB solution $\hat{u}_0$ is L-periodic in x with $L = 2\pi/k_0 > \pi$ and

$$\lim_{t\to\pm\infty} \hat{u}_0(x,t) = 1 - 2\lambda_0^2 \pm ik_0\lambda_0 = \left(\sqrt{1-\lambda_0^2} \pm i\lambda_0\right)^2,$$

from which it follows that $\lim_{t\to\pm\infty} |\hat{u}_0(x,t)| = 1$. The complementary transformation (2.12) gives

$$|\hat{u}_0(x,t)|^2 = 1 + \lambda_0 k_0^2 \frac{\cosh(\sigma_0 t)\cos(k_0 x) - \lambda_0}{\left[\cosh(\sigma_0 t) - \lambda_0 \cos(k_0 x)\right]^2},$$

which is consistent with the exact solution (4.2).

4.1. **Lax spectrum at AB.** For the Lax system (2.2)–(2.3), we consider both L-periodic and L-antiperiodic eigenfunctions $\varphi = \varphi(x,t)$ in x. We use the Darboux transformation (2.13) and the result of Lemma 1 to determine the Lax spectrum for AB, which is illustrated in Figure 2.

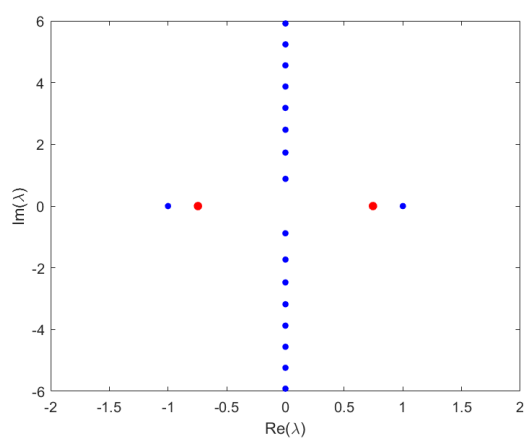

FIGURE 2. The union $\Sigma_{AB}^{(P)} \cup \Sigma_{AB}^{(A)}$ of the Lax spectra in $L^2_{\rm per}(0,L)$ and $L^2_{\rm antiper}(0,L)$ for AB. The red dots represent the eigenvalues $\{+\lambda_0, -\lambda_0\}$.

In the space of L-antiperiodic functions, we show that Lax spectrum of AB consists of the same eigenvalues (3.9) as for the constant-amplitude solution $u = 1$. The only difference between the two spectra is that the eigenvalues $\{+\lambda_0, -\lambda_0\}$ are geometrically double for $u = 1$, while they are geometrically simple and algebraically double for $u = \hat{u}_0$.

Lemma 2. *Consider AB given by (4.2) and assume $L \notin \pi\mathbb{N}_{\rm odd}$. The spectrum of the ZS spectral problem (2.2) with $u = \hat{u}_0$ in $L^2_{\rm antiper}(0, L)$ consists of isolated eigenvalues*

$$\Sigma^{(A)}_{\rm AB} = \{\pm\lambda^{(A)}_m, \ \ m \in \mathbb{N}_{\rm odd}\}, \quad \lambda^{(A)}_m := \sqrt{1 - \frac{\pi^2}{L^2} m^2},$$

with the following properties:

(1) *For each $m \in \mathbb{N}_{\rm odd}$, $m \neq 1$, the eigenvalues $\pm\lambda^{(A)}_m$ are geometrically and algebraically double.*
(2) *The eigenvalues $\lambda^{(A)}_1 = \lambda_0$ and $-\lambda^{(A)}_1 = -\lambda_0$ are geometrically simple and algebraically double with associated eigenfunctions $\varphi = (\hat{p}_0, \hat{q}_0)^T$ and $\varphi = (-\bar{\hat{q}}_0, \bar{\hat{p}}_0)^T$ and generalized eigenfunctions $\varphi_g = (\varphi_{1,1}, \varphi_{1,2})^T$ and $\varphi_g = (-\bar{\varphi}_{1,2}, \bar{\varphi}_{1,1})^T$, where $\varphi_0 = (\hat{p}_0, \hat{q}_0)^T$ and $\varphi_1 = (\varphi_{1,1}, \varphi_{1,2})^T$ are given by* (A.7) *and* (A.10) *in Appendix A.*

Proof. The Darboux matrix $D(\lambda)$ given by (2.14) is L-periodic in x and invertible for every $\lambda \neq \pm\lambda_0$. It follows from the relation (2.13) that there is one-to-one correspondence between the L-antiperiodic solutions of the Lax systems with $u = 1$ and $u = \hat{u}_0$ when $\lambda \neq \pm\lambda_0$. Consequently, with the exception of $m = 1$, the L-antiperiodic Lax spectrum for $u = \hat{u}_0$ is the same as the L-antiperiodic Lax spectrum for $u = 1$ in (3.9) and the property (1) holds. The linearly independent eigenfunctions for the eigenvalues $\lambda = \pm\lambda^{(A)}_m$ are given in the form

$$\hat{\varphi} := \varphi + \frac{1}{\lambda - \lambda_0} \left[\begin{array}{c} \hat{p}_0 \\ \hat{q}_0 \end{array} \right] [-q_0 \ \ p_0]\, \varphi, \quad \hat{\phi} := \phi + \frac{1}{\lambda - \lambda_0} \left[\begin{array}{c} \hat{p}_0 \\ \hat{q}_0 \end{array} \right] [-q_0 \ \ p_0]\, \phi, \tag{4.3}$$

where the two linearly independent eigenfunctions φ and ϕ are given by (3.2) if $0 < L < \pi m$ and by (3.4) if $L > \pi m$. The marginal case $L = \pi m$ is excluded by the assumption.

For $\lambda = \lambda_0$, transformation (2.11) gives the eigenfunction $\varphi_0 = (\hat{p}_0, \hat{q}_0)^T$ of the Lax system with $u = \hat{u}_0$ and it is easy to check that φ_0 is L-antiperiodic in x. For $\lambda = -\lambda_0$ we have the eigenfunction $\varphi = (-\bar{\hat{q}}_0, \bar{\hat{p}}_0)^T$ due to the symmetry in Remark 1. Hence $\{+\lambda_0, -\lambda_0\}$ belong to the L-antiperiodic Lax spectrum for $u = \hat{u}_0$. It remains to show that λ_0 is geometrically simple and algebraically double, the result for $-\lambda_0$ following then by the symmetry of the Lax system.

For this part of the proof, we rely on the explicit computation of the expansion into Laurent series of the 2×2 matrix solution $\hat{\Phi}(\lambda)$ to the Lax system with $u = \hat{u}_0$ given in Appendix A. The vector ϕ_0 given by (A.11) is a second linearly independent solution to the Lax system (2.2)–(2.3) for $u = \hat{u}_0$ and $\lambda = \lambda_0$. Since it is not L-antiperiodic in x, we deduce that λ_0 is geometrically simple. Next, φ_1 given by (A.10) is L-antiperiodic and satifies $(\mathcal{L} - \lambda_0 I)\varphi_1 = \varphi_0$, whereas φ_2 given by (A.13) satisfies $(\mathcal{L} - \lambda_0 I)\varphi_2 = \varphi_1$, but it is not L-antiperiodic. This implies that λ_0 is algebraically double and completes the proof. □

Remark 10. *For an alternative proof that λ_0 is algebraically double, we can check the Fredholm condition for the eigenfunction φ_0 and the first generalized eigenfunction φ_1. Taking the eigenfunction $\varphi^*_0 = (\bar{\hat{q}}_0, \bar{\hat{p}}_0)^T$ of the adjoint problem $(\mathcal{L}^* - \lambda_0 I)\, \varphi^*_0 = 0$ and the inner product $\langle \cdot, \cdot \rangle$ in*

$L^2(0,L)$, we find that

$$\langle \varphi_0^*, \varphi_0 \rangle = \lambda_0^2 \int_0^L \frac{\cosh(\sigma_0 t - ik_0 x) - \lambda_0}{(\cosh(\sigma_0 t) - \lambda_0 \cos(k_0 x))^2} dx = 0$$

and

$$\begin{aligned} \langle \varphi_0^*, \varphi_1 \rangle &= \frac{\lambda_0}{2} \int_0^L \frac{\cos(k_0 x)}{\cosh(\sigma_0 t) - \lambda_0 \cos(k_0 x)} dx \\ &\quad + \frac{2\lambda_0^2}{k_0^2} \int_0^L \frac{(\cosh(\sigma_0 t + ik_0 x) - \lambda_0)(\cosh(\sigma_0 t - ik_0 x) - \lambda_0)}{(\cosh(\sigma_0 t) - \lambda_0 \cos(k_0 x))^2} dx \\ &= \frac{2\lambda_0^2}{k_0^2} \int_0^L \frac{\cosh^2(\sigma_0 t) + \lambda_0^2 \cos^2(k_0 x) - 2\lambda_0^2}{(\cosh(\sigma_0 t) - \lambda_0 \cos(k_0 x))^2} dx = \frac{2\lambda_0^2}{k_0^2} L \neq 0. \end{aligned}$$

Since $\langle \varphi_0^*, \varphi_0 \rangle = 0$, there exists the generalized eigenfunction φ_1 satisfying $(\mathcal{L} - \lambda_0 I)\varphi_1 = \varphi_0$ in $L^2_{\text{antiper}}(0,L)$. Since $\langle \varphi_0^*, \varphi_1 \rangle \neq 0$, there is no the second generalized eigenfunction φ_2 satisfying $(\mathcal{L} - \lambda_0 I)\varphi_2 = \varphi_1$ in $L^2_{\text{antiper}}(0,L)$. This implies that λ_0 is algebraically double.

For L-periodic solutions, we show that Lax spectrum of AB consists of the same eigenvalues (3.8) as for the constant-amplitude solution $u = 1$. Moreover, algebraic multiplicitilies of the eigenvalues coincide.

Lemma 3. *Consider AB given by (4.2) and assume $L \notin \pi \mathbb{N}_{\text{even}}$. The spectrum of the ZS spectral problem (2.2) with $u = \hat{u}_0$ in $L^2_{\text{per}}(0,L)$ consists of isolated eigenvalues*

$$\Sigma_{\text{AB}}^{(P)} = \{\pm \lambda_m^{(P)}, \ \ m \in \{0, \mathbb{N}_{\text{even}}\}\}, \quad \lambda_m^{(P)} := \sqrt{1 - \frac{\pi^2}{L^2} m^2},$$

with the following properties:

(1) *For each $m \in \mathbb{N}_{\text{even}}$, the eigenvalues $\pm \lambda_m^{(P)}$ are geometrically and algebraically double.*
(2) *The eigenvalues $\lambda_0^{(P)} = 1$ and $-\lambda_0^{(P)} = -1$ are algebraically simple with associated eigenfunctions $\varphi = (\hat{\varphi}_1, \hat{\varphi}_2)^T$ and $\varphi = (-\bar{\hat{\varphi}}_2, \bar{\hat{\varphi}}_1)^T$ respectively, where $\hat{\varphi} = (\hat{\varphi}_1, \hat{\varphi}_2)^T$ is given by*

$$\hat{\varphi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \frac{p_0 + q_0}{1 - \lambda_0} \begin{bmatrix} \hat{p}_0 \\ \hat{q}_0 \end{bmatrix}. \tag{4.4}$$

Proof. As in the proof of Lemma 2, the set of eigenvalues and their geometric and algebraic multiplicities are found from the Darboux matrix $D(\lambda)$ in (2.14) and the transformation (2.13). Since $D(\lambda)$ is L-periodic in x and invertible for every $\lambda = \pm \lambda_m^{(P)}$, there is one-to-one correspondence between the L-periodic solutions of the Lax systems with $u = 1$ and $u = \hat{u}_0$. Moreover, the explicit expressions (4.3) for eigenfunctions $\hat{\varphi}$ and $\hat{\phi}$ hold for every $\lambda = \pm \lambda_m^{(P)}$.

For the eigenvalue $\lambda = \lambda_0^{(P)} = 1$, only one linearly independent eigenfunction $\varphi = \hat{\varphi} = (\hat{\varphi}_1, \hat{\varphi}_2)^T$ in $L^2_{\text{per}}(0,L)$ exists in the form (4.4). In order to check the algebraic multiplicity of $\lambda = 1$, we take eigenfunction $\varphi^* = (\bar{\hat{\varphi}}_2, \bar{\hat{\varphi}}_1)^T$ of the adjoint problem $(\mathcal{L}^* - I)\varphi^* = 0$ and compute

the scalar product

$$\begin{aligned}
\langle \varphi^*, \varphi \rangle &= -2 \int_0^L \left[1 + \frac{2\lambda_0}{1-\lambda_0} \frac{p_0 \bar{q}_0 + \bar{p}_0 q_0 + |p_0|^2 + |q_0|^2}{|p_0|^2 + |q_0|^2} + \frac{4\lambda_0^2}{(1-\lambda_0)^2} \frac{p_0 q_0 (\bar{p}_0 + \bar{q}_0)^2}{(|p_0|^2 + |q_0|^2)^2} \right] dx \\
&= -2 \frac{1+\lambda_0}{1-\lambda_0} \int_0^L \left[1 + 2\lambda_0 \frac{\cosh(\sigma_0 t)\cos(k_0 x) - \lambda_0 - i\lambda_0 \sinh(\sigma_0 t)\sin(k_0 x)}{[\cosh(\sigma_0 t) - \lambda_0 \cos(k_0 x)]^2} \right] dx \\
&= -2 \frac{1+\lambda_0}{1-\lambda_0} L.
\end{aligned}$$

Since $\langle \varphi^*, \varphi \rangle \neq 0$, there exists no generalized eigenfunction satisfying $(\mathcal{L} - I)\hat{\varphi}_g = \hat{\varphi}$ in $L^2_{\rm per}(0, L)$ so that the eigenvalue $\lambda = 1$ is algebraically simple. The result for $\lambda = -1$ is a consequence of the symmetry of the Lax system in Remark 1. □

4.2. **Linearized NLS equation at AB.** As in the case of the constant-amplitude solution $u = 1$, we construct L-periodic solutions of the linearized NLS equation at AB from the L-periodic and L-antiperiodic solutions of the Lax equations. These solutions are generated by the eigenvalues $\lambda_m^{(A)}$ and $\lambda_m^{(P)}$ in Lemmas 2 and 3, respectively.

We focus on the solutions related to the positive eigenvalues $\lambda_1^{(A)} = \lambda_0$ and $\lambda_0^{(P)} = 1$. By the symmetry of the Lax system, the negative eigenvalues $-\lambda_1^{(A)} = -\lambda_0$ and $-\lambda_0^{(P)} = -1$ provide the same solutions up to the sign change. The particular goal is to identify six linearly independent solutions of the linearized NLS equation at AB which correspond to the six linearly independent solutions in the decomposition (3.26) with $m = 0$ and $m = 1$ for the solutions of the linearized NLS equation at $u = 1$. The correspondence is established by showing that the solutions constructed for AB become identical to the ones for $u = 1$ asymptotically as $t \to \pm\infty$.

The following theorem presents the main result of these computations.

Theorem 1. *Consider AB given by (4.2). Solutions of the Lax system (2.2)–(2.3) with $u = \hat{u}_0$ for the eigenvalues $\lambda = 1$ and $\lambda = \lambda_0$ generate the following six linearly independent L-periodic solutions of the linearized NLS equation (2.6) at AB:*

1. *the solutions v_1 in (4.6) and v_2 in (4.7), which are asymptotically equivalent to the solutions $v^-_{\lambda(k_1)}$ and $\widetilde{v}_0^+$ respectively, in the decomposition (3.26);*
2. *the solution w_2 in (4.11), which is asymptotically equivalent to the solution $v^-_{-\lambda(k_1)}$ in the decomposition (3.26);*
3. *the solution v in (4.15), which is asymptotically equivalent to the solution $\widetilde{v}_0^-$ in the decomposition (3.26);*
4. *the solutions v in (4.19) and (4.20), which are asymptotically equivalent to the solutions $v^+_{-\lambda(k_1)}$ and $v^+_{\lambda(k_1)}$ respectively, in the decomposition (3.26).*

The six solutions in this theorem are computed explicitly in the next three subsections.

Remark 11. *Due to the two exponentially growing solutions in item (4) of Theorem 1, AB is linearly unstable. This agrees with the main conclusion of* [19] *based on symbolic computations. For periods $L \in (\pi, 2\pi)$ the eigenvalues λ_0 and 1 are the only positive eigenvalues of the Lax system, see Figure 2. For larger periods $L > 2\pi$, there are additional positive eigenvalues which lead to exponentially growing solutions for the linearized NLS equation at AB.*

4.2.1. *Solutions related to* $\lambda = 1$. Recall from Lemma 3 that $\lambda = 1$ is an algebraically simple eigenvalue in $\Sigma_{\rm AB}^{(P)}$ associated with eigenfunction $\hat{\varphi}$ given by (4.4). The second linearly independent solution of the Lax system (2.2)–(2.3) for $u = \hat{u}_0$ and $\lambda = 1$ is obtained from the second vector in (3.3) by using the transformation formula (2.13)–(2.14) with $\lambda = 1$:

$$\hat{\phi} = \left[\begin{array}{c} x + it + 1 \\ -x - it \end{array} \right] - \frac{(x+it)(p_0+q_0)+q_0}{1-\lambda_0} \left[\begin{array}{c} \hat{p}_0 \\ \hat{q}_0 \end{array} \right]. \tag{4.5}$$

By using the L-periodic eigenfunction $\hat{\varphi}$ in (4.4) in Pair I in Table 1, we obtain the following two L-periodic solutions of the linearized NLS equation (2.6):

$$v_1(x,t) = -\frac{2\lambda_0(1+\lambda_0)}{1-\lambda_0} \frac{\sin(k_0 x)\left[k_0\cosh(\sigma_0 t) + 2i\lambda_0 \sinh(\sigma_0 t)\right]}{\left[\cosh(\sigma_0 t) - \lambda_0\cos(k_0 x)\right]^2}. \tag{4.6}$$

and

$$\begin{aligned} v_2(x,t) &= \frac{2i(1+\lambda_0)}{1-\lambda_0} \left[\frac{ik_0\lambda_0 \sinh(\sigma_0 t)\cosh(\sigma_0 t)}{\left[\cosh(\sigma_0 t) - \lambda_0\cos(k_0 x)\right]^2} \right. \\ &\quad \left. + \frac{(1-2\lambda_0^2)\cosh^2(\sigma_0 t) - \lambda_0^2\cos^2(k_0 x) + 2\lambda_0^2}{\left[\cosh(\sigma_0 t) - \lambda_0\cos(k_0 x)\right]^2} \right]. \end{aligned} \tag{4.7}$$

As $t \to \pm\infty$, the periodic solution v_1 decays to 0, whereas the periodic solution v_2 approaches a nonzero constant. These two solutions are asymptotically equivalent to the solutions $v^-_{\lambda(k_1)}$ and $\widetilde{v}_0^+$ in the decomposition (3.26).

By using both the L-periodic eigenfunction $\hat{\varphi}$ in (4.4) and the non-periodic solution $\hat{\phi}$ in (4.5) in Pair II in Table 1, we obtain the following two non-periodic solutions of the linearized NLS equation (2.6):

$$v_3(x,t) = xv_1(x,t) + tv_2(x,t) + f_1(x,t), \tag{4.8}$$

$$v_4(x,t) = xv_2(x,t) - tv_1(x,t) + f_2(x,t), \tag{4.9}$$

where the periodic parts f_1 and f_2 are given by

$$\begin{aligned} f_1(x,t) &= \frac{1+\lambda_0}{1-\lambda_0} \left[1 + \frac{4i\lambda_0^2}{k_0}\sinh(\sigma_0 t) \frac{\lambda_0\cosh(\sigma_0 t) - \cos(k_0 x)}{\left[\cosh(\sigma_0 t) - \lambda_0\cos(k_0 x)\right]^2} \right. \\ &\quad + 2\lambda_0^2 \frac{\cosh^2(\sigma_0 t) - \cos^2(k_0 x) - i\sin(k_0 x)\sinh(\sigma_0 t)}{\left[\cosh(\sigma_0 t) - \lambda_0\cos(k_0 x)\right]^2} \\ &\quad \left. - \lambda_0 k_0 \frac{\cosh(\sigma_0 t)\sin(k_0 x)}{\left[\cosh(\sigma_0 t) - \lambda_0\cos(k_0 x)\right]^2} \right]. \end{aligned}$$

and

$$\begin{aligned} f_2(x,t) &= i\frac{1+\lambda_0}{1-\lambda_0} \left[1 - \frac{k_0\lambda_0}{1-\lambda_0^2} \frac{\sin(k_0 x)}{\cosh(\sigma_0 t) - \lambda_0\cos(k_0 x)} + \frac{i\lambda_0 k_0 \cosh(\sigma_0 t)\sinh(\sigma_0 t)}{\left[\cosh(\sigma_0 t) - \lambda_0\cos(k_0 x)\right]^2} \right. \\ &\quad + \frac{4\lambda_0(1-\lambda_0)\cos(k_0 x)}{\cosh(\sigma_0 t) - \lambda_0\cos(k_0 x)} - \lambda_0^2 \frac{\cosh(2\sigma_0 t) + \cos(2k_0 x)}{\left[\cosh(\sigma_0 t) - \lambda_0\cos(k_0 x)\right]^2} \\ &\quad \left. + \frac{\lambda_0(1+\lambda_0)(1-\lambda_0^2)\cos(k_0 x)^2}{\left[\cosh(\sigma_0 t) - \lambda_0\cos(k_0 x)\right]^2} \frac{2\lambda_0^3\cosh(\sigma_0 t)\cos(k_0 x)}{\left[\cosh(\sigma_0 t) - \lambda_0\cos(k_0 x)\right]^2} \right]. \end{aligned}$$

Both solutions grow linearly in x and are not L-periodic. As $t \to \pm\infty$, the non-periodic solution v_3 becomes asymptotically periodic, because v_1 decays to 0, and could represent v_0^- in the decomposition (3.26). However, one needs to cancel the polynomial term in x by using a linear combination with other solutions of the linearized NLS equation (2.6).

Finally, by using the non-periodic solution (4.5) in Pair III in Table 1, we obtain two other non-periodic solutions of the linearized NLS equation (2.6), which are quadratic with respect to x. As is described in the recent symbolic computations in [19], such quadratic solutions in x play no role in the proof of Theorem 1.

4.2.2. *Solutions related to* $\lambda = \lambda_0$. By Lemma 2, the eigenvalue λ_0 is geometrically simple with L-antiperiodic eigenfunction $\varphi_0 = (\hat{p}_0, \hat{q}_0)^T$ given by (A.7). A second linearly independent solution of the Lax system is the non-periodic solution ϕ_0 given by (A.11) in Appendix A.

By using Pair I in Table 1 with φ_0, we obtain the following two L-periodic solutions of the linearized NLS equation (2.6):

$$w_1 = \hat{p}_0^2 - \bar{\hat{q}}_0^2 = \frac{\lambda_0^2 \sin(k_0 x)\left[k_0 \cosh(\sigma_0 t) + 2i\lambda_0 \sinh(\sigma_0 t)\right]}{2\left[\cosh(\sigma_0 t) - \lambda_0 \cos(k_0 x)\right]^2} = -\lambda_0 k_0^{-2} \frac{\partial \hat{u}_0}{\partial x} \tag{4.10}$$

and

$$w_2 = i(\hat{p}_0^2 + \bar{\hat{q}}_0^2) = \frac{\lambda_0^2 \left[k_0 \sinh(\sigma_0 t)\cos(k_0 x) + 2i\lambda_0 \cosh(\sigma_0 t)\cos(k_0 x) - 2i\right]}{2\left[\cosh(\sigma_0 t) - \lambda_0 \cos(k_0 x)\right]^2} = -k_0^{-2} \frac{\partial \hat{u}_0}{\partial t}. \tag{4.11}$$

These are neutral modes generated by the translational symmetries of the NLS equation (2.1) in x and t. Note that w_1 is proportional to the solution v_1 in (4.6),

$$v_1 = -\frac{4(1+\lambda_0)}{\lambda_0(1-\lambda_0)} w_1. \tag{4.12}$$

As $t \to \pm\infty$, the two periodic solutions w_1 and w_2 decay to 0. These two solutions are asymptotically equivalent to the solutions $v^-_{\lambda(k_1)}$ and $v^-_{-\lambda(k_1)}$ in the decomposition (3.26).

Next, we record the following algebraic computations:

$$\begin{aligned}
&-\bar{q}_0 p_+(\lambda_0) + p_0 \bar{q}_+(\lambda_0) = 4i \sin(k_0 x),\\
&-\bar{q}_0 p_-(\lambda_0) + p_0 \bar{q}_-(\lambda_0) = 0,\\
&-\bar{q}_0 p_+(\lambda_0) - p_0 \bar{q}_+(\lambda_0) = 4\lambda_0 \sinh(\sigma_0 t) - 2ik_0 \cosh(\sigma_0 t),\\
&-\bar{q}_0 p_-(\lambda_0) - p_0 \bar{q}_-(\lambda_0) = 4\left[\lambda_0 \cosh(\sigma_0 t) - \cos(k_0 x)\right] - 2ik_0 \sinh(\sigma_0 t).
\end{aligned}$$

Then, by using Pair II in Table 1 with the L-antiperiodic eigenfunction φ_0 in (A.7) and the non-periodic solution ϕ_0 in (A.11), we obtain the following two non-periodic solutions of the linearized NLS equation (2.6):

$$w_3(x,t) = -4\lambda_0 x w_1(x,t) + 4(1-2\lambda_0^2) t w_2(x,t) + g_1(x,t), \tag{4.13}$$

$$w_4(x,t) = -4\lambda_0 x w_2(x,t) - 4(1-2\lambda_0^2) t w_1(x,t) + g_2(x,t), \tag{4.14}$$

where the periodic parts g_1 and g_2 are given by

$$\begin{aligned}
g_1(x,t) \;=\;& 4k_0^{-1}\left[\cosh(\sigma_0 t)\sin(k_0 x) w_1(x,t) + \sinh(\sigma_0 t)\cos(k_0 x) w_2(x,t)\right]\\
&+\lambda_0 \frac{2\lambda_0 \cosh(\sigma_0 t) - 2\cos(k_0 x) - ik_0 \sinh(\sigma_0 t)}{\cosh(\sigma_0 t) - \lambda_0 \cos(k_0 x)}
\end{aligned}$$

and

$$g_2(x,t) = 4k_0^{-1}\left[\cosh(\sigma_0 t)\sin(k_0 x)w_2(x,t) - \sinh(\sigma_0 t)\cos(k_0 x)w_1(x,t)\right].$$

Note that the components g_1 and g_2 are bounded in t as $t \to \pm\infty$. In view of (4.12), the linear combination

$$v(x,t) = \lambda_0^2 \frac{1-\lambda_0}{1+\lambda_0} v_3(x,t) - w_3(x,t) = t s_0(x,t) + f_0(x,t), \tag{4.15}$$

where s_0 and f_0 are L-periodic in x and bounded as $t \to \pm\infty$, e.g.

$$\begin{aligned} s_0 &= 2i\lambda_0^2 \left[1 - 2\lambda_0^2 + \frac{ik_0\lambda_0 \sinh(\sigma_0 t)}{\cosh(\sigma_0 t) - \lambda_0 \cos(k_0 x)} \right. \\ &\quad \left. + (1-\lambda_0^2) \frac{ik_0 \sinh(\sigma_0 t)\cos(k_0 x) + 2(1-\lambda_0^2\cos^2(k_0 x))}{[\cosh(\sigma_0 t) - \lambda_0\cos(k_0 x)]^2} \right]. \end{aligned}$$

As $t \to \pm\infty$, the solution v in (4.15) is asymptotically equivalent to the solution $\widetilde{v}_0^-$ in the decomposition (3.26).

Remark 12. *Since v_2 and w_2 are not linearly dependent from each other, there is no a linear combination of v_4 and w_4 which would be L-periodic in x.*

Finally, by using Pair III in Table 1 with the non-periodic solution ϕ_0 in (A.11), we obtain two non-periodic solutions which are quadratic in x. Again, such quadratic solutions in x play no role in the proof of Theorem 1.

4.2.3. *Solutions related to the generalized eigenfunction at $\lambda = \lambda_0$.* With the account of v_1, v_2, w_1, w_2, the relation (4.12), and the linear combination (4.15), it remains to obtain two L-periodic solutions of the linearized NLS equation (2.6) at AB, which would be asymptotically equivalent to the remaining solutions $v^+_{\lambda(k_1)}$ and $v^+_{-\lambda(k_1)}$ in the decomposition (3.26). These solutions will be constructed from linear combinations of non-periodic solutions that grows linearly in x, just as the solution v in (4.15).

By Lemma 2, the eigenvalue λ_0 is algebraically double with the generalized eigenfunction φ_1 given by (A.10) in addition to the eigenfunction φ_0 given by (A.7). It is natural to expect additional solutions for the linearized NLS equation to be obtained from the eigenfunction φ_0 and the generalized eigenfunction φ_1. It is surprising, however, that this is not the case. As a result, we have to use the eigenfunction φ_0 and the generalized eigenfunction φ_1 together with the non-periodic solutions ϕ_0 and ϕ_1 given by (A.11) and (A.14).

By using the expansion of the 2×2 matrix solution $\hat{\Phi}(\lambda)$ computed in Appendix A, we write

$$\hat{\Phi}(\lambda) = \left[2ik_0\varphi(\lambda), \phi(\lambda)\right],$$

where

$$\varphi(\lambda) = \frac{\varphi_0}{\lambda-\lambda_0} + \varphi_1 + \varphi_2(\lambda-\lambda_0) + \mathcal{O}((\lambda-\lambda_0)^2), \tag{4.16}$$

$$\phi(\lambda) = \phi_0 + \phi_1(\lambda-\lambda_0) + \mathcal{O}((\lambda-\lambda_0)^2). \tag{4.17}$$

Here φ_0, φ_1, and φ_2 are given by (A.7), (A.10), and (A.13), whereas ϕ_0 and ϕ_1 are given by (A.11) and (A.14).

Both columns of $\hat{\Phi}(\lambda)$ being solutions of the Lax system (2.2)–(2.3), the three pairs in Table 1 give solutions $v(\lambda)$ of the linearized NLS equation (2.6) at AB. Expanding $v(\lambda)$ at $\lambda = \lambda_0$

generates a set of possible solutions to the linearized NLS equation (2.6) at AB. It turns out that the L-periodic solutions and the linearly growing in x solutions obtained from Pair I in Table 1 are all linear combinations of the previously obtained solutions and that the solutions obtained from Pair III in Table 1 are all at least quadratic in x. As a result, the new suitable solutions to the linearized NLS equation (2.6) must be obtained by using Pair II in Table 1.

Using Pair II in Table 1 with the two columns in the matrix $\hat{\Phi}(\lambda)$ expanded at $\lambda = \lambda_0$, we obtain the following expansions

$$v = (2ik_0)\left[\frac{w_\pm}{\lambda - \lambda_0} + v_\pm + \mathcal{O}(\lambda - \lambda_0)\right], \tag{4.18}$$

where each term of the expansion gives a solution v to the linearized NLS equation (2.6) at AB. It follows that $w_+ = w_3$ and $w_- = w_4$ were previously obtained in (4.13) and (4.14) respectively. The next corrections in (4.18) give two new solutions:

$$\begin{aligned} v_+ &= \varphi_{0,1}\phi_{1,1} - \bar{\varphi}_{0,2}\bar{\phi}_{1,2} + \varphi_{1,1}\phi_{0,1} - \bar{\varphi}_{1,2}\bar{\phi}_{0,2}, \\ v_- &= i\varphi_{0,1}\phi_{1,1} + i\bar{\varphi}_{0,2}\bar{\phi}_{1,2} + i\varphi_{1,1}\phi_{0,1} + i\bar{\varphi}_{1,2}\bar{\phi}_{0,2}, \end{aligned}$$

where the first subscript stands for φ_0, φ_1, ϕ_0, and ϕ_1 and the second subscript stands for the first and second components of the 2-vectors. For further computations of $v_\pm$ we obtain

$$\begin{aligned} p_0 p_+(\lambda_0) + \bar{q}_0\bar{q}_+(\lambda_0) &= 2\cos(k_0 x)\left[2\lambda_0 \sinh(\sigma_0 t) - ik_0\cosh(\sigma_0 t)\right], \\ p_0 p_+(\lambda_0) - \bar{q}_0\bar{q}_+(\lambda_0) &= -2\sin(k_0 x)\left[k_0\sinh(\sigma_0 t) + 2i\lambda_0\cosh(\sigma_0 t)\right]. \end{aligned}$$

After substitution of (A.7), (A.10), (A.11), and (A.14) into $v_\pm$, we obtain

$$v_\pm(x,t) = xr_\pm(x,t) + ts_\pm(x,t) + f_\pm(x,t),$$

where the L-periodic parts are computed explicitly:

$$\begin{aligned} r_+ &= -\frac{8}{k_0^2}(3 - 2\lambda_0^2)w_1, \\ r_- &= -\frac{8}{k_0^2}(1 - 4\lambda_0^2)w_2 + \frac{2\lambda_0^2}{(1+\lambda_0)^2}v_2, \end{aligned}$$

$$\begin{aligned} s_+ &= \frac{4}{k_0}(1 - 2\lambda_0^2)(\hat{p}_0 p_+(\lambda_0) - \bar{\hat{q}}_0\bar{q}_+(\lambda_0)) + \frac{16}{k_0^2}(1 - 2\lambda_0^2)\sinh(\sigma_0 t)\sin(k_0 x)w_1 \\ &\quad + \frac{8}{k_0^2}(1 - 2\lambda_0^2)\left(2\cosh(\sigma_0 t)\cos(k_0 x) - \lambda_0\right)w_2 - 8\lambda_0 w_2, \\ s_- &= \frac{4i}{k_0}(1 - 2\lambda_0^2)(\hat{p}_0 p_+(\lambda_0) + \bar{\hat{q}}_0\bar{q}_+(\lambda_0)) + \frac{16}{k_0^2}(1 - 2\lambda_0^2)\sinh(\sigma_0 t)\sin(k_0 x)w_2 \\ &\quad - \frac{8}{k_0^2}(1 - 2\lambda_0^2)\left(2\cosh(\sigma_0 t)\cos(k_0 x) - \lambda_0\right)w_1 + 8\lambda_0 w_1, \end{aligned}$$

and

$$\begin{aligned}
f_+ &= \frac{1}{2ik_0}\left(p_0p_+(\lambda_0)+\bar{q}_0\bar{q}_+(\lambda_0)\right)+\frac{i}{k_0}(\hat{p}_0p_+(\lambda_0)-\bar{\hat{q}}_0\bar{q}_+(\lambda_0))\\
&\quad+\frac{2}{k_0^2}\left(\cosh(\sigma_0t)\cos(k_0x)-\lambda_0\right)(p_0\hat{p}_0-\bar{q}_0\bar{\hat{q}}_0)-\frac{2i}{k_0^2}\sinh(\sigma_0t)\sin(k_0x)(p_0\hat{p}_0+\bar{q}_0\bar{\hat{q}}_0)\\
&\quad+\frac{2}{k_0^2}\sinh(\sigma_0t)\cos(k_0x)(\hat{p}_0p_+(\lambda_0)-\bar{\hat{q}}_0\bar{q}_+(\lambda_0))\\
&\quad-\frac{2i}{k_0^2}\cosh(\sigma_0t)\sin(k_0x)(\hat{p}_0p_+(\lambda_0)+\bar{\hat{q}}_0\bar{q}_+(\lambda_0))\\
&\quad+\frac{4}{k_0^3}\cosh(2\sigma_0t)\sin(2k_0x)w_1+\frac{4}{k_0^3}\sinh(2\sigma_0t)\cos(2k_0x)w_2,\\
f_- &= \frac{1}{2k_0}\left(p_0p_+(\lambda_0)-\bar{q}_0\bar{q}_+(\lambda_0)\right)-\frac{1}{k_0}(\hat{p}_0p_+(\lambda_0)+\bar{\hat{q}}_0\bar{q}_+(\lambda_0))\\
&\quad+\frac{2i}{k_0^2}\left(\cosh(\sigma_0t)\cos(k_0x)-\lambda_0\right)(p_0\hat{p}_0+\bar{q}_0\bar{\hat{q}}_0)+\frac{2}{k_0^2}\sinh(\sigma_0t)\sin(k_0x)(p_0\hat{p}_0-\bar{q}_0\bar{\hat{q}}_0)\\
&\quad+\frac{2i}{k_0^2}\sinh(\sigma_0t)\cos(k_0x)(\hat{p}_0p_+(\lambda_0)+\bar{\hat{q}}_0\bar{q}_+(\lambda_0))\\
&\quad+\frac{2}{k_0^2}\cosh(\sigma_0t)\sin(k_0x)(\hat{p}_0p_+(\lambda_0)-\bar{\hat{q}}_0\bar{q}_+(\lambda_0))\\
&\quad+\frac{4}{k_0^3}\cosh(2\sigma_0t)\sin(2k_0x)w_2-\frac{4}{k_0^3}\sinh(2\sigma_0t)\cos(2k_0x)w_1.
\end{aligned}$$

The x-growing part of v_+ is cancelled in the linear combination

$$\begin{aligned}
v(x,t) &= k_0^2v_+(x,t)-\frac{2\lambda_0(3-2\lambda_0^2)(1-\lambda_0)}{1+\lambda_0}v_3(x,t)\\
&= ts_1(x,t)+k_0^2f_+(x,t)-\frac{2\lambda_0(3-2\lambda_0^2)(1-\lambda_0)}{1+\lambda_0}f_1(x,t),
\end{aligned}\tag{4.19}$$

where

$$s_1(x,t) = 4i\lambda_0(7-10\lambda_0^2)\frac{(2\lambda_0^2-1)\cosh(\sigma_0t)-i\lambda_0k_0\sinh(\sigma_0t)-\lambda_0\cos(k_0x)}{\cosh(\sigma_0t)-\lambda_0\cos(k_0x)},$$

and $f_+(x,t)$, $f_1(x,t)$ are L-periodic in x. Note that $ts_1(x,t)$ is irreducible in the sense that there are no other solutions of the linearized NLS equation (2.6) with the same behavior as $ts_1(x,t)$. On the other hand, $s_1(x,t)$ and $f_1(x,t)$ are bounded as $t\to\pm\infty$, whereas $k_0^2f_+(x,t)$ is unbounded. As $t\to\pm\infty$, we deduce that the exponentially growing component of $v(x,t)$ is given by

$$v(x,t)\sim -4k_0^2(1-4\lambda_0^2)\cosh(\sigma_0t)\cos(k_0x)-8i\lambda_0k_0(3-4\lambda_0^2)\sinh(\sigma_0t)\cos(k_0x).$$

We conclude that the solution v in (4.19) is L-periodic and asymptotically equivalent to the mode $v^+_{-\lambda(k_1)}$ in the decomposition (3.26) as $t\to\pm\infty$.

The x-growing part of v_- is cancelled in the linear combination

$$\begin{aligned} v(x,t) &= k_0^2 v_-(x,t) - \frac{2(1-4\lambda_0^2)}{\lambda_0} w_4(x,t) - \frac{8\lambda_0^2(1-\lambda_0)}{1+\lambda_0} v_4(x,t) \\ &= t s_2(x,t) + k_0^2 f_-(x,t) - \frac{2(1-4\lambda_0^2)}{\lambda_0} g_2(x,t) - \frac{8\lambda_0^2(1-\lambda_0)}{1+\lambda_0} f_2(x,t), \end{aligned} \tag{4.20}$$

where

$$s_2(x,t) = -\lambda_0 k_0^2 (2\lambda_0^2+1) \frac{\sin(k_0 x)[k_0 \cosh(\sigma_0 t) + 2i\lambda_0 \sinh(\sigma_0 t)]}{[\cosh(\sigma_0 t) - \lambda_0 \cos(k_0 x)]^2},$$

and $f_-(x,t)$, $f_2(x,t)$, $g_2(x,t)$ are L-periodic in x. Again, the solution $v(x,t)$ grows exponentially in time as $t \to \pm\infty$ due to the unbounded component $k_0^2 f_-(x,t)$, according to

$$v(x,t) \sim -4k_0^2(1-4\lambda_0^2)\sinh(\sigma_0 t)\sin(k_0 x) - 8i\lambda_0 k_0 (3-4\lambda_0^2)\cosh(\sigma_0 t)\sin(k_0 x).$$

We conclude that the solution v in (4.20) is L-periodic and asymptotically equivalent to the mode $v^+_{\lambda(k_1)}$ in the decomposition (3.26) as $t \to \pm\infty$. This completes the proof of the theorem.

4.2.4. *Solutions related to other eigenvalues.* We conclude this section with some comments on the solutions generated by the remaining eigenvalues in the Lax spectrum. These are the geometrically double eigenvalues $\{\lambda_m^{(P)}\}_{m\in\mathbb{N}_{\rm even}}$ for L-periodic solutions and $\{\lambda_m^{(A)}\}_{m\in\mathbb{N}_{\rm odd}\setminus\{1\}}$ for L-antiperiodic solutions. We exclude the case $L \in \pi\mathbb{N}$ when 0 is an eigenvalue of higher algebraic multiplicity and the two eigenfunctions alone were not enough to obtain the decomposition (3.26) for the constant solution $u = 1$. Then, using Pairs I and III in Table 1, from the associated eigenfunctions we obtain L-periodic solutions of the linearized NLS equation (2.6) at AB which are asymptotically equivalent to the solutions $\{v^\pm_{\pm\lambda(k_m)}\}_{m\in\mathbb{N}\setminus\{0,1\}}$, in the decomposition (3.26). Pair II in Table 1 generates two L-periodic solutions which are linear combinations of the solutions v_1, v_2, and w_2 from Theorem 1. Together with the other three solutions from Theorem 1, the resulting set of solutions is asymptotically equivalent to the one in the decomposition (3.26). While we do not attempt to prove completeness of this set, we refer to [19, Section 4] for a recent discussion of this question.

5. Kuznetsov–Ma breather (KMB)

Here we apply the same procedure of Section 4 for KMB. Since KMB is localized in x, we have to consider the Lax spectrum and bounded solutions of the linearized NLS equation in the function space $L^2(\mathbb{R})$.

Let $\lambda_0 \in (1,\infty)$ and define the particular solution $\varphi = (p_0, q_0)^T$ of the Lax system (2.2)–(2.3) with $u = 1$ and $\lambda = \lambda_0$:

$$\left\{ \begin{aligned} p_0(x,t) &= \sqrt{\lambda_0 + \frac{1}{2}\beta_0}\; e^{\frac{1}{2}(\beta_0 x + i\alpha_0 t)} - \sqrt{\lambda_0 - \frac{1}{2}\beta_0}\; e^{-\frac{1}{2}(\beta_0 x + i\alpha_0 t)}, \\ q_0(x,t) &= -\sqrt{\lambda_0 - \frac{1}{2}\beta_0}\; e^{\frac{1}{2}(\beta_0 x + i\alpha_0 t)} + \sqrt{\lambda_0 + \frac{1}{2}\beta_0}\; e^{-\frac{1}{2}(\beta_0 x + i\alpha_0 t)}, \end{aligned} \right. \tag{5.1}$$

where $\beta_0 = 2\sqrt{\lambda_0^2 - 1}$ and $\alpha_0 = \lambda_0\beta_0$. Notice that p_0 and q_0 in (5.1) are related symbolically to the ones for AB in (4.1) through the equalities $k_0 = i\beta_0$ and $\sigma_0 = i\alpha_0$. Elementary computations

give

$$\begin{aligned} |p_0|^2 + |q_0|^2 &= 4\left[\lambda_0 \cosh(\beta_0 x) - \cos(\alpha_0 t)\right] \\ |p_0|^2 - |q_0|^2 &= 2\beta_0 \sinh(\beta_0 x) \\ p_0 \bar{q}_0 &= -2\cosh(\beta_0 x) + 2\lambda_0 \cos(\alpha_0 t) + i\beta_0 \sin(\alpha_0 t), \end{aligned} \tag{5.2}$$

so that the one-fold Darboux transformation (2.10) yields the formula for KMB:

$$\hat{u}_0(x,t) = -1 + \frac{2(\lambda_0^2 - 1)\cos(\alpha_0 t) + i\alpha_0 \sin(\alpha_0 t)}{\lambda_0 \cosh(\beta_0 x) - \cos(\alpha_0 t)}. \tag{5.3}$$

The complementary transformation (2.12) gives a consistent relation

$$|\hat{u}_0(x,t)|^2 = 1 + \alpha_0\beta_0 \frac{\lambda_0 - \cosh(\beta_0 x)\cos(\alpha_0 t)}{(\lambda_0 \cosh(\beta_0 x) - \cos(\alpha_0 t))^2},$$

which can also be derived from (5.3). KMB is periodic in t with period $T = 2\pi/\alpha_0$ and localized in x with $\lim_{x\to\pm\infty} \hat{u}_0(x,t) = -1$.

5.1. **Lax spectrum at KMB.** As for AB, we use the Darboux matrix (2.14) to construct the bounded solutions of the Lax system (2.2)–(2.3) with $u = \hat{u}_0$ from the bounded solutions of the Lax system with $u = 1$ and then determine the Lax spectrum at KMB in $L^2(\mathbb{R})$. The Lax spectrum $\Sigma_{\rm KMB}$ is shown in Figure 3 where the red dots show isolated eigenvalues $\{+\lambda_0, -\lambda_0\}$.

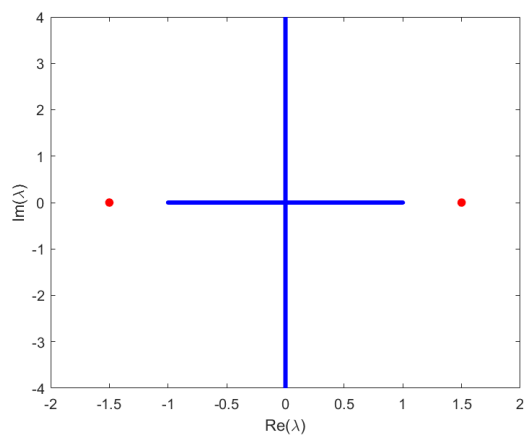

FIGURE 3. The Lax spectrum $\Sigma_{\rm KMB}$ in $L^2(\mathbb{R})$ for KMB.

The following lemma gives precisely the spectrum $\Sigma_{\rm KMB}$.

Lemma 4. *Consider KMB given by (5.3). The spectrum of the ZS spectral problem (2.2) with $u = \hat{u}_0$ in $L^2(\mathbb{R})$ is the set*

$$\Sigma_{\rm KMB} = i\mathbb{R} \cup [-1,1] \cup \{\lambda_0, -\lambda_0\}, \tag{5.4}$$

with the following properties:

(1) *For each $\lambda \in i\mathbb{R} \cup (-1,1)$, there exist two linearly independent bounded solutions.*
(2) *For $\lambda = 1$ and $\lambda = -1$, there exists only one bounded solution.*
(3) *The eigenvalues $\lambda = \lambda_0$ and $\lambda = -\lambda_0$ are algebraically simple with associated eigenfunctions $\varphi = (\hat{p}_0, \hat{q}_0)^T$ and $\varphi = (\bar{\hat{q}}_0, -\bar{\hat{p}}_0)^T$ respectively.*

Proof. The Darboux matrix $D(\lambda)$ given by (2.14) is invertible for $\lambda \neq \pm\lambda_0$. Moreover, both $D(\lambda)$ and its inverse are bounded in x for $\lambda \neq \pm\lambda_0$. As a result, we have a one-to-one correspondence between the bounded solutions of the Lax systems with $u = 1$ and $u = \hat{u}_0$ for $\lambda \neq \pm\lambda_0$. This implies that, up to the values $\pm\lambda_0$, the ZS spectral problems (2.2) with $u = 1$ and $u = \hat{u}_0$ have the same continuous spectrum Σ_0 given by (3.7), so that properties (1) and (2) hold.

It remains to prove (3). Due to the symmetry property of the Lax system, it is enough to show the result for λ_0.

The vector $\varphi = (p_0, q_0)^T$ given by (5.1) is a solution of the Lax system (2.2)–(2.3) with $u = 1$ and $\lambda = \lambda_0$. The Darboux transformation (2.11) gives the solution $\varphi = (\hat{p}_0, \hat{q}_0)^T$ of the Lax system (2.2)–(2.3) with $u = \hat{u}_0$ and $\lambda = \lambda_0$. From the formulas (5.1) and (5.2) we find that $\hat{p}_0(x,t), \hat{q}_0(x,t) \to 0$ as $|x| \to \infty$ exponentially fast, hence $\varphi = (\hat{p}_0, \hat{q}_0)^T$ is an eigenfunction in $L^2(\mathbb{R})$ associated with λ_0. Furthermore, the Lax system (2.2)–(2.3) having zero-trace, the Wronskian of any two solutions is constant both in x and t. Since one solution is decaying to zero as $|x| \to \infty$, another linearly independent solution is necessarily growing at infinity. Consequently, λ_0 is geometrically simple.

For the algebraic multiplicity, we use the form $(\mathcal{L} - \lambda_0 I)\,\varphi_0 = 0$ with the eigenfunction $\varphi_0 = (\hat{p}_0, \hat{q}_0)^T \in L^2(\mathbb{R})$ and show that the linear nonhomogeneous equation

$$(\mathcal{L} - \lambda_0 I)\,\psi_0 = \varphi_0, \tag{5.5}$$

does not have the generalized eigenfunction $\psi_0 \in L^2(\mathbb{R})$. The solvability condition for this equation is given by the Fredholm condition $\langle \varphi_0^*, \varphi_0 \rangle = 0$, where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R})$ and φ_0^* is an eigenfunction of the adjoint problem $(\mathcal{L}^* - \lambda_0 I)\,\varphi_0^* = 0$. A direct calculation shows that $\varphi_0^* = (\bar{\hat{q}}_0, \bar{\hat{p}}_0)^T$, and then we compute

$$\begin{aligned}
\langle \varphi_0^*, \varphi_0 \rangle &= -\int_{\mathbb{R}} \frac{8\lambda_0^2 p_0 q_0}{(|p_0|^2 + |q_0|^2)^2} dx \\
&= \lambda_0^2 \int_{\mathbb{R}} \frac{\cosh(\beta_0 x + i\alpha_0 t) - \lambda_0}{(\lambda_0 \cosh(\beta_0 x) - \cos(\alpha_0 t))^2} dx \\
&= -\lambda_0 \beta_0^{-1} \frac{\lambda_0 \sinh(\beta_0 x) + i \sin(\alpha_0 t)}{\lambda_0 \cosh(\beta_0 x) - \cos(\alpha_0 t)} \Bigg|_{x \to -\infty}^{x \to +\infty} = -2\lambda_0 \beta_0^{-1}.
\end{aligned}$$

Consequently, $\langle \varphi_0^*, \varphi_0 \rangle \neq 0$ so that λ_0 is a simple eigenvalue and property (3) holds. □

5.2. **Linearized NLS equation at KMB.** As in the case of the constant-amplitude solution $u = 1$, we construct bounded solutions of the linearized NLS equation (2.6) at KMB from the bounded solutions of the Lax equations for $\lambda \in \Sigma_{\rm KMB}$ in Lemma 4. Recall that the solutions are unbounded in x when $\lambda \notin \Sigma_{\rm KMB}$.

Here we focus on solutions generated by the eigenvalue λ_0, and in particular on those which are decaying to zero as $|x| \to \infty$. The eigenvalue $\lambda = -\lambda_0$ produces the same solutions of the linearized NLS equation, due to the symmetry in Remark 1. The following theorem provides the main result of these computations.

Theorem 2. *Consider KMB given by (5.3). The eigenvalue λ_0 of the Lax system generates three linearly independent exponentially decaying solutions of the linearized NLS equation (2.6). These solutions are proportional to the three derivatives of KMB with respect to x, t, and λ_0.*

Remark 13. *Besides the three exponentially decaying solutions in Theorem 2, we also find three linearly independent bounded solutions which are asymptotically constant as $x \to \pm\infty$. These six solutions are the analgue for KMB of the six solutions given in Theorem 1 for AB.*

5.2.1. *Solutions related to $\lambda = \lambda_0$.* Since λ_0 is a simple eigenvalue in Σ_{KMB}, the eigenfunction $\varphi_0 = (\hat{p}_0, \hat{q}_0)^T$ provides an exponentially decaying solution of the Lax system (2.2)–(2.3) with $\lambda = \lambda_0$. The second linearly independent solution is exponentially growing in x. According to Remark 17 in Appendix A this second solution is given by:

$$\phi_0 = \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} + 4\left[-\lambda_0 x + i(1-2\lambda_0^2)t + \beta_0^{-1}\sinh(\beta_0 x + i\alpha_0 t)\right]\begin{bmatrix} \hat{p}_0 \\ \hat{q}_0 \end{bmatrix}. \tag{5.6}$$

By using Pair I in Table 1 with φ_0, we obtain the solutions

$$w_1(x,t) = \hat{p}_0^2 - \bar{\hat{q}}_0^2 = -\frac{\lambda_0^2 \sinh(\beta_0 x)(\beta_0\cos(\alpha_0 t) + 2i\lambda_0 \sin(\alpha_0 t))}{2(\lambda_0\cosh(\beta_0 x) - \cos(\alpha_0 t))^2} \tag{5.7}$$

and

$$w_2(x,t) = i\hat{p}_0^2 + i\bar{\hat{q}}_0^2 = \frac{i\lambda_0^2(2\lambda_0\cosh(\beta_0 x)\cos(\alpha_0 t) - 2 + i\beta_0\cosh(\beta_0 x)\sin(\alpha_0 t))}{2(\lambda_0\cosh(\beta_0 x) - \cos(\alpha_0 t))^2}, \tag{5.8}$$

which are periodic in t and exponentially decaying in x. It turns out that these solutions are proportional to the derivatives of $\hat{u}_0$ with respect to x and t,

$$w_1 = \lambda_0\beta_0^{-2}\frac{\partial \hat{u}_0}{\partial x}, \quad w_2 = \beta_0^{-2}\frac{\partial \hat{u}_0}{\partial t}.$$

Hence these solutions are generated by the symmetries of the NLS equation (2.1) with respect to translation in x and t.

While Pair III in Table 1 with ϕ_0 gives exponentially growing solutions, Pair II with φ_0 and ϕ_0 gives two bounded solutions:

$$w_3(x,t) = -4\lambda_0 x w_1(x,t) + 4(1-2\lambda_0^2)t w_2(x,t) + f_1(x,t), \tag{5.9}$$

$$w_4(x,t) = -4\lambda_0 x w_2(x,t) - 4(1-2\lambda_0^2)t w_1(x,t) + f_2(x,t), \tag{5.10}$$

where w_1 and w_2 are given by (5.7) and (5.8), respectively, and

$$\begin{aligned} f_1(x,t) &= 2\lambda_0\cos(\alpha_0 t)\frac{2\lambda_0\cos(\alpha_0 t) - (1+\lambda_0^2)\cosh(\beta_0 x)}{\left[\lambda_0\cosh(\beta_0 x) - \cos(\alpha_0 t)\right]^2} \\ &\quad + 4i\lambda_0\beta_0^{-1}\sin(\alpha_0 t)\frac{(2\lambda_0^2-1)\cos(\alpha_0 t) - \lambda_0^3\cosh(\beta_0 x)}{\left[\lambda_0\cosh(\beta_0 x) - \cos(\alpha_0 t)\right]^2}, \\ f_2(x,t) &= 4i\lambda_0^2\beta_0^{-1}\frac{\sinh(\beta_0 x)}{\lambda_0\cosh(\beta_0 x) - \cos(\alpha_0 t)}. \end{aligned}$$

Here f_1 is exponentially decreasing as $|x| \to \infty$, whereas f_2 is bounded but not decaying as $|x| \to \infty$, and both f_1 and f_2 are periodic in t. Consequently, w_3 is also exponentially decaying in x, and a direct computation shows that it is proportional to the derivative of $\hat{u}_0$ with respect to λ_0,

$$\frac{d\hat{u}_0}{d\lambda_0} = x\beta_0^{-1}\beta_0'(\lambda_0)\frac{\partial \hat{u}_0}{\partial x} + t\alpha_0^{-1}\alpha_0'(\lambda_0)\frac{\partial \hat{u}_0}{\partial t} + \frac{\partial \hat{u}_0}{\partial \lambda_0} = -\lambda_0^{-1}v_3.$$

In this computation, λ_0 is an arbitrary parameter and we write $\beta_0 = \beta_0(\lambda_0) = 2\sqrt{\lambda_0^2 - 1}$ and $\alpha_0 = \alpha_0(\lambda_0) = \lambda_0\beta_0(\lambda_0)$. The solution w_4 is asymptotically constant, with

$$\lim_{x\to\pm\infty} w_4(x,t) = \pm 4i\lambda_0\beta_0^{-1}. \tag{5.11}$$

The solutions w_1, w_2, and w_3 are the three linearly independent exponentially decaying solutions in Theorem 2.

5.2.2. *Solutions related to* $\lambda \in \Sigma_0$. First, we consider the solutions of the linearized NLS equation which are asymptotically constant, but not decaying to 0. These solutions are obtained from Pairs I and II in Table 1 for $\lambda = 1$ and from Pair II for any $\lambda \in \Sigma_0$. We are looking for a suitably chosen linear combination of these solutions with w_4 which might lead to a fourth exponentially decaying solution of the linearized NLS equation (2.6). We show below that this is not the case.

For $\lambda = 1$, the Lax system has the bounded solution $\hat{\varphi}$ in (4.4) and the unbounded solution $\hat{\phi}$ in (4.5) in which (p_0, q_0) is given by (5.1). Using Pair I of Table 1 with $\hat{\varphi}$ we find the solutions

$$v_1(x,t) = \frac{2\lambda_0(1+\lambda_0)}{1-\lambda_0}\frac{\sinh(\beta_0 x)(\beta_0\cos(\alpha_0 t) + 2i\lambda_0\sin(\alpha_0 t))}{[\lambda_0\cosh(\beta_0 x) - \cos(\alpha_0 t)]^2} \tag{5.12}$$

and

$$\begin{aligned} v_2(x,t) \;=\; & \frac{2i}{(1-\lambda_0)^2}\left[\frac{-\lambda_0\cosh(\beta_0 x) + (2\lambda_0^2 - 1)\cos(\alpha_0 t) + 2i\lambda_0\beta_0\sin(\alpha_0 t)}{\lambda_0\cosh(\beta_0 x) - \cos(\alpha_0 t)}\right. \\ & \left. +\lambda_0^2\frac{\beta_0^2\sinh^2(\beta_0 x) + (-2\cosh(\beta_0 x) + 2\lambda_0\cos(\alpha_0 t) + i\beta_0\sin(\alpha_0 t))^2}{4\left[\lambda_0\cosh(\beta_0 x) - \cos(\alpha_0 t)\right]^2}\right]. \end{aligned} \tag{5.13}$$

The solution v_1 is proportional to w_1 given in (5.7),

$$v_1 = -\frac{4(1+\lambda_0)}{\lambda_0(1-\lambda_0)}w_1,$$

whereas the solution v_2 is asymptotically constant with

$$\lim_{x\to\pm\infty} v_2(x,t) = -\frac{2i(1+\lambda_0)}{1-\lambda_0}.$$

By using Pair II with $\hat{\varphi}$ and $\hat{\phi}$, we find the bounded solution

$$v_3(x,t) = \left(x + \frac{1}{2}\right)v_1(x,t) + tv_2(x,t) + g_1(x,t), \tag{5.14}$$

where

$$g_1(x,t) = \frac{1}{(1-\lambda_0)^2}\left[1 + \lambda_0^2\frac{\beta_0^2\sinh^2(\beta_0 x) + (-2\cosh(\beta_0 x) + 2\lambda_0\cos(\alpha_0 t) + i\beta_0\sin(\alpha_0 t))^2}{4\left[\lambda_0\cosh(\beta_0 x) - \cos(\alpha_0 t)\right]^2}\right],$$

and a solution $v_4(x,t)$ which is unbounded in x. The solution v_3 grows linearly in t and

$$\lim_{x\to\pm\infty} v_3(x,t) = -t\,\frac{2i(1+\lambda_0)}{1-\lambda_0} + \frac{\lambda_0^2}{(1-\lambda_0)^2}.$$

Pair III gives two unbounded solutions. The three bounded, but not decaying to 0, solutions v_2, v_3, and w_4 are linearly independent. Comparing their limits at $x = \pm\infty$ we conclude that there is no linear combination of these solutions which could lead to a localized solution.

Using Pair II in Table 1 with the two linearly independent solutions of the Lax system for $\lambda \in i\mathbb{R} \cup (-1,1)$ we do not find any new solutions. After some computations, we obtain that all these solutions are linear combinations of the exponentially decaying solutions w_1, w_2, obtained from the eigenvalue $\lambda = \lambda_0$, and the asymptotically constant solution v_2 obtained from $\lambda = 1$.

The remaining solutions of the linearized NLS equation (2.6) are obtained using Pairs I and III in Table 1 for $\lambda \in i\mathbb{R} \cup (-1,1)$. Since the Darboux matrix $D(\lambda)$ in (2.14) is invertible with constant limits as $x \to \pm\infty$, these solutions are asymptotically the linear combinations of the solutions found for $u = 1$. For $\lambda \in i\mathbb{R} \cup (-1,1)$ the latter solutions are asymptotically periodic in x with wavenumber $k = k(\lambda) = 2\sqrt{1-\lambda^2} > 0$. By analogy with the case $u = 1$, we denote by $\hat{v}^{\pm}_{\pm\lambda(k)}(x,t)$ the four solutions of the linearized NLS equation at KMB for $k \in (0,\infty)\backslash\{2\}$. Although only two linearly independent solutions are obtained for $k = 2$, the point $k = 2$ is of measure zero in the continuous spectrum $\Sigma_{\rm KMB}$.

5.2.3. *Localized solutions.* Based on these explicit computations, we expect that a solution $v \in C^0(\mathbb{R}, L^2(\mathbb{R}))$ of the linearized NLS equation (2.6) at KMB can be uniquely expressed in the linear superposition form

$$\begin{aligned} v(x,t) &= c_1 v_1(x,t) + c_2 v_2(x,t) + c_3 v_3(x,t) \\ &\quad + \int_0^\infty \left[\hat{c}^+_k \hat{v}^+_{\lambda(k)}(x,t) + \hat{c}^-_k \hat{v}^-_{\lambda(k)}(x,t) + \hat{c}^+_{-k} \hat{v}^+_{-\lambda(k)}(x,t) + \hat{c}^-_{-k} \hat{v}^-_{-\lambda(k)}(x,t) \right] dk, \end{aligned} \tag{5.15}$$

where the coefficients c_1, c_2, c_3, and $\hat{c}^{\pm}_{\pm k}$ are uniquely determined by the coefficients by the initial condition $v(\cdot,0) = v_0 \in L^2(\mathbb{R})$. A rigorous justification of this formula requires an additional completeness proof which is outside the scope of this paper.

Remark 14. *The decomposition (5.15) precisely shows how many linearly independent solutions of the linearized NLS equation (2.6) at KMB correspond to the point and continuous parts of the Lax spectrum $\Sigma_{\rm KMB}$. Interestingly, this decomposition is different from the complete set of solutions of the linearized NLS equation at the NLS soliton* [20, 21] *where perturbations to a single NLS soliton are decomposed over four exponentially decaying solutions which correspond to translations of the NLS soliton over four parameters and the four continuous families of eigenfunctions of the continuous spectrum. Here, we only found three exponentially decaying solutions.*

Remark 15. *It follows from (5.15) that the linear instability of KMB is related to the continuous spectrum Σ_0 in $\Sigma_{\rm KMB}$ with exactly the same growth rate as the one of the constant-amplitude background $u = 1$. This is in agreement with the numerical computation of unstable modes for KMB in* [14]*, where KMB was truncated on a spatially periodic domain $[-L, L]$. According to Figs. 1,2, and 3 in* [14]*, the number of unstable modes of KMB depends on the period T for every fixed L. In the limit $T \to 0$ ($\lambda_0 \to \infty$), the number of unstable modes corresponds to those of the constant-amplitude background $u = 1$. However, for each fixed L, the number of unstable modes decreases as T decreases. Our analysis corresponds to the case $L = \infty$, when the unstable modes form a continuous spectrum which is independent of period T. Indeed, the results in* [14] *showed that the number of unstable modes increases when L increases.*

6. Conclusion

We have classified solutions of the linearized NLS equation (2.6) at two breather solutions of the NLS equation (2.1) given by AB and KMB. In the case of AB, our results agree with the symbolic computations in [19] where exponentially growing in time and spatially periodic solutions of the linearized NLS equation were discovered. In the case of KMB, we provide the set of solutions for characterizing the linear instability of breathers which was not achieved in the previous work [35] due to lack of spectral mapping properties. In both cases, the question of completeness was left opened and is the future open problem of highest priority.

Among further directions, it is worth mentioning that AB and KMB are particular solutions of the third-order Lax-Novikov equation

$$u''' + 6|u|^2 u' + 2ic(u'' + 2|u|^2 u) + 4bu' + 8iau = 0, \tag{6.1}$$

for $a = c = 0$. More general solutions of the third-order Lax–Novikov equation with $a = c = 0$ are represented by the double-periodic solutions which are periodic both in x and t [2, 10]. Linear instabilities of the double-periodic solutions were recently explored in [27] by utilizing the Floquet theory both in x and t. The linear unstable bands of the double-periodic solutions should correspond to the linear unstable modes of AB and KMB in the case of degeneration of the double-periodic solutions, this limiting procedure is still to be studied in future.

Overall, characterizing instability of breathers on the constant-amplitude background is a more difficult problem compared to characterizing of the modulation instability of travelling periodic waves in the NLS equation [11, 15]. Further understanding of the linear and nonlinear instability of breathers will provide better clarity of the formation of complex rogue wave patterns and integrable turbulence in the framework of the NLS equation (1.1).

Appendix A. Computation of $\hat{\Phi}(\lambda)$ near $\lambda = \lambda_0$

Let $u = \hat{u}_0$ be AB given by (4.2) and consider the 2×2 matrix solution $\Phi(\lambda)$ to the Lax system (2.2)–(2.3) with $u = 1$ in the form:

$$\Phi(\lambda) = \begin{bmatrix} p_+(\lambda) & p_-(\lambda) \\ q_+(\lambda) & q_-(\lambda) \end{bmatrix}, \tag{A.1}$$

where

$$\begin{aligned} p_\pm(\lambda) &:= \sqrt{\lambda - \frac{i}{2}k(\lambda)}\, e^{-\frac{1}{2}ik(\lambda)x + \frac{1}{2}\sigma(\lambda)t} \pm \sqrt{\lambda + \frac{i}{2}k(\lambda)}\, e^{\frac{1}{2}ik(\lambda)x - \frac{1}{2}\sigma(\lambda)t}, \\ q_\pm(\lambda) &:= -\sqrt{\lambda + \frac{i}{2}k(\lambda)}\, e^{-\frac{1}{2}ik(\lambda)x + \frac{1}{2}\sigma(\lambda)t} \mp \sqrt{\lambda - \frac{i}{2}k(\lambda)}\, e^{\frac{1}{2}ik(\lambda)x - \frac{1}{2}\sigma(\lambda)t}, \end{aligned}$$

with $k(\lambda) := 2\sqrt{1 - \lambda^2}$, $\sigma(\lambda) := \lambda k(\lambda)$, and $\lambda \in \mathbb{R}$.

Remark 16. *The solution (4.1) used for the construction of $\hat{u}_0$ corresponds to the second column of $\Phi(\lambda)$ evaluated at $\lambda = \lambda_0$, so that $p_0 = p_-(\lambda_0)$ and $q_0 = q_-(\lambda_0)$.*

The 2×2 matrix solution $\hat{\Phi}(\lambda)$ of the Lax system (2.2)–(2.3) with $u = \hat{u}_0$ is given by the transformation (2.13) and (2.14), or explicitly by

$$\hat{\Phi}(\lambda) = \Phi(\lambda) + \frac{1}{\lambda - \lambda_0} \begin{bmatrix} \hat{p}_0 \\ \hat{q}_0 \end{bmatrix} [-q_0 \;\; p_0]\, \Phi(\lambda). \tag{A.2}$$

Expanding $\hat{\Phi}(\lambda)$ into Laurent series near the simple pole at $\lambda = \lambda_0$, we write

$$\hat{\Phi}(\lambda) = \frac{\hat{\Phi}_{-1}}{\lambda - \lambda_0} + \hat{\Phi}_0 + \hat{\Phi}_1(\lambda - \lambda_0) + \mathcal{O}((\lambda - \lambda_0)^2), \tag{A.3}$$

where

$$\begin{aligned} \hat{\Phi}_{-1} &:= \begin{bmatrix} \hat{p}_0 \\ \hat{q}_0 \end{bmatrix} [-q_0 \;\; p_0]\, \Phi(\lambda_0), \\ \hat{\Phi}_0 &:= \Phi(\lambda_0) + \begin{bmatrix} \hat{p}_0 \\ \hat{q}_0 \end{bmatrix} [-q_0 \;\; p_0]\, \Phi'(\lambda_0), \\ \hat{\Phi}_1 &:= \Phi'(\lambda_0) + \frac{1}{2} \begin{bmatrix} \hat{p}_0 \\ \hat{q}_0 \end{bmatrix} [-q_0 \;\; p_0]\, \Phi''(\lambda_0). \end{aligned}$$

Further expanding the eigenvalue problem

$$(\mathcal{L} - \lambda I)\hat{\Phi}(\lambda) = 0,$$

into Laurent series near the simple pole at $\lambda = \lambda_0$, we find successively at orders $\mathcal{O}((\lambda - \lambda_0)^{-1})$, $\mathcal{O}(1)$ and $\mathcal{O}(\lambda - \lambda_0)$ the equalities

$$(\mathcal{L} - \lambda_0 I)\hat{\Phi}_{-1} = 0, \quad (\mathcal{L} - \lambda_0 I)\hat{\Phi}_0 = \hat{\Phi}_{-1}, \quad (\mathcal{L} - \lambda_0 I)\hat{\Phi}_1 = \hat{\Phi}_0. \tag{A.4}$$

We use these equalities to identify the eigenfunctions and the generalized eigenfunctions of the Lax system for $\lambda = \lambda_0$ from the columns of $\hat{\Phi}_{-1}$, $\hat{\Phi}_0$, and $\hat{\Phi}_1$.

We record the following algebraic computations:

$$\begin{aligned} -q_0 p_+(\lambda_0) + p_0 q_+(\lambda_0) &= 2ik_0, \\ -q_0 p_-(\lambda_0) + p_0 q_-(\lambda_0) &= 0, \\ -q_0 p_+(\lambda_0) - p_0 q_+(\lambda_0) &= 4\sinh(\sigma_0 t - ik_0 x), \\ -q_0 p_-(\lambda_0) - p_0 q_-(\lambda_0) &= 4\cosh(\sigma_0 t - ik_0 x) - 4\lambda_0. \end{aligned} \tag{A.5}$$

By using the first two lines of (A.5), we obtain

$$\hat{\Phi}_{-1} = [2ik_0\varphi_0, 0], \tag{A.6}$$

where

$$\varphi_0 = \begin{bmatrix} \hat{p}_0 \\ \hat{q}_0 \end{bmatrix} = \frac{\lambda_0}{2\left[\cosh(\sigma_0 t) - \lambda_0 \cos(k_0 x)\right]} \begin{bmatrix} -\bar{q}_0 \\ \bar{p}_0 \end{bmatrix}, \tag{A.7}$$

is precisely the L-antiperiodic eigenfunction of the Lax system (2.2)–(2.3) for $u = \hat{u}_0$ and $\lambda = \lambda_0$.

Next, using the equalities

$$\frac{d}{d\lambda}\sqrt{\lambda \pm \frac{i}{2}k(\lambda)} = \mp\frac{i}{k}\sqrt{\lambda \pm \frac{i}{2}k(\lambda)}$$

we compute the derivatives

$$\begin{aligned} p'_\pm(\lambda) &= k(\lambda)^{-1}\left[i + 2i\lambda x + 2(1 - 2\lambda^2)t\right] p_\mp(\lambda), \\ q'_\pm(\lambda) &= k(\lambda)^{-1}\left[-i + 2i\lambda x + 2(1 - 2\lambda^2)t\right] q_\mp(\lambda). \end{aligned} \tag{A.8}$$

which together with the four equalities of (A.5), lead to

$$[-q_0 \;\; p_0]\, \Phi'(\lambda_0) = 4ik_0^{-1}\left[\cosh(\sigma_0 t - ik_0 x) - \lambda_0, k_0(i\lambda_0 x + (1 - 2\lambda_0^2)t) + \sinh(\sigma_0 t - ik_0 x)\right].$$

Then we compute

$$\hat{\Phi}_0 = [2ik_0\varphi_1, \phi_0], \tag{A.9}$$

where

$$\varphi_1 = \frac{1}{2ik_0} \begin{bmatrix} p_+(\lambda_0) \\ q_+(\lambda_0) \end{bmatrix} + \frac{2}{k_0^2} \left[\cosh(\sigma_0 t - ik_0 x) - \lambda_0 \right] \begin{bmatrix} \hat{p}_0 \\ \hat{q}_0 \end{bmatrix}. \tag{A.10}$$

is the L-antiperiodic generalized eigenfunction satisfying $(\mathcal{L} - \lambda_0 I)\varphi_1 = \varphi_0$, and

$$\phi_0 = \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} + 4 \left[-\lambda_0 x + i(1 - 2\lambda_0^2)t + ik_0^{-1} \sinh(\sigma_0 t - ik_0 x) \right] \begin{bmatrix} \hat{p}_0 \\ \hat{q}_0 \end{bmatrix}, \tag{A.11}$$

is a second linearly independent solution to the Lax system (2.2)–(2.3) for $u = \hat{u}_0$ and $\lambda = \lambda_0$, which is not L-antiperiodic in x.

Finally, by differentiating (A.8) in λ and using (A.5) we obtain

$$\begin{aligned} -q_0 p_+''(\lambda_0) + p_0 q_+''(\lambda_0) &= 2ik_0^{-1} \left[4(i\lambda_0 x + (1 - 2\lambda_0^2)t)^2 - 1 \right] \\ &\quad + 16ik_0^{-2} \left[i\lambda_0 x + (1 - 2\lambda_0^2)t \right] \sinh(\sigma_0 t - ik_0 x) \\ &\quad + 16i\lambda_0 k_0^{-3} \left[\cosh(\sigma_0 t - ik_0 x) - \lambda_0 \right]. \end{aligned}$$

and

$$\begin{aligned} -q_0 p_-''(\lambda_0) + p_0 q_-''(\lambda_0) &= 16ik_0^{-2} \left[i\lambda_0 x + (1 - 2\lambda_0^2)t \right] \cosh(\sigma_0 t - ik_0 x) \\ &\quad + 16i\lambda_0 k_0^{-3} \sinh(\sigma_0 t - ik_0 x) + 4i(ix - 4\lambda_0 t). \end{aligned}$$

Then we compute

$$\hat{\Phi}_1 = [2ik_0 \varphi_2, \phi_1], \tag{A.12}$$

which consists of non-L-antiperiodic functions,

$$\begin{aligned} \varphi_2 &= \frac{1}{k_0^2}(\lambda_0 x - i(1 - 2\lambda_0^2)t) \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} + \frac{1}{2k_0^2} \begin{bmatrix} p_0 \\ -q_0 \end{bmatrix} \\ &\quad + \frac{1}{2k_0^2} \left[4(i\lambda_0 x + (1 - 2\lambda_0^2)t)^2 - 1 \right] \begin{bmatrix} \hat{p}_0 \\ \hat{q}_0 \end{bmatrix} \\ &\quad + \frac{4}{k_0^3}(i\lambda_0 x + (1 - 2\lambda_0^2)t) \sinh(\sigma_0 t - ik_0 x) \begin{bmatrix} \hat{p}_0 \\ \hat{q}_0 \end{bmatrix} \\ &\quad + \frac{4\lambda_0}{k_0^4} \left[\cosh(\sigma_0 t - ik_0 x) - \lambda_0 \right] \begin{bmatrix} \hat{p}_0 \\ \hat{q}_0 \end{bmatrix}. \end{aligned} \tag{A.13}$$

satisfying $(\mathcal{L} - \lambda_0 I)\varphi_2 = \varphi_1$, and

$$\begin{aligned} \phi_1 &= \frac{2}{k_0}(i\lambda_0 x + (1 - 2\lambda_0^2)t) \begin{bmatrix} p_+(\lambda_0) \\ q_+(\lambda_0) \end{bmatrix} + \frac{i}{k_0} \begin{bmatrix} p_+(\lambda_0) \\ -q_+(\lambda_0) \end{bmatrix} \\ &\quad + \frac{8i}{k_0^2}(i\lambda_0 x + (1 - 2\lambda_0^2)t) \cosh(\sigma_0 t - ik_0 x) \begin{bmatrix} \hat{p}_0 \\ \hat{q}_0 \end{bmatrix} + \frac{8i\lambda_0}{k_0^3} \sinh(\sigma_0 t - ik_0 x) \begin{bmatrix} \hat{p}_0 \\ \hat{q}_0 \end{bmatrix} \\ &\quad + 2i(ix - 4\lambda_0 t) \begin{bmatrix} \hat{p}_0 \\ \hat{q}_0 \end{bmatrix}. \end{aligned} \tag{A.14}$$

satisfying $(\mathcal{L} - \lambda_0 I)\phi_1 = \phi_0$.

Remark 17. *These computations can be transferred to KMB given by (5.3) by taking $k_0 = i\beta_0$ and $\sigma_0 = i\alpha_0$. In particular, we obtain two linearly independent solutions of the Lax system*

(2.2)–(2.3) with $u = \hat{u}_0$ and $\lambda = \lambda_0$ for KMB from (A.7) and (A.11). The first one is the solution $\varphi_0 = (\hat{p}_0, \hat{q}_0)^T$ in Lemma 4 and the second one is the solution ϕ_0 given by (5.6).

Remark 18. *Derivatives of the matrix solution $\hat{\Phi}(\lambda)$ in λ at $\lambda = \pm\lambda_0$ were computed in the recent work* [19] *without discussing geometric and algebraic multiplicities of the eigenvalue λ_0.*

References

[1] M.J. Ablowitz, D.J. Kaup, A.C. Newell, and H. Segur, "The inverse scattering transform–Fourier analysis for nonlinear problems", Stud. Appl. Math. **53** (1974), 249–315.

[2] N.N. Akhmediev, V.M. Eleonskii, and N.E. Kulagin, "Exact first-order solutions of the nonlinear Schrödinger equation", Theor. Math. Phys. **72** (1987), 809–818.

[3] M.A. Alejo, L. Fanelli, and C. Munoz, "The Akhmediev breather is unstable", Sao Paulo J. Math. Sci. **13** (2019), 391–401.

[4] M.A. Alejo, L. Fanelli, and C. Munoz, "Stability and instability of breathers in the $U(1)$ Sasa–Satsuma and nonlinear Schrödinger models", Nonlinearity **34** (2021), 3429–3484.

[5] M.A. Alejo, L. Fanelli, and C. Munoz, "Review on the stability of the Peregrine and related breathers", Front. Phys. **8** (2020), 591995 (8 pages).

[6] D. Bilman and P.D. Miller, "A robust inverse scattering transform for the focusing nonlinear Schrödinger equation", Comm. Pure Appl. Math. **72** (2019), 1722–1805.

[7] G. Biondini and G. Kovacic, "Inverse scattering transform for the focusing nonlinear Schröodinger equation with nonzero boundary conditions", J. Math. Phys. **55** (2014), 031506 (22 pages).

[8] A. Calini and C.M. Schober, "Dynamical criteria for rogue waves in nonlinear Schrödinger models", Nonlinearity **25** (2012) R99–R116.

[9] A. Calini, C.M. Schober, and M. Strawn, "Linear instability of the Peregrine breather: Numerical and analytical investigations", Appl. Numer. Math. **141** (2019), 36–43.

[10] J. Chen, D.E. Pelinovsky, and R.E. White, "Rogue waves on the double-periodic background in the focusing nonlinear Schrödinger equation", Physical Review E **100** (2019), 052219 (18 pages).

[11] J. Chen, D.E. Pelinovsky, and R.E. White, "Periodic standing waves in the focusing nonlinear Schrodinger equation: Rogue waves and modulation instability", Physica D **405** (2020), 132378 (13 pages).

[12] A. Contreras and D.E. Pelinovsky, "Stability of multi-solitons in the cubic NLS equation", J. Hyper. PDEs **11** (2014), 329–353.

[13] F. Copie, S. Randoux, and P. Suret, "The physics of the one-dimensional nonlinear Schrödinger equation in fiber optics: Rogue waves, modulation instability and self-focusing phenomena", Reviews in Physics **5** (2020), 100037 (17 pages)

[14] J. Cuevas–Maraver, P.G. Kevrekidis, D.J. Frantzeskakis, N.I. Karachalios, M. Haragus, and G. James, "Floquet analysis of Kuznetsov–Ma breathers: A path towards spectral stability of rogue waves", Phys. Rev. E **96** (2017), 012202 (8 pages).

[15] B. Deconinck and B. L. Segal, "The stability spectrum for elliptic solutions to the focusing NLS equation", Physica D **346** (2017) 1–19

[16] J.M. Dudley, G. Genty, A. Mussot, A. Chabchoub, and F. Dias, "Rogue waves and analogies in optics and oceanography", Nature Reviews **1** (2019), 675.

[17] G. Fibich, *The Nonlinear Schrödinger Equation: Singular Solutions and Optical Collapse*, Springer-Verlag (New York, 2015).

[18] J. Garnier and K. Kalimeris, "Inverse scattering perturbation theory for the nonlinear Schrödinger equation with non-vanishing background", J. Phys. A: Math. Theor. **45** (2012), 035202 (13pp).

[19] P.G. Grinevich and P.M. Santini, "The linear and nonlinear instability of the Akhmediev breather", Nonlinearity **34** (2021), 8331–8358.

[20] D.J. Kaup, "A perturbation expansion for the Zakharov-Shabat inverse scattering transform", SIAM J. Appl. Math. **31** (1976), 121–133.

[21] D.J. Kaup, "Closure of the squared Zakharov-Shabat eigenstates", J. Math. Anal. Appl. **54** (1976), 849–864.

[22] C. Kharif, E. Pelinovsky, and A. Slunyaev, *Rogue Waves in the Ocean* (Springer, Heidelberg, 2009).

[23] M. Klaus, D.E. Pelinovsky, and V.M. Rothos, "Evans function for Lax operators with algebraically decaying potentials", J. Nonlin. Sci. **16** (2006), 1–44.
[24] C. Klein and M. Haragus, "Numerical study of the stability of the Peregrine breather", Ann. Math. Sci. Appl. **2** (2017), 217–239.
[25] E.A. Kuznetsov, "Solitons in a parametrically unstable plasma," Sov. Phys. Dokl. **22** (1977), 507–508.
[26] Y.-C. Ma, "The perturbed plane-wave solutions of the cubic Schrödinger equation," Stud. Appl. Math. **60** (1979), 43–58.
[27] D.E. Pelinovsky, "Instability of double-periodic solutions in the nonlinear Schrodinger equation", Frontiers in Physics **9** (2021) 599146 (10 pages)
[28] D.H. Peregrine, "Water waves, nonlinear Schrödinger equations and their solutions", J. Austral. Math. Soc. B. **25** (1983), 16–43.
[29] C. Sulem and P.L. Sulem, *The Nonlinear Schrödinger Equation*, Springer-Verlag (New York, 1999).
[30] J. Sullivan, E.G. Charalampidis, J. Cuevas-Maraver, P.G. Kevrekidis, and N.I. Karachalios, "Kuznetsov–Ma breather-like solutions in the Salerno model", Eur. Phys. J. Plus **135** (2020) 607 (12 pages).
[31] M. Tajiri and Y. Watanabe, "Breather solutions to the focusing nonlinear Schrödinger equation", Phys. Rev. E **57** (1998), 3510–3519.
[32] S. Wabnitz (editor), *Nonlinear Guided Wave Optics: A Testbed for Extreme Waves* (Iop Publishing Ltd, Bristol, 2018)
[33] V.E. Zakharov and A.A. Gelash, "Nonlinear stage of modulation instability", Phys. Rev. Lett. **111** (2013), 054101 (5 pages).
[34] V.E. Zakharov and A.B. Shabat, "Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media", Soviet Physics JETP **34** (1972), 62–69.
[35] J. Zweck, Y. Latushkin, J.L. Marzuola, and C.K.R.T. Jones, "The essential spectrum of periodically stationary solutions of the complex Ginzburg-Landau equation", J. Evol. Equ. **21** (2021), 3313–3329.

(M. Haragus) FEMTO-ST institute, Univ. Bourgogne Franche-Comté, CNRS, 15b avenue des Montboucons, 25030 Besançon cedex, France
Email address: mharagus@univ-fcomte.fr

(D.E. Pelinovsky) Department of Mathematics, McMaster University, Hamilton, Ontario, Canada, L8S 4K1
Email address: dmpeli@math.mcmaster.ca