# Rotational symmetry and rotating waves in planar integro-difference equations

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# Rotational symmetry and rotating waves in planar integro-difference equations


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**Abstract**

Mathematical models often possess symmetries, either because of actual symmetries of the situation being modelled, or as approximations. It is well-known that these symmetries often impose restrictions on the solutions to these models. In this paper, we investigate the role of rotational symmetry in certain integro-difference equations, and study the existence of rotating wave solutions to these equations. We perform explicit computations in the case where the integration kernel is a Gaussian distribution, which often occurs in applications.

# 1 Introduction

In the theory of dynamical systems, it is by now well-known that symmetries play an important role in shaping the dynamics and bifurcations of the system [15, 16]. In this paper, we will be interested in investigating the consequences of rotational symmetries in discrete-time dynamical systems given by iteration of an integro-difference equation of the form

$$v(x) = \mathcal{N}[u](x) = \int_{\Omega} \kappa(x,y) F(x, u(y))\, dy \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^m$, $u$ and $v$ are functions which go from $\Omega$ into $\mathbb{R}^N$, $F : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is some (generally nonlinear) function and $\kappa : \Omega \times \Omega \longrightarrow \mathbb{R}$ is such that

$$\int_{\Omega} \kappa(x,y)\, dy = 1, \ \ \forall x \in \Omega. \tag{1.2}$$

Systems such as (1.2) have been of interest to mathematicians and scientists for many years, see for example [6, 7, 24, 25, 40, 41]. For example, in many applications, $N = 1$ and $u$ represents the distribution in space $\Omega$ of a generation of a certain population, and $v$ represents the subsequent distribution in space of the next generation after the population has reproduced locally (under a reproduction law modelled through the function $F(x,\cdot)$), and then spread out in space according to the distribution kernel $\kappa$ [24, 25, 40]. The function $\kappa(x,y) \geq 0$ might represent the density of the probability that an individual initially at point $y$ moves to the point $x$ after one generation. Subsequently one is interested in the discrete-time dynamical system generated through iteration of the map $\mathcal{N}$ in (1.1), i.e. given an initial distribution $u_0(x)$, compute $u_n(x)$ (for integer $n \geq 1$) through the recursion $u_n = \mathcal{N}[u_{n-1}]$. It is then quite natural to want to characterize the limiting behavior of orbits of this integro-difference system as $n \to \infty$. For further information about integro-difference equations and their role as mathematical models for a variety of situations in ecology, as well as an extensive literature review of the field, the reader is invited to consult [26].

The scope of this paper is not to focus on any one particular application, but rather to perform a mathematical analysis of the role played by any rotational symmetries which may be present in (1.1). To this end, we will assume that, $\Omega = \mathbb{R}^2$, that the image of $F$ is in $\mathbb{R}^2$, and the kernel $\kappa$ and the function $F$ are homogeneous in space such that we can write $F(x,u) = f(u)$ and $\kappa(x,y) = k(|x-y|)$, where $|z|$ is the Euclidean norm of $z \in \mathbb{R}^2$. It is well-known that the fact that the Euclidean norm is invariant under translations implies that (1.1) has an important symmetry:

**Proposition 1.1** *Let $u_0$ be a given function from $\mathbb{R}^2$ into $\mathbb{R}^2$ such that*

$$u_1(x) = \int_{\mathbb{R}^2} k(|x-y|) f(u_0(y))\, dy$$

*exists. Let $p \in \mathbb{R}^2$ be given, and define $v_0(x) = u_0(x-p)$. Then*

$$v_1(x) = \int_{\mathbb{R}^2} k(|x-y|) f(v_0(y))\, dy$$

*exists and is such that $v_1(x) = u_1(x-p)$.*

This translation symmetry is fundamental to proving the existence of *travelling waves*, i.e. orbits $\{u_{n+1}\} = \{\mathcal{N}[u_n]\}$ of the discrete-time dynamical system (1.1), such that $u_{n+1}(x) = u_n(x-p)$ for a given fixed $p \in \mathbb{R}^2$, as was outlined in the pioneering papers of Weinberger [39, 40].

Surprisingly, very little attention has been paid to the fact that the Euclidean norm is also invariant under rigid rotations and that this fact could lead to solutions compatible with this symmetry. It is this aspect that we wish to explore in this paper.

We define $\mathbf{SO}(2)$ to be the group of all rigid rotations on the plane, then $\mathbf{SO}(2)$ acts on $\mathbb{R}^2$ according to the standard action

$$\theta \in \mathbf{SO}(2) \Longrightarrow \theta \cdot z = R_\theta z,$$

where $R_\theta$ is the rotation matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The natural phase-space for the discrete-time dynamical system (1.1) is a space of functions from $\mathbb{R}^2$ into $\mathbb{R}^2$, for example $C_b(\mathbb{R}^2, \mathbb{R}^2)$ the space of bounded and continuous functions with supremum norm. We distinguish two natural actions of $\mathbf{SO}(2)$ on such a space [16]:

$$(\theta \cdot u)(z) = G(\theta) \cdot u(R_{-\theta} \cdot z), \quad \theta \in \mathbf{SO}(2)\,, \quad u \in C_b(\mathbb{R}^2, \mathbb{R}^2), \tag{1.3}$$

where $G(\theta)$ can be either the identity matrix, or the rotation matrix $R_\theta$.

The kernel $k(|x-y|)$ is compatible with either representations of this symmetry (since both representations rotate the space variable). We will assume that the local reaction dynamics given by the function $F(x, u) = f(u)$ are compatible with the form $G(\theta) = R_\theta$ of this symmetry in the sense that

$$\forall \theta \in \mathbf{SO}(2), \;\; f(R_\theta \cdot u) = R_\theta \cdot f(u), \;\; \forall u \in \mathbb{R}^2, \tag{1.4}$$

i.e. $f$ is $\mathbf{SO}(2)$-equivariant in the sense of [16]. In this case, it also (trivially) respects the equivariance symmetry in the case where $G(\theta)$ is the identity matrix. This leads to:

**Proposition 1.2** *Suppose $f$ satisfies the condition (1.4) and let $u_0$ be a given function from $\mathbb{R}^2$ into $\mathbb{R}^2$ such that*

$$u_1(x) = \int_{\mathbb{R}^2} k(|x-y|) f(u_0(y))\, dy$$

*exists. Let $\theta \in \mathbf{SO}(2)$ be given, and define $v_0(x) = G(\theta) \cdot u_0(R_{-\theta} \cdot x)$. Then*

$$v_1(x) = \int_{\mathbb{R}^2} k(|x-y|) f(v_0(y))\, dy$$

*exists and is such that $v_1(x) = G(\theta) \cdot u_1(R_{-\theta} \cdot x)$.*

**Proof** This is a simple computation:

$$\begin{aligned} v_1(x) = G(\theta) \cdot u_1(R_{-\theta} \cdot x) &= \int_{\mathbb{R}^2} k(|R_{-\theta} \cdot x - y|) G(\theta) \cdot f(u_0(y))\, dy \\ &= \int_{\mathbb{R}^2} k(|R_{-\theta} \cdot (x - \tilde{y})|) f(G(\theta) \cdot u_0(R_{-\theta} \cdot \tilde{y}))\, d\tilde{y} \\ &= \int_{\mathbb{R}^2} k(|x-y|) f(v_0(y))\, dy, \end{aligned}$$

where we have used the rotational invariance of the norm and the fact that the determinant of a rotation matrix is equal to one. ∎

In the language of dynamical systems, we formulate this property as follows. Suppose $X$ is an appropriate space of functions from $\mathbb{R}^2$ into $\mathbb{R}^2$, for example $X = C_b(\mathbb{R}^2, \mathbb{R}^2)$, so that for all $u \in X$, the function

$$v(x) = \mathcal{N}[u](x) = \int_{\mathbb{R}^2} k(|x-y|) f(u(y))\, dy \tag{1.5}$$

exists and is an element of $X$. We may view $\mathcal{N}$ in (1.5) as a map $\mathcal{N} : X \longrightarrow X$. Considering the action (1.3) of $\mathbf{SO}(2)$ on $X$, Proposition 1.2 is equivalent to the equivariance equality

$$\theta \circ \mathcal{N} = \mathcal{N} \circ \theta, \quad \forall \theta \in \mathbf{SO}(2). \tag{1.6}$$

If we then consider the dynamical system on $X$ defined by iterations of the map $\mathcal{N}$ in (1.5), equation (1.6) implies that the group $\mathbf{SO}(2)$ maps orbits of this dynamical system into other orbits of the dynamical system.

The study of group equivariant dynamical systems has enjoyed a rich development over the past decades, see for example [1, 8, 9, 10, 13, 14, 16, 23]. An interesting class of orbits which may exist for $\mathbf{SO}(2)$-equivariant dynamical systems are so-called *rotating waves*. In the context of the discrete dynamical system on $X$ defined above, a rotating wave corresponds to $u^* \in X$ such that there exists $\theta_* \in \mathbf{SO}(2)$ with

$$\mathcal{N}[u^*] = \theta_* \cdot u^*,$$

which using (1.6) leads to

$$\mathcal{N}^j[u^*] = (\mathcal{N}^{j-1} \circ \mathcal{N})[u^*] = (\mathcal{N}^{j-1} \circ \theta_*)[u^*] = (\theta_* \circ \mathcal{N}^{j-1})[u^*] \cdots = (\theta_*)^j \cdot u^*, \quad \forall j = 1, 2, 3, \ldots,$$

i.e. the time-orbit of $u^*$ is contained in its $\mathbf{SO}(2)$ orbit. Note that because there are two possible representations in (1.3), there are two distinct types of rotating wave solutions depending on the form of the matrix $G(\theta)$. In the case where $G(\theta)$ is the identity matrix, we will use the terminology *untwisted* rotating wave, whereas in the case where $G(\theta) = R_\theta$ we will use the terminology *twisted* rotating wave.

The existence of rotating waves for (1.5) (untwisted or twisted) is not necessarily guaranteed by the $\mathbf{SO}(2)$ symmetry, however we may establish a sufficient condition in the form of a fixed-point equation:

$$G(\alpha) \cdot u^*(R_{-\alpha} \cdot x) = \int_{\mathbb{R}^2} k(|x - y|) f(u^*(y))\, dy, \quad \text{for some } \alpha \in \mathbf{SO}(2),$$

or equivalently

$$u^*(x) = \int_{\mathbb{R}^2} k(|R_\alpha \cdot x - y|) f(G(-\alpha) \cdot u^*(y))\, dy. \tag{1.7}$$

The function $u^*$ is called the *wave profile* of the rotating wave. It should be noted that because of the translation symmetry of our integral operator, we are free to set the center of rotation of the rotating wave at the origin of physical space, as we have done in (1.7).

As mentioned earlier, although there has been considerable attention paid to the existence of travelling waves for (1.1), there are relatively few studies of its rotating wave solutions. This is somewhat surprising since the existence and dynamics of rotating waves in partial differential reaction-diffusion systems (mostly spiral waves) has been studied at great length over the past decades [2, 3, 4, 5, 14, 32, 33, 34, 42]. Many models of cardiac electrophysiology are in the form of reaction diffusion PDEs, and rotating spiral solutions represent pathologies such as arrhythmias [11, 20, 27, 29, 30, 31]. We do however note that for the related problem of integro-differential equations, there have been some numerical and analytical studies of spiral waves [19, 37].

The study of the existence of travelling wave solutions of (1.5) typically relies heavily on properties of the kernel $k$, but also on the internal dynamics of the discrete-time dynamical system

$$u \longmapsto f(u). \tag{1.8}$$

In particular equilibrium points (and their local asymptotic stability) of this system often correspond to limiting properties (as $x \to \pm\infty$) of the wave profile $u^*$. Of course it is possible for systems such as (1.8) to have limiting states more complicated than equilibrium points. For example there may exist a period-2 cycle (or more generally a period-$p$ cycle) in the dynamics of (1.8). This question was analyzed in some detail [6, 7] by studying the second-iterate map $\mathcal{N}^2$ in (1.5) and looking for travelling waves of the resulting operator connecting two fixed points of the second iterate $f^2$ in (1.8). See also [12, 17]. Therefore, it is expected that limiting states of (1.8) will also play a role in describing rotating waves, i.e. solutions of (1.7).

The equivariance condition (1.4) imposes strong restrictions on the functional form of the mapping $f$ in (1.8). As is shown in [16], $f$ must have the form

$$f(u) = f(u_1, u_2) = \mathscr{A}_1(u_1^2 + u_2^2) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \mathscr{A}_2(u_1^2 + u_2^2) \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix} \tag{1.9}$$

where $\mathscr{A}_1$ and $\mathscr{A}_2$ are real-valued functions. Rather than studying the problem (1.5) in the full generality (1.9), we will limit ourselves to the following representative normal form of (1.9) (which, as we will see, already leads to an analysis and computations that are quite involved but manageable):

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \longmapsto ((1+\beta) - u_1^2 - u_2^2) \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{1.10}$$

where $\beta$ is a real parameter and $\omega \in (0, \pi)$ for which it is possible to completely describe the dynamics analytically. We note that despite its apparent simplicity, this mapping is in fact a special case of the generic cubic truncated normal form for the Naimark-Sacker bifurcation from an equilibrium point with eigenvalues of its linearization crossing the unit circle (when $\beta = 0$) at $e^{\pm i\omega}$ (assuming the non-resonance conditions $e^{i\ell\omega} \neq 1$ for $\ell = 1, 2, 3, 4$) [21]. Adopting polar coordinates $(u_1, u_2) = (\eta\cos\psi\,,\ \eta\sin\psi)$, (1.10) reduces to

$$\begin{pmatrix} \eta \\ \psi \end{pmatrix} \longmapsto \begin{pmatrix} \eta(1 + \beta - \eta^2) \\ \psi + \omega \end{pmatrix} \tag{1.11}$$

for which the dynamics of (1.10) become clearer. The origin $(u_1, u_2) = 0$ is a fixed point for all $\beta$ real, locally stable when $\beta < 0$ and locally unstable when $\beta > 0$. For $\beta > 0$, there is a locally asymptotically stable closed invariant circle of radius $\eta = \sqrt{\beta}$ on which the dynamics of (1.10) reduces to a rigid rotation around the origin through angle $\omega$. See figure 1 for a summary of this discussion.

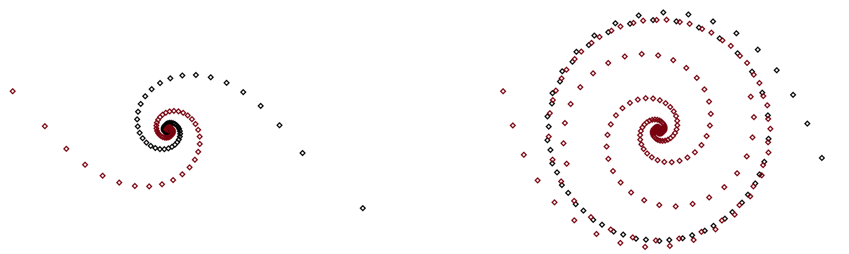

Figure 1: Phase diagrams for the dynamical system (1.10)-(1.11) pre-(left panel) and post-(right panel) a Neimark-Sacker bifurcation of the equilibrium point at the origin. In the left panel, the equilibrium is stable, whereas in the right panel, the equilibrium is unstable and there is a stable closed invariant curve.

Taking into account all these considerations, we will therefore focus our attention in this paper to the fixed-point problem (1.7) which reduces to

$$u^*(x) = \iint_{\mathbb{R}^2} k\left(|R_\alpha x - y|\right) G(-\alpha) R_\omega (1 + \beta - |u^*(y)|^2) u^*(y)\, dy_1 dy_2, \tag{1.12}$$

where $x = (x_1, x_2)^T$, $y = (y_1, y_2)^T$, $u^* = (u_1, u_2)^T$.

# 2 Polar coordinates representation and pinwheel solutions

It will be useful to adopt polar coordinates for both variables $x = x_1 + ix_2 = re^{i\varphi}$ and $y = y_1 + iy_2 = \rho e^{is}$ so that if we write $u = u_1 + iu_2$, the integral operator $\mathcal{N}$ in (1.5) is

$$v(re^{i\varphi}) = \mathcal{N}[u](re^{i\varphi}) = \int_0^\infty \int_0^{2\pi} k(|re^{i\varphi} - \rho e^{is}|) f(u(\rho e^{is}))\, \rho\, ds d\rho\,. \tag{2.1}$$

If we write $u(re^{i\varphi})$ as a Fourier series

$$u(re^{i\varphi}) = \sum_{\ell \in \mathbb{Z}} \mathscr{C}_\ell(r) e^{i\ell\varphi},$$

then the operator (2.1) admits as invariant subspaces each of the Fourier modes, as follows:

**Proposition 2.1** *If $u(re^{i\varphi}) = \mathscr{C}_\ell(r)e^{i\ell\varphi}$ for some $\ell \in \mathbb{Z}$, then*

$$\mathcal{N}[u](re^{i\varphi}) = \mathscr{B}_\ell(r)e^{i\ell\varphi},$$

*where*

$$\mathscr{B}_\ell(r) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} k(|r - \rho e^{is}|)\, e^{i\ell s} f(\mathscr{C}_\ell(\rho))\, \rho\, ds d\rho\,. \tag{2.2}$$

**Proof** Let $\ell' \in \mathbb{Z}$ be given. Then

$$\begin{aligned}
\int_0^{2\pi} e^{-i\ell'\varphi}\mathcal{N}[u](re^{i\varphi})\, d\varphi &= \int_0^{2\pi}\int_0^{\infty}\int_0^{2\pi} k(|re^{i\varphi} - \rho e^{is}|)e^{-i\ell'\varphi} f(\mathscr{C}_\ell(\rho)e^{i\ell s})\, \rho\, ds d\rho d\varphi \\
&= \int_0^{2\pi}\int_0^{\infty}\int_0^{2\pi} k(|r - \rho e^{i(s-\varphi)}|)e^{i\ell(s-\varphi)}e^{i(\ell-\ell')\varphi} f(\mathscr{C}_\ell(\rho))\, \rho\, ds d\rho d\varphi \\
&= \left(\int_0^{2\pi} e^{i(\ell-\ell')\varphi} d\varphi\right)\left(\int_0^{2\pi}\int_0^{\infty} k(|r - \rho e^{i\tilde{s}}|)\, e^{i\ell\tilde{s}} f(\mathscr{C}_\ell(\rho))\, \rho\, d\tilde{s} d\rho\right) \\
&= \begin{cases} 0 & \text{if } \ell' \neq \ell \\ 2\pi\mathscr{B}_\ell(r) & \text{if } \ell' = \ell \end{cases}
\end{aligned}$$

where $\mathscr{B}_\ell(r)$ is as in (2.2). ∎

We will exploit Proposition 2.1 and search for solutions $u^*$ of the fixed-point problem (1.12) of the form

$$u^*(re^{i\varphi}) = P(r)e^{im\varphi}\,, \tag{2.3}$$

where $m \in \{1, 2, 3, \ldots\}$ is the degree of rotational symmetry. If $P(r) \geq 0$ is real-valued, we call the solution a *pinwheel* solution (see figure 2). The perhaps better-known class of rotating wave solution, the *spiral wave* (see figure 2), which requires a complex-valued $P(r)$, will not be addressed in this paper.

**Remark 2.2** *For rotating wave profiles of the form (2.3), the difference between twisted and untwisted rotating waves is easy to describe. We have*

$$G(\alpha) \cdot u^*(R_{-\alpha} \cdot x) = e^{i\delta\alpha}P(r)e^{im(\varphi-\alpha)} = P(r)e^{im\varphi}e^{-i(m-\delta)\alpha}, \;\; \delta \in \{0, 1\},$$

*which implies that for the untwisted action ($\delta = 0$), the wave profile $u^*$ is rotated through the origin by an angle $m\alpha$, whereas for the twisted action the angle is $(m-1)\alpha$.*

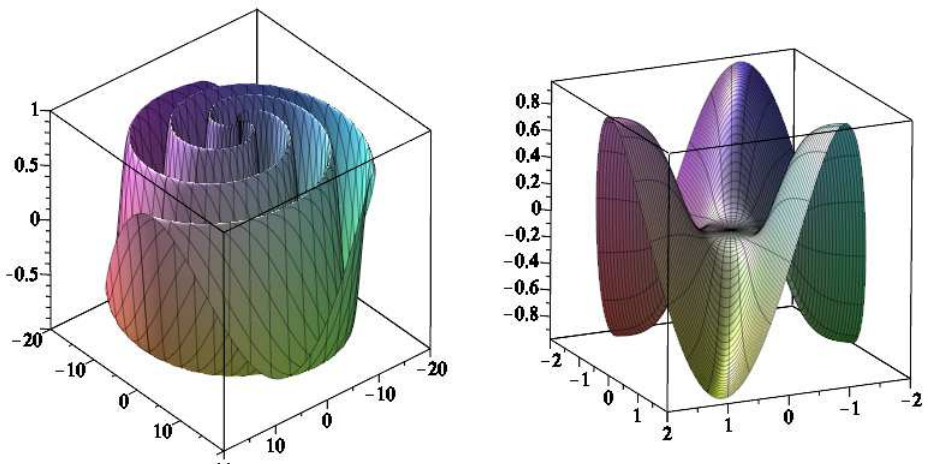


Figure 2: Examples of spatial profiles of rotating waves. 3-armed spiral wave (left panel) and 3-armed pinwheel wave (right panel)

Setting

$$g(z,\beta) = z(1+\beta-z^2) \tag{2.4}$$

and substituting (2.3) in (1.12) using the polar representation (2.1) we get after some simplification

$$P(r) = \int_0^\infty \int_0^{2\pi} k(|re^{i(\varphi+\alpha)} - \rho e^{is}|)e^{i(\omega-\delta\alpha)}e^{-im\varphi}e^{ims}g(P(\rho),\beta)\rho\, ds d\rho\,. \tag{2.5}$$

where $\delta = 0$ leads to untwisted rotating waves, and $\delta = 1$ leads to twisted rotating waves. Using the $2\pi$-periodicity of the integrand and setting $\tilde{s} = s - \varphi - \alpha$, (2.1) becomes (upon dropping the tildes)

$$P(r) = \int_0^\infty \int_0^{2\pi} k(|r - \rho e^{is}|)\, e^{ims}e^{i(\omega+(m-\delta)\alpha)}g(P(\rho),\beta)\rho\, ds d\rho\,, \tag{2.6}$$

The term $k(|r-\rho e^{is}|)$ which appears in (2.6) is equal to

$$k(|r-\rho e^{is}|) = k(\sqrt{r^2+\rho^2-2r\rho\cos s}).$$

We note that for any integrable function $h$, we have

$$\int_0^{2\pi} h(\cos(s))\, \sin ms\, ds = 0$$

Therefore

$$A_m(r,\rho) \equiv \int_0^{2\pi} k(|r-\rho e^{is}|)\, e^{ims}\, ds \;=\; \int_0^{2\pi} k(\sqrt{r^2+\rho^2-2r\rho\cos s})\, \cos ms\, ds \\ = 2\int_0^{\pi} k(\sqrt{r^2+\rho^2-2r\rho\cos s})\, \cos ms\, ds \tag{2.7}$$

is a real-valued function, and we note that $A_m(r,\rho) = A_m(\rho, r)$. Since the left-hand side of (2.6) is real, we are immediately led to the compatibility condition for the rotational frequency $\alpha$ of the rotating wave

$$(m-\delta)\alpha + \omega = 2\ell\pi \quad \ell \in \mathbb{Z}, \tag{2.8}$$

and have reduced (2.1) to the following fixed-point problem for the function $P$:

$$P(r) = \int_0^{\infty} \rho A_m(r,\rho)\, g(P(\rho),\beta)\, d\rho\,, \tag{2.9}$$

where $A_m(r,\rho) = A_m(\rho, r)$ is as in (2.7).

Since we have assumed that $\omega \in (0,\pi)$, condition (2.8) implies that there are no twisted solutions to (2.1) of the form (2.3) with $m = 1$. Untwisted solutions with $m = 1$ are possible provided $\alpha = 2\ell\pi - \omega$.

To the extent possible we will keep the analysis general, but in one important case of integration kernel $k$ in (2.9), i.e. the Gaussian kernel

$$k(t) = \frac{1}{2\pi\sigma^2} e^{-\frac{t^2}{2\sigma^2}}, \quad \sigma > 0\,, \tag{2.10}$$

explicit computations are possible and we will elaborate on these computations throughout the sequel.

The analysis required to prove the existence of solutions to (2.9) will depend on an appropriate choice of function space, and on properties of the function $A_m$. We will further explore these issues in the next sections.

# 3 Hypotheses on the integration kernel

In this section we will discuss some general assumptions on the integration kernel $k$ for the fixed point problems (2.1) and (2.9), and then in the next section we will analyse the specific case of the Gaussian kernel (2.10).

We first note that the function $A_m(r,\rho)$ defined in (2.7) is such that when $r=0$ or $\rho=0$, we get

$$A_m(0,t) = A_m(t,0) = \int_0^{2\pi} k(|t|)\, \cos ms\, ds = 0, \ \ \forall t \geq 0. \tag{3.1}$$

We will suppose

**Hypothesis 3.1** *The function $k:[0,\infty)\to\mathbb{R}$ and the function $A_m$ in (2.7) are such that $k$ is positive and $C^2$ on $(0,\infty)$, and*

*(i)* $\displaystyle\iint_{\mathbb{R}^2} k(\sqrt{x_1^2+x_2^2})\, dx_1\, dx_2 = 1.$

*(ii) The expected value of $\rho$ for the kernel $\rho A_m(r,\rho)$ satisfies*

$$\int_0^\infty \rho A_m(r,\rho)\rho\, d\rho = \int_0^\infty \rho^2 A_m(r,\rho)\, d\rho = rK(r), \tag{3.2}$$

*where $K$ is bounded on $[0,\infty)$.*

*(iii) $A_m$ satisfies the inequalities $A_m(r,\rho)>0$ for all $r>0$, $\rho>0$ and*

$$\det\begin{pmatrix} \rho\, A_m(r,\rho) & \dfrac{\partial(\rho\, A_m)}{\partial\rho}(r,\rho) \\ \dfrac{\partial(\rho\, A_m)}{\partial r}(r,\rho) & \dfrac{\partial^2(\rho\, A_m)}{\partial r\partial\rho}(r,\rho) \end{pmatrix} > 0, \quad \forall r>0, \rho>0\,, \tag{3.3}$$

*and as such $\rho\, A_m(r,\rho)$ is of total positivity class $TP_2$ (see [18]).*

*(iv) For any fixed $r>0$, the function $\rho\dfrac{\partial A_m}{\partial r}(r,\rho)$ is absolutely integrable on $[0,\infty)$. Moreover, $\displaystyle\lim_{r\to 0^+}\rho\frac{\partial A_m}{\partial r}\equiv \mathcal{G}(\rho)$ is such that $\mathcal{G}(\rho)\geq 0$ and is integrable. More strongly, we will suppose that*

$$\sup_{r\in[0,\infty)}\int_0^\infty \rho\left|\frac{\partial A_m}{\partial r}(r,\rho)\right| d\rho < \infty\,. \tag{3.4}$$

*Consequently, if we define $M(r)$ by*

$$M(r) = \int_0^\infty \rho A_m(r,\rho)\, d\rho\,, \tag{3.5}$$

*then*

$$\int_0^\infty \rho\frac{\partial A_m}{\partial r}(r,\rho)\, d\rho = \frac{d}{dr}\left(\int_0^\infty \rho A_m(r,\rho)\, d\rho\right) = M'(r) \tag{3.6}$$

*is bounded.*

*(v) If $M(r)$ is as in (3.5), then $M'(r) > 0$ for all $r > 0$, and $M'(0) \geq 0$.*

The function $M(r)$ defined in (3.5) will serve an important purpose in our analysis. We note that item (i) in the above hypothesis and the condition $A_m \geq 0$ (item (iii) above) guarantee that $M(0) = 0$, $M(r) \geq 0$ for $r > 0$, and

$$|M(r)| = \int_0^\infty \rho A_m(r,\rho)\, d\rho \leq \int_0^\infty \int_0^{2\pi} |k(|r - \rho e^{is}|)e^{ims}|\rho\, ds d\rho \leq 1 \tag{3.7}$$

so $M$ is positive, bounded and strictly increasing (item (v) in the above hypothesis) on $[0, \infty)$.

# 4 Gaussian kernel

In the case where the integration kernel $k$ is as in (2.10), we will show that all elements of Hypothesis 3.1 are satisfied (clearly item (i) is satisfied, so we will focus on the other items). We compute that

$$A_m(r,\rho) = \frac{2e^{-\frac{r^2+\rho^2}{2\sigma^2}}}{2\pi\sigma^2} \int_0^\pi e^{\frac{r\rho\cos s}{\sigma^2}} \cos ms\, ds = \frac{1}{\sigma^2}\, e^{-\frac{r^2+\rho^2}{2\sigma^2}} \mathcal{I}_m\left(\frac{r\rho}{\sigma^2}\right), \tag{4.8}$$

where for $\nu \geq 0$, $\mathcal{I}_\nu(t)$ is the modified Bessel function of order $\nu$ (see [38]), which in terms of the ordinary Bessel function of order $\nu$, $\mathcal{J}_\nu$, is given by

$$\mathcal{I}_\nu(t) = (i)^{-\nu} \mathcal{J}_\nu(it) \geq 0, \;\; \forall x \geq 0.$$

For any integer $\nu \geq 0$, the function $\mathcal{I}_\nu(t)$ is strictly increasing, $\mathcal{I}_0(0) = 1$, $\mathcal{I}_\nu(0) = 0$ for $\nu > 0$, and we have that for large positive real $t$ and for $\nu \geq 0$ [38]

$$\mathcal{I}_\nu(t) \sim \frac{e^t}{\sqrt{2\pi t}} \sum_{\ell=0}^\infty (-1)^\ell \frac{\Gamma(\nu + \ell + \frac{1}{2})}{\ell!\Gamma(\nu - \ell + \frac{1}{2})(2t)^\ell}\,, \tag{4.9}$$

so that asymptotically the function $A_m(r,\rho)$ behaves like

$$A_m(r,\rho) \sim \frac{1}{\sqrt{r\rho}} \frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{(r-\rho)^2}{2\sigma^2}},$$

which resembles a "travelling" Gaussian (in $(r,\rho)$-space) with decaying amplitude (see left panel of Figure 3). The function $M(r)$ in (3.5) can also be computed analytically, using

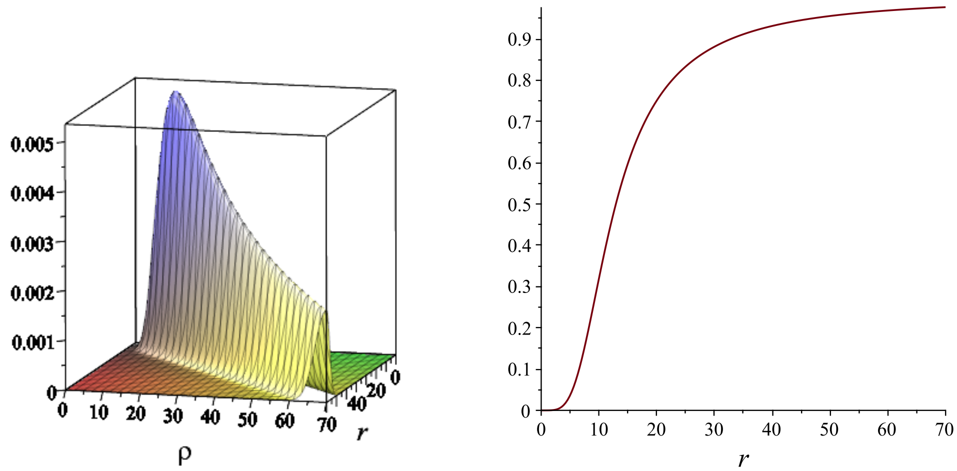


Figure 3: The function $A_5(r,\rho)$ (left panel) and the function $M(r)$ (right panel) with $k$ as in (2.10) and $\sigma = 3$.

formula (3) on page 394 of [38] and relationships between confluent hypergeometric functions and Bessel functions (e.g pages 100-105 of [38] and Chapter 13 of [28])

$$M(r) = \int_0^\infty \rho A_m(r,\rho)\,d\rho = \frac{\sqrt{2\pi}}{4\sigma}\, r e^{-\frac{r^2}{4\sigma^2}} \left( \mathcal{I}_{\frac{m-1}{2}}\left(\frac{r^2}{4\sigma^2}\right) + \mathcal{I}_{\frac{m+1}{2}}\left(\frac{r^2}{4\sigma^2}\right)\right). \tag{4.10}$$

See the right panel in Figure 3, and note in particular that inequality (3.7) is manifested in that figure. We note that a straightforward computation using recurrence relations involving modified Bessel functions and their derivatives (for example, see page 79 of [38]):

$$M'(r) = \frac{\sqrt{2\pi} m}{4\sigma}\, e^{-\frac{r^2}{4\sigma^2}} \left( \mathcal{I}_{\frac{m-1}{2}}\left(\frac{r^2}{4\sigma^2}\right) - \mathcal{I}_{\frac{m+1}{2}}\left(\frac{r^2}{4\sigma^2}\right)\right) > 0, \ \ \forall r > 0 \tag{4.11}$$

where we have used the inequality [36]

$$\mathcal{I}_\nu(t) < \mathcal{I}_{\nu-1}(t), \ \ \forall \nu > \frac{1}{2}. \tag{4.12}$$

Moveover, $M'(0) = 0$ unless $m = 1$ in which case we have $M'(0) = \frac{\sqrt{2\pi}}{4\sigma} > 0$. So item (v) in the hypothesis is satisfied.

We now compute

$$\det\begin{pmatrix} \rho\, A_m(r,\rho) & \dfrac{\partial(\rho\, A_m)}{\partial\rho}(r,\rho) \\ \dfrac{\partial(\rho A_m)}{\partial r}(r,\rho) & \dfrac{\partial^2(\rho A_m)}{\partial r\partial\rho}(r,\rho) \end{pmatrix} = \frac{\rho^2}{\sigma^6}\, e^{-\frac{r^2+\rho^2}{\sigma^2}} Q(r,\rho;m,\sigma)$$

where

$$Q(r,\rho;m,\sigma) = t\,\mathcal{I}_m(t)^2 - t\,\mathcal{I}_{m+1}(t)^2 - 2m\,\mathcal{I}_m(t)\mathcal{I}_{m+1}(t), \quad t = \frac{r\rho}{\sigma^2}. \tag{4.13}$$

We thus want to show that

$$\mathcal{Q}(t;m) = t\,\mathcal{I}_m(t)^2 - t\,\mathcal{I}_{m+1}(t)^2 - 2m\,\mathcal{I}_m(t)\mathcal{I}_{m+1}(t) > 0, \;\; \forall t > 0. \tag{4.14}$$

Using the recurrence relation (see page 79 of [38]),

$$\mathcal{I}_{m+1}(t) = \mathcal{I}_{m-1}(t) - \frac{2m}{t}\,\mathcal{I}_m(t)$$

$\mathcal{Q}(t;m)$ in (4.14) can be rewritten as

$$\mathcal{Q}(t;m) = \mathcal{I}_{m-1}(t)^2\left(t\left(\frac{\mathcal{I}_m(t)}{\mathcal{I}_{m-1}(t)}\right)^2 + 2m\left(\frac{\mathcal{I}_m(t)}{\mathcal{I}_{m-1}(t)}\right) - t\right).$$

For $t > 0$, the positive root of the quadratic $tw^2 + 2mw - t$ is $w_+ = \dfrac{-m+\sqrt{m^2+t^2}}{t}$. In [22], it is shown that

$$\frac{\mathcal{I}_m(t)}{\mathcal{I}_{m-1}(t)} > w_+,$$

from which it follows that $\mathcal{Q}(t;m) > 0$ for all $t > 0$, and thus in (4.13) we have $Q(r,\rho;m,\sigma) > 0$ for all $r > 0$, $\rho > 0$. So the kernel $\rho A_m(r,\rho)$ for the fixed point problem (2.9) is of total positivity class $TP_2$, i.e. item (iii) is satisfied.

Next we check Hypothesis 3.1 (ii). In this case, we compute using formula (3) on page 394 of [38]

$$\int_0^\infty A_m(r,\rho)\rho^2\,d\rho = \frac{1}{\sigma^2}\int_0^\infty e^{-\frac{r^2+\rho^2}{2\sigma^2}}\mathcal{I}_m\left(\frac{r\rho}{\sigma^2}\right)\rho^2\,d\rho = rK(r),$$

where

$$K(r) = \frac{\sqrt{2\pi}}{4\sigma}e^{-\frac{r^2}{4\sigma^2}}\left(r\mathcal{I}_{\frac{m+2}{2}}\left(\frac{r^2}{4\sigma^2}\right) + \left[r + \frac{2\sigma^2(1+m)}{r}\right]\mathcal{I}_{\frac{m}{2}}\left(\frac{r^2}{4\sigma^2}\right)\right). \tag{4.15}$$

Note that for $m = 1$, this formula can be simplified to

$$K(r) = 1$$

using Bessel function identities. For $m > 1$, the leading term in the Taylor expansion of $\mathcal{I}_{\frac{m}{2}}(r^2/4\sigma^2)$ is $\sim r^m$, so that for $m > 1$,

$$\lim_{r\to 0} K(r) = 0.$$

Also, for $r > 0$ we have

$$K'(r) = \frac{\sqrt{2\pi}\sigma(m^2-1)}{2r^2} e^{-\frac{r^2}{4\sigma^2}} \mathcal{I}_{\frac{m}{2}}\left(\frac{r^2}{4\sigma^2}\right) > 0.$$

So $K(r)$ is positive, increasing (strictly increasing for $m > 1$), and using the asymptotic formula (4.9) we get

$$\lim_{r\to\infty} K(r) = 1,$$

i.e. $K(r)$ is bounded, see Figure 4 for an example.

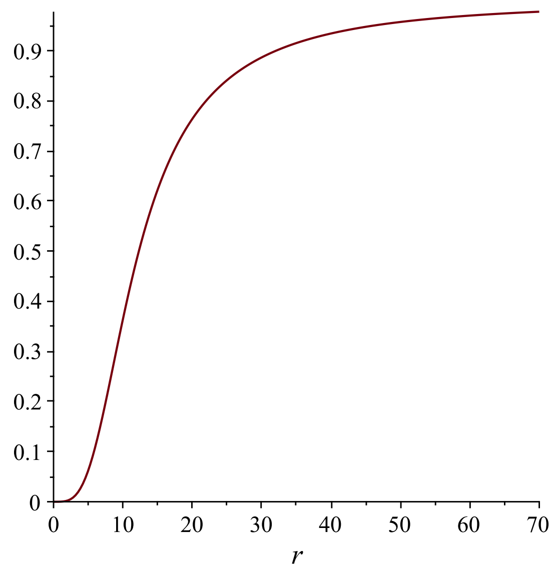


Figure 4: The function $K(r)$ in (4.15) in the case where $m = 5$ and $\sigma = 3$.

For Hypothesis 3.1 (iv), we leave the somewhat lengthy computation to the appendix.

# 5 Functional setup

Proving the existence of solutions to the fixed point equation

$$P(r) = \int_0^\infty \rho A_m(r,\rho)\, g(P(\rho),\beta)\, d\rho\,, \tag{5.1}$$

where $g(P,\beta) = P(1+\beta-P^2)$, requires an appropriate choice of function space for the function $P : [0,\infty) \to \mathbb{R}$.

Throughout, we will assume that $\beta$ is small enough and in particular is less than 1/2. We note the following obvious property

**Proposition 5.1** *If $P : [0,\infty) \to \mathbb{R}$ is continuous and such that*

*(i)* $P(t) \geq 0$ *for all* $t \geq 0$ *and* $P(0) = 0$,

*(ii)* $P(t)$ *is continuous and increasing on* $[0,\infty)$,

*(iii)* $\sup\limits_{t\in[0,\infty)} P(t) \leq \sqrt{\beta}$

*Then the function $\tilde{P}(t)$ defined by $\tilde{P}(t) = (g \circ P)(t)$ also satisfies all properties (i)-(iii) above.*

**Proof** This is a simple consequence of the fact that $g(P,\beta)$ is continuous, increasing and greater than $P$ on $[0,\sqrt{\beta}]$, with $g(0,\beta) = 0$ and $g(\sqrt{\beta},\beta) = \sqrt{\beta}$. ∎

Therefore, if $X \equiv C_b([0,\infty),\mathbb{R})$ denotes the Banach space of bounded and continuous functions from $[0,\infty)$ into $\mathbb{R}$ endowed with supremum norm $\|\ \|_\infty$, we will consider the following closed subspace of $X$

$$X_{\sqrt{\beta}} = \{P \in X \;:\; P(0) = 0,\; P \text{ is increasing},\; \|\, P \,\|_\infty \leq \sqrt{\beta}\,\}. \tag{5.2}$$

Proposition 5.1 is equivalent to saying that the space $X_{\sqrt{\beta}}$ is invariant under the mapping $P \mapsto g(P,\beta)$.

**Theorem 5.2** *If $P \in X_{\sqrt{\beta}}$ and if we define*

$$Z(r) \equiv T[P](r) = \int_0^\infty \rho A_m(r,\rho)\, g(P(\rho),\beta)\, d\rho$$

*then $Z \in X_{\sqrt{\beta}}$.*

**Proof** Given Proposition 5.1, it will suffice to show that if $L$ is the linear operator defined by

$$L[v](r) = \int_0^\infty \rho A_m(r,\rho) v(\rho)\, d\rho, \tag{5.3}$$

and if $P \in X_{\sqrt{\beta}}$, then $\tilde{Z} = L[P]$ belongs to $X_{\sqrt{\beta}}$. From (3.1) we obviously have $\tilde{Z}(0) = 0$. The fact that $\rho A_m(r,\rho)$ satisfies Hypothesis 3.1 implies that $\tilde{Z}(r) \geq 0$ for all $r \geq 0$. It also follows that $\tilde{Z}$ is continuous (in fact, $\tilde{Z}$ is $C^2$ on $(0,\infty)$) and

$$|\tilde{Z}(r)| \leq M(r) \| P \|_\infty \leq \sqrt{\beta}, \;\; \forall r \geq 0,$$

where $M(r)$ is as in (3.5). Thus $\| \tilde{Z} \|_\infty \leq \sqrt{\beta}$. It thus remains to be shown that $\tilde{Z}$ is increasing. Since $\tilde{Z}(0) = 0$ and $\tilde{Z}$ is $C^2$ on $(0,\infty)$, it will suffice to show that $\tilde{Z}'(r) > 0$ for all $r > 0$.

We will exploit the fact that $A_m$ satisfies Hypothesis 3.1 (iii), which implies that the kernel $\rho A_m(r,\rho)$ is of total positivity class $TP_2$. Kernels $\mathcal{F}(r,\rho)$ of class $TP_2$ satisfy an important property (called the *variation diminishing property* in the literature), which is as follows: suppose the function $\zeta(\rho)$ changes sign once, then the function

$$\tilde{\zeta}(r) = \int_0^\infty \mathcal{F}(r,\rho)\, \zeta(\rho)\, d\rho \tag{5.4}$$

changes sign at most once. Moreover, if $\tilde{\zeta}(r)$ changes sign exactly once, then $\tilde{\zeta}(r)$ and $\zeta(\rho)$ must have the same arrangements of signs as $r$ and $\rho$ respectively traverse $\mathbb{R}^+$ from left to right [18].

For any $a$ such that $a > 0$, consider the horizontal line $\rho = a$. If the function $P(\rho) - a$ vanishes, then it does so either at one isolated root, or on an interval $[\rho_1, \rho_2]$ (in the case where $P$ is constant on the interval). Either way, there is at most one sign change (necessarily from negative to positive) as $\rho$ traverses $\mathbb{R}^+$ from left to right. Since $\rho A_m(r,\rho)$ is $TP_2$, the same remark on sign changes holds for the function

$$\tilde{Z}(r) - aM(r) = \int_0^\infty \rho A_m(r,\rho)\, (P(\rho) - a)\, d\rho$$

(where $M$ is as in (3.5)). We compute (see Hypothesis 3.1 (iv))

$$\lim_{r\to 0^+} \tilde{Z}'(r) = \int_0^\infty \rho \left( \lim_{r \to 0^+} \frac{\partial A_m}{\partial r}(r,\rho) \right) P(\rho)\, d\rho \geq 0, \tag{5.5}$$

Suppose $r_0 > 0$ is such that $\tilde{Z}'(r_0) = 0$, and define $\tilde{a} = \tilde{Z}(r_0)/M(r_0) > 0$. Then the function

$$\mathcal{W}(r) = \tilde{Z}(r) - \tilde{a}M(r) = \int_0^\infty \rho A_m(r,\rho)\, (P(\rho) - \tilde{a})\, d\rho$$

has a root at $r_0$, and $\mathcal{W}'(r_0) = -\tilde{a}M'(r_0) < 0$. Therefore $\mathcal{W}$ goes from positive to negative as $r$ passes through $r_0$, which is a contradiction. We conclude that $\tilde{Z}'$ can have no root on $(0, \infty)$. Using (5.5) and the fact that $\tilde{Z}(0) = 0$, $\tilde{Z}(r) \geq 0$, we conclude that $\tilde{Z}'(r) > 0$ for all $r > 0$, i.e. $\tilde{Z}$ is (strictly) increasing. ∎

Therefore, we may define the nonlinear operator

$$\begin{aligned} T : X_{\sqrt{\beta}} &\longrightarrow X_{\sqrt{\beta}} \\ P \longmapsto T[P](r) &= \int_0^\infty \rho A_m(r,\rho)\, g(P(\rho),\beta)\, d\rho \end{aligned} \tag{5.6}$$

and we will be interested in proving the existence of a non-trivial fixed point for $T$ in $X_{\sqrt{\beta}}$.

# 6 Lower bounds

Of course it is clear that the function $P(r) = 0$ is a fixed point for the operator $T$ in (5.6). We are obviously interested in non-trivial fixed points. To this end, we will need to impose an additional hypothesis on the kernel function $\rho A_m(r, \rho)$:

**Hypothesis 6.1** *There exists a $\beta_0 > 0$ such that for every $\beta \in (0, \beta_0)$, there exists a bounded positive function $\phi(r;\beta)$ from $[0,\infty)$ into $\mathbb{R}$ such that $0 < \|\, \phi(\cdot;\beta)\,\|_\infty < \sqrt{\beta}$ and such that*

$$T[\phi(\cdot;\beta)](r;\beta) \geq \phi(r;\beta), \;\; \forall r \in [0,\infty), \forall \beta \in (0,\beta_0). \tag{6.7}$$

We note that in the case of an integration kernel of the form $\mathcal{K}(r,\rho) = \mathscr{K}(|r-\rho|)$, it is shown in [17] how to construct a such a lower-bound $\phi(r;\beta)$ under some general conditions, which unfortunately we can not exploit here since the integration kernel $\rho A_m(r,\rho)$ is not of the form $\mathscr{K}(|r-\rho|)$.

While condition (6.7) may be difficult to verify in practice, the following gives a sufficient condition which presumably is easier to check.

**Proposition 6.2** *Consider the linear operator $L$ defined by*

$$L[v](r) = \int_0^\infty \rho A_m(r,\rho) v(\rho)\, d\rho$$

*as in (5.3). Suppose there exists a $\beta_1 > 0$ and a bounded positive function $v : [0,\infty) \times [0,\beta_1) \longrightarrow \mathbb{R}$ with $0 < \|\, v(\cdot;\beta)\,\|_\infty \leq \sqrt{\beta}$ and such that*

$$\frac{v(r;\beta)}{L[v](r;\beta)} \leq D(\beta), \;\; \forall r \geq 0, \forall \beta \in (0,\beta_1),$$

*where $D$ is a $C^2$ smooth positive function with $D(0) = 1$ and $D'(0) \leq 0$. Then the function $\phi(r; \beta) = e^{-\sqrt{\beta}} v(r; \beta)$ satisfies the conditions of Hypothesis 6.1.*

**Proof** Since $v$ is positive it follows that $\phi = e^{-\sqrt{\beta}} v$ is positive and

$$\| \phi(\cdot; \beta) \|_\infty = e^{-\sqrt{\beta}} \| v(\cdot; \beta) \|_\infty \leq \sqrt{\beta}\, e^{-\sqrt{\beta}} < \sqrt{\beta}.$$

Using the fact that the integration kernel $\rho A_m$ is positive, we may write

$$L[\phi^3] = \int_0^\infty \rho A_m(r, \rho) \phi(\rho; \beta)^3 \, d\rho \leq \| \phi \|_\infty^2 L[\phi] \leq \beta e^{-2\sqrt{\beta}} L[\phi].$$

Now

$$\begin{aligned} T[\phi] - \phi &= (1+\beta) L[\phi] - \phi - L[\phi^3] = L[\phi] \left( (1+\beta) - \frac{\phi}{L[\phi]} - \frac{L[\phi^3]}{L[\phi]} \right) \\ &= D(\beta) L[\phi] \left( \frac{(1+\beta)}{D(\beta)} - \frac{v}{D(\beta) L[v]} - \frac{L[\phi^3]}{D(\beta) L[\phi]} \right). \end{aligned} \tag{6.8}$$

From the hypotheses on the function $D$, we claim that there exists a $\beta_0 > 0$ such that

$$\frac{(1+\beta)}{D(\beta)} - \frac{\beta e^{-2\sqrt{\beta}}}{D(\beta)} \geq 1, \ \ \forall \beta \in (0, \beta_0).$$

To verify the claim, we note that

$$\frac{(1+\beta)}{D(\beta)} - \frac{\beta e^{-2\sqrt{\beta}}}{D(\beta)} = 1 - D'(0)\beta + 2\beta^{\frac{3}{2}} + O(\beta^2)$$

which is increasing for small enough $\beta > 0$ since $D'(0) \leq 0$ (note that the claim holds even in the case where $D'(0) = 0$ because $2\beta^{\frac{3}{2}} > 0$).

We may then write

$$\frac{(1+\beta)}{D(\beta)} - \frac{L[\phi^3]}{D(\beta) L[\phi]} \geq \frac{(1+\beta)}{D(\beta)} - \frac{\beta e^{-2\sqrt{\beta}}}{D(\beta)} \geq 1 \geq \frac{v}{D(\beta) L[v]}, \ \ \forall r \in [0, \infty), \ \forall \beta \in (0, \beta_0).$$

Using (6.8), we conclude $T[\phi] - \phi \geq 0$, or equivalently that (6.7) is satisfied. ∎

The hypotheses of Proposition 6.2 are easy to verify in the case where $k$ is a Gaussian as in (2.10) and consequently $A_m(r, \rho)$ is as in (4.8). We will use formula (3) on page 394 of [38], which after simplification leads to the following formula (for given constant $\lambda > 0$)

$$\int_0^\infty \rho A_m(r, \rho) \rho^m e^{-\lambda \rho^2} \, d\rho = \frac{1}{(1 + 2\lambda\sigma^2)^{1+m}} \, r^m e^{-\frac{\lambda r^2}{1 + 2\lambda\sigma^2}} \ . \tag{6.9}$$

**Proposition 6.3** *For $m \geq 1$ integer and $\beta > 0$, $r \geq 0$, consider*

$$v(r;\beta) = \beta^{m+\frac{1}{2}} \left(\frac{2e}{m}\right)^{\frac{m}{2}} r^m e^{-\beta^2 r^2}.$$

*Then $v$ satisfies the hypotheses of Proposition 6.2.*

**Proof** The function $v(r;\beta)$ is clearly positive and bounded. It attains a maximal value of $\sqrt{\beta}$ at $r_0 = \dfrac{1}{\beta}\sqrt{\dfrac{m}{2}}$, so we have $\| v \|_\infty = \sqrt{\beta}$. A simple computation using (6.9) yields

$$\frac{v(r;\beta)}{L[v](r;\beta)} = (1+2\beta^2\sigma^2)^{m+1} e^{-\frac{2\beta^4\sigma^2 r^2}{1+2\beta^2\sigma^2}} \leq (1+2\beta^2\sigma^2)^{m+1} \equiv D(\beta)$$

where we note that $D(0) = 1$ and $D'(0) = 0$.

**Remark 6.4** *From Propositions 6.2 and 6.3, it follows that in the case where k is a Gaussian as in (2.10) and consequently $A_m(r,\rho)$ is as in (4.8), Hypothesis 6.1 is satisfied, using the lower bound*

$$\phi(r;\beta) = e^{-\sqrt{\beta}} \left( \beta^{m+\frac{1}{2}} \left(\frac{2e}{m}\right)^{\frac{m}{2}} r^m e^{-\beta^2 r^2} \right),$$

*whose maximum value is $\sqrt{\beta} e^{-\sqrt{\beta}}$.*

# 7 Main result

We are now ready to state and prove the main result of the paper concerning the existence of rotating pinwheel solutions to (1.12) via the existence of non-trivial fixed points for the operator $T$ in (5.6).

For small $\beta > 0$, recall the definition of the metric space $X_{\sqrt{\beta}}$ in (5.2), and note that from Theorem 5.2 we have $T(X_{\sqrt{\beta}}) \subset X_{\sqrt{\beta}}$. Let $\phi(r;\beta)$ be the lower bound such as in Hypothesis 6.1. Consider the Banach space

$$Z_{\frac{1}{r+1}} = \left\{ p \in C([0,\infty), \mathbb{R}) \; : \; \sup_{r\in[0,\infty)} \left| \frac{p(r)}{r+1} \right| < \infty \right\}$$

with norm $\|p\|_{\frac{1}{r+1}} = \sup\limits_{r\in[0,\infty)} \left|\dfrac{p(r)}{r+1}\right|$. The set

$$\mathcal{X}_{\sqrt{\beta},\phi} = \left\{ p \in Z_{\frac{1}{r+1}} \cap X_{\sqrt{\beta}} \; : \; \phi(r;\beta) \leq p(r) \leq \sqrt{\beta}, \; \forall r \in [0,\infty) \right\} \tag{7.10}$$

is a non-empty, closed and convex subset of $Z_{\frac{1}{r+1}}$, invariant under the nonlinear operator $T$.

**Proposition 7.1** *The operator $T$ is continuous on $\mathcal{X}_{\sqrt{\beta},\phi} \subset Z_{\frac{1}{r+1}}$.*

**Proof** Note that

$$\int_0^\infty \rho A_m(r,\rho)\left(\frac{\rho+1}{r+1}\right)\,d\rho = \frac{M(r)}{r+1} + \frac{rK(r)}{r+1} \leq \Phi \equiv 1 + K_0, \;\; \forall r \in [0,\infty), \tag{7.11}$$

where the function $M$ is as in (3.5), the function $K$ is as in Hypothesis 3.1 (ii) and $K_0$ is an upper bound for $K(r)$. From this and the Lipschitz continuity (with Lipschitz constant $\mathcal{L} > 0$) of the function $u \longmapsto g(u;\beta) = (1+\beta)u - u^3$ on $(u,\beta) \in [0,\sqrt{\beta}] \times [0,1/2]$, we have that for all $p_1, p_2 \in \mathcal{X}_{\sqrt{\beta},\phi}$,

$$\begin{aligned}
\left|\frac{T[p_1](r) - T[p_2](r)}{r+1}\right| &\leq \frac{1}{r+1}\int_0^\infty \rho A_m(r,\rho)\left|(1+\beta)(p_1(\rho) - p_2(\rho)) - (p_1(\rho)^3 - p_2(\rho)^3)\right|\,d\rho \\
&\leq \mathcal{L}\int_0^\infty \rho A_m(r,\rho)\left(\frac{\rho+1}{r+1}\right)\left|\frac{p_1(\rho)-p_2(\rho)}{\rho+1}\right|\,d\rho \\
&\leq \mathcal{L}\Phi\|p_1 - p_2\|_{\frac{1}{r+1}} \quad \forall r \in [0,\infty),
\end{aligned}$$

and so $T : \mathcal{X}_{\sqrt{\beta},\phi} \longrightarrow \mathcal{X}_{\sqrt{\beta},\phi}$ is continuous. ∎

**Proposition 7.2** *The set $\mathcal{U} = T(\mathcal{X}_{\sqrt{\beta},\phi})$ is precompact in $Z_{\frac{1}{r+1}}$.*

**Proof** It is clear that for any $p \in \mathcal{U}$ we have $p = T(Y_p)$ for some $Y_p \in \mathcal{X}_{\sqrt{\beta},\phi}$, and for any $r \geq 0$ we have

$$|p(r)| \leq \sqrt{\beta}.$$

Considering Hypothesis 3.1 (vi), let $\mathcal{M} > 0$ be such that

$$\sup_{r\in[0,\infty)} \int_0^\infty \rho\left|\frac{\partial A_m}{\partial r}(r,\rho)\right|\,d\rho \leq \mathcal{M}.$$

Then for all $p \in \mathcal{U}$ we may write (since $p$ is $C^1$)

$$|p'(r)| = \left|\int_0^\infty \rho\frac{\partial A_m}{\partial r}(r,\rho)g(Y_p(\rho),\beta)\,d\rho\right| \leq \|g(Y_p,\beta)\|_\infty \int_0^\infty \rho\left|\frac{\partial A_m}{\partial r}(r,\rho)\right|\,d\rho \leq \mathcal{M}\sqrt{\beta}.$$

Therefore the number $\mathcal{M}\sqrt{\beta}$ is a uniform global Lipschitz constant for all elements of $\mathcal{U}$, and we conclude that $\mathcal{U}$ is equicontinuous.

Let $\{p_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of elements of $\mathcal{U}$. Then there is a subsequence $\{p_{n_\ell}\}$ which converges uniformly on compact subsets of $[0,\infty)$ to a continuous function $p : [0,\infty) \longrightarrow \mathbb{R}$. Since each $p_n$ is such that

(i) $p_n(0) = 0$

(ii) $p_n$ is increasing

(iii) $\phi(r;\beta) \le p_n(r) \le \sqrt{\beta}$

then the limit function $p$ enjoys the same properties because these are preserved by pointwise convergence, i.e. $p$ belongs to $\mathcal{X}_{\sqrt{\beta},\phi}$.

Since $\|\phi(\cdot;\beta)\|_\infty < \sqrt{\beta}$, we have

$$\lim_{r\to\infty} \left| \frac{\sqrt{\beta} - \phi(r,\beta)}{r+1} \right| = 0.$$

Thus, for any $\varepsilon > 0$, there exists $R > 0$ such that

$$0 \le \left| \frac{p_{n_\ell}(r) - p(r)}{r+1} \right| \le \frac{\sqrt{\beta} - \phi(r;\beta)}{r+1} < \varepsilon/2, \ \ \forall r > R, \forall \ell \in \mathbb{N}.$$

Since $\{p_{n_\ell} - p\}$ converges uniformly to 0 on $[0, R]$, then the same holds for $\left\{\frac{p_{n_\ell}-p}{r+1}\right\}$, so there exists $N \in \mathbb{N}$ such that

$$\left| \frac{p_{n_\ell}(r) - p(r)}{r+1} \right| < \varepsilon/2, \ \ \forall r \in [0, R], \ \ \forall \ell > N.$$

It follows that

$$\|p_{n_\ell} - p\|_{\frac{1}{r+1}} = \sup_{r\in[0,\infty)} \left| \frac{p_{n_\ell}(r) - p(r)}{r+1} \right| \le \varepsilon/2 < \varepsilon, \ \ \forall \ell > N,$$

i.e. $\lim_{\ell\to\infty} p_{n_\ell} = p$ in $Z_{\frac{1}{r+1}}$. We thus conclude that $\mathcal{U} = T(\mathcal{X}_{\sqrt{\beta},\phi})$ is pre-compact in $Z_{\frac{1}{r+1}}$. ∎

We can now give the main result of this paper in the form of a theorem and a corollary.

**Theorem 7.3** *Consider the two component integro-difference equation on the $x = (x_1, x_2)$ plane (written in complex notation $u = u_R + iu_I$),*

$$u_{n+1}(x) = \mathcal{N}[u_n](x) = \iint_{\mathbb{R}^2} k(|x-y|)e^{i\omega}u_n(y)(1+\beta-|u_n(y)|^2)\,dy_1dy_2, \ \ n = 0,1,2,\ldots, \tag{7.12}$$

*where $\beta > 0$ is a small-enough parameter, $\omega \in (0, \pi)$. Let $m > 0$ be an integer and suppose that the kernel $k$ satisfies the conditions of Hypotheses 3.1 and 6.1 for that value of $m$.*

*There exists a function* $U^*(x) = U^*(re^{i\varphi}) = P^*(r)e^{im\varphi}$ *which is a rotating-wave solution to the system (7.12) in the sense that*

$$e^{i\delta\alpha}U^*(e^{-i\alpha}x) = \iint_{\mathbb{R}^2} k(|x-y|)e^{i\omega}U^*(y)(1+\beta-|U^*(y)|^2)\,dy_1dy_2, \quad \delta \in \{0,1\},$$

*where* $\alpha$ *satisfies the compatibility condition (2.8). The rotating wave is untwisted if* $\delta = 0$*, and twisted if* $\delta = 1$ *(in which case* $m \neq 1$*). Furthermore, the radial-shape function* $P^*$ *is* $C^2$ *smooth and such that*

*(i)* $P^*(0) = 0$

*(ii)* $P^*$ *is increasing*

*(iii)* $\lim_{r\to\infty} P^*(r) \in [\tau(\beta), \sqrt{\beta}]$ *where* $0 < \tau(\beta) = \|\phi(\cdot;\beta)\|_\infty < \sqrt{\beta}$*, where* $\phi(r;\beta)$ *is as in Hypothesis 6.1.*

**Proof** This is a simple application of Schauder's fixed point theorem, which given Proposition 7.2 guarantees the existence of a $P^*_\beta \in \mathcal{X}_{\sqrt{\beta},\phi}$ which satisfies all the properties (i)-(iii) in the statement of the theorem (because these properties are satisfied by all elements of $\mathcal{X}_{\sqrt{\beta},\phi}$) and such that $P^*_\beta = T[P^*_\beta]$, i.e.

$$P^*_\beta(r) = \int_0^\infty \rho A_m(r,\rho)[(1+\beta)P^*_\beta(\rho) - P^*_\beta(\rho)^3]\,d\rho.$$

∎

**Corollary 7.4** *Consider the two component integro-difference equation on the* $x = (x_1, x_2)$ *plane (written in complex notation* $u = u_R + iu_I$*),*

$$u_{n+1}(x) = \mathcal{N}[u_n](x) = \iint_{\mathbb{R}^2} k(|x-y|)e^{i\omega}u_n(y)(1+\beta-|u_n(y)|^2)\,dy_1dy_2, \quad n = 0,1,2,\ldots, \tag{7.13}$$

*where* $\beta > 0$ *is a small-enough parameter,* $\omega \in (0,\pi)$ *and* $k$ *is the Gaussian*

$$k(t) = \frac{1}{2\pi\sigma^2}\,e^{-\frac{t^2}{2\sigma^2}}, \ \sigma > 0.$$

*For any integer* $m > 0$*, let* $\alpha$ *satisfy the compatibility condition (2.8). There exists a function* $U^*(x) = U^*(re^{i\varphi}) = P^*(r)e^{im\varphi}$ *which is a rotating-wave solution (untwisted if* $\delta = 0$ *and twisted if* $\delta = 1$*,* $m \neq 1$*) to the system (7.13) in the sense that*

$$e^{i\delta\alpha}U^*(e^{-i\alpha}x) = \iint_{\mathbb{R}^2} k(|x-y|)e^{i\omega}U^*(y)(1+\beta-|U^*(y)|^2)\,dy_1dy_2.$$

*Furthermore, the radial-shape function* $P^*$ *is* $C^2$ *smooth and such that* *(i)* $P^*(0) = 0$

*(ii)* $P^*$ *is increasing*

*(iii)* $\lim_{r\to\infty} P^*(r) \in [\sqrt{\beta}e^{-\sqrt{\beta}}, \sqrt{\beta}]$

*In particular, the amplitude of the shape function* $P^*(r)$ *satisfies* $\|P\|_\infty \sim \sqrt{\beta}$ *as* $\beta \to 0$.

**Proof** This is a consequence of Section 4, Appendix A, Proposition 6.3 and the subsequent Remark 6.4. ∎

# 8 Radial stability in the Gaussian case, and a uniqueness result

In the case of the Gaussian kernel, we will prove a partial stability result for the rotating wave solution found in Corollary 7.4. That is, we will provide sufficient conditions to guarantee that the fixed point $P^*$ of the operator

$$T[P](r) = \int_0^\infty \frac{\rho}{\sigma^2} e^{-\frac{r^2+\rho^2}{2\sigma^2}} \mathcal{I}_m\left(\frac{r\rho}{\sigma^2}\right)\left((1+\beta)P(\rho) - P(\rho)^3\right) d\rho \tag{8.1}$$

is locally asymptotically stable in $C_b([0,\infty),\mathbb{R})$. At this point, we do not have any results concerning stability with respect to angular perturbations.

The important technical tools used in our analysis are the ordering (4.12) of the Bessel functions and the fact that the lower bound $\phi(r;\beta)$ found in Remark 6.4 is independent of the parameter $\sigma$ in the Gaussian distribution.

We note that the linearization of the operator $T$ in (8.1) at the fixed point $P = P^*$ is given by

$$\mathscr{L}[v](r) = \int_0^\infty \frac{\rho}{\sigma^2} e^{-\frac{r^2+\rho^2}{2\sigma^2}} \mathcal{I}_m\left(\frac{r\rho}{\sigma^2}\right)(1+\beta-3P^*(\rho)^2)v(\rho)\, d\rho.$$

Since $\|P^*\|_\infty \leq \sqrt{\beta}$, for small $\beta > 0$ the term $1+\beta-3P^*(\rho)^2$ is positive, so the norm of the operator $\mathscr{L}$ on the space $X = C_b([0,\infty),\mathbb{R})$ satisfies

$$\|\mathscr{L}\|_\infty \leq \sup_{r\in[0,\infty)} \left(\int_0^\infty \frac{\rho}{\sigma^2} e^{-\frac{r^2+\rho^2}{2\sigma^2}} \mathcal{I}_m\left(\frac{r\rho}{\sigma^2}\right)(1+\beta-3P^*(\rho)^2)\, d\rho\right). \tag{8.2}$$

The main result is the following:

**Theorem 8.1** *Let $\beta > 0$ be fixed and small enough. Then for sufficiently large $\sigma > 0$, we have $\| \mathscr{L} \|_\infty < 1$ in (8.2).*

**Proof** We first will prove the following lemmas which provide a uniform bound for the term $\dfrac{\rho}{\sigma} e^{-\frac{r^2+\rho^2}{2\sigma^2}} \mathcal{I}_m \left( \dfrac{r\rho}{\sigma^2} \right)$ which appears in the integral in (8.2)

**Lemma 8.2** *For all $t \geq 0$ and all $m \geq 1$, we have*

$$te^{-\frac{t^2}{2}} \mathcal{I}_m \left( \frac{t^2}{2} \right) \leq \frac{1}{\sqrt{\pi}}.$$

**Proof of Lemma 8.2**: Because of the ordering (4.12), we need only prove the result for $m = 1$. Therefore, let $\xi(t) = te^{-\frac{t^2}{2}} \mathcal{I}_1 \left( \frac{t^2}{2} \right)$. Clearly $\xi(0) = 0$, $\xi$ is positive and the asymptotic formula (4.9) leads to

$$\lim_{t \to \infty} \xi(t) = \frac{1}{\sqrt{\pi}}.$$

The result will follow if we can prove that $\xi$ is increasing. We compute

$$\xi'(t) = e^{-\frac{t^2}{2}} \mathcal{I}_1 \left( \frac{t^2}{2} \right) \left[ t^2 \frac{\mathcal{I}_0 \left( \frac{t^2}{2} \right)}{\mathcal{I}_1 \left( \frac{t^2}{2} \right)} - (t^2 + 1) \right]$$

which is positive because of formula (10) from [35] (see also (A.5) in the Appendix) which allows us to write

$$\frac{t^2}{2} \frac{\mathcal{I}_0 \left( \frac{t^2}{2} \right)}{\mathcal{I}_1 \left( \frac{t^2}{2} \right)} > \frac{t^2}{2} + \frac{1}{2}.$$

∎

**Lemma 8.3** *For all $r \geq 0$, $\rho \geq 0$, $m \geq 1$ and $\sigma > 0$, we have*

$$\frac{\rho}{\sigma} e^{-\frac{r^2+\rho^2}{2\sigma^2}} \mathcal{I}_m \left( \frac{r\rho}{\sigma^2} \right) \leq \frac{1}{\sqrt{\pi}}.$$

**Proof of Lemma 8.3**: Again, we only need to prove the result for $m = 1$. Set $\tilde{r} = r/\sigma$ and $\tilde{\rho} = \rho/\sigma$, then upon dropping the tildes the result follows if we can show that

$$\rho e^{-\frac{r^2+\rho^2}{2}} \mathcal{I}_1(r\rho) \leq \frac{1}{\sqrt{\pi}}, \quad \forall r, \rho \geq 0.$$

Write $r = t\cos\eta$, $\rho = t\sin\eta$ where $t \geq 0$ and $\eta \in [0, \pi/2]$. Then using the fact that $\mathcal{I}_1$ is an increasing function, we have

$$\rho e^{-\frac{r^2+\rho^2}{2}} \mathcal{I}_1(r\rho) = t\sin\eta\, e^{-\frac{t^2}{2}} \mathcal{I}_1\left(\sin 2\eta\,\frac{t^2}{2}\right) \leq t\, e^{-\frac{t^2}{2}} \mathcal{I}_1\left(\frac{t^2}{2}\right)$$

and we invoke Lemma 8.2 for the conclusion of the proof of this lemma. ∎

Let $\phi(r;\beta)$ be as in Remark 6.4. This function attains a maximum value of $e^{-\sqrt{\beta}}\sqrt{\beta}$ at $r = \dfrac{\sqrt{2m}}{2\beta}$. Since $P^*(r) \geq \phi(r;\beta)$ and $P^*$ is increasing, then we conclude that

$$P^*(r) \geq \mathscr{S}(r) \equiv \begin{cases} \phi(r;\beta) & \text{if} \quad r \leq \dfrac{\sqrt{2m}}{2\beta} \\ e^{-\sqrt{\beta}}\sqrt{\beta} & \text{if} \quad r > \dfrac{\sqrt{2m}}{2\beta} \end{cases} \tag{8.3}$$

Therefore

$$\begin{aligned}
&\int_0^\infty \frac{\rho}{\sigma^2} e^{-\frac{r^2+\rho^2}{2\sigma^2}} \mathcal{I}_m\left(\frac{r\rho}{\sigma^2}\right)(1+\beta-3P^*(\rho)^2)\,d\rho \leq \int_0^\infty \frac{\rho}{\sigma^2} e^{-\frac{r^2+\rho^2}{2\sigma^2}} \mathcal{I}_m\left(\frac{r\rho}{\sigma^2}\right)(1+\beta-3\mathscr{S}(\rho)^2)\,d\rho \\
&= \int_0^{\frac{\sqrt{2m}}{2\beta}} \frac{1}{\sigma}\left[\frac{\rho}{\sigma} e^{-\frac{r^2+\rho^2}{2\sigma^2}} \mathcal{I}_m\left(\frac{r\rho}{\sigma^2}\right)\right](1+\beta-3\phi(\rho;\beta)^2)\,d\rho \\
&\qquad\qquad + \int_{\frac{\sqrt{2m}}{2\beta}}^\infty \frac{\rho}{\sigma^2} e^{-\frac{r^2+\rho^2}{2\sigma^2}} \mathcal{I}_m\left(\frac{r\rho}{\sigma^2}\right)(1+\beta-3\beta e^{-2\sqrt{\beta}})\,d\rho \\
&\leq \frac{1}{\sigma}\frac{1}{\sqrt{\pi}}\mathscr{G}(m,\beta) + \int_0^\infty \frac{\rho}{\sigma^2} e^{-\frac{r^2+\rho^2}{2\sigma^2}} \mathcal{I}_m\left(\frac{r\rho}{\sigma^2}\right)(1+\beta-3\beta e^{-2\sqrt{\beta}})\,d\rho \\
&= \frac{1}{\sigma}\frac{1}{\sqrt{\pi}}\mathscr{G}(m,\beta) + M(r)(1+\beta-3\beta e^{-2\sqrt{\beta}}) \leq \frac{1}{\sigma}\frac{1}{\sqrt{\pi}}\mathscr{G}(m,\beta) + (1+\beta-3\beta e^{-2\sqrt{\beta}}),
\end{aligned}$$

where $M(r)$ is as in (3.5) and (4.10) (we recall that $M(r) \leq 1$ for all $r \geq 0$), and

$$\mathscr{G}(m,\beta) \equiv \int_0^{\frac{\sqrt{2m}}{2\beta}} (3\beta e^{-2\sqrt{\beta}} - 3\phi(\rho;\beta)^2)\,d\rho$$

is finite (and computable in closed form expression). Thus, if

$$\sigma > \frac{\mathscr{G}(m,\beta)}{\sqrt{\pi}\beta(3e^{-2\sqrt{\beta}}-1)} \tag{8.4}$$

we conclude from (8.2) that

$$\| \mathscr{L} \|_\infty < 1.$$

∎

**Remark 8.4** *We make the following observations about the proof of the previous theorem:*

*(i) We can compute a closed form expression for $\mathscr{G}(m, \beta)$ in terms of the $\Gamma$ function*

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt$$

*and the incomplete $\Gamma$ function:*

$$\Gamma(a, z) = \int_z^\infty e^{-t}\, t^{a-1}\, dt\,.$$

*We have*

$$\mathscr{G}(m, \beta) =$$

$$\frac{3\sqrt{2}\, e^{-2\sqrt{\beta}}}{4(2m+1)} \left(4m^{3/2} + 2m^{-m} e^m \Gamma\left(\frac{3}{2} + m, m\right) - m^{-m} e^m\, \Gamma\left(m + \frac{1}{2}\right)(2m+1)\right). \tag{8.5}$$

*We have illustrated in figure 5 the curves (as a function of $\beta$) given by the right-hand side of inequality (8.4) for $m = 1$ and $m = 4$.*

*(ii) As is clear from the proof of the previous theorem, condition (8.4) is sufficient, but not necessary for stability of the fixed point. We have performed numerical simulations to better characterize the stability of the fixed point $P^*$. The results will be presented in the next section. See figure 11 and the description in the caption.*

*(iii) Indeed the inequality (8.3) is far from optimal, but is valid for all elements $p$ of the space $\mathcal{X}_{\sqrt{\beta}}$ in (5.2) which are such that $p \geq \phi$. In fact, we can use this observation to improve on Corollary 7.4.*

**Proposition 8.5** *Suppose $\sigma > 0$ and $\beta > 0$ are such that*

$$\sup_{r \in [0,\infty)} \left( \int_0^\infty \frac{\rho}{\sigma^2} e^{-\frac{r^2+\rho^2}{2\sigma^2}} \mathcal{I}_m\left(\frac{r\rho}{\sigma^2}\right) (1 + \beta - 3\mathscr{S}(\rho)^2)\, d\rho \right) = \gamma < 1,$$

*where $\mathscr{S}$ is as in (8.3). Define the set*

$$Y_{\sqrt{\beta},\phi} \equiv \left\{ p \in \mathcal{X}_{\sqrt{\beta}} \,|\, p(r) \geq \phi(r;\beta),\ \forall r \in [0,\infty) \right\}$$

*Then the operator $T$ in (8.1) has a* unique *fixed point in $Y_{\sqrt{\beta},\phi}$.*

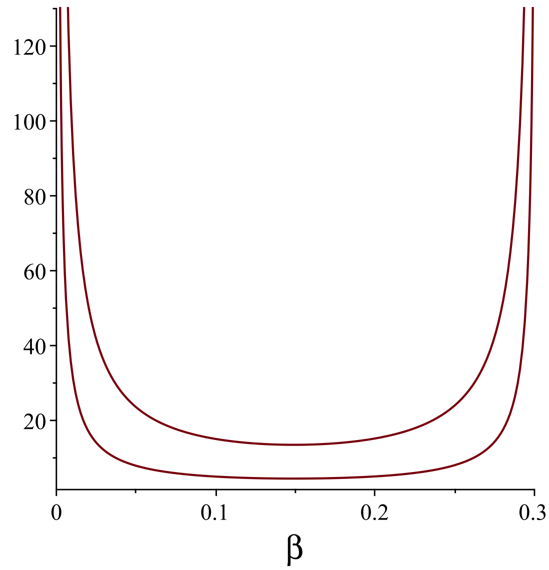


Figure 5: The stability bounds given by the right-hand side (8.4) for $m = 1$ (bottom curve) and $m = 4$ (top curve)

**Proof** The space $Y_{\sqrt{\beta},\phi}$ is closed in $C_b([0,\infty),\mathbb{R})$ and $T$ maps $Y_{\sqrt{\beta},\phi}$ into itself. Let $p_1, p_2 \in Y_{\sqrt{\beta},\phi}$ be given, then

$$
\begin{aligned}
&|T[p_1](r) - T[p_2](r)| \leq \\
&||p_1 - p_2||_\infty \int_0^\infty \frac{\rho}{\sigma^2} e^{-\frac{r^2+\rho^2}{2\sigma^2}} \mathcal{I}_m\left(\frac{r\rho}{\sigma^2}\right)\left(1 + \beta - p_1(\rho)^2 - p_1(\rho)p_2(\rho) - p_2(\rho)^2\right) d\rho \leq \\
&||p_1 - p_2||_\infty \int_0^\infty \frac{\rho}{\sigma^2} e^{-\frac{r^2+\rho^2}{2\sigma^2}} \mathcal{I}_m\left(\frac{r\rho}{\sigma^2}\right)\left(1 + \beta - 3\mathscr{S}(\rho)^2\right) d\rho \leq \gamma ||p_1 - p_2||_\infty.
\end{aligned}
$$

So $T$ is a contraction on $Y_{\sqrt{\beta},\phi}$ and we get the conclusion from the Banach Fixed Point Theorem. ∎

# 9 Simulations

We present the results of iterations of the nonlinear operator $T$ defined by (8.1). The numerical method we have used is explained below and was implemented in the symbolic programming language Maple.

We have used a spatial discretization of the $\rho$ and the $r$ axes with 10000 intervals of length 1/400, which gives values of $r$ and $\rho$ between 0 and 25. For fixed values of the parameters $m$ and $\sigma$, we compute once a $10000 \times 10000$ matrix $\mathscr{M}$ such that

$$\mathscr{M}[i,j] = \frac{\rho[j]}{\sigma^2} e^{-\frac{r[i]^2+\rho[j]^2}{2\sigma^2}} \mathcal{I}_m\left(\frac{r[i]\rho[j]}{\sigma^2}\right).$$

The initial condition $P_0(r)$ is written as a 10000 dimensional vector and the vector $v_0 = (1+\beta)P_0 - P_0^3$ is computed. Then for every $i$ value between 1 and 10000, the value $P_1(r[i])$ is computed by using a trapezoidal rule to sum the points

$$\{\mathscr{M}[i,j]v_0[j],\ j = 1, 2, \dots, 10000\}.$$

with spatial distancing 1/400. Since the true integral is an integral from 0 to $\infty$, obviously our implementation will give problematic results near the endpoints at $\rho = 25$ and at $r = 25$. For this reason, for purposes of viewing $P_j[r]$, we use only the first 8000 datapoints, corresponding to $r$ values between 0 and 20. The iteration process is halted (i.e. considered to have converged) when the maximum value of $|P_{n+1}(r) - P_n(r)|/\sqrt{\beta}$ for $r$ between 0 and 20 is less than $5 \times 10^{-5}$. We report here the results of two such simulations: one for parameter values $\beta = 0.2$, $\sigma = 0.5$, $m = 4$, and another for parameter values $\beta = 0.08$, $\sigma = 0.3$ and $m = 1$. See figures 6 and 7 for a summary of the results.

For both simulations we have reported on here, we have numerically computed the function

$$\mathscr{L}_{63}(r) = \int_0^\infty \frac{\rho}{\sigma^2} e^{-\frac{r^2+\rho^2}{2\sigma^2}} \mathcal{I}_m\left(\frac{r\rho}{\sigma^2}\right)(1+\beta-3P_{63}(\rho)^2)\,d\rho \tag{9.1}$$

i.e. we have used $P_{63}$ as an approximation of the fixed point $P^*$, and the results are reported in figure 10. As we can see, the maximum value of $\mathscr{L}_{63}(r)$ is bounded away from one which would indicate that the linearization $\mathscr{L}$ of the operator $T$ at the fixed point $P^*$ is a contraction for the chosen parameter values. Note that the value of $\sigma$ for each of these simulations is considerably less than the lower bound given in (8.4) (taking into account formula (8.5)): in the first case the right hand side of inequality (8.4) yields a value of $\sim 15.4$ whereas for the second case the value is $\sim 5.9$.

In figure 11, we have numerically computed the fixed point $P^*$ up to an accuracy of $5 \times 10^{-5}$ (by the iteration procedure described above) for values of $\beta$ between 0 and 0.1, and then computed the maximum value of the linearized operator

$$\Lambda(r) = \int_0^\infty \frac{\rho}{\sigma^2} e^{-\frac{r^2+\rho^2}{2\sigma^2}} \mathcal{I}_m\left(\frac{r\rho}{\sigma^2}\right)(1+\beta-3P^*(\rho)^2)\,d\rho\,, \tag{9.2}$$

which suggests (as expected) that the bifurcating solution is stable for all $\beta > 0$ small enough.

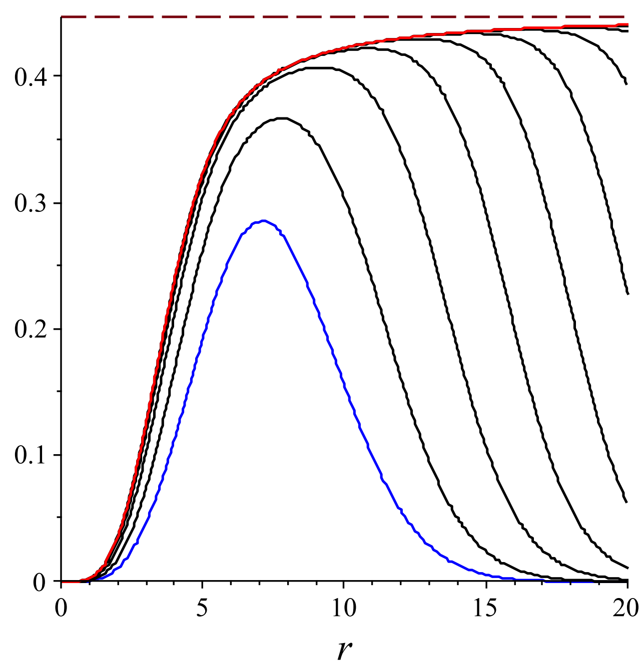


Figure 6: For the dynamical system $P_{j+1}(r) = T[P_j](r)$ where $T$ is as in (8.1), we show here the iterations $P_{7n}$ for $n \in \{0,1,2,3,4,5,6,7,8,9\}$. The parameter values are $\beta = 0.2$, $\sigma = 0.5$ and $m = 4$. The initial condition is $P_0(r) = \phi(r;\beta)$ as in Remark 6.4 and is in blue. The iterate $P_{63}$ is in solid red and the dashed horizontal redline is at height $\sqrt{\beta} = \sqrt{0.2}$. The residual ratio $\max_{r\in[0,20]} |P_{64}(r) - P_{63}(r)|/\sqrt{\beta}$ is less than $5 \times 10^{-5}$. The ordering of the curves is such that $P_{7(n+1)}(r) \geq P_{7n}(r)$.

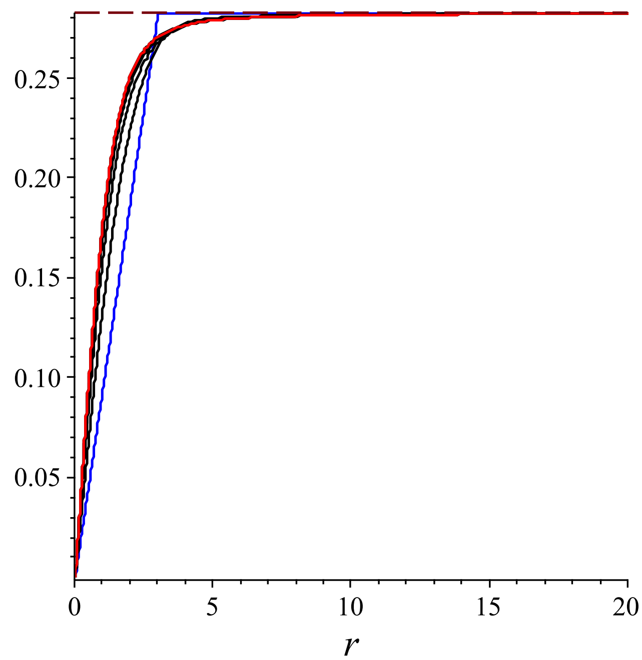


Figure 7: For the dynamical system $P_{j+1}(r) = T[P_j](r)$ where $T$ is as in (8.1), we show here the iterations $P_{7n}$ for $n \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The parameter values are $\beta = 0.08$, $\sigma = 0.3$ and $m = 1$. The initial condition is $P_0(r) = \frac{\sqrt{\beta}\, r}{3}$ for $r \in [0, 3]$ and $= \sqrt{\beta}$ for $r \geq 3$, and is in blue. The iterate $P_{63}$ is in solid red and the dashed horizontal redline is at height $\sqrt{\beta} = \sqrt{0.08}$. The residual ratio $\max_{r \in [0,20]} |P_{64}(r) - P_{63}(r)|/\sqrt{\beta}$ is less than $5 \times 10^{-5}$. The ordering of the curves is such that the "steep intial part" of the curve $y = P_{7(n+1)}(r)$ is to the left of that of $y = P_{7n}(r)$.

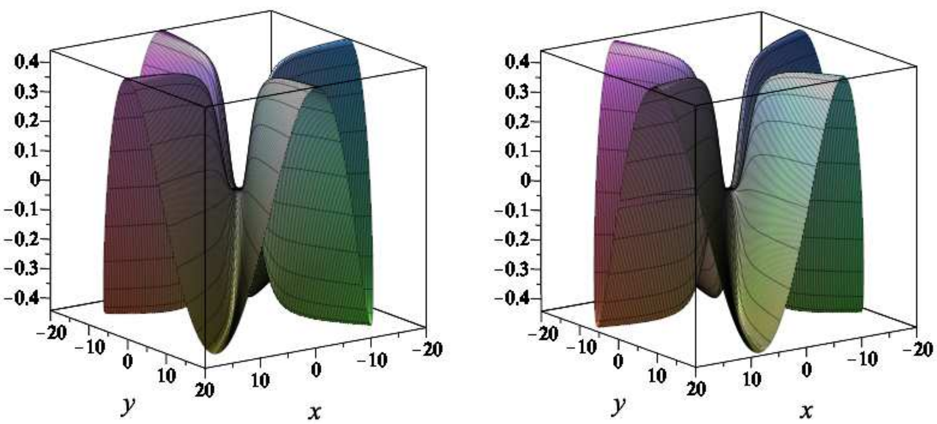


Figure 8: The real and imaginary parts (left panel and right panel respectively) of the function $u(re^{i\varphi}) = P(r)e^{4i\varphi}$, where $P(r)$ is the function $P_{63}(r)$ from figure 6.

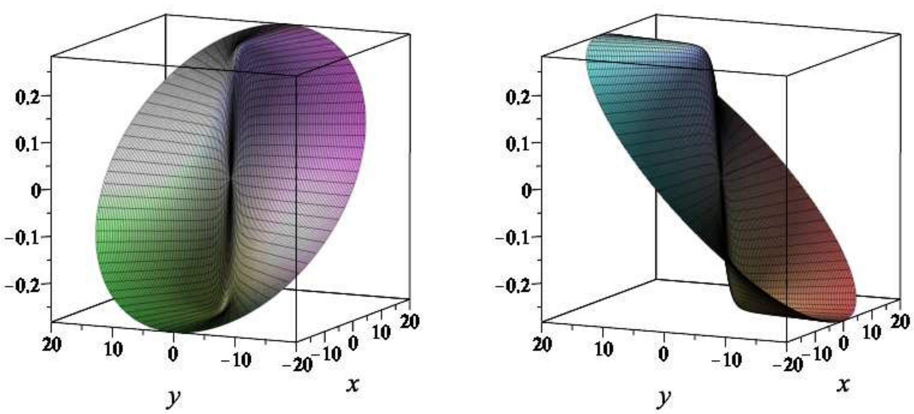


Figure 9: The real and imaginary parts (left panel and right panel respectively) of the function $u(re^{i\varphi}) = P(r)e^{i\varphi}$, where $P(r)$ is the function $P_{63}(r)$ from figure 7.

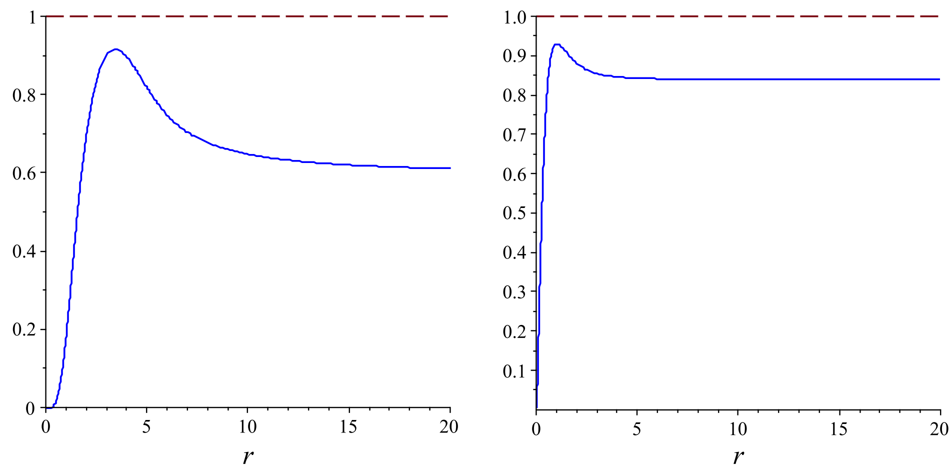


Figure 10: The function $\mathscr{L}_{63}(r)$ in (9.1) for $P_{63}(r)$ as in figure 6 (left panel) and $P_{63}(r)$ as in figure 7 (right panel).

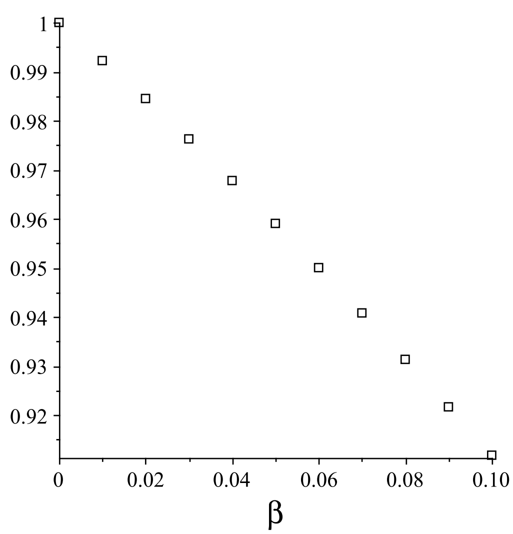


Figure 11: Graph of the maximum of the function $\Lambda(r)$ in (9.2) for values of $\beta = 0.01$ to $0.1$ in increments of $0.01$. The parameter values of the distribution are $m = 1$ and $\sigma = 0.5$.

# 10 Concluding remarks

We have established the existence of rotating waves in a class of $\mathbf{SO}(2)$-equivariant discrete time dynamical system defined by an integro-difference equation, with special attention focussed on the important case of a Gaussian kernel. In physical space, the solutions have the form of a pinwheel (see for example the right panel of figure 2 and figures 8-9). In this case, the analysis essentially reduced to proving the existence of a pair $(P(r), \alpha)$ which solves the system (2.8)-(2.9):

$$\begin{aligned} \int_0^\infty \rho A_m(r,\rho) g(P(\rho),\beta)\, d\rho &= P(r) \\ (m-\delta)\alpha + \omega &= 0 \pmod{2\pi}, \quad \delta \in \{0,1\}. \end{aligned} \tag{10.1}$$

In contrast, spiral wave solutions to (2.1), for example of the ansatz form $u^*(re^{i\varphi}) = P(r)e^{i(r+m\varphi)}$, would need to satisfy for $(P(r), \alpha)$ the system

$$\begin{aligned} \int_0^\infty \rho A_m(r,\rho) \cos(\rho - r + \omega + (m-\delta)\alpha) g(P(\rho),\beta)\, d\rho &= P(r) \\ \int_0^\infty \rho A_m(r,\rho) \sin(\rho - r + \omega + (m-\delta)\alpha) g(P(\rho),\beta)\, d\rho &= 0, \end{aligned}$$

which has similarities with (10.1), but is sufficiently different (for example, the kernels oscillate and change sign) that our techniques here don't immediately apply. This is currently being investigated.

We wish to make a few comments on our choice (1.10) for the local dynamics in the integro-difference equation, which has the expression (1.11) in polar coordinates. As previously noted, the dynamics of this map are a combination of expansion (or contraction) in the $\eta$-direction, coupled with a uniform (in $\eta$) rotation of angle $\omega$ around the origin. However, we note that the general form of the cubic truncation would be (after rescaling and in complex notation)

$$u \longmapsto e^{i\omega}(1+\beta-(1+id_2)|u|^2)u\,, \tag{10.2}$$

where the coefficient $d_2$ is not zero in general and the real part of the cubic coefficient has been rescaled to 1. In polar coordinates, mapping (10.2) becomes [21]

$$\begin{pmatrix} \eta \\ \psi \end{pmatrix} \longmapsto \begin{pmatrix} \eta(1+\beta-\eta^2) + O(\eta^4) \\ \psi + \omega - d_2\eta^2 + O(\eta^3) \end{pmatrix}$$

and should be compared to (1.11). In particular, although the dynamics consist of rigid rotations around the origin, the angle of rotation is now dependent on $\eta$, and following the approach in this paper, the relevant equations to solve (instead of (10.1)) would be

$$
\begin{aligned}
\int_0^\infty \rho A_m(r,\rho)(C_1(1+\beta)P(\rho)+(-C_1+d_2S_1)P(\rho)^3)\,d\rho &= P(r) \\
\int_0^\infty \rho A_m(r,\rho)(S_1(1+\beta)P(\rho)+(-S_1-d_2C_1)P(\rho)^3),d\rho &= 0,
\end{aligned}
\tag{10.3}
$$

where $C_1=\cos(\omega+(m-\delta)\alpha)$ and $S_1=\sin(\omega+(m-\delta)\alpha)$, which reduce to (10.1) when $d_2=0$. Although clearly more complicated structurally than (10.1), we believe that our results in this paper will be critical to a full analysis of (10.3). Another important remark is that we have chosen the local dynamics to possess the necessary symmetry with respect to the twisted action (1.4), which consequently imposed the algebraic restrictions (1.9). It is clear from our analysis that these algebraic restrictions were fully exploited and immensely simplified the subsequent analysis. Such simplified computations would not have been possible for a *general* form of $f$ (i.e. different than (1.9)). The functional form (1.9) is not required for equivariance in the case of the untwisted action. Therefore, we believe that a full analysis of the untwisted case would be considerably more difficult, although we are confident that our analysis here would prove beneficial.

Another possible generalization to our analysis would be to allow the kernel $\kappa(x,y)$ in (1.1) to be a $2\times 2$ matrix of functions

$$
\kappa(x,y)=\begin{pmatrix} k_1(|x-y|) & -k_2(|x-y|) \\ k_2(|x-y|) & k_1(|x-y|) \end{pmatrix},
$$

which would also lead to the $\mathbf{SO}(2)$ symmetry (1.6). While of great potential interest, this falls outside the scope of this paper and will be the subject of future investigation.

The techniques used in this paper would also prove the existence of rotating pinwheel solutions to integro-differential equations of the form

$$
\frac{\partial u}{\partial t}(x,t)=-(1+i\omega)u(x,t)+\iint_{\mathbb{R}^2} k(|x-y|)(1+\beta-|u(y,t)|^2)u(y,t)\,dy_1dy_2. \tag{10.4}
$$

Indeed, using polar coordinates, a simple computation shows that (10.4) has rotating wave solutions of the form $u(re^{i\varphi},t)=P(r)e^{im(\varphi+\alpha t)}$ if $(P(r),\alpha)$ satisfy (10.1) with $m\alpha+\omega=0$.

Whereas we have determined the existence of rotating wave solutions in invariant spaces (corresponding to Fourier modes), and characterized their stability within these invariant spaces (in the case of a Gaussian kernel), we have not investigated stability properties with respect to perturbations which are not in the form of a pure Fourier mode, nor have we investigated existence of rotating wave solutions outside of these invariant subspaces. While these are important questions, they fall outside the scope of this paper.

Finally, we note that while we have used the origin of the physical two-dimensional space as the center of rotation for the rotating waves, the translation equivariance of our system implies that any point can be made to correspond to the center of rotation, therefore there exist rotating wave solutions rotating about any point in physical space. Initial conditions would determine which is observed.

# 11 Compliance with Ethical Standards and Data Availability

This work has been partly funded by the Natural Sciences and Engineering Research Council of Canada in the form of a Discovery Grant to the author. The author declares that there are no potential or actual conflicts of interest, and the research did not involve any Human or Animal participants.

The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

# A Proof that Hypothesis 3.1 (iv) is satisfied by the Gaussian distribution

We start with the computation

$$\rho\frac{\partial A_m}{\partial r}(r,\rho) = \frac{\rho}{r\sigma^4}e^{-\frac{r^2+\rho^2}{2\sigma^2}}\left(r\rho\,\mathcal{I}_{m-1}\left(\frac{r\rho}{\sigma^2}\right) - (r^2+m\sigma^2)\,\mathcal{I}_m\left(\frac{r\rho}{\sigma^2}\right)\right) \tag{A.1}$$

which we rewrite using the identity

$$\mathcal{I}_{m+1}(t) = \mathcal{I}_{m-1}(t) - \frac{2m}{t}\mathcal{I}_m(t)$$

as

$$\rho\frac{\partial A_m}{\partial r}(r,\rho) = g_1(r,\rho) + g_2(r,\rho) + g_3(r,\rho) \tag{A.2}$$

where

$$\begin{aligned}
g_1(r,\rho) &= -\frac{r\rho}{\sigma^4}e^{-\frac{r^2+\rho^2}{2\sigma^2}}\mathcal{I}_m\left(\frac{r\rho}{\sigma^2}\right)\\
g_2(r,\rho) &= \frac{\rho^2}{\sigma^4}e^{-\frac{r^2+\rho^2}{2\sigma^2}}\mathcal{I}_{m+1}\left(\frac{r\rho}{\sigma^2}\right)\\
g_3(r,\rho) &= \frac{m\rho}{r\sigma^2}e^{-\frac{r^2+\rho^2}{2\sigma^2}}\mathcal{I}_m\left(\frac{r\rho}{\sigma^2}\right).
\end{aligned}$$

Since $\mathcal{I}_m(t) \sim t^m$ for small $t$, we have that (uniformly in $\rho \geq 0$) the dominant term in (A.2) for small $r > 0$ is the term $g_3$ which is of order $r^{m-1}$. Since $g_3$ is positive, we have

**Lemma A.1** *There exists an $R > 0$ such that $\rho \dfrac{\partial A_m}{\partial r}(r,\rho) \geq 0$ for all $r \in [0, R]$ and for all $\rho \geq 0$.*

Consequently, if we write

$$\mathscr{G}(r) \equiv \int_0^\infty \rho \left| \frac{\partial A_m}{\partial r}(r,\rho) \right| d\rho,$$

then for all $r \in [0, R]$ we have

$$\mathscr{G}(r) = \int_0^\infty \rho \frac{\partial A_m}{\partial r}(r,\rho)\, d\rho = M'(r) \leq K_1(m,R) < \infty$$

where $M(r)$ is as in (4.10) and $M'(r)$ in (4.11) is bounded by some positive number $K_1(m,R)$.

For $r > R$, we write

$$\begin{aligned} \rho \left| \frac{\partial A_m}{\partial r}(r,\rho) \right| &= |g_1(r,\rho) + g_2(r,\rho) + g_3(r,\rho)| \leq |g_1(r,\rho) + g_2(r,\rho)| + |g_3(r,\rho)| \\ &= |g_1(r,\rho) + g_2(r,\rho)| + g_3(r,\rho). \end{aligned}$$

We easily compute

$$\int_0^\infty g_3(r,\rho)\, d\rho = \frac{\sqrt{2\pi} m}{4\sigma}\, e^{-\frac{r^2}{4\sigma^2}} \left( \mathcal{I}_{\frac{m-1}{2}}\left(\frac{r^2}{4\sigma^2}\right) + \mathcal{I}_{\frac{m+1}{2}}\left(\frac{r^2}{4\sigma^2}\right) \right)$$

which is bounded above for $r \geq 0$ by some positive constant $K_2(m)$ because of the asymptotic formula (4.9). Therefore, it remains to show that

$$\int_0^\infty |g_1(r,\rho) + g_2(r,\rho)|\, d\rho = \int_0^\infty \left| \frac{\rho^2}{\sigma^4} e^{-\frac{r^2+\rho^2}{2\sigma^2}} \mathcal{I}_{m+1}\left(\frac{r\rho}{\sigma^2}\right) - \frac{r\rho}{\sigma^4} e^{-\frac{r^2+\rho^2}{2\sigma^2}} \mathcal{I}_m\left(\frac{r\rho}{\sigma^2}\right) \right| d\rho$$

is bounded on $r \in [R, \infty)$. The asymptotic formula (4.9) will be very useful here. In fact, we have

**Lemma A.2** *Let $m \geq 1$ be an integer. Then for all $t > 0$ we have*

$$\mathcal{I}_m(t) \leq \frac{e^t}{\sqrt{2\pi t}}. \tag{A.3}$$

**Proof** The function $\mathscr{F}(t) \equiv \dfrac{e^t}{\sqrt{2\pi t}\,\mathcal{I}_m(t)}$ is positive for $t > 0$, has a vertical asymptote at $t = 0$, and $\lim_{t\to\infty} \mathscr{F}(t) = 1$ (from (4.9)). We compute the derivative, which after some simplification gives

$$\mathscr{F}'(t) = \frac{\sqrt{2}e^t}{2t^{3/2}\sqrt{\pi}\,\mathcal{I}_m(t)}\left(\left(t + m - \frac{1}{2}\right) - t\frac{\mathcal{I}_{m-1}(t)}{\mathcal{I}_m(t)}\right) \tag{A.4}$$

and claim that $\mathscr{F}'(t) < 0$ for all $t > 0$, from which it follows that $\mathscr{F}(t) \geq 1$ for all $t > 0$ from the above asymptotic behaviour of $\mathscr{F}$ as $t \to 0^+$ and as $t \to \infty$. To prove the claim, we use formula (10) from [35] :

$$t\frac{\mathcal{I}_\nu(t)}{\mathcal{I}_{\nu+1}(t)} > \frac{2\nu+1}{2} + \frac{\sqrt{(2\nu+1)^2 + 4(t^2+\nu+\frac{1}{2})}}{2}$$

which for $\nu = m - 1$ reduces to

$$t\frac{\mathcal{I}_{m-1}(t)}{\mathcal{I}_m(t)} > m - \frac{1}{2} + \frac{1}{2}\sqrt{4t^2 + 4\left(m - \frac{1}{2}\right) + (2m-1)^2} > m - \frac{1}{2} + \frac{1}{2}\cdot 2t = m + t - \frac{1}{2}. \tag{A.5}$$

So from (A.4) we conclude that $\mathscr{F}'(t) < 0$ and the lemma is proved. ∎

Now, we write for $r \in [R, \infty)$

$$\begin{aligned}
|g_1(r,\rho) + g_2(r,\rho)| &= \frac{\rho e^{-\frac{r^2+\rho^2}{2\sigma^2}}}{\sigma^4}\left|\rho\mathcal{I}_{m+1}\left(\frac{r\rho}{\sigma^2}\right) - r\mathcal{I}_m\left(\frac{r\rho}{\sigma^2}\right)\right| \\
&\leq \frac{\rho e^{-\frac{r^2+\rho^2}{2\sigma^2}}}{\sigma^4}\left(|\rho - r|\,\mathcal{I}_{m+1}\left(\frac{r\rho}{\sigma^2}\right) + r\left|\mathcal{I}_{m+1}\left(\frac{r\rho}{\sigma^2}\right) - \mathcal{I}_m\left(\frac{r\rho}{\sigma^2}\right)\right|\right) \\
&= \frac{\rho e^{-\frac{r^2+\rho^2}{2\sigma^2}}}{\sigma^4}\left(|\rho - r|\,\mathcal{I}_{m+1}\left(\frac{r\rho}{\sigma^2}\right) + r\left(\mathcal{I}_m\left(\frac{r\rho}{\sigma^2}\right) - \mathcal{I}_{m+1}\left(\frac{r\rho}{\sigma^2}\right)\right)\right),
\end{aligned}$$

where we have used (4.12).

We note that

$$0 \leq \mathcal{A}(r) \equiv \int_0^\infty \frac{\rho\, r\, e^{-\frac{r^2+\rho^2}{2\sigma^2}}}{\sigma^4}\left(\mathcal{I}_m\left(\frac{r\rho}{\sigma^2}\right) - \mathcal{I}_{m+1}\left(\frac{r\rho}{\sigma^2}\right)\right) d\rho = \frac{\sqrt{2\pi}\, r^2\, e^{-\frac{r^2}{4\sigma^2}}}{4\sigma^3}\,\mathcal{E}(r)$$

where

$$\mathcal{E}(r) = \mathcal{I}_{\frac{m-1}{2}}\left(\frac{r^2}{4\sigma^2}\right) + \mathcal{I}_{\frac{m+1}{2}}\left(\frac{r^2}{4\sigma^2}\right) - \mathcal{I}_{\frac{m}{2}}\left(\frac{r^2}{4\sigma^2}\right) - \mathcal{I}_{\frac{m+2}{2}}\left(\frac{r^2}{4\sigma^2}\right).$$

We conclude that $\mathcal{A}(r)$ is bounded on $[R,\infty)$ since $\mathcal{A}(0)=0$ and (using (4.9))

$$\lim_{r\to\infty}\mathcal{A}(r)=0.$$

So our final step is to prove the boundedness on $[R,\infty)$ of the term

$$\mathcal{B}(r)\equiv\int_0^\infty\frac{\rho\,e^{-\frac{r^2+\rho^2}{2\sigma^2}}}{\sigma^4}\,|\rho-r|\,\mathcal{I}_{m+1}\left(\frac{r\rho}{\sigma^2}\right)\,d\rho. \tag{A.6}$$

It follows from Lemma A.2 that

$$\begin{aligned}\frac{\rho\,e^{-\frac{r^2+\rho^2}{2\sigma^2}}}{\sigma^4}\,|\rho-r|\,\mathcal{I}_{m+1}\left(\frac{r\rho}{\sigma^2}\right) &\le \frac{\rho}{\sqrt{2\pi}\,\sigma^3}\,\frac{|\rho-r|}{\sqrt{r\rho}}\,e^{-\frac{(r-\rho)^2}{2\sigma^2}}\le\frac{1}{\sqrt{2\pi}\,\sigma^3}\,\frac{\sqrt{|\rho-r|+r}}{\sqrt{r}}\,|\rho-r|\,e^{-\frac{(r-\rho)^2}{2\sigma^2}}\\ &\le \frac{1}{\sqrt{2\pi R}\,\sigma^3}\,|\rho-r|^{3/2}\,e^{-\frac{(r-\rho)^2}{2\sigma^2}}+\frac{1}{\sqrt{2\pi}\,\sigma^3}\,|\rho-r|\,e^{-\frac{(r-\rho)^2}{2\sigma^2}}\ ,\end{aligned}$$

and so from (A.6) we have

$$\mathcal{B}(r)\le\int_0^\infty\frac{1}{\sqrt{2\pi R}\,\sigma^3}\,|\rho-r|^{3/2}\,e^{-\frac{(r-\rho)^2}{2\sigma^2}}\,d\rho+\int_0^\infty\frac{1}{\sqrt{2\pi}\,\sigma^3}\,|\rho-r|\,e^{-\frac{(r-\rho)^2}{2\sigma^2}}\,d\rho. \tag{A.7}$$

So our final step will be to show the boundedness (in $r\in[R,\infty)$) of both integrals in (A.7).

We note that for every $t\ge 0$, we can show using simple calculus arguments that $2\,e^{t^2/2}\ge 2^{3/4}\,t^{3/2}$ from which it follows (using $t\to\frac{t}{\sqrt{2}\,\sigma}$) that

$$\frac{t^{3/2}}{\sigma^3}\,e^{-\frac{t^2}{2\sigma^2}}\le\frac{2}{\sigma^{3/2}}\,e^{-\frac{t^2}{4\sigma^2}}\ ,$$

so for the first term in (A.7) we may write

$$\begin{aligned}\int_0^\infty\frac{1}{\sqrt{2\pi R}\,\sigma^3}\,|\rho-r|^{3/2}\,e^{-\frac{(r-\rho)^2}{2\sigma^2}}\,d\rho &\le \frac{2}{\sqrt{2\pi R}\,\sigma^{\frac{3}{2}}}\int_0^\infty e^{-\frac{(r-\rho)^2}{4\sigma^2}}\,d\rho\\ &\le \frac{2}{\sqrt{2\pi R}\,\sigma^{\frac{3}{2}}}\int_{-\infty}^\infty e^{-\frac{(r-\rho)^2}{4\sigma^2}}\,d\rho=\frac{2\sqrt{2}}{\sqrt{R\,\sigma}}.\end{aligned}$$

A similar argument is used to show the boundedness of the second integral in (A.7) using the relation (for $t\ge 0$)

$$\sqrt{2}\,t\le 2\,e^{t^2/2}\Longrightarrow\frac{t\,e^{-\frac{t^2}{2\sigma^2}}}{\sigma}\le 2\,e^{-\frac{t^2}{4\sigma^2}}$$

which leads to

$$\int_0^\infty \frac{1}{\sqrt{2\pi}\,\sigma^3}\,|\rho - r|\,e^{-\frac{(r-\rho)^2}{2\sigma^2}}\,d\rho \leq \frac{2}{\sqrt{2\pi}\,\sigma^2}\int_0^\infty e^{-\frac{(r-\rho)^2}{4\sigma^2}}\,d\rho \leq \frac{2}{\sqrt{2\pi}\,\sigma^2}\int_{-\infty}^\infty e^{-\frac{(r-\rho)^2}{4\sigma^2}}\,d\rho = \frac{2\sqrt{2}}{\sigma}$$

Thus we conclude that the function $\mathcal{B}(r)$ in (A.6) is bounded on $r \in [R, \infty)$, and thus Hypothesis 3.1 (iv) is satisfied by the Gaussian.

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