

A geometrically nonlinear Cosserat (micropolar) curvy shell model via Gamma convergence

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Abstract

Using Γ -convergence arguments, we construct a nonlinear membrane-like Cosserat shell model on a curvy reference configuration starting from a geometrically nonlinear, physically linear three-dimensional isotropic Cosserat model. Even if the theory is of order $O(h)$ in the shell thickness h , by comparison to the membrane shell models proposed in classical nonlinear elasticity, beside the change of metric, the membrane-like Cosserat shell model is still capable to capture the transverse shear deformation and the Cosserat-curvature due to remaining Cosserat effects. We formulate the limit problem by scaling both unknowns, the deformation and the microrotation tensor, and by expressing the parental three-dimensional Cosserat energy with respect to a fictitious flat configuration. The model obtained via Γ -convergence is similar to the membrane (no $O(h^3)$ flexural terms, but still depending on the Cosserat-curvature) Cosserat shell model derived via a derivation approach but these two models do not coincide. Comparisons to other shell models are also included.

Key words: dimensional reduction, curved reference configuration, membrane shell model, Gamma-convergence, nonlinear scaling, microrotations, Cosserat theory, Cosserat shell, micropolar shell, generalized continua, multiplicative split.

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1 Introduction

If a three-dimensional elastic body is very thin in one direction, it has special load-bearing capacities. Due to the geometry, it is always tempting to try to come up with simplified equations for this situation. The ensuing theory is subsumed under the name shell theory. We speak of a flat shell problem if the reference configuration is flat, i.e., the undeformed configuration is given by $\Omega_h = \omega \times [-\frac{h}{2}, \frac{h}{2}]$, with $\omega \subset \mathbb{R}^2$ and $h \ll 1$, and of a shell (or curvy shell) if the reference configuration is curvy, in the sense that the undeformed configuration is given by $\Omega_\xi = \Theta(\Omega_h)$, with Θ a C^1 -diffeomorphism $\Theta: \mathbb{R}^3 \rightarrow \mathbb{R}^3$.

There are many different ways to mathematically describe the response of shells and of obtaining two-dimensional field equations. One method is called the *derivation approach*. The idea of this method is reducing the dimension of a given 3 dimensional model to 2 dimensions through physically reasonable constitution assumptions on the kinematics [46]. The last author has introduced this derivation procedure based on the geometrically nonlinear Cosserat model in his habilitation thesis [50, 51]. The other approach is the *intrinsic approach* which from the beginning views the shell as a two-dimensional surface and refers to methods from differential geometry [3, 5, 44]. The *asymptotic method* seeks, by using the formal expansion of the three-dimensional solution in power series in terms of a small thickness parameter to establish two-dimensional equations. Moreover, the *direct approach* [43] assumes that the shell is a two-dimensional medium which has additional extrinsic directors in the concept of a restricted Cosserat surface ([11, 14, 15, 21, 22, 23, 26, 29, 44, 59]). Of course, the intrinsic approach is related to the direct approach. More information regarding to this method can be found in [50, 52, 53, 54].

One of the most famous shell theories is the *Reissner-Mindlin membrane-bending model* which is an extension of the *Kirchhoff-Love* membrane-bending model [10] (the Koiter model [9]). The kinematic assumptions in this theory are that straight lines normal to the reference mid-surface remain straight and normal to the mid-surface after deformation. The Reissner-Mindlin theory can be applied for thick plates and it does not require the cross-section to be perpendicular to the axial axes after deformation, i.e. it includes transverse shear. A serious

drawback of both these theories is that a geometrically nonlinear, physically linear membrane-bending model is typically not well-posed ([42]) and needs specific modifications [8, 9] to re-establish well-posedness.

There is another powerful tool that one can use to perform the dimensional reduction namely Γ -convergence. In this case, a given 3D model is dimensionally reduced via physically reasonable assumptions on the scaling of the energy.

In this regard, one of the first advances in finite elasticity was the derivation of a nonlinear membrane model (energy scaling with h) which is given in [47]. After that, the idea of Γ -convergence was developed in [30, 31, 32, 33], where different scalings on the applied forces are considered, see also [16, 60].

A notorious property of the Γ -limit model based on classical elasticity is its de-coupling of the limit into either a membrane-like (scaling with h) or bending-like problem (scaling with h^3), see e.g. [12, 45].

In this paper we will use the idea of Γ -convergence to deduce our two-dimensional curvy shell model from a 3-dimensional geometrically nonlinear Cosserat model ([56]). This work is a challenging extension of the Cosserat membrane Γ -limit for flat shells, which was previously obtained by Neff and Chelminski in [55], to the situation of shells with initial curvature.

The Cosserat model was introduced in 1909 by the Cosserat brothers [24, 25, 26]. They imposed a principle of least action, combining the classical deformation $\varphi: \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and an independent triad of orthogonal directors, the microrotation $\bar{R}: \Omega \subset \mathbb{R}^3 \rightarrow \text{SO}(3)$. Invariance of the energy under superposed rigid body motions (left-invariance under $\text{SO}(3)$) allowed them to conclude the suitable form of the energy as $W = W(\bar{R}^T D\varphi, \bar{R}^T \partial_{x_1} \bar{R}, \bar{R}^T \partial_{x_2} \bar{R}, \bar{R}^T \partial_{x_3} \bar{R})$. The balance of force equation appears by taking variations w.r.t φ and balance of angular momentum follows from taking variations of $\bar{R} \in \text{SO}(3)$. Here, as additional structural assumption we will assume material isotropy, i.e., right-invariance of the energy under $\text{SO}(3)$. In addition we will only consider a physically linear version of the model (quadratic energy in suitable strains) which allows a complete and definite representation of the energy, see eq. (3.5).

In the geometric description of shells the normal to the midsurface and the tangent plane appear naturally and the Darboux-Frenet-frame can be used. The underlying Cosserat model immediately generalizes this concept in that the additional microrotation field R can replace the Darboux-Frenet frame. The third column of the microrotation matrix R generalizes the normal in a Kirchhoff-Love model and the director in a Reissner-Mindlin model. Note that the Cosserat model allows for global minimizers [51].

Concerning now the thin shell Γ -limit, we choose the nonlinear scaling and concentrate on a $O(h)$ -model, i.e. the membrane response. Since, however, the 3-D Cosserat model already features curvature terms (derivatives of the microrotations), these terms "survive" the Γ -limit procedure and scale with h , while dedicated bending-like terms scaling with h^3 do not appear¹.

The major difficulty compared to the flat shell Γ -limit in [55] is therefore the incorporation of the curved reference configuration. This problem is solved by introducing a multiplicative decomposition of the appearing fields into elastic and (compatible) permanent parts. The permanent parts encode the geometry of the curved surface given by Θ . In this way, we are able to avoid completely the use of the intrinsic geometry of the curved shell.

The related Cosserat shell model in [35, 36] is obtained by the derivation approach. There, the 2-dimensional model depends on the deformation of the midsurface $m: \omega \rightarrow \mathbb{R}^3$ and the microrotation of the shell $\bar{Q}_{e,s}: \omega \rightarrow \text{SO}(3)$ for $\omega \subset \mathbb{R}^2$, the same as here. The resulting reduced energy contains a membrane part, membrane-bending part and bending-curvature part, while the Cosserat Γ -limit model obtained in this paper contains only the membrane energy and the curvature energy separately. The membrane part is a combination of the shell energy and transverse shear energy and the curvature part includes the 2-dimensional Cosserat-curvature energy of the shell.

The present paper consists of 6 sections. After some notations in Section 2, we start by introducing the three dimensional isotropic nonlinear Cosserat model on the curved reference configuration Ω_ξ formulated in terms of the deformation φ_ξ and microrotation \bar{R}_ξ . Then we transfer the problem to a variational problem defined on the fictitious flat configuration Ω_h . For this goal, the diffeomorphism $\Theta: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ will help us to transfer the deformation from Ω_h to Ω_c (the deformed configuration), Θ encodes the geometry of the curved reference configuration. For applying Γ -convergence arguments we need to transform our problem from Ω_h to a domain with fixed thickness Ω_1 . This action depends on the type of scaling of the variables, which is introduced in

¹Observe that the surviving Cosserat curvature is not related to the change of curvature tensor, which measures the change of mean curvature and Gauß curvature of the surface, see Acharya [1], Anicic and Legér [10] as well as the recent work by Silhavy [61] and [37, 38, 39, 41]).

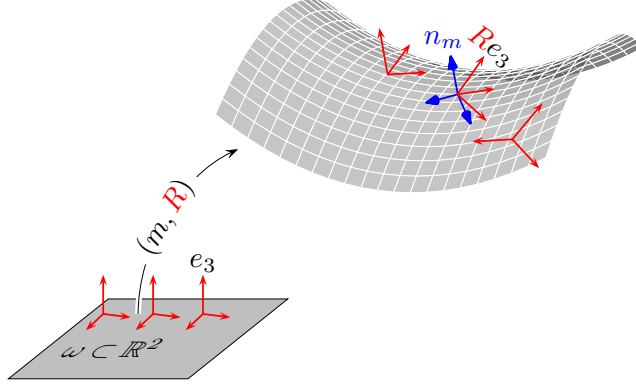


Figure 1: The mapping $m: \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ describes the deformation of a flat midsurface $\omega \subset \mathbb{R}^2$. The Frenet-Darboux frame (in blue, trièdre caché) is tangent to the midsurface m . The independent orthogonal frame mapped by $R \in \text{SO}(3)$ is the trièdre mobile (in red, not necessary tangent to the midsurface). Both fields m and R are coupled in the variational problem. This picture describes the situation of a flat Cosserat shell.

Section 4. Next, we propose the admissible sets on which the Γ -convergence will be studied. We also obtain the family of functionals which are depending on the thickness h . From Section 6 on, we start to discuss the construction of the Γ -limit for the family of functionals I_h . After lengthy calculations, in Subsections 6.2 and 6.3 we get the homogenized membrane and curvature energies. The main result of this work is presented in Section 7, where we prove Theorem (7.1) on the Γ -limit. In Section 8 we extend the Γ -limit theorem to the situation when external loads are present. Finally, in Section 9, we compare our model with other models: a Cosserat flat shell model obtained via Γ -convergence, a Cosserat shell model obtained via the derivation approach, a 6-parameter shell model, a Cosserat shell model up to $O(h^5)$, the Reissner-Mindlin membrane bending model and Aganovic and Neff's model.

2 Notation

Let $a, b \in \mathbb{R}^3$. We denote the scalar product on \mathbb{R}^3 with $\langle a, b \rangle_{\mathbb{R}^3}$ and the associated vector norm with $\|a\|_{\mathbb{R}^3}^2 = \langle a, a \rangle_{\mathbb{R}^3}$. The set of real-valued 3×3 second order tensors is denoted by $\mathbb{R}^{3 \times 3}$, where the elements are shown in capital letters. The standard Euclidean scalar product on $\mathbb{R}^{3 \times 3}$ is given by $\langle X, Y \rangle_{\mathbb{R}^{3 \times 3}} = \text{tr}(XY^T)$, and the associated norm is $\|X\|^2 = \langle X, X \rangle_{\mathbb{R}^{3 \times 3}}$. If $\mathbb{1}_3$ denotes the identity matrix in $\mathbb{R}^{3 \times 3}$, then we have $\text{tr}(X) = \langle X, \mathbb{1}_3 \rangle$. For an arbitrary matrix $X \in \mathbb{R}^{3 \times 3}$, we define $\text{sym}(X) = \frac{1}{2}(X + X^T)$ and $\text{skew}(X) = \frac{1}{2}(X - X^T)$ as the symmetric and skew-symmetric parts, respectively and the deviatoric part is defined as $\text{dev } X = X - \frac{1}{n} \text{tr}(X) \mathbb{1}_n$, for all $X \in \mathbb{R}^{n \times n}$. We let $\text{Sym}(n)$ and $\text{Sym}^+(n)$ denote the symmetric and positive definite symmetric tensors, respectively. We consider the decomposition $X = \text{sym}(X) + \text{skew}(X)$ and the spaces

$$\begin{aligned} \text{GL}(3) &:= \{X \in \mathbb{R}^{3 \times 3} \mid \det X \neq 0\}, & \text{GL}^+(3) &:= \{X \in \mathbb{R}^{3 \times 3} \mid \det X > 0\}, \\ \text{SO}(3) &:= \{X \in \mathbb{R}^{3 \times 3} \mid X^T X = \mathbb{1}_3, \det X = 1\}, & \mathfrak{so}(3) &:= \{A \in \mathbb{R}^{3 \times 3} \mid A^T = -A\}, \\ \mathfrak{sl}(n) &:= \{X \in \mathbb{R}^{n \times n} \mid \text{tr}(X) = 0\}, & \text{O}(3) &:= \{X \in \text{GL}(3) \mid X^T X = \mathbb{1}_3\}. \end{aligned}$$

The *canonical identification* of $\mathfrak{so}(3)$ and \mathbb{R}^3 is denoted by $\text{axl } A \in \mathbb{R}^3$, for $A \in \mathfrak{so}(3)$. We have the following identities

$$\underbrace{\text{axl} \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix}}_{=A} := \begin{pmatrix} -\gamma \\ \beta \\ -\alpha \end{pmatrix}, \quad \|A\|_{\mathbb{R}^{3 \times 3}}^2 = 2 \|\text{axl } A\|_{\mathbb{R}^3}^2. \quad (2.1)$$

We use the *orthogonal Cartan-decomposition of the Lie-algebra* $\mathfrak{gl}(3)$ of all three by three matrices with real components

$$\mathfrak{gl}(3) = \{\mathfrak{sl}(3) \cap \text{Sym}(3)\} \oplus \mathfrak{so}(3) \oplus \mathbb{R} \cdot \mathbf{1}, \quad X = \text{dev sym } X + \text{skew } X + \frac{1}{3} \text{tr}(X) \mathbf{1} \quad \forall X \in \mathfrak{gl}(3). \quad (2.2)$$

A matrix having the three column vectors $A_1, A_2, A_3 \in \mathbb{R}^3$ will be written as $(A_1 | A_2 | A_3)$.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and $\Gamma \subset \partial\Omega$ be a smooth subset of the boundary of Ω . In the two dimensional case, we assume that $\omega \subset \mathbb{R}^2$ with Lipschitz boundary $\partial\omega$ and γ is also a smooth subset of $\partial\omega$.

Assume that $\varphi \in C^1(\Omega, \mathbb{R}^3)$, then for the vector $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ one can write $\nabla_x \varphi = (\partial_{x_1} \varphi | \partial_{x_2} \varphi | \partial_{x_3} \varphi)$. The standard volume element is $dx dy dz = dV = d\omega dz$. The mapping $m: \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is the deformation of the midsurface and $\nabla m := \nabla_{(x_1, x_2)} m$, is its gradient. We may write $m(x_1, x_2) = (x_1, x_2, 0) + v(x_1, x_2)$, where $v: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is the displacement of the midsurface.

For $1 \leq p < \infty$, we consider the *Lebesgue spaces* $L^p(\Omega) = \{f: \Omega \rightarrow \mathbb{R} \mid \|f\|_{L^p(\Omega)} < \infty\}$ and their corresponding norms $\|f\|_{L^p(\Omega)} := \left(\int_{\Omega} |f|^p dx \right)^{\frac{1}{p}}$. For $p \in [1, \infty]$, we define the *Sobolev spaces* $W^{1,p}(\Omega) = \{u \in L^p(\Omega) \mid Du \in L^p(\Omega)\}$, $\|u\|_{W^{1,p}(\Omega)}^p = \|u\|_{L^p(\Omega)}^p + \|Du\|_{L^p(\Omega)}^p$, where Du is the weak derivative of u . In the case $p = 2$, we set $H^1(\Omega) = W^{1,2}(\Omega)$, where $H^1(\Omega) = \{\varphi \in L^2(\Omega) \mid \nabla \varphi \in L^2(\Omega)\}$. For the energy function W we define DW as the Fréchet derivative of W and $D^2W(F) \cdot (H, H)$ denotes the bilinear form of second derivatives.

3 The geometrically nonlinear three dimensional Cosserat model

3.1 The variational problem defined on the thin curved reference configuration

Let us consider an elastic material which in its reference configuration fills the three dimensional *shell-like thin* domain $\Omega_{\xi} \subset \mathbb{R}^3$, i.e., we assume that there exists a C^1 -diffeomorphism $\Theta: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with $\Theta(x_1, x_2, x_3) := (\xi_1, \xi_2, \xi_3)$ such that $\Theta(\Omega_h) = \Omega_{\xi}$ and $\omega_{\xi} = \Theta(\omega \times \{0\})$, where $\Omega_h \subset \mathbb{R}^3$ with $\Omega_h = \omega \times [-\frac{h}{2}, \frac{h}{2}]$, and $\omega \subset \mathbb{R}^2$ a bounded domain with Lipschitz boundary $\partial\omega$. The scalar $0 < h \ll 1$ is called *thickness* of the shell, while the domain Ω_h is called *fictitious flat Cartesian configuration* of the body. We consider the following diffeomorphism $\Theta: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ which is used to describe the curved surface of the shell

$$\Theta(x_1, x_2, x_3) = y_0(x_1, x_2) + x_3 n_0(x_1, x_2), \quad (3.1)$$

where $y_0: \omega \rightarrow \mathbb{R}^3$ is a $C^2(\omega)$ -function and $n_0 = \frac{\partial_{x_1} y_0 \times \partial_{x_2} y_0}{\|\partial_{x_1} y_0 \times \partial_{x_2} y_0\|}$ is the unit normal vector on ω_{ξ} . Remark that

$$\nabla_x \Theta(x_3) = (\nabla y_0 | n_0) + x_3 (\nabla n_0 | 0) \quad \forall x_3 \in \left(-\frac{h}{2}, \frac{h}{2}\right), \quad \nabla_x \Theta(0) = (\nabla y_0 | n_0), \quad [\nabla_x \Theta(0)]^{-T} e_3 = n_0, \quad (3.2)$$

and $\det \nabla_x \Theta(0) = \det(\nabla y_0 | n_0) = \sqrt{\det[(\nabla y_0)^T \nabla y_0]}$ represents the surface element.

In the following we identify the *Weingarten map (or shape operator)* on $y_0(\omega)$ with its associated matrix by $L_{y_0} = I_{y_0}^{-1} \Pi_{y_0}$, where $I_{y_0} := [\nabla y_0]^T \nabla y_0 \in \mathbb{R}^{2 \times 2}$ and $\Pi_{y_0} := -[\nabla y_0]^T \nabla n_0 \in \mathbb{R}^{2 \times 2}$ are the matrix representations of the *first fundamental form (metric)* and the *second fundamental form* of the surface $y_0(\omega)$, respectively. Then, the *Gauß curvature* K of the surface $y_0(\omega)$ is determined by $K = \det L_{y_0}$ and the *mean curvature* H through $2H := \text{tr}(L_{y_0})$. We denote the principal curvatures of the surface by κ_1 and κ_2 .

We note that $\det \nabla \Theta(x_3) = 1 - 2H x_3 + K x_3^2 = (1 - \kappa_1 x_3)(1 - \kappa_2 x_3) > 0$. Therefore, $1 - 2H x_3 + K x_3^2 > 0$, $\forall x_3 \in [-h/2, h/2]$ if and only if $1 > \kappa_1 x_3$ and $1 > \kappa_2 x_3$, for all $x_3 \in [-h/2, h/2]$. These conditions are equivalent with $|\kappa_1| \frac{h}{2} < 1$ and $|\kappa_2| \frac{h}{2} < 1$, i.e., equivalent with

$$h \max\left\{ \sup_{(x_1, x_2) \in \omega} |\kappa_1|, \sup_{(x_1, x_2) \in \omega} |\kappa_2| \right\} < 2. \quad (3.3)$$

We assume that after a deformation process given by the function $\varphi_{\xi}: \Omega_{\xi} \rightarrow \mathbb{R}^3$, the curvy reference configuration Ω_{ξ} is mapped to the deformed configuration $\Omega_c = \varphi_{\xi}(\Omega_{\xi})$.

In the Cosserat theory, each point of the reference body is endowed with three independent orthogonal directors, i.e., with a matrix $\bar{R}_{\xi}: \Omega_{\xi} \rightarrow \text{SO}(3)$ called the *microrotation* tensor. Let us remark that while the

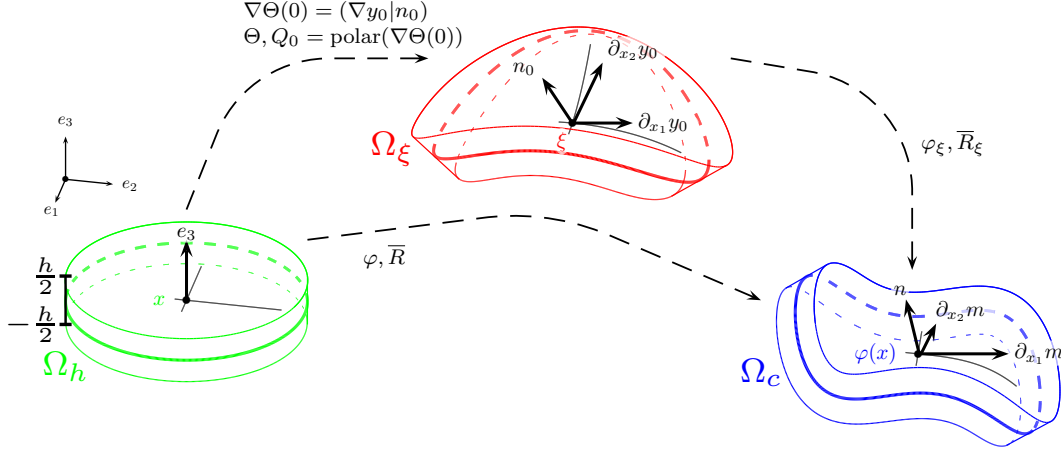


Figure 2: Kinematics of the 3D-Cosserat model. In each point $\xi \in \Omega_\xi$ of the curvy reference configuration, there is the deformation $\varphi_\xi: \Omega_\xi \rightarrow \mathbb{R}^3$ and the microrotation $\bar{R}_\xi: \Omega_\xi \rightarrow \text{SO}(3)$. We introduce a fictitious flat configuration Ω_h and refer all fields to that configuration. This introduces a multiplicative split of the total deformation $\varphi: \Omega_h \rightarrow \mathbb{R}^3$ and total rotation $\bar{R}: \Omega_h \rightarrow \text{SO}(3)$ into “elastic” parts ($\varphi_\xi: \Omega_\xi \rightarrow \mathbb{R}^3$ and $\bar{R}_\xi: \Omega_\xi \rightarrow \text{SO}(3)$) and compatible “plastic” parts (given by $\Theta: \Omega_h \rightarrow \Omega_\xi$ and $Q_0: \Omega_h \rightarrow \text{SO}(3)$). The “intermediate” configuration Ω_ξ is compatible by construction.

tensor $\text{polar}(\nabla_\xi \varphi_\xi) \in \text{SO}(3)$ of the polar decomposition of $F_\xi := \nabla_\xi \varphi_\xi = \text{polar}(\nabla_\xi \varphi_\xi) \sqrt{(\nabla_\xi \varphi_\xi)^T \nabla_\xi \varphi_\xi}$ is not independent of φ_ξ , the tensor \bar{R}_ξ in the Cosserat theory is independent of $\nabla \varphi_\xi$. In other words, in general, $\bar{R}_\xi \neq \text{polar}(\nabla_\xi \varphi_\xi)$.

In a geometrical nonlinear and physically linear Cosserat elastic 3D model, the deformation φ_ξ and the microrotation \bar{R}_ξ are the solutions of the following *nonlinear minimization problem* on Ω_ξ :

$$I(\varphi_\xi, F_\xi, \bar{R}_\xi, \alpha_\xi) = \int_{\Omega_\xi} \left[W_{\text{mp}}(\bar{U}_\xi) + W_{\text{curv}}(\alpha_\xi) \right] dV_\xi - \Pi(\varphi_\xi, \bar{R}_\xi) \mapsto \min. \quad \text{w.r.t } (\varphi_\xi, \bar{R}_\xi), \quad (3.4)$$

where

$$\begin{aligned} F_\xi &:= \nabla_\xi \varphi_\xi \in \mathbb{R}^{3 \times 3} && \text{(the deformation gradient),} \\ \bar{U}_\xi &:= \bar{R}_\xi^T F_\xi \in \mathbb{R}^{3 \times 3} && \text{(the non-symmetric Biot-type stretch tensor),} \\ \alpha_\xi &:= \bar{R}_\xi^T \text{Curl}_\xi \bar{R}_\xi \in \mathbb{R}^{3 \times 3} && \text{(the second order dislocation density tensor [49]),} \end{aligned} \quad (3.5)$$

$$W_{\text{mp}}(\bar{U}_\xi) := \mu \|\text{dev sym}(\bar{U}_\xi - \mathbb{1}_3)\|^2 + \mu_c \|\text{skew}(\bar{U}_\xi - \mathbb{1}_3)\|^2 + \frac{\kappa}{2} [\text{tr}(\text{sym}(\bar{U}_\xi - \mathbb{1}_3))]^2 \quad \text{(physically linear),}$$

$$W_{\text{curv}}(\alpha_\xi) := \mu L_c^2 (a_1 \|\text{dev sym} \alpha_\xi\|^2 + a_2 \|\text{skew} \alpha_\xi\|^2 + a_3 [\text{tr}(\alpha_\xi)]^2) \quad \text{(quadratic curvature energy),}$$

and $dV(\xi)$ denotes the volume element in the Ω_ξ -configuration. The total stored energy can be seen by $W = W_{\text{mp}} + W_{\text{curv}}$, with W_{mp} as strain energy and W_{curv} as curvature energy. Clearly, W depends on the deformation gradient $F_\xi = \nabla_\xi \varphi_\xi$ and the microrotation \bar{R}_ξ . The parameters μ and λ are the *Lamé constants* of classical isotropic elasticity, $\kappa = \frac{2\mu + 3\lambda}{3}$ is the *infinitesimal bulk modulus*, $\mu_c > 0$ is the *Cosserat couple modulus* and $L_c > 0$ is the *internal length* and responsible for *size effects* in the sense that smaller samples are relatively stiffer than larger samples. If not stated otherwise, we assume that $\mu > 0$, $\kappa > 0$, $\mu_c > 0$. We also assume that $a_1 > 0$, $a_2 > 0$ and $a_3 > 0$, which assures the *coercivity* and *convexity* of the curvature energy [55].

The *external loading potential* $\Pi(\varphi_\xi, \bar{R}_\xi)$ is given by

$$\Pi(\varphi_\xi, \bar{R}_\xi) = \Pi_f(\varphi_\xi) + \Pi_c(\bar{R}_\xi),$$

where

$$\begin{aligned}\Pi_f(\varphi_\xi) &:= \int_{\Omega_\xi} \langle f, u_\xi \rangle dV_\xi = \text{potential of external applied body forces } f, \\ \Pi_c(\bar{R}_\xi) &:= \int_{\Gamma_\xi} \langle c, \bar{R}_\xi \rangle dS_\xi = \text{potential of external applied boundary couple forces } c,\end{aligned}$$

with $u_\xi = \varphi_\xi - \xi$ the displacement vector. We will assume that the external loads satisfy in regularity condition:

$$f \in L^2(\Omega_\xi, \mathbb{R}^3), \quad c \in L^2(\Gamma_\xi, \mathbb{R}^3), \quad \bar{R}_\xi \in L^2(\Omega_\xi, \mathbb{R}^3). \quad (3.6)$$

For simplicity, we consider only Dirichlet-type boundary conditions on $\Gamma_\xi = \gamma_\xi \times [-\frac{h}{2}, \frac{h}{2}]$, $\gamma_\xi \subset \partial\omega_\xi$, i.e., we assume that $\varphi_\xi = \varphi_\xi^d$ on Γ_ξ , where φ_ξ^d is a given function on Γ_ξ .

In [50] existence of minimizers is shown for positive Cosserat couple modulus $\mu_c > 0$. The case $\mu_c = 0$ can be handled as well with a slight modification of the curvature energy. The form of the curvature energy W_{curv} is not that originally considered in [48]. Indeed, Neff [48] used a curvature energy expressed in terms of the *third order curvature tensor* $\mathfrak{A}_\xi = (\bar{R}_\xi^T \nabla(\bar{R}_\xi \cdot e_1) | \bar{R}_\xi^T \nabla(\bar{R}_\xi \cdot e_2) | \bar{R}_\xi^T \nabla(\bar{R}_\xi \cdot e_3))$. The new form of the energy based on the *second order dislocation density tensor* α_ξ simplifies considerably the representation by allowing to use the orthogonal decomposition

$$\bar{R}_\xi^T \text{Curl}_\xi \bar{R}_\xi = \alpha_\xi = \text{dev sym } \alpha_\xi + \text{skew } \alpha_\xi + \frac{1}{3} \text{tr}(\alpha_\xi) \mathbb{1}_3. \quad (3.7)$$

Moreover, it yields an equivalent control of spatial derivatives of rotations [49] and allows us to write the curvature energy in a fictitious Cartesian configuration in terms of the so-called *wryness tensor* [27, 49]

$$\Gamma_\xi := \left(\text{axl}(\bar{R}_\xi^T \partial_{\xi_1} \bar{R}_\xi) | \text{axl}(\bar{R}_\xi^T \partial_{\xi_2} \bar{R}_\xi) | \text{axl}(\bar{R}_\xi^T \partial_{\xi_3} \bar{R}_\xi) \right) \in \mathbb{R}^{3 \times 3}, \quad (3.8)$$

since (see [49]) the following close relationship between the *wryness tensor* and the *dislocation density tensor* holds

$$\alpha_\xi = -\Gamma_\xi^T + \text{tr}(\Gamma_\xi) \mathbb{1}_3, \quad \text{or equivalently,} \quad \Gamma_\xi = -\alpha_\xi^T + \frac{1}{2} \text{tr}(\alpha_\xi) \mathbb{1}_3. \quad (3.9)$$

For infinitesimal strains this formula is well-known under the name Nye's formula, and $-\Gamma$ is also called Nye's curvature tensor [57]. Our choice of the *second order dislocation density tensor* α_ξ has some further implications, e.g., the coupling between the membrane part, the membrane-bending part, the bending-curvature part and the curvature part of the energy of the shell model is transparent and will coincide with shell-bending curvature tensors elsewhere considered [28].

Within our assumptions on the constitutive coefficients, together with the orthogonal Cartan-decomposition of the Lie-algebra $\mathfrak{gl}(3)$ and with the definition

$$\begin{aligned}W_{\text{mp}}(X) &:= W_{\text{mp}}^\infty(\text{sym } X) + \mu_c \|\text{skew } X\|^2 \quad \forall X \in \mathbb{R}^{3 \times 3}, \\ W_{\text{mp}}^\infty(S) &= \mu \|S\|^2 + \frac{\lambda}{2} [\text{tr}(S)]^2 \quad \forall S \in \text{Sym}(3),\end{aligned} \quad (3.10)$$

it follows that there exist positive constants c_1^+, c_2^+, C_1^+ and C_2^+ such that for all $X \in \mathbb{R}^{3 \times 3}$ the following inequalities hold

$$\begin{aligned}C_1^+ \|S\|^2 &\geq W_{\text{mp}}^\infty(S) \geq c_1^+ \|S\|^2 & \forall S \in \text{Sym}(3), \\ C_1^+ \|\text{sym } X\|^2 + \mu_c \|\text{skew } X\|^2 &\geq W_{\text{mp}}(X) \geq c_1^+ \|\text{sym } X\|^2 + \mu_c \|\text{skew } X\|^2 & \forall X \in \mathbb{R}^{3 \times 3}, \\ C_2^+ \|X\|^2 &\geq W_{\text{curv}}(X) \geq c_2^+ \|X\|^2 & \forall X \in \mathbb{R}^{3 \times 3}.\end{aligned} \quad (3.11)$$

Here, c_1^+ and C_1^+ denote respectively the smallest and the largest eigenvalues of the quadratic form $W_{\text{mp}}^\infty(X)$. Hence, they are independent of μ_c . Both W_{mp} and W_{curv} are quadratic, convex and coercive functions of \bar{U}_ξ and α_ξ , respectively.

The regularity condition of the external loads allows us to conclude that

$$|\Pi_f(\varphi_\xi)| = \left| \int_{\Omega_\xi} \langle f, u_\xi \rangle dV_\xi \right| \leq \|f\|_{L^2(\Omega_\xi)} \|u_\xi\|_{L^2(\Omega_\xi)}, \quad (3.12)$$

which implies that

$$|\Pi_f(\varphi_\xi)| = \left| \int_{\Omega_\xi} \langle f, u_\xi \rangle dV_\xi \right| \leq \|f\|_{L^2(\Omega_\xi)} \|u_\xi\|_{H^1(\Omega_\xi)}. \quad (3.13)$$

Similarly we have

$$|\Pi_c(\overline{R}_\xi)| = \left| \int_{\Gamma_\xi} \langle c, \overline{R}_\xi \rangle dS_\xi \right| \leq \|c\|_{L^2(\Gamma_\xi)} \|\overline{R}_\xi\|_{L^2(\Gamma_\xi)}. \quad (3.14)$$

Note that $\|\overline{R}_\xi\|^2 = 3$. By using the fact that $\|\overline{R}_\xi\|_{L^2(\Gamma_\xi)}^2 = (3 \text{ area } \Gamma_\xi)$, we get

$$|\Pi(\varphi_\xi, \overline{R}_\xi)| \leq \|f\|_{L^2(\Omega_\xi)} \|u_\xi\|_{H^1(\Omega_\xi)} + \|c\|_{L^2(\Gamma_\xi)} (3 \text{ area } \Gamma_\xi)^{\frac{1}{2}}. \quad (3.15)$$

This boundedness will be later used in the subject of Γ -convergence.

3.2 Transformation of the problem from Ω_ξ to the fictitious flat configuration Ω_h

The first step in our shell model is to transform the problem to a variational problem defined on the fictitious flat configuration $\Omega_h = \omega \times \left[-\frac{h}{2}, \frac{h}{2}\right]$. This process is going to be done with the help of the diffeomorphism Θ . To this aim, we define the mapping

$$\varphi: \Omega_h \rightarrow \Omega_c, \quad \varphi(x_1, x_2, x_3) = \varphi_\xi(\Theta(x_1, x_2, x_3)).$$

The function φ maps Ω_h (fictitious flat Cartesian configuration) into Ω_c (deformed current configuration). Moreover, we consider the *elastic microrotation* $\overline{Q}_e: \Omega_h \rightarrow \text{SO}(3)$ similarly defined by

$$\overline{Q}_e(x_1, x_2, x_3) := \overline{R}_\xi(\Theta(x_1, x_2, x_3)), \quad (3.16)$$

and the *elastic Biot-type stretch tensor* $\overline{U}_e: \Omega_h \rightarrow \mathbb{R}^{3 \times 3}$ is then given by

$$\overline{U}_e(x_1, x_2, x_3) := \overline{U}_\xi(\Theta(x_1, x_2, x_3)). \quad (3.17)$$

We also have the polar decomposition $\nabla_x \Theta = Q_0 U_0$, where

$$Q_0 = \text{polar}(\nabla_x \Theta) = \text{polar}([\nabla_x \Theta]^{-T}) \in \text{SO}(3) \quad \text{and} \quad U_0 \in \text{Sym}^+(3). \quad (3.18)$$

Now by using (3.16), we define the *total microrotation* tensor

$$\overline{R}: \Omega_h \rightarrow \text{SO}(3), \quad \overline{R}(x_1, x_2, x_3) = \overline{Q}_e(x_1, x_2, x_3) Q_0(x_1, x_2, x_3). \quad (3.19)$$

By applying the chain rule for φ one obtains

$$\nabla_x \varphi(x_1, x_2, x_3) = \nabla_\xi \varphi_\xi(\Theta(x_1, x_2, x_3)) \nabla_x \Theta(x_1, x_2, x_3), \quad (3.20)$$

or equivalently the *multiplicative decomposition*

$$F_\xi(\Theta(x_1, x_2, x_3)) = F(x_1, x_2, x_3) [\nabla_x \Theta(x_1, x_2, x_3)]^{-1}. \quad (3.21)$$

Finally the *elastic non-symmetric stretch tensor* expressed on Ω_h can now be expressed as

$$\overline{U}_e = \overline{Q}_e^T F[\nabla_x \Theta]^{-1} = Q_0 \overline{R}^T F[\nabla_x \Theta]^{-1}. \quad (3.22)$$

Note that $\partial_{x_k} \bar{Q}_e = \sum_{i=1}^3 \partial_{\xi_i} \bar{R}_\xi \partial_{x_k} \xi_i$, $\partial_{\xi_k} \bar{R}_\xi = \sum_{i=1}^3 \partial_{x_i} \bar{Q}_e \partial_{\xi_k} x_i$ and

$$\begin{aligned} \bar{R}_\xi^T \partial_{\xi_k} \bar{R}_\xi &= \sum_{i=1}^3 (\bar{Q}_e^T \partial_{x_i} \bar{Q}_e) \partial_{\xi_k} x_i = \sum_{i=1}^3 (\bar{Q}_e^T \partial_{x_i} \bar{Q}_e) ([\nabla_x \Theta]^{-1})_{ik}, \\ \text{axl}(\bar{R}_\xi^T \partial_{\xi_k} \bar{R}_\xi) &= \sum_{i=1}^3 \text{axl}(\bar{Q}_e^T \partial_{x_i} \bar{Q}_e) ([\nabla_x \Theta]^{-1})_{ik}. \end{aligned} \quad (3.23)$$

Thus, we have from the chain rule

$$\begin{aligned} \Gamma_\xi &= \left(\sum_{i=1}^3 \text{axl}(\bar{Q}_e^T \partial_{x_i} \bar{Q}_e) ([\nabla_x \Theta]^{-1})_{i1} \mid \sum_{i=1}^3 \text{axl}(\bar{Q}_e^T \partial_{x_i} \bar{Q}_e) ([\nabla_x \Theta]^{-1})_{i2} \mid \sum_{i=1}^3 \text{axl}(\bar{Q}_e^T \partial_{x_i} \bar{Q}_e) ([\nabla_x \Theta]^{-1})_{i3} \right) \\ &= \left(\text{axl}(\bar{Q}_e^T \partial_{x_1} \bar{Q}_e) \mid \text{axl}(\bar{Q}_e^T \partial_{x_2} \bar{Q}_e) \mid \text{axl}(\bar{Q}_e^T \partial_{x_3} \bar{Q}_e) \right) [\nabla_x \Theta]^{-1}. \end{aligned} \quad (3.24)$$

We recall again Nye's formula

$$\alpha_\xi = -\Gamma_\xi^T + \text{tr}(\Gamma_\xi) \mathbb{1}_3, \quad \text{or} \quad \Gamma_\xi = -\alpha_\xi^T + \frac{1}{2} \text{tr}(\alpha_\xi) \mathbb{1}_3. \quad (3.25)$$

Define

$$\Gamma_e := \left(\text{axl}(\bar{Q}_e^T \partial_{x_1} \bar{Q}_e) \mid \text{axl}(\bar{Q}_e^T \partial_{x_2} \bar{Q}_e) \mid \text{axl}(\bar{Q}_e^T \partial_{x_3} \bar{Q}_e) \right), \quad \alpha_e := \bar{Q}_e^T \text{Curl}_x \bar{Q}_e. \quad (3.26)$$

Using Nye's formula for α_e and Γ_e , we deduce (see [35])

$$\begin{aligned} \alpha_\xi &= [\nabla_x \Theta]^{-T} \alpha_e - \frac{1}{2} \text{tr}(\alpha_e) [\nabla_x \Theta]^{-T} - \text{tr}([\nabla_x \Theta]^{-T} \alpha_e) \mathbb{1}_3 + \frac{1}{2} \text{tr}(\alpha_e) \text{tr}([\nabla_x \Theta]^{-1}) \mathbb{1}_3 \\ &= [\nabla_x \Theta]^{-T} \alpha_e - \text{tr}(\alpha_e^T [\nabla_x \Theta]^{-1}) \mathbb{1}_3 - \frac{1}{2} \text{tr}(\alpha_e) \left([\nabla_x \Theta]^{-T} - \text{tr}([\nabla_x \Theta]^{-1}) \mathbb{1}_3 \right). \end{aligned} \quad (3.27)$$

However, we will not use this formula to rewrite the curvature energy in the fictitious Cartesian configuration Ω_h , since it is easier to use (from (3.9))

$$\begin{aligned} \text{sym } \alpha_\xi &= -\text{sym } \Gamma_\xi + \text{tr}(\Gamma_\xi) \mathbb{1}_3 = -\text{sym}(\Gamma_e [\nabla_x \Theta]^{-1}) + \text{tr}(\Gamma_e [\nabla_x \Theta]^{-1}) \mathbb{1}_3, \\ \text{dev sym } \alpha_\xi &= -\text{dev sym } \Gamma_\xi = -\text{dev sym}(\Gamma_e [\nabla_x \Theta]^{-1}), \\ \text{skew } \alpha_\xi &= -\text{skew } \Gamma_\xi = -\text{skew}(\Gamma_e [\nabla_x \Theta]^{-1}), \\ \text{tr}(\alpha_\xi) &= -\text{tr}(\Gamma_\xi) + 3 \text{tr}(\Gamma_\xi) = 2 \text{tr}(\Gamma_\xi) = 2 \text{tr}(\Gamma_e [\nabla_x \Theta]^{-1}), \end{aligned} \quad (3.28)$$

for expressing the curvature energy in terms of $\Gamma_e [\nabla_x \Theta]^{-1}$ as

$$W_{\text{curv}}(\alpha_\xi) = \mu L_c^2 \left(a_1 \|\text{dev sym}(\Gamma_e [\nabla_x \Theta]^{-1})\|^2 + a_2 \|\text{skew}(\Gamma_e [\nabla_x \Theta]^{-1})\|^2 + 4 a_3 [\text{tr}(\Gamma_e [\nabla_x \Theta]^{-1})]^2 \right). \quad (3.29)$$

Note that using

$$\bar{Q}_e^T \partial_{x_i} \bar{Q}_e = Q_0 \bar{R}^T \partial_{x_i} (\bar{R} Q_0^T) = Q_0 (\bar{R}^T \partial_{x_i} \bar{R}) Q_0^T - Q_0 (Q_0^T \partial_{x_i} Q_0) Q_0^T, \quad i = 1, 2, 3, \quad (3.30)$$

and the invariance ([35], relation (3.12))

$$\text{axl}(Q A Q^T) = Q \text{axl}(A) \quad \forall Q \in \text{SO}(3) \quad \text{and} \quad \forall A \in \mathfrak{so}(3), \quad (3.31)$$

we obtain the following form of the wryness tensor defined on Ω_h

$$\begin{aligned} \Gamma(x_1, x_2, x_3) &:= \Gamma_\xi(\Theta(x_1, x_2, x_3)) = \Gamma_e [\nabla_x \Theta]^{-1} \\ &= Q_0 \left[\left(\text{axl}(\bar{R}^T \partial_{x_1} \bar{R}) \mid \text{axl}(\bar{R}^T \partial_{x_2} \bar{R}) \mid \text{axl}(\bar{R}^T \partial_{x_3} \bar{R}) \right) \right. \\ &\quad \left. - \left(\text{axl}(Q_0^T \partial_{x_1} Q_0) \mid \text{axl}(Q_0^T \partial_{x_2} Q_0) \mid \text{axl}(Q_0^T \partial_{x_3} Q_0) \right) \right] [\nabla_x \Theta]^{-1}. \end{aligned} \quad (3.32)$$

Now the minimization problem on the curved reference configuration Ω_ξ is transformed to the fictitious flat Cartesian configuration Ω_h as follows

$$I = \int_{\Omega_h} \left[W_{\text{mp}}(\bar{U}_e) + \widetilde{W}_{\text{curv}}(\Gamma) \right] \det(\nabla_x \Theta) dV - \widetilde{\Pi}(\varphi, \bar{Q}_e) \mapsto \min. \quad \text{w.r.t. } (\varphi, \bar{Q}_e), \quad (3.33)$$

where

$$\begin{aligned} W_{\text{mp}}(\bar{U}_e) &= \mu \|\text{sym}(\bar{U}_e - \mathbb{1}_3)\|^2 + \mu_c \|\text{skew}(\bar{U}_e - \mathbb{1}_3)\|^2 + \frac{\lambda}{2} [\text{tr}(\text{sym}(\bar{U}_e - \mathbb{1}_3))]^2 \\ &= \mu \|\text{dev sym}(\bar{U}_e - \mathbb{1}_3)\|^2 + \mu_c \|\text{skew}(\bar{U}_e - \mathbb{1}_3)\|^2 + \frac{\kappa}{2} [\text{tr}(\text{sym}(\bar{U}_e - \mathbb{1}_3))]^2, \\ \widetilde{W}_{\text{curv}}(\Gamma) &= \mu L_c^2 (a_1 \|\text{dev sym } \Gamma\|^2 + a_2 \|\text{skew } \Gamma\|^2 + 4a_3 [\text{tr}(\Gamma)]^2) \\ &= \mu L_c^2 (b_1 \|\text{sym } \Gamma\|^2 + b_2 \|\text{skew } \Gamma\|^2 + b_3 [\text{tr}(\Gamma)]^2), \end{aligned} \quad (3.34)$$

where $b_1 = a_1, b_2 = a_2, b_3 = \frac{12a_3 - a_1}{3}$ and $\widetilde{\Pi}(\varphi, \bar{Q}_e) = \widetilde{\Pi}_f(\varphi) + \widetilde{\Pi}_c(\bar{Q}_e)$, with the following forms

$$\begin{aligned} \widetilde{\Pi}_f(\varphi) &:= \Pi_f(\varphi_\xi) = \int_{\Omega_\xi} \langle f, u_\xi \rangle dV_\xi = \int_{\Omega_h} \langle \tilde{f}, \tilde{u} \rangle dV, \\ \widetilde{\Pi}_c(\bar{Q}_e) &:= \Pi_c(\bar{R}_\xi) = \int_{\Gamma_\xi} \langle c, \bar{R}_\xi \rangle dS_\xi = \int_{\Gamma_h} \langle \tilde{c}, \bar{Q}_e \rangle dS, \end{aligned} \quad (3.35)$$

with $\tilde{u}(x_i) = \varphi(x_i) - \Theta(x_i)$ the displacement vector, $\bar{R} = \bar{Q}_e Q_0$ the total microrotation, the vector fields \tilde{f} and \tilde{c} can be determined in terms of f and c , respectively, for instance (see [20, Theorem 1.3.-1])

$$\tilde{f}(x) = f(\Theta(x)) \det(\nabla_x \Theta), \quad \tilde{c}(x) = c(\Theta(x)) \det(\nabla_x \Theta). \quad (3.36)$$

Note that regarding to the regularity condition (3.6), the following regularity conditions will hold as well

$$\tilde{f} \in L^2(\Omega_h, \mathbb{R}^3), \quad \tilde{c} \in L^2(\Gamma_h, \mathbb{R}^3), \quad \bar{Q}_e \in L^2(\Gamma_h, \mathbb{R}^3). \quad (3.37)$$

The Dirichlet-type boundary conditions (in the sense of the traces) on $\Gamma_\xi = \gamma_\xi \times [-\frac{h}{2}, \frac{h}{2}]$, $\gamma_\xi \subset \partial\omega_\xi$, read on the boundary $\Gamma_h = \gamma \times [-\frac{h}{2}, \frac{h}{2}]$, $\gamma = \Theta^{-1}(\gamma_\xi) \subset \partial\omega$, as $\varphi = \varphi_d^h$ on Γ_h , where $\varphi_d^h = \Theta^{-1}(\varphi_\xi^h)$.

4 Construction of the family of functionals I_{h_j}

4.1 Nonlinear scaling for the deformation gradient and the microrotation

In order to apply the methods of Γ -convergence, the first step is to transform our problem further from Ω_h to a *domain* with fixed thickness $\Omega_1 = \omega \times [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}^3$, $\omega \subset \mathbb{R}^2$. For this goal, scaling of the variables (dependent/independent) would be the first step. However, it is important to know which kind of scaling is suitable for our variables. In this paper we introduce only the *nonlinear scaling*, although in linear models, a concept of *linear scaling* is used as well ([55, 56]). For a vector field $z: \Omega_h \rightarrow \mathbb{R}^3$ we consider the *nonlinear scaling* $z^{\natural}: \Omega_1 \rightarrow \mathbb{R}^3$, where only the independent variables will be scaled

$$\begin{aligned} x_1 &= \eta_1, & x_2 &= \eta_2, & x_3 &= h\eta_3, \\ z^{\natural}\left(x_1, x_2, \frac{1}{h}x_3\right) &:= z(x_1, x_2, x_3), & \text{nonlinear scaling.} \end{aligned} \quad (4.1)$$

Consequently, the gradient of $z(x) = (z_1(x), z_2(x), z_3(x))$ with respect to $x = (x_1, x_2, x_3)$ can be expressed in terms of the derivative of z^{\natural} with respect to $\eta = (\eta_1, \eta_2, \eta_3)$

$$\nabla_x z(x_1, x_2, x_3) = \left(\partial_{\eta_1} z^{\natural}(\eta_1, \eta_2, \eta_3) \mid \partial_{\eta_2} z^{\natural}(\eta_1, \eta_2, \eta_3) \mid \frac{1}{h} \partial_{\eta_3} z^{\natural}(\eta_1, \eta_2, \eta_3) \right) := \nabla_{\eta}^h z^{\natural}(\eta). \quad (4.2)$$

For more details about scaling of the variable we refer to [56]. In all our computations the mark \cdot^{\natural} indicates the nonlinear scaling.

In a first step we will apply the nonlinear scaling to the deformation. For $\Omega_1 = \omega \times \left[-\frac{1}{2}, \frac{1}{2}\right] \subset \mathbb{R}^3$, $\omega \subset \mathbb{R}^2$, we define the scaling transformations

$$\begin{aligned} \zeta: \eta \in \Omega_1 &\mapsto \mathbb{R}^3, & \zeta(\eta_1, \eta_2, \eta_3) &:= (\eta_1, \eta_2, h\eta_3), \\ \zeta^{-1}: x \in \Omega_h &\mapsto \mathbb{R}^3, & \zeta^{-1}(x_1, x_2, x_3) &:= (x_1, x_2, \frac{x_3}{h}), \end{aligned} \quad (4.3)$$

with $\zeta(\Omega_1) = \Omega_h$. By using the relation (4.1) and above transformations we obtain the formula for the transformed deformation φ as

$$\begin{aligned} \varphi(x_1, x_2, x_3) &= \varphi^\sharp(\zeta^{-1}(x_1, x_2, x_3)) \quad \forall x \in \Omega_h; \quad \varphi^\sharp(\eta) = \varphi(\zeta(\eta)) \quad \forall \eta \in \Omega_1, \\ \nabla_x \varphi(x_1, x_2, x_3) &= \begin{pmatrix} \partial_{\eta_1} \varphi_1^\sharp(\eta) & \partial_{\eta_2} \varphi_1^\sharp(\eta) & \frac{1}{h} \partial_{\eta_3} \varphi_1^\sharp(\eta) \\ \partial_{\eta_1} \varphi_2^\sharp(\eta) & \partial_{\eta_2} \varphi_2^\sharp(\eta) & \frac{1}{h} \partial_{\eta_3} \varphi_2^\sharp(\eta) \\ \partial_{\eta_1} \varphi_3^\sharp(\eta) & \partial_{\eta_2} \varphi_3^\sharp(\eta) & \frac{1}{h} \partial_{\eta_3} \varphi_3^\sharp(\eta) \end{pmatrix} = \nabla_\eta^h \varphi^\sharp(\eta) = F_h^\sharp. \end{aligned} \quad (4.4)$$

Now we will do the same process for the microrotation tensor $\overline{Q}_e^\sharp: \Omega_1 \rightarrow \text{SO}(3)$

$$\overline{Q}_e(x_1, x_2, x_3) = \overline{Q}_e^\sharp(\zeta^{-1}(x_1, x_2, x_3)) \quad \forall x \in \Omega_h; \quad \overline{Q}_e^\sharp(\eta) = \overline{Q}_e(\zeta(\eta)), \quad \forall \eta \in \Omega_1,$$

as well as for $\nabla_x \Theta(x)$, the matrices of its polar decomposition $\nabla_x \Theta(x) = Q_0(x)U_0(x)$, in the sense that

$$(\nabla_x \Theta)^\sharp(\eta) = (\nabla_x \Theta)(\zeta(\eta)), \quad Q_0^\sharp(\eta) = Q_0(\zeta(\eta)), \quad U_0^\sharp(\eta) = U_0(\zeta(\eta)). \quad (4.5)$$

We also define $\overline{R}^\sharp: \Omega_1 \rightarrow \text{SO}(3)$

$$\overline{R}(x_1, x_2, x_3) = \overline{R}^\sharp(\zeta^{-1}(x_1, x_2, x_3)) \quad \forall x \in \Omega_h; \quad \overline{R}^\sharp(\eta) = \overline{R}(\zeta(\eta)), \quad \forall \eta \in \Omega_1.$$

With this, the non-symmetric stretch tensor expressed in a point of Ω_1 is given by

$$\overline{U}_e^\sharp = \overline{Q}_e^{\sharp, T} F_h^\sharp [(\nabla_x \Theta)^\sharp]^{-1} = \overline{Q}_e^{\sharp, T} \nabla_\eta^h \varphi^\sharp(\eta) [(\nabla_x \Theta)^\sharp]^{-1}. \quad (4.6)$$

Since for $\eta_3 = 0$ their values expressed in terms of $(\eta_1, \eta_2, 0)$ and $(x_1, x_2, 0)$ coincide, we will omit the sign $^\sharp$ and we will understand from the context the variables into discussion, i.e.,

$$\begin{aligned} (\nabla_x \Theta)(0) &:= (\nabla y_0 | n_0) = (\nabla_x \Theta)^\sharp(\eta_1, \eta_2, 0) \equiv (\nabla_x \Theta)(x_1, x_2, 0), \\ Q_0(0) &:= Q_0^\sharp(\eta_1, \eta_2, 0) \equiv Q_0(x_1, x_2, 0), & U_0(0) &:= U_0^\sharp(\eta_1, \eta_2, 0) \equiv U_0(x_1, x_2, 0). \end{aligned}$$

Therefore, we have

$$\overline{Q}_e^\sharp(\eta) = \overline{R}^\sharp(\eta)(Q_0^\sharp(\eta))^T, \quad \overline{U}_e^\sharp(\eta) = \overline{Q}_e^{\sharp, T}(\eta) F_h^\sharp(\eta) [(\nabla_x \Theta)^\sharp]^{-1} = Q_0^\sharp(\eta) \overline{R}^{\sharp, T}(\eta) F_h^\sharp(\eta) [(\nabla_x \Theta)^\sharp]^{-1}, \quad (4.7)$$

and

$$\Gamma_h^\sharp = \left(\text{axl}(\overline{Q}_{e,h}^{\sharp, T} \partial_{\eta_1} \overline{Q}_{e,h}^\sharp) \mid \text{axl}(\overline{Q}_{e,h}^{\sharp, T} \partial_{\eta_2} \overline{Q}_{e,h}^\sharp) \mid \frac{1}{h} \text{axl}(\overline{Q}_{e,h}^{\sharp, T} \partial_{\eta_3} \overline{Q}_{e,h}^\sharp) \right) [(\nabla_x \Theta)^\sharp]^{-1}. \quad (4.8)$$

4.2 Transformation of the problem from Ω_h to a fixed domain Ω_1

The next step, in order to apply the Γ -convergence technique, is to transform the minimization problem onto the *fixed domain* Ω_1 , which is independent from the thickness h . According to the results from the previous subsection, we have the following minimization problem on Ω_1

$$\begin{aligned} I_h^\sharp(\varphi^\sharp, \nabla_\eta^h \varphi^\sharp, \overline{Q}_e^\sharp, \Gamma_h^\sharp) &= \int_{\Omega_1} \left(W_{\text{mp}}(\overline{U}_h^\sharp) + \widetilde{W}_{\text{curv}}(\Gamma_h^\sharp) \right) \det(\nabla_\eta \zeta(\eta)) \det((\nabla_x \Theta)^\sharp) dV_\eta - \Pi_h^\sharp(\varphi^\sharp, \overline{Q}_e^\sharp) \\ &= \underbrace{\int_{\Omega_1} h \left[\left(W_{\text{mp}}(\overline{U}_h^\sharp) + \widetilde{W}_{\text{curv}}(\Gamma_h^\sharp) \right) \det((\nabla_x \Theta)^\sharp) \right] dV_\eta - \Pi_h^\sharp(\varphi^\sharp, \overline{Q}_e^\sharp)}_{:= J_h^\sharp(\varphi^\sharp, \nabla_\eta^h \varphi^\sharp, \overline{Q}_e^\sharp, \Gamma_h^\sharp)} \mapsto \min \text{ w.r.t } (\varphi^\sharp, \overline{Q}_e^\sharp), \end{aligned} \quad (4.9)$$

where

$$\begin{aligned} W_{\text{mp}}(\bar{U}_h^\natural) &= \mu \|\text{sym}(\bar{U}_h^\natural - \mathbb{1}_3)\|^2 + \mu_c \|\text{skew}(\bar{U}_h^\natural - \mathbb{1}_3)\|^2 + \frac{\lambda}{2} [\text{tr}(\text{sym}(\bar{U}_h^\natural - \mathbb{1}_3))]^2, \\ \widetilde{W}_{\text{curv}}(\Gamma_h^\natural) &= \mu L_c^2 \left(a_1 \|\text{dev sym } \Gamma_h^\natural\|^2 + a_2 \|\text{skew } \Gamma_h^\natural\|^2 + a_3 [\text{tr}(\Gamma_h^\natural)]^2 \right), \end{aligned} \quad (4.10)$$

with $\Pi_h^\natural(\varphi^\natural, \bar{Q}_e^\natural) = \Pi_f^\natural(\varphi^\natural) + \Pi_c^\natural(\bar{Q}_e^\natural)$,

$$\begin{aligned} \Pi_f^\natural(\varphi^\natural) &:= \widetilde{\Pi}_f(\varphi) = \int_{\Omega_h} \langle \tilde{f}, \tilde{u} \rangle dV = \int_{\Omega_1} \langle \tilde{f}^\natural, \tilde{u}^\natural \rangle \det(\nabla_\eta \zeta(\eta)) dV_\eta = h \int_{\Omega_1} \langle \tilde{f}^\natural, \tilde{u}^\natural \rangle dV_\eta, \\ \Pi_c^\natural(\bar{Q}_e^\natural) &:= \widetilde{\Pi}_c(\bar{Q}_e) = \int_{\Gamma_h} \langle \tilde{c}, \bar{Q}_e \rangle dS = \int_{\Gamma_1} \langle \tilde{c}^\natural, \bar{Q}_e^\natural \rangle \det(\nabla_\eta \zeta(\eta)) dS_\eta = h \int_{\Gamma_1} \langle \tilde{c}^\natural, \bar{Q}_e^\natural \rangle dS_\eta, \end{aligned} \quad (4.11)$$

with $\tilde{f}^\natural(\eta) = \tilde{f}(\zeta(\eta))$, $\tilde{u}^\natural(\eta) = \tilde{u}(\zeta(\eta))$, $\tilde{c}^\natural(\eta) = \tilde{c}(\zeta(\eta))$ and $\bar{Q}_e^\natural(\eta) = \bar{Q}_e(\zeta(\eta))$. Here we recall that regarding to the regularity condition (3.37), it holds

$$\tilde{f}^\natural \in L^2(\Omega_1, \mathbb{R}^3), \quad \tilde{c}^\natural \in L^2(\Gamma_1, \mathbb{R}^3), \quad \bar{Q}_e^\natural \in L^2(\Gamma_1, \mathbb{R}^3). \quad (4.12)$$

Therefore, we may write

$$\begin{aligned} |\Pi_f^\natural(\varphi^\natural)| &= |h \int_{\Omega_1} \langle \tilde{f}^\natural, \tilde{u}^\natural \rangle dV_\eta| \leq h \|\tilde{f}^\natural\|_{L^2(\Omega_1)} \|\tilde{u}^\natural\|_{L^2(\Omega_1)}, \\ |\Pi_c^\natural(\bar{Q}_e^\natural)| &= |h \int_{\Gamma_1} \langle \tilde{c}^\natural, \bar{Q}_e^\natural \rangle dS_\eta| \leq h \|\tilde{c}^\natural\|_{L^2(\Gamma_1)} \|\bar{Q}_e^\natural\|_{L^2(\Gamma_1)}, \end{aligned} \quad (4.13)$$

and consequently

$$|\Pi_h^\natural(\varphi^\natural, \bar{Q}_e^\natural)| \leq h \left[\|\tilde{f}^\natural\|_{L^2(\Omega_1)} \|\tilde{u}^\natural\|_{L^2(\Omega_1)} + \|\tilde{c}^\natural\|_{L^2(\Gamma_1)} \|\bar{Q}_e^\natural\|_{L^2(\Gamma_1)} \right]. \quad (4.14)$$

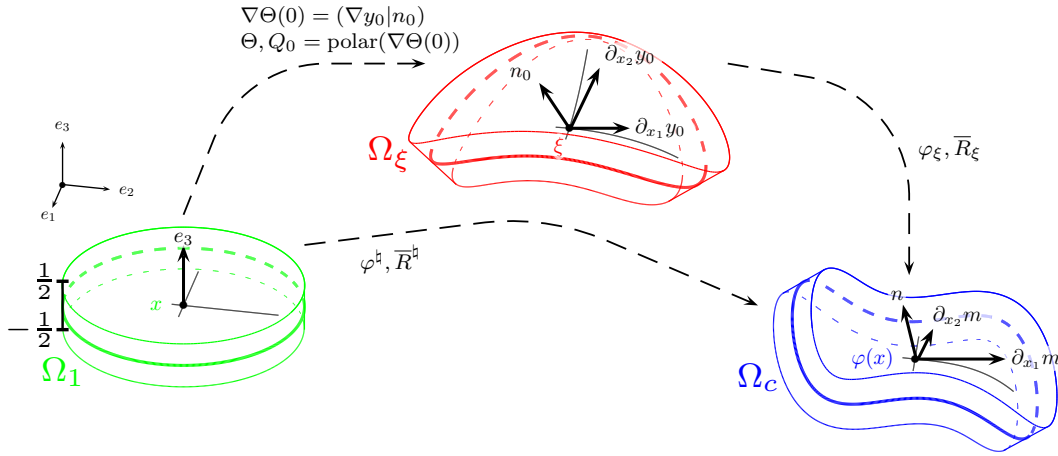


Figure 3: The complete picture of the involved domains. Ω_1 is the fictitious flat domain with unit thickness, Ω_ξ denotes the curved reference configuration, Ω_c is the current deformed configuration. Again, the reference configuration Ω_ξ takes on the role of a compatible intermediate configuration in the multiplicative decomposition.

The Dirichlet-type boundary conditions (in the sense of the trace) on $\Gamma_h = \gamma \times [-\frac{h}{2}, \frac{h}{2}]$, $\gamma = \Theta^{-1}(\gamma_\xi) \subset \partial\omega$, read on the boundary $\Gamma_1 = \gamma \times [-\frac{1}{2}, \frac{1}{2}]$ as $\varphi^\natural = \varphi_d^\natural$ on Γ_1 , where $\varphi_d^\natural = \Theta^{-1}(\varphi_d^h)$.

5 Equi-coercivity and compactness of the family of energy functionals

5.1 The set of admissible solutions

Due to the scaling, we have obtained a family of functionals

$$J_h^\natural(\varphi^\natural, \nabla_\eta^h \varphi^\natural, \overline{Q}_e^\natural, \Gamma_h^\natural) = \int_{\Omega_1} h \left[\left(W_{\text{mp}}(\overline{U}_h^\natural) + \widetilde{W}_{\text{curv}}(\Gamma_h^\natural) \right) \det((\nabla_x \Theta)^\natural) \right] dV_\eta, \quad (5.1)$$

depending on the thickness h . The next step is to prepare a suitable space X on which the existence of Γ -convergence will be studied. As already mentioned, for applying the Γ -limit techniques we need to work with a separable and metrizable space X . Since working in $H^1(\Omega_1, \mathbb{R}^3) \times H^1(\Omega_1, \text{SO}(3))$ means to consider the weak topology, which does not give rise to a metric space, we introduce the following spaces:

$$\begin{aligned} X &:= \{(\varphi^\natural, \overline{Q}_e^\natural) \in L^2(\Omega_1, \mathbb{R}^3) \times L^2(\Omega_1, \text{SO}(3))\}, \\ X' &:= \{(\varphi^\natural, \overline{Q}_e^\natural) \in H^1(\Omega_1, \mathbb{R}^3) \times H^1(\Omega_1, \text{SO}(3))\}, \\ X_\omega &:= \{(\varphi, \overline{Q}_e) \in L^2(\omega, \mathbb{R}^3) \times L^2(\omega, \text{SO}(3))\}, \\ X'_\omega &:= \{(\varphi, \overline{Q}_e) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, \text{SO}(3))\}. \end{aligned} \quad (5.2)$$

We also consider the following admissible sets

$$\begin{aligned} \mathcal{S}' &:= \{(\varphi, \overline{Q}_e) \in H^1(\Omega_1, \mathbb{R}^3) \times H^1(\Omega_1, \text{SO}(3)) \mid \varphi|_{\Gamma_1}(\eta) = \varphi_d^\natural(\eta)\}, \\ \mathcal{S}'_\omega &:= \{(\varphi, \overline{Q}_e) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, \text{SO}(3)) \mid \varphi|_{\partial\omega}(\eta_1, \eta_2) = \varphi_d^\natural(\eta_1, \eta_2, 0)\}. \end{aligned} \quad (5.3)$$

By the embedding theorem ([19], Theorem 6.1-3), the embedding $X' \subset X$ is true and clearly² $X_\omega \subset X$, $X'_\omega \subset X'$.

The functionals in our analysis are obtained by extending the functionals J_h (respectively I_h) to the entire space X and to take their averages over the thickness, through

$$\begin{aligned} \mathcal{I}_h^\natural(\varphi^\natural, \nabla_\eta^h \varphi^\natural, \overline{Q}_e^\natural, \Gamma_h^\natural) &= \begin{cases} \frac{1}{h} I_h^\natural(\varphi^\natural, \nabla_\eta^h \varphi^\natural, \overline{Q}_e^\natural, \Gamma_h^\natural) & \text{if } (\varphi^\natural, \overline{Q}_e^\natural) \in \mathcal{S}', \\ +\infty & \text{else in } X. \end{cases} \\ &= \begin{cases} \frac{1}{h} J_h^\natural(\varphi^\natural, \nabla_\eta^h \varphi^\natural, \overline{Q}_e^\natural, \Gamma_h^\natural) - \frac{1}{h} \Pi_h^\natural(\varphi^\natural, \overline{Q}_e^\natural) & \text{if } (\varphi^\natural, \overline{Q}_e^\natural) \in \mathcal{S}', \\ +\infty & \text{else in } X. \end{cases} \end{aligned} \quad (5.4)$$

The main aim of the current paper is to find the Γ -limit of the family of functional $\mathcal{I}_h^\natural(\varphi^\natural, \nabla_\eta^h \varphi^\natural, \overline{Q}_e^\natural, \Gamma_h^\natural)$, i.e., to obtain an energy functional expressed only in terms of the weak limit of a subsequence of $(\varphi_{h_j}^\natural, \overline{Q}_{e, h_j}^\natural) \in X$, when h_j goes to zero. In other words, as we will see, to construct an energy function depending only on quantities defined on the midsurface of the shell-like domain, see Figure 4.

As a first step we consider the functionals

$$\mathcal{J}_h^\natural(\varphi^\natural, \nabla_\eta^h \varphi^\natural, \overline{Q}_e^\natural, \Gamma_h^\natural) = \begin{cases} \frac{1}{h} J_h^\natural(\varphi^\natural, \nabla_\eta^h \varphi^\natural, \overline{Q}_e^\natural, \Gamma_h^\natural) & \text{if } (\varphi^\natural, \overline{Q}_e^\natural) \in \mathcal{S}', \\ +\infty & \text{else in } X. \end{cases} \quad (5.5)$$

5.2 Equi-coercivity and compactness of the family \mathcal{J}_h^\natural

Theorem 5.1. *Assume that the initial configuration is defined by a continuous injective mapping $y_0 : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ which admits an extension to $\overline{\omega}$ into $C^2(\overline{\omega}; \mathbb{R}^3)$ such that $\det[\nabla_x \Theta(0)] \geq a_0 > 0$ on $\overline{\omega}$, where a_0 is a positive constant, and assume that the boundary data satisfies the conditions*

$$\varphi_d^\natural = \varphi_d|_{\Gamma_1} \text{ (in the sense of traces) for } \varphi_d \in H^1(\Omega_1, \mathbb{R}^3). \quad (5.6)$$

²Since $\infty > \int_\omega |\varphi|^2 dx dy = \int_\omega \int_{-1/2}^{1/2} |\varphi|^2 dz dx dy = \int_{\Omega_1} |\varphi|^2 dV$, which means any element from X_ω , belongs to X as well.

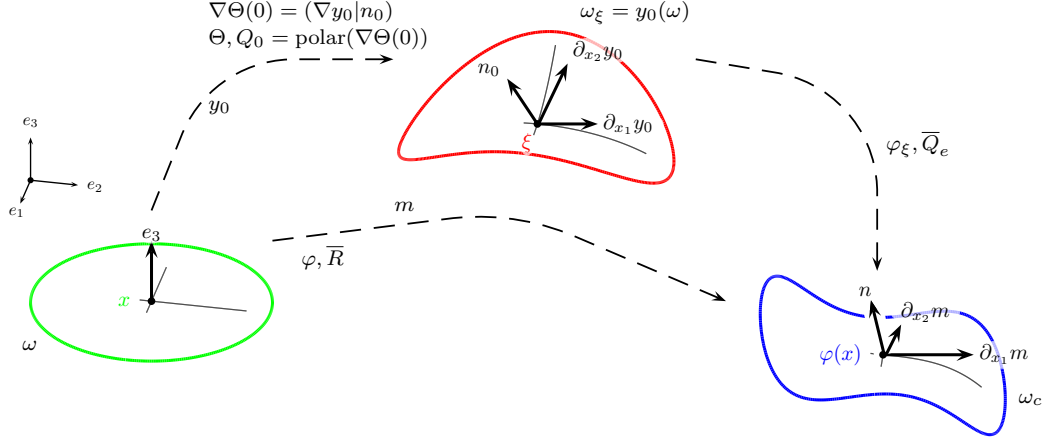


Figure 4: Kinematics of the dimensionally reduced Cosserat shell model. All fields are referred to two-dimensional surfaces. The geometry of the curved surface ω_ξ is fully encoded by the map Θ . Instead of the elastic deformation starting from ω_ξ , the total deformation m from the fictitious flat midsurface ω is considered, likewise for the total rotation \bar{R} .

Consider a sequence $(\varphi_{h_j}^\natural, \bar{Q}_{e,h_j}^\natural) \in X$, such that the energy functionals $\mathcal{J}_{h_j}^\natural(\varphi_{h_j}^\natural, \bar{Q}_{e,h_j}^\natural)$ are bounded as $h_j \rightarrow 0$. Let the constitutive parameters satisfy

$$\mu > 0, \quad \kappa > 0, \quad \mu_c > 0, \quad a_1 > 0, \quad a_2 > 0, \quad a_3 > 0. \quad (5.7)$$

Then the sequence $(\varphi_{h_j}^\natural, \bar{Q}_{e,h_j}^\natural)$ admits a subsequence which is weakly convergent to $(\varphi_0^\natural, \bar{Q}_{e,0}^\natural) \in X_\omega$.

Proof. Consider the sequence $(\varphi_{h_j}^\natural, \bar{Q}_{e,h_j}^\natural) \in X$, such that the energy functionals $\mathcal{J}_{h_j}^\natural(\varphi_{h_j}^\natural, \bar{Q}_{e,h_j}^\natural)$ are bounded as $h_j \rightarrow 0$. Obviously this implies that $(\varphi_{h_j}^\natural, \bar{Q}_{e,h_j}^\natural) \in \mathcal{S}'$ for all h_j . We have

$$\begin{aligned} 2\left(\|\bar{U}_{h_j}^\natural - \mathbb{1}_3\|^2 + \|\mathbb{1}_3\|^2\right) &\geq (\|\bar{U}_{h_j}^\natural - \mathbb{1}_3\| + \|\mathbb{1}_3\|)^2 \geq \|\bar{U}_{h_j}^\natural\|^2 = \|\bar{Q}_e^{\natural,T} \nabla_\eta^{h_j} \varphi_{h_j}^\natural [(\nabla_x \Theta)^\natural(\eta)]^{-1}\|^2 \\ &= \langle \bar{Q}_e^{\natural,T} \nabla_\eta^{h_j} \varphi_{h_j}^\natural [(\nabla_x \Theta)^\natural(\eta)]^{-1}, \bar{Q}_e^{\natural,T} \nabla_\eta^{h_j} \varphi_{h_j}^\natural [(\nabla_x \Theta)^\natural(\eta)]^{-1} \rangle = \|\nabla_\eta^{h_j} \varphi_{h_j}^\natural [(\nabla_x \Theta)^\natural(\eta)]^{-1}\|^2. \end{aligned} \quad (5.8)$$

Thus, we deduce with (5.8)₁

$$\|\bar{U}_{h_j}^\natural - \mathbb{1}_3\|^2 \geq \frac{1}{2} \|\nabla_\eta^{h_j} \varphi_{h_j}^\natural [(\nabla_x \Theta)^\natural(\eta)]^{-1}\|^2 - 3. \quad (5.9)$$

But

$$\|\nabla_\eta^{h_j} \varphi_{h_j}^\natural\| = \|\nabla_\eta^{h_j} \varphi_{h_j}^\natural [(\nabla_x \Theta)^\natural(\eta)]^{-1} [(\nabla_x \Theta)^\natural(\eta)]\| \leq \|\nabla_\eta^{h_j} \varphi_{h_j}^\natural [(\nabla_x \Theta)^\natural(\eta)]^{-1}\| \cdot \|(\nabla_x \Theta)^\natural(\eta)\| \quad (5.10)$$

and we obtain

$$\|\nabla_\eta^{h_j} \varphi_{h_j}^\natural [(\nabla_x \Theta)^\natural(\eta)]^{-1}\| \geq \|\nabla_\eta^{h_j} \varphi_{h_j}^\natural\| \frac{1}{\|(\nabla_x \Theta)^\natural(\eta)\|}. \quad (5.11)$$

From the formula $[(\nabla_x \Theta)^\natural(\eta)] = (\nabla y_0 | n_0) + h_j \eta_3 (\nabla n_0 | 0)$ we get

$$\begin{aligned} \|(\nabla_x \Theta)^\natural(\eta)\| &\leq \|(\nabla y_0 | n_0)\| + h_j |\eta_3| \|(\nabla n_0 | 0)\| \leq \|(\nabla y_0 | n_0)\| + h_j \|(\nabla n_0 | 0)\| \\ &< \|(\nabla y_0 | n_0)\| + \|(\nabla n_0 | 0)\|. \end{aligned} \quad (5.12)$$

since $h_j \ll 1$. Thus

$$\frac{1}{\|(\nabla_x \Theta)^\natural(\eta)\|} \geq \frac{1}{\|(\nabla y_0 | n_0)\| + \|(\nabla n_0 | 0)\|}. \quad (5.13)$$

Moreover, since $y_0 \in C^2(\bar{\omega}; \mathbb{R}^3)$, it follows for h_j small enough that there exists $c_1 > 0$ such that $\frac{1}{\|(\nabla_x \Theta)^\sharp(\eta)\|} \geq c_1$. Therefore, from (5.9) and (5.11), we get that there exist $c_1, c_2 > 0$ such that

$$\|\bar{U}_{h_j}^\sharp - \mathbb{1}_3\|^2 \geq \frac{c_1}{2} \|\nabla_\eta^{h_j} \varphi_{h_j}^\sharp\|^2 - c_2. \quad (5.14)$$

From the hypothesis we have

$$\begin{aligned} \infty > \mathcal{J}_{h_j}^\sharp(\varphi_{h_j}^\sharp, \bar{Q}_{e, h_j}^\sharp) &\geq \int_{\Omega_1} \left(W_{\text{mp}}(\bar{U}_{h_j}^\sharp) + \widetilde{W}_{\text{curv}}(\Gamma_{h_j}^\sharp) \right) \det((\nabla_x \Theta)^\sharp) dV_\eta \\ &\geq \int_{\Omega_1} W_{\text{mp}}(\bar{U}_{h_j}^\sharp) \det((\nabla_x \Theta)^\sharp) dV_\eta \geq \min(c_1^+, \mu_c) \int_{\Omega_1} \|\bar{U}_{h_j}^\sharp - \mathbb{1}_3\|^2 \det((\nabla_x \Theta)^\sharp) dV_\eta, \end{aligned} \quad (5.15)$$

where c_1^+ denotes the smallest eigenvalue of the quadratic form $W_{\text{mp}}^\infty(X)$.

Let us recall that $\det(\nabla_x \Theta(x_3)) = \det(\nabla y_0|n_0) \left[1 - 2x_3 H + x_3^2 K \right] = \det(\nabla y_0|n_0)(1 - \kappa_1 x_3)(1 - \kappa_2 x_3)$, where H, K are the mean curvature and Gauß curvature, respectively. But $(1 - \kappa_1 x_3)(1 - \kappa_2 x_3) > 0, \forall x_3 \in [-h_j/2, h_j/2]$ if and only if h_j satisfies the hypothesis (3.3). Therefore, there exists a constant $c > 0$ such that

$$\det(\nabla_x \Theta(x_3)) \geq c \det(\nabla y_0|n_0) \quad \forall x_3 \in [-h/2, h/2]. \quad (5.16)$$

Due to the hypothesis $\det[\nabla_x \Theta(0)] \geq a_0 > 0$ this implies that there exists a constant $c > 0$ such that

$$\det(\nabla_x \Theta(x_3)) \geq c \quad \forall x_3 \in [-h_j/2, h_j/2], \quad (5.17)$$

which means that $\det(\nabla_x \Theta(x_3)^\sharp) \geq c \quad \forall x_3 \in [-1/2, 1/2]$.

Hence, from (5.15), (5.14) and (5.17), it follows that for small enough h_j there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$\begin{aligned} \infty > \mathcal{J}_{h_j}^\sharp(\varphi_{h_j}^\sharp, \bar{Q}_{e, h_j}^\sharp) &\geq c_1 \int_{\Omega_1} \|\nabla_\eta^{h_j} \varphi_{h_j}^\sharp\|^2 dV_\eta - c_2 \\ &\geq c_1 \int_{\Omega_1} \left(\|\partial_{\eta_1} \varphi_{h_j}^\sharp\|^2 + \|\partial_{\eta_2} \varphi_{h_j}^\sharp\|^2 + \frac{1}{h_j^2} \|\partial_{\eta_3} \varphi_{h_j}^\sharp\|^2 \right) dV_\eta - c_2. \end{aligned} \quad (5.18)$$

Furthermore, due to the hypothesis on h_j , it is clear that there exists $c > 0$ such that

$$\|\partial_{\eta_1} \varphi_{h_j}^\sharp\|^2 + \|\partial_{\eta_2} \varphi_{h_j}^\sharp\|^2 + \frac{1}{h_j^2} \|\partial_{\eta_3} \varphi_{h_j}^\sharp\|^2 \geq c \left(\|\partial_{\eta_1} \varphi_{h_j}^\sharp\|^2 + \|\partial_{\eta_2} \varphi_{h_j}^\sharp\|^2 + \|\partial_{\eta_3} \varphi_{h_j}^\sharp\|^2 \right), \quad (5.19)$$

which implies the existence of $c_1, c_2 > 0$ such that

$$\infty > \mathcal{J}_{h_j}^\sharp(\varphi_{h_j}^\sharp, \bar{Q}_{e, h_j}^\sharp) \geq c_1 \int_{\Omega_1} \underbrace{\left(\|\partial_{\eta_1} \varphi_{h_j}^\sharp\|^2 + \|\partial_{\eta_2} \varphi_{h_j}^\sharp\|^2 + \|\partial_{\eta_3} \varphi_{h_j}^\sharp\|^2 \right)}_{=:\|\nabla_\eta^{h_j} \varphi_{h_j}^\sharp\|^2} dV_\eta - c_2. \quad (5.20)$$

We also obtain, applying the Poincaré-inequality [58], that there exists a constant $C > 0$ such that

$$\begin{aligned} \|\nabla_\eta^{h_j} \varphi_{h_j}^\sharp\|_{L^2(\omega)}^2 &= \|\nabla_\eta^{h_j} \varphi_{h_j}^\sharp - \nabla_\eta^{h_j} \varphi_d + \nabla_\eta^{h_j} \varphi_d\|_{L^2(\omega)}^2 \\ &\geq (\|\nabla_\eta^{h_j} (\varphi_{h_j}^\sharp - \varphi_d)\|_{L^2(\omega)} - \|\nabla_\eta^{h_j} \varphi_d\|_{L^2(\omega)})^2 \\ &= \|\nabla_\eta^{h_j} (\varphi_{h_j}^\sharp - \varphi_d)\|_{L^2(\omega)}^2 - 2\|\nabla_\eta^{h_j} (\varphi_{h_j}^\sharp - \varphi_d)\|_{L^2(\omega)} \|\nabla_\eta^{h_j} \varphi_d\|_{L^2(\omega)} + \|\nabla_\eta^{h_j} \varphi_d\|_{L^2(\omega)}^2 \\ &\geq C \|\varphi_{h_j}^\sharp - \varphi_d\|_{H^1(\omega)}^2 - 2\|\varphi_{h_j}^\sharp - \varphi_d\|_{H^1(\omega)} \|\nabla_\eta^{h_j} \varphi_d\|_{L^2(\omega)} + \|\nabla_\eta^{h_j} \varphi_d\|_{L^2(\omega)}^2 \\ &\geq C \|\varphi_{h_j}^\sharp - \varphi_d\|_{H^1(\omega)}^2 - \frac{1}{\varepsilon} \|\varphi_{h_j}^\sharp - \varphi_d\|_{H^1(\omega)}^2 - \varepsilon \|\nabla_\eta^{h_j} \varphi_d\|_{L^2(\omega)}^2 + \|\nabla_\eta^{h_j} \varphi_d\|_{L^2(\omega)}^2 \quad \forall \varepsilon > 0, \end{aligned} \quad (5.21)$$

where we have used Young's and Poincaré's inequality. Therefore, by choosing $\varepsilon > 0$ small enough, (5.21) ensures the existence of constants $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that

$$\begin{aligned} \|\nabla_{\eta}^{h_j} \varphi_{h_j}^{\natural}\|_{L^2(\omega)}^2 &\geq c_1 \|\varphi_{h_j}^{\natural} - \varphi_d\|_{H^1(\omega)}^2 - c_2 \geq \frac{c_1}{2} 2 (\|\varphi_{h_j}^{\natural}\|_{H^1(\omega)} - \|\varphi_d\|_{H^1(\omega)})^2 - c_2 \\ &\geq \frac{c_1}{2} \|\varphi_{h_j}^{\natural}\|_{H^1(\omega)}^2 + \frac{c_1}{2} \|\varphi_d\|_{H^1(\omega)}^2 - c_2. \end{aligned} \quad (5.22)$$

Thus, there exists $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that

$$\|\nabla_{\eta}^{h_j} \varphi_{h_j}^{\natural}\|_{L^2(\omega)}^2 \geq \frac{c_1}{2} \|\varphi_{h_j}^{\natural}\|_{H^1(\omega)}^2 - c_2, \quad (5.23)$$

which implies the uniform bound for $\varphi_{h_j}^{\natural}$ in \mathcal{S}' . On the other hand, since

$$\|\partial_{\eta_1} \varphi_{h_j}^{\natural}\|^2 + \|\partial_{\eta_2} \varphi_{h_j}^{\natural}\|^2 + \frac{1}{h_j^2} \|\partial_{\eta_3} \varphi_{h_j}^{\natural}\|^2 \geq \frac{1}{h_j^2} \|\partial_{\eta_3} \varphi_{h_j}^{\natural}\|^2, \quad (5.24)$$

from (5.18) it results that $\frac{1}{h_j^2} \|\partial_{\eta_3} \varphi_{h_j}^{\natural}\|^2$ is bounded, i.e., there is $c > 0$, such that

$$\|\partial_{\eta_3} \varphi_{h_j}^{\natural}\|_{L^2(\Omega)} \leq c h_j. \quad (5.25)$$

This means that $\partial_{\eta_3} \varphi_{h_j}^{\natural} \rightarrow 0$ strongly in $L^2(\Omega)$, when $h_j \rightarrow 0$.

Hence, considering $(\varphi_{h_j}^{\natural}, \overline{Q}_{e,h_j}^{\natural}) \in X$, such that the energy functionals $\mathcal{J}_{h_j}^{\natural}(\varphi_{h_j}^{\natural}, \overline{Q}_{e,h_j}^{\natural})$ are bounded, it follows that any limit point φ_0^{\natural} of $\varphi_{h_j}^{\natural}$ for the weak topology of $L^2(\Omega_1, \mathbb{R}^3)$ (which exists due to its uniform boundedness in $H^1(\omega, \mathbb{R}^3)$) satisfies

$$\partial_{\eta_3} \varphi_0^{\natural} = 0 \quad \Rightarrow \quad \varphi_0^{\natural} \in H^1(\omega, \mathbb{R}^3). \quad (5.26)$$

Similar arguments for the curvature energy imply that there exists $c > 0$ such that

$$\begin{aligned} \infty &> \mathcal{J}_{h_j}^{\natural}(\varphi_{h_j}^{\natural}, \overline{Q}_{e,h_j}^{\natural}) \geq \int_{\Omega_1} \widetilde{W}_{\text{curv}}(\Gamma_{h_j}^{\natural}) \det((\nabla_x \Theta)^{\natural}) dV_{\eta} \geq \int_{\Omega_1} c \|\Gamma_{h_j}^{\natural}\|^2 \det((\nabla_x \Theta)^{\natural}) dV_{\eta} \\ &= c \int_{\Omega_1} \left\| \left(\text{axl}(\overline{Q}_{e,h_j}^{\natural,T} \partial_{\eta_1} \overline{Q}_{e,h_j}^{\natural}) \mid \text{axl}(\overline{Q}_{e,h_j}^{\natural,T} \partial_{\eta_2} \overline{Q}_{e,h_j}^{\natural}) \mid \frac{1}{h_j} \text{axl}(\overline{Q}_{e,h_j}^{\natural,T} \partial_{\eta_3} \overline{Q}_{e,h_j}^{\natural}) \right) [(\nabla_x \Theta)^{\natural}(\eta)]^{-1} \right\|^2 \det((\nabla_x \Theta)^{\natural}) dV_{\eta}. \end{aligned} \quad (5.27)$$

In the next step, as in the deduction of (5.8)–(5.18), it will be shown that for $a_1, a_2, a_3 > 0$ there exists $c > 0$ such that

$$\begin{aligned} \infty &> c \int_{\Omega_1} \left(\|\text{axl}(\overline{Q}_{e,h_j}^{\natural,T} \partial_{\eta_1} \overline{Q}_{e,h_j}^{\natural})\|^2 + \|\text{axl}(\overline{Q}_{e,h_j}^{\natural,T} \partial_{\eta_2} \overline{Q}_{e,h_j}^{\natural})\|^2 + \frac{1}{h_j^2} \|\text{axl}(\overline{Q}_{e,h_j}^{\natural,T} \partial_{\eta_3} \overline{Q}_{e,h_j}^{\natural})\|^2 \right) dV_{\eta} \\ &= \frac{c}{2} \int_{\Omega_1} \left(\|\overline{Q}_{e,h_j}^{\natural,T} \partial_{\eta_1} \overline{Q}_{e,h_j}^{\natural}\|^2 + \|\overline{Q}_{e,h_j}^{\natural,T} \partial_{\eta_2} \overline{Q}_{e,h_j}^{\natural}\|^2 + \frac{1}{h_j^2} \|\overline{Q}_{e,h_j}^{\natural,T} \partial_{\eta_3} \overline{Q}_{e,h_j}^{\natural}\|^2 \right) dV_{\eta} \\ &= \frac{c}{2} \int_{\Omega_1} \left(\|\partial_{\eta_1} \overline{Q}_{e,h_j}^{\natural}\|^2 + \|\partial_{\eta_2} \overline{Q}_{e,h_j}^{\natural}\|^2 + \frac{1}{h_j^2} \|\partial_{\eta_3} \overline{Q}_{e,h_j}^{\natural}\|^2 \right) dV_{\eta}. \end{aligned} \quad (5.28)$$

With the same argument as in the strain part, we deduce

$$\infty > c \int_{\Omega_1} \left(\|\partial_{\eta_1} \overline{Q}_{e,h_j}^{\natural}\|^2 + \|\partial_{\eta_2} \overline{Q}_{e,h_j}^{\natural}\|^2 + \|\partial_{\eta_3} \overline{Q}_{e,h_j}^{\natural}\|^2 \right) dV_{\eta}, \quad (5.29)$$

where $c > 0$. Hence, it follows that $\partial_{\eta_i} \overline{Q}_{e,h_j}^{\natural}$ is bounded in $L^2(\Omega_1, \mathbb{R}^{3 \times 3})$, for $i = 1, 2, 3$. Since $\overline{Q}_{e,h_j}^{\natural} \in \text{SO}(3)$, we have $\|\overline{Q}_{e,h_j}^{\natural}\|^2 = 3$ and therefore $\overline{Q}_{e,h_j}^{\natural}$ is bounded in $L^2(\Omega_1, \mathbb{R}^{3 \times 3})$. Hence, we can infer that the sequence $\overline{Q}_{e,h_j}^{\natural}$ is bounded in $H^1(\Omega_1, \text{SO}(3))$, independently from h_j .

Therefore, there is a subsequence from $\overline{Q}_{e,h_j}^{\natural}$ which is weakly convergent (without relabeling) to $\overline{Q}_{e,0}^{\natural}$. That is

$$\overline{Q}_{e,h_j}^{\natural} \rightharpoonup \overline{Q}_{e,0}^{\natural} \quad \text{in } H^1(\Omega_1, \text{SO}(3)). \quad (5.30)$$

In addition, from (5.28), we also obtain that there exists $c > 0$ such that $ch_j > \|\partial_{\eta_3} \overline{Q}_{e,h_j}^{\natural}\|_{L^2(\Omega_1, \text{SO}(3))}$. This means that $\partial_{\eta_3} \overline{Q}_{e,h_j}^{\natural} \rightarrow 0$ strongly in $L^2(\Omega_1, \text{SO}(3))$, when $h_j \rightarrow 0$. Hence, considering $(\varphi_{h_j}^{\natural}, \overline{Q}_{e,h_j}^{\natural}) \in X$, such that the energy functionals $\mathcal{J}_{h_j}^{\natural}(\varphi_{h_j}^{\natural}, \overline{Q}_{e,h_j}^{\natural})$ are bounded, it follows that any limit point $\overline{Q}_{e,0}^{\natural}$ of $\overline{Q}_{e,h_j}^{\natural}$ for the weak topology of X satisfies

$$\partial_{\eta_3} \overline{Q}_{e,0}^{\natural} = 0 \quad \Rightarrow \quad \overline{Q}_{e,0}^{\natural} \in H^1(\omega, \text{SO}(3)). \quad (5.31)$$

From (5.26), (5.31) and due to the continuity of the trace operator we obtain that considering $(\varphi_{h_j}^{\natural}, \overline{Q}_{e,h_j}^{\natural}) \in X$, such that the energy functionals $\mathcal{J}_{h_j}^{\natural}(\varphi_{h_j}^{\natural}, \overline{Q}_{e,h_j}^{\natural})$ are bounded, it follows that any limit point $(\varphi_0^{\natural}, \overline{Q}_{e,0}^{\natural})$ for the weak topology of X belongs to \mathcal{S}'_{ω} (since actually, such a sequence belongs to \mathcal{S}'). ■

Since the embedding $X' \subset X$ is compact, it follows that the set of the sequence of energies due to the scaling is a subset of X' , and hence, we have obtained that the family of energy functionals J_h^{\natural} is equi-coercive with respect to X .

6 The construction of the Γ -limit J_0 of the rescaled energies

In this section we construct the Γ -limit of the rescaled energies

$$\mathcal{J}_h^{\natural}(\varphi^{\natural}, \nabla_{\eta}^h \varphi^{\natural}, \overline{Q}_e^{\natural}, \Gamma_h^{\natural}) = \begin{cases} \frac{1}{h} J_h^{\natural}(\varphi^{\natural}, \nabla_{\eta}^h \varphi^{\natural}, \overline{Q}_e^{\natural}, \Gamma_h^{\natural}) & \text{if } (\varphi^{\natural}, \overline{Q}_e^{\natural}) \in \mathcal{S}', \\ +\infty & \text{else in } X, \end{cases} \quad (6.1)$$

with

$$J_h^{\natural}(\varphi^{\natural}, \nabla_{\eta}^h \varphi^{\natural}, \overline{Q}_e^{\natural}, \Gamma_h^{\natural}) = \int_{\Omega_1} h \left[\left(W_{\text{mp}}(\overline{U}_h^{\natural}) + \widetilde{W}_{\text{curv}}(\Gamma_h^{\natural}) \right) \det((\nabla_x \Theta)^{\natural}) \right] dV_{\eta}. \quad (6.2)$$

6.1 Auxiliary optimization problem

For $\varphi^{\natural} : \Omega_1 \rightarrow \mathbb{R}^3$ and $\overline{Q}_e^{\natural} : \Omega_1 \rightarrow \text{SO}(3)$ we associate the non fully dimensional reduced elastic shell stretch tensor

$$\overline{U}_{\varphi^{\natural}, \overline{Q}_e^{\natural}} := \overline{Q}_e^{\natural, T} (\nabla_{(\eta_1, \eta_2)} \varphi^{\natural} | 0) [(\nabla_x \Theta)^{\natural}]^{-1}, \quad (6.3)$$

and the non fully dimensional reduced elastic shell strain tensor

$$\mathcal{E}_{\varphi^{\natural}, \overline{Q}_e^{\natural}} := (\overline{Q}_e^{\natural, T} \nabla_{(\eta_1, \eta_2)} \varphi^{\natural} - (\nabla y_0)^{\natural} | 0) [(\nabla_x \Theta)^{\natural}]^{-1} = \overline{U}_{\varphi^{\natural}, \overline{Q}_e^{\natural}} - ((\nabla y_0)^{\natural} | 0) [(\nabla_x \Theta)^{\natural}]^{-1}. \quad (6.4)$$

Here, "non-fully" means that the introduced quantities still depend on η_3 and h , because the elements $\nabla_{(\eta_1, \eta_2)} \varphi^{\natural}$ still depend on η_3 and $\overline{Q}_e^{\natural, T}$ depends on h .

For reaching our goal we need to solve the following optimization problem: for $\varphi^{\natural} : \Omega_1 \rightarrow \mathbb{R}^3$ and $\overline{Q}_e^{\natural} : \Omega_1 \rightarrow \text{SO}(3)$, we determine a vector $d^* \in \mathbb{R}^3$ through

$$W_{\text{mp}}^{\text{hom}, \natural}(\mathcal{E}_{\varphi^{\natural}, \overline{Q}_e^{\natural}}) = W_{\text{mp}} \left(\overline{Q}_e^{\natural, T} (\nabla_{(\eta_1, \eta_2)} \varphi^{\natural} | d^*) [(\nabla_x \Theta)^{\natural}]^{-1} \right) := \inf_{c \in \mathbb{R}^3} W_{\text{mp}} \left(\overline{Q}_e^{\natural, T} (\nabla_{(\eta_1, \eta_2)} \varphi^{\natural} | c) [(\nabla_x \Theta)^{\natural}]^{-1} \right). \quad (6.5)$$

The motivation for this optimization problem is to minimize the effect of the derivative in the η_3 -direction in the local energy W_{mp} . Due to the coercivity and continuity of the energy W_{mp} , it is clear that this function is well

defined and the infimum is attained. Note that φ^\natural and \bar{Q}_e^\natural depend on η_3 and h . Hence $W_{\text{mp}}^{\text{hom},\natural}(\mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural})$ depends on η_3 and h . While it is not immediately clear why $W_{\text{mp}}(\bar{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural|d^*)[(\nabla_x\Theta)^\natural]^{-1})$ can be expressed as a function of $\mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural}$, this aspect will be clarified in the rest of this subsection.

We do some lengthy but straightforward calculations in Appendix A.1 and after using the fact that $[\nabla_x\Theta]^{-T}e_3 = n_0$ and $[(\nabla_x\Theta)^\natural]^{-T}e_3 = n_0$, as well, we obtain the minimizer d^* from (6.5) as

$$d^* = \left(1 - \frac{\lambda}{2\mu + \lambda} \langle \mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural}, \mathbb{1}_3 \rangle \right) \bar{Q}_e^\natural n_0 + \frac{\mu_c - \mu}{\mu_c + \mu} \bar{Q}_e^\natural \mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural}^T n_0. \quad (6.6)$$

In terms of $\bar{Q}_e^\natural = \bar{R}^\natural Q_0^{\natural,T}$ we obtain the following expression for d^*

$$\begin{aligned} d^* &= \left(1 - \frac{\lambda}{2\mu + \lambda} \langle (Q_0^\natural \bar{R}^{\natural,T} \nabla_{(\eta_1,\eta_2)}\varphi^\natural - (\nabla y_0)^\natural | 0) [(\nabla_x\Theta)^\natural]^{-1}, \mathbb{1}_3 \rangle \right) \bar{R}^\natural Q_0^{\natural,T} n_0 \\ &\quad + \frac{\mu_c - \mu}{\mu_c + \mu} \bar{R}^\natural Q_0^{\natural,T} \left((Q_0^\natural \bar{R}^{\natural,T} \nabla_{(\eta_1,\eta_2)}\varphi^\natural - (\nabla y_0)^\natural | 0) [(\nabla_x\Theta)^\natural]^{-1} \right)^T n_0. \end{aligned} \quad (6.7)$$

Inserting d^* in the strain energy $W_{\text{mp}}(\bar{U}_h^\natural) = \mu \|\text{sym}(\bar{U}_h^\natural - \mathbb{1}_3)\|^2 + \mu_c \|\text{skew}(\bar{U}_h^\natural - \mathbb{1}_3)\|^2 + \frac{\lambda}{2} [\text{tr}(\text{sym}(\bar{U}_h^\natural - \mathbb{1}_3))]^2$ and using (A.36), (A.41) and (A.42), we obtain the explicit form of the homogenized energy for the membrane part

$$\begin{aligned} W_{\text{mp}}^{\text{hom},\natural}(\mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural}) &= \mu \|\text{sym} \mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural}\|^2 + \frac{\mu}{2} \frac{(\mu_c - \mu)^2}{(\mu_c + \mu)^2} \|\mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural}^T n_0\|^2 + \frac{\mu(\mu_c - \mu)}{(\mu_c + \mu)} \|\mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural}^T n_0\|^2 \\ &\quad + \frac{\mu \lambda^2}{(2\mu + \lambda)^2} \text{tr}(\mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural})^2 + \mu_c \|\text{skew}(\bar{Q}_e^{\natural,T}(\nabla_{(\eta_1,\eta_2)}\varphi^\natural | 0) [(\nabla_x\Theta)^\natural]^{-1})\|^2 \\ &\quad + \frac{\mu_c}{2} \frac{(\mu_c - \mu)^2}{(\mu_c + \mu)^2} \|\mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural}^T n_0\|^2 - \frac{\mu_c(\mu_c - \mu)}{(\mu_c + \mu)} \|\mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural}^T n_0\|^2 + \frac{2\mu^2\lambda}{(2\mu + \lambda)^2} \text{tr}(\mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural})^2, \end{aligned} \quad (6.8)$$

and finally

$$W_{\text{mp}}^{\text{hom},\natural}(\mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural}) = W_{\text{shell}}(\mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural}) - \frac{(\mu_c - \mu)^2}{2(\mu_c + \mu)} \|\mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural}^T n_0\|^2, \quad (6.9)$$

where

$$W_{\text{shell}}(X) = \mu \|\text{sym} X\|^2 + \mu_c \|\text{skew} X\|^2 + \frac{\lambda\mu}{\lambda + 2\mu} [\text{tr} X]^2.$$

Using the orthogonal decomposition in the tangential plane and in the normal direction, gives

$$X = X^\parallel + X^\perp, \quad X^\parallel := A_{y_0} X, \quad X^\perp := (\mathbb{1}_3 - A_{y_0}) X, \quad (6.10)$$

and we deduce that for all $X = (*|*|0) \cdot [\nabla_x\Theta(0)]^{-1}$ we have the following split in the expression of the considered quadratic forms

$$W_{\text{shell}}(X) = \mu \|\text{sym} X^\parallel\|^2 + \mu_c \|\text{skew} X^\parallel\|^2 + \frac{\mu + \mu_c}{2} \|X^\perp\|^2 + \frac{\lambda\mu}{\lambda + 2\mu} [\text{tr}(X)]^2. \quad (6.11)$$

Moreover, using that for all $X = (*|*|0) [\nabla_x\Theta(0)]^{-1}$, it holds that

$$\text{tr}(X^\perp) = \text{tr}((\mathbb{1}_3 - A_{y_0})X) = \text{tr}(X) - \text{tr}(A_{y_0}X) = \text{tr}(X) - \text{tr}(X A_{y_0}) = 0, \quad (6.12)$$

we obtain

$$\begin{aligned} W_{\text{shell}}(\mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural}) &= \mu \|\text{sym} \mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural}^\parallel\|^2 + \mu_c \|\text{skew} \mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural}^\parallel\|^2 + \frac{\lambda\mu}{\lambda + 2\mu} [\text{tr}(\mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural}^\parallel)]^2 + \frac{\mu + \mu_c}{2} \|\mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural}^\perp\|^2 \\ &= \mu \|\text{sym} \mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural}^\parallel\|^2 + \mu_c \|\text{skew} \mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural}^\parallel\|^2 + \frac{\lambda\mu}{\lambda + 2\mu} [\text{tr}(\mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural}^\parallel)]^2 + \frac{\mu + \mu_c}{2} \|\mathcal{E}_{\varphi^\natural,\bar{Q}_e^\natural}^T n_0\|^2. \end{aligned} \quad (6.13)$$

Therefore, the homogenized energy for the membrane part is

$$\begin{aligned}
W_{\text{mp}}^{\text{hom},\mathfrak{h}}(\mathcal{E}_{\varphi^{\mathfrak{h}},\overline{Q}_e^{\mathfrak{h}}}) &= \mu \|\text{sym } \mathcal{E}_{\varphi^{\mathfrak{h}},\overline{Q}_e^{\mathfrak{h}}}^{\parallel}\|^2 + \mu_c \|\text{skew } \mathcal{E}_{\varphi^{\mathfrak{h}},\overline{Q}_e^{\mathfrak{h}}}^{\parallel}\|^2 + \frac{\lambda \mu}{\lambda + 2\mu} [\text{tr}(\mathcal{E}_{\varphi^{\mathfrak{h}},\overline{Q}_e^{\mathfrak{h}}}^{\parallel})]^2 \\
&\quad + \frac{\mu + \mu_c}{2} \|\mathcal{E}_{\varphi^{\mathfrak{h}},\overline{Q}_e^{\mathfrak{h}}}^T n_0\|^2 - \frac{(\mu_c - \mu)^2}{2(\mu_c + \mu)} \|\mathcal{E}_{\varphi^{\mathfrak{h}},\overline{Q}_e^{\mathfrak{h}}}^T n_0\|^2 \\
&= \mu \|\text{sym } \mathcal{E}_{\varphi^{\mathfrak{h}},\overline{Q}_e^{\mathfrak{h}}}^{\parallel}\|^2 + \mu_c \|\text{skew } \mathcal{E}_{\varphi^{\mathfrak{h}},\overline{Q}_e^{\mathfrak{h}}}^{\parallel}\|^2 + \frac{\lambda \mu}{\lambda + 2\mu} [\text{tr}(\mathcal{E}_{\varphi^{\mathfrak{h}},\overline{Q}_e^{\mathfrak{h}}}^{\parallel})]^2 + \frac{2\mu \mu_c}{\mu_c + \mu} \|\mathcal{E}_{\varphi^{\mathfrak{h}},\overline{Q}_e^{\mathfrak{h}}}^T n_0\|^2 \quad (6.14) \\
&= W_{\text{shell}}(\mathcal{E}_{\varphi^{\mathfrak{h}},\overline{Q}_e^{\mathfrak{h}}}^{\parallel}) + \frac{2\mu \mu_c}{\mu_c + \mu} \|\mathcal{E}_{\varphi^{\mathfrak{h}},\overline{Q}_e^{\mathfrak{h}}}^{\perp}\|^2.
\end{aligned}$$

6.2 Homogenized membrane energy

Now, we will be able to propose the form of the homogenized membrane energy. To each pair $(m, \overline{Q}_{e,0})$, where $m : \omega \rightarrow \mathbb{R}^3$, $\overline{Q}_{e,0} : \omega \rightarrow \text{SO}(3)$, we associate the *elastic shell strain tensor*

$$\mathcal{E}_{m,s} := (\overline{Q}_{e,0}^T \nabla m - \nabla y_0 | 0) [\nabla_x \Theta(0)]^{-1}, \quad (6.15)$$

and we define the homogenized energy

$$W_{\text{mp}}^{\text{hom}}(\mathcal{E}_{m,s}) := \inf_{\tilde{d} \in \mathbb{R}^3} W_{\text{mp}}\left(\overline{Q}_{e,0}^T (\nabla m | \tilde{d}) [(\nabla_x \Theta)(0)]^{-1}\right) = \inf_{\tilde{d} \in \mathbb{R}^3} W_{\text{mp}}\left(\mathcal{E}_{m,s} - (0 | 0 | \tilde{d}) [(\nabla_x \Theta)(0)]^{-1}\right). \quad (6.16)$$

Direct calculations as in the previous subsection (6.1) show us that the infimum is attained for

$$\tilde{d}^* = \left(1 - \frac{\lambda}{2\mu + \lambda} \langle \mathcal{E}_{m,s}, \mathbb{1}_3 \rangle\right) \overline{Q}_{e,0} n_0 + \frac{\mu_c - \mu}{\mu_c + \mu} \overline{Q}_{e,0} \mathcal{E}_{m,s}^T n_0, \quad (6.17)$$

and

$$\begin{aligned}
W_{\text{mp}}^{\text{hom}}(\mathcal{E}_{m,s}) &= \mu \|\text{sym } \mathcal{E}_{m,s}^{\parallel}\|^2 + \mu_c \|\text{skew } \mathcal{E}_{m,s}^{\parallel}\|^2 + \frac{\lambda \mu}{\lambda + 2\mu} [\text{tr}(\mathcal{E}_{m,s}^{\parallel})]^2 + \frac{2\mu \mu_c}{\mu_c + \mu} \|\mathcal{E}_{m,s}^T n_0\|^2 \quad (6.18) \\
&= W_{\text{shell}}(\mathcal{E}_{m,s}^{\parallel}) + \frac{2\mu \mu_c}{\mu_c + \mu} \|\mathcal{E}_{m,s}^{\perp}\|^2,
\end{aligned}$$

where

$$W_{\text{shell}}(\mathcal{E}_{m,s}^{\parallel}) = \mu \|\text{sym } \mathcal{E}_{m,s}^{\parallel}\|^2 + \mu_c \|\text{skew } \mathcal{E}_{m,s}^{\parallel}\|^2 + \frac{\lambda \mu}{\lambda + 2\mu} [\text{tr}(\mathcal{E}_{m,s}^{\parallel})]^2. \quad (6.19)$$

Note that $W_{\text{mp}}^{\text{hom},\mathfrak{h}}(\mathcal{E}_{\varphi^{\mathfrak{h}},\overline{Q}_e^{\mathfrak{h}}})$ constructed in (6.14) depends on η_3 and h , while $W_{\text{mp}}^{\text{hom}}(\mathcal{E}_{m,s})$ in (6.18) does not depend on η_3 and h , since $\overline{Q}_{e,0}$ and $[(\nabla_x \Theta)(0)]$ do not depend on η_3 and h .

6.3 Homogenized curvature energy

We define the homogenized curvature energy as

$$\begin{aligned}
\widetilde{W}_{\text{curv}}^{\text{hom},\mathfrak{h}}(\mathcal{K}_e^{\mathfrak{h}}) &:= \widetilde{W}_{\text{curv}}\left(\text{axl}(\overline{Q}_e^{\mathfrak{h},T} \partial_{\eta_1} \overline{Q}_e^{\mathfrak{h}}) \mid \text{axl}(\overline{Q}_e^{\mathfrak{h},T} \partial_{\eta_2} \overline{Q}_e^{\mathfrak{h}}) \mid \text{axl}(A^*)\right) [(\nabla_x \Theta)^{\mathfrak{h}}]^{-1} \\
&= \inf_{A \in \mathfrak{so}(3)} \widetilde{W}_{\text{curv}}\left(\text{axl}(\overline{Q}_e^{\mathfrak{h},T} \partial_{\eta_1} \overline{Q}_e^{\mathfrak{h}}) \mid \text{axl}(\overline{Q}_e^{\mathfrak{h},T} \partial_{\eta_2} \overline{Q}_e^{\mathfrak{h}}) \mid \text{axl}(A)\right) [(\nabla_x \Theta)^{\mathfrak{h}}]^{-1}, \quad (6.20)
\end{aligned}$$

where

$$\mathcal{K}_e^{\mathfrak{h}} := \left(\text{axl}(\overline{Q}_e^{\mathfrak{h},T} \partial_{\eta_1} \overline{Q}_e^{\mathfrak{h}}) \mid \text{axl}(\overline{Q}_e^{\mathfrak{h},T} \partial_{\eta_2} \overline{Q}_e^{\mathfrak{h}}) \mid 0\right) [(\nabla_x \Theta)^{\mathfrak{h}}]^{-1},$$

represents a not fully reduced elastic shell bending-curvature tensor, in the sense that it still depends on η_3 and h , since $\overline{Q}_e^{\mathfrak{h}} = \overline{Q}_e^{\mathfrak{h}}(\eta_1, \eta_2, \eta_3)$. Therefore, $\widetilde{W}_{\text{curv}}^{\text{hom},\mathfrak{h}}(\mathcal{K}_e^{\mathfrak{h}})$ given by the above definitions still depends on η_3 and h .

As in the case of the homogenized membrane part in (4.10), from which we obtained the unknown d^* , one can explicitly determine the infinitesimal microrotation $A^* \in \mathfrak{so}(3)$ as well. Ghiba et. al, in [40] obtained the homogenized quadratic curvature energy (see Appendix A.4 for its explicit form). Presently, it is enough to see that $\widetilde{W}_{\text{curv}}^{\text{hom}}$ is uniquely defined and has the other requirements like remaining convex in its argument and having the same growth as $\widetilde{W}_{\text{curv}}$. Therefore,

$$\widetilde{W}_{\text{curv}}(\Gamma_h^{\sharp}) \geq \widetilde{W}_{\text{curv}}^{\text{hom}, \sharp}(\mathcal{K}_e^{\sharp}), \quad (6.21)$$

i.e.,

$$\begin{aligned} \widetilde{W}_{\text{curv}}\left(\left(\text{axl}(\overline{Q}_{e,h}^{\sharp,T} \partial_{\eta_1} \overline{Q}_{e,h}^{\sharp}) \mid \text{axl}(\overline{Q}_{e,h}^{\sharp,T} \partial_{\eta_2} \overline{Q}_{e,h}^{\sharp}) \mid \frac{1}{h} \text{axl}(\overline{Q}_{e,h}^{\sharp,T} \partial_{\eta_3} \overline{Q}_{e,h}^{\sharp})\right) [(\nabla_x \Theta)^{\sharp}]^{-1}\right) \\ \geq \widetilde{W}_{\text{curv}}^{\text{hom}, \sharp}\left(\left(\text{axl}(\overline{Q}_e^{\sharp,T} \partial_{\eta_1} \overline{Q}_e^{\sharp}) \mid \text{axl}(\overline{Q}_e^{\sharp,T} \partial_{\eta_2} \overline{Q}_e^{\sharp}) \mid 0\right) [(\nabla_x \Theta)^{\sharp}]^{-1}\right), \end{aligned} \quad (6.22)$$

where this relation will help us in Subsection 7.1 to show the lim inf condition for the curvature energy.

In order to construct the Γ -limit, we have to define a homogenized curvature energy. This energy will be expressed in terms of the elastic shell bending-curvature tensor

$$\mathcal{K}_{e,s} := \left(\text{axl}(\overline{Q}_{e,0}^T \partial_{x_1} \overline{Q}_{e,0}) \mid \text{axl}(\overline{Q}_{e,0}^T \partial_{x_2} \overline{Q}_{e,0}) \mid 0\right) [\nabla_x \Theta(0)]^{-1} \notin \text{Sym}(3) \quad \text{elastic shell bending-curvature tensor},$$

which will be defined for any $\overline{Q}_{e,0} : \omega \rightarrow \text{SO}(3)$. For $\overline{Q}_{e,0} : \omega \rightarrow \text{SO}(3)$, we set

$$\begin{aligned} \widetilde{W}_{\text{curv}}^{\text{hom}}(\mathcal{K}_{e,s}) &:= \widetilde{W}_{\text{curv}}^* \left(\text{axl}(\overline{Q}_{e,0}^T \partial_{\eta_1} \overline{Q}_{e,0}) \mid \text{axl}(\overline{Q}_{e,0}^T \partial_{\eta_2} \overline{Q}_{e,0}) \mid \text{axl}(A^*)\right) [(\nabla_x \Theta)^{\sharp}(0)]^{-1} \\ &= \inf_{A \in \mathfrak{so}(3)} \widetilde{W}_{\text{curv}} \left(\text{axl}(\overline{Q}_{e,0}^T \partial_{\eta_1} \overline{Q}_{e,0}) \mid \text{axl}(\overline{Q}_{e,0}^T \partial_{\eta_2} \overline{Q}_{e,0}) \mid \text{axl}(A)\right) [(\nabla_x \Theta)^{\sharp}(0)]^{-1}. \end{aligned} \quad (6.23)$$

Again note that while $\widetilde{W}_{\text{curv}}^{\text{hom}, \sharp}(\mathcal{K}_e^{\sharp})$ (previously constructed) depends on η_3 and h , $\widetilde{W}_{\text{curv}}^{\text{hom}}(\mathcal{K}_{e,s})$ does not depend on η_3 and h , since $\overline{Q}_{e,0}$ and $[(\nabla_x \Theta)(0)]$ do not depend on η_3 and h .

7 Γ -convergence of \mathcal{J}_{h_j}

We are now ready to formulate the main result of this paper

Theorem 7.1. *Assume that the initial configuration of the curved shell is defined by a continuous injective mapping $y_0 : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ which admits an extension to $\overline{\omega}$ into $C^2(\overline{\omega}; \mathbb{R}^3)$ such that for*

$$\Theta(x_1, x_2, x_3) = y_0(x_1, x_2) + x_3 n_0(x_1, x_2)$$

we have $\det[\nabla_x \Theta(0)] \geq a_0 > 0$ on $\overline{\omega}$, where a_0 is a constant, and assume that the boundary data satisfy the conditions

$$\varphi_d^{\sharp} = \varphi_d|_{\Gamma_1} \text{ (in the sense of traces) for } \varphi_d \in H^1(\Omega_1; \mathbb{R}^3). \quad (7.1)$$

Let the constitutive parameters satisfy

$$\mu > 0, \quad \kappa > 0, \quad \mu_c > 0, \quad a_1 > 0, \quad a_2 > 0, \quad a_3 > 0. \quad (7.2)$$

Then, for any sequence $(\varphi_{h_j}^{\sharp}, \overline{Q}_{e,h_j}^{\sharp}) \in X$ such that $(\varphi_{h_j}^{\sharp}, \overline{Q}_{e,h_j}^{\sharp}) \rightarrow (\varphi_0, \overline{Q}_{e,0})$ as $h_j \rightarrow 0$, the sequence of functionals $\mathcal{J}_{h_j} : X \rightarrow \overline{\mathbb{R}}$ Γ -converges to the limit energy functional $\mathcal{J}_0 : X \rightarrow \overline{\mathbb{R}}$ defined by

$$\mathcal{J}_0(m, \overline{Q}_{e,0}) = \begin{cases} \int_{\omega} [W_{\text{mp}}^{\text{hom}}(\mathcal{E}_{m, \overline{Q}_{e,0}}) + \widetilde{W}_{\text{curv}}^{\text{hom}}(\mathcal{K}_{e,s})] \det(\nabla y_0 | n_0) \, d\omega & \text{if } (m, \overline{Q}_{e,0}) \in \mathcal{S}'_{\omega}, \\ +\infty & \text{else in } X, \end{cases} \quad (7.3)$$

where

$$\begin{aligned}
m(x_1, x_2) &:= \varphi_0(x_1, x_2) = \lim_{h_j \rightarrow 0} \varphi_{h_j}^{\natural}(x_1, x_2, \frac{1}{h_j}x_3), & \overline{Q}_{e,0}(x_1, x_2) &= \lim_{h_j \rightarrow 0} \overline{Q}_{e,h_j}^{\natural}(x_1, x_2, \frac{1}{h_j}x_3), \\
\mathcal{E}_{m, \overline{Q}_{e,0}} &= (\overline{Q}_{e,0}^T \nabla m - \nabla y_0 | 0) [\nabla_x \Theta(0)]^{-1}, \\
\mathcal{K}_{e,s} &= \left(\text{axl}(\overline{Q}_{e,0}^T \partial_{x_1} \overline{Q}_{e,0}) | \text{axl}(\overline{Q}_{e,0}^T \partial_{x_2} \overline{Q}_{e,0}) | 0 \right) [\nabla_x \Theta(0)]^{-1} \notin \text{Sym}(3),
\end{aligned} \tag{7.4}$$

and

$$\begin{aligned}
W_{\text{mp}}^{\text{hom}}(\mathcal{E}_{m, \overline{Q}_{e,0}}) &= \mu \|\text{sym } \mathcal{E}_{m, \overline{Q}_{e,0}}^{\parallel}\|^2 + \mu_c \|\text{skew } \mathcal{E}_{m, \overline{Q}_{e,0}}^{\parallel}\|^2 + \frac{\lambda \mu}{\lambda + 2\mu} [\text{tr}(\mathcal{E}_{m, \overline{Q}_{e,0}}^{\parallel})]^2 + \frac{2\mu \mu_c}{\mu_c + \mu} \|\mathcal{E}_{m, \overline{Q}_{e,0}}^T n_0\|^2 \\
&= W_{\text{shell}}(\mathcal{E}_{m, \overline{Q}_{e,0}}^{\parallel}) + \frac{2\mu \mu_c}{\mu_c + \mu} \|\mathcal{E}_{m, \overline{Q}_{e,0}}^{\perp}\|^2, \\
\widetilde{W}_{\text{curv}}^{\text{hom}}(\mathcal{K}_{e,s}) &= \inf_{A \in \mathfrak{so}(3)} \widetilde{W}_{\text{curv}} \left(\text{axl}(\overline{Q}_{e,0}^T \partial_{\eta_1} \overline{Q}_{e,0}) | \text{axl}(\overline{Q}_{e,0}^T \partial_{\eta_2} \overline{Q}_{e,0}) | \text{axl}(A) \right) [(\nabla_x \Theta)^{\natural}(0)]^{-1} \\
&= \mu L_c^2 \left(b_1 \|\text{sym } \mathcal{K}_{e,s}^{\parallel}\|^2 + b_2 \|\text{skew } \mathcal{K}_{e,s}^{\parallel}\|^2 + \frac{b_1 b_3}{(b_1 + b_3)} \text{tr}(\mathcal{K}_{e,s}^{\parallel})^2 + \frac{2b_1 b_2}{b_1 + b_2} \|\mathcal{K}_{e,s}^{\perp}\|^2 \right).
\end{aligned} \tag{7.5}$$

Proof. The first part of the proof is represented by the proof of equi-coercivity and compactness of the family of energy functionals which are already done. The rest of the proof will be divided into two parts which make the subjects of the following two subsections. \blacksquare

7.1 Step 1 of the proof: The lim-inf condition

In this section we prove the following lemma

Lemma 7.2. *In the hypothesis of Theorem 7.1, for any sequence $(\varphi_{h_j}^{\natural}, \overline{Q}_{e,h_j}^{\natural}) \in X$ such that $(\varphi_{h_j}^{\natural}, \overline{Q}_{e,h_j}^{\natural}) \rightarrow (\varphi_0^{\natural}, \overline{Q}_{e,0}^{\natural})$ for $h_j \rightarrow 0$, i.e.,*

$$\varphi_{h_j}^{\natural} \rightharpoonup \varphi_0^{\natural} \quad \text{in } L^2(\Omega_1, \mathbb{R}^3), \quad \overline{Q}_{e,h_j}^{\natural} \rightarrow \overline{Q}_{e,0}^{\natural} \quad \text{in } L^2(\Omega_1, \text{SO}(3)), \tag{7.6}$$

we have

$$\mathcal{J}_0(\varphi_0^{\natural}, \overline{Q}_{e,0}^{\natural}) \leq \liminf_{h_j \rightarrow 0} \mathcal{J}_{h_j}^{\natural}(\varphi_{h_j}^{\natural}, \overline{Q}_{e,h_j}^{\natural}). \tag{7.7}$$

Proof. It is clear that we may restrict our proof to sequences $(\varphi_{h_j}^{\natural}, \overline{Q}_{e,h_j}^{\natural}) \in \mathcal{S}' \subset X'$, i.e., to sequences in which the functionals $\mathcal{J}_{h_j}^{\natural}(\varphi_{h_j}^{\natural}, \overline{Q}_{e,h_j}^{\natural})$ are finite, since otherwise the statement is satisfied. In addition, any $(\varphi_{h_j}^{\natural}, \overline{Q}_{e,h_j}^{\natural})$ such that $\mathcal{J}_{h_j}^{\natural}(\varphi_{h_j}^{\natural}, \overline{Q}_{e,h_j}^{\natural}) < \infty$ is uniformly bounded in X' . Therefore, there exists a subsequence (not relabeled) which is weakly convergent in X' . Due to the strong convergence of the original sequence, the considered subsequence is weakly convergent to $(\varphi^{\natural}, \overline{Q}_{e,0}^{\natural})$, i.e.,

$$\varphi_{h_j}^{\natural} \rightharpoonup \varphi_0^{\natural} \quad \text{in } L^2(\Omega_1, \mathbb{R}^3), \quad \overline{Q}_{e,h_j}^{\natural} \rightharpoonup \overline{Q}_{e,0}^{\natural} \quad \text{in } L^2(\Omega_1, \text{SO}(3)). \tag{7.8}$$

Therefore, we have the weak convergence $(\varphi_{h_j}^{\natural}, \overline{Q}_{e,h_j}^{\natural})$ (without relabeling it) to $(\varphi_0^{\natural}, \overline{Q}_{e,0}^{\natural})$ in $H^1(\omega, \mathbb{R}^3) \times H^1(\omega, \text{SO}(3))$. For $\overline{U}_h^{\natural} = \overline{Q}_e^{\natural,T} \nabla_{\eta}^h \varphi^{\natural} [(\nabla_x \Theta)^{\natural}]^{-1}$ we have

$$W_{\text{mp}}(\overline{U}_h^{\natural}) = \mu \|\text{sym}(\overline{U}_h^{\natural} - \mathbb{1}_3)\|^2 + \mu_c \|\text{skew}(\overline{U}_h^{\natural} - \mathbb{1}_3)\|^2 + \frac{\lambda}{2} [\text{tr}(\text{sym}(\overline{U}_h^{\natural} - \mathbb{1}_3))]^2, \tag{7.9}$$

while for $\mathcal{E}_{\varphi^{\natural}, \overline{Q}_e^{\natural}} = \mathcal{E}_{\varphi^{\natural}, \overline{Q}_e^{\natural}}^{\parallel} + \mathcal{E}_{\varphi^{\natural}, \overline{Q}_e^{\natural}}^{\perp}$ with $\mathcal{E}_{\varphi^{\natural}, \overline{Q}_e^{\natural}} = (\overline{Q}_e^{\natural,T} \nabla_{(\eta_1, \eta_2)} \varphi^{\natural} - [\nabla y_0]^{\natural} | 0) [(\nabla_x \Theta)^{\natural}]^{-1}$ we have

$$W_{\text{mp}}^{\text{hom}, \natural}(\mathcal{E}_{\varphi^{\natural}, \overline{Q}_e^{\natural}}) = \mu \|\text{sym } \mathcal{E}_{\varphi^{\natural}, \overline{Q}_e^{\natural}}^{\parallel}\|^2 + \mu_c \|\text{skew } \mathcal{E}_{\varphi^{\natural}, \overline{Q}_e^{\natural}}^{\parallel}\|^2 + \frac{\lambda \mu}{\lambda + 2\mu} [\text{tr}(\mathcal{E}_{\varphi^{\natural}, \overline{Q}_e^{\natural}}^{\parallel})]^2 + \frac{2\mu \mu_c}{\mu_c + \mu} \|\mathcal{E}_{\varphi^{\natural}, \overline{Q}_e^{\natural}}^{\perp}\|^2.$$

Hence, for the sequence $(\varphi_{h_j}^\natural, \overline{Q}_{e,h_j}^\natural) \in H^1(\Omega_1, \mathbb{R}^3) \times H^1(\Omega_1, \text{SO}(3))$ where $(\varphi_{h_j}^\natural, \overline{Q}_{e,h_j}^\natural) \rightarrow (\varphi_0^\natural, \overline{Q}_{e,0}^\natural)$ with $J_{h_j}^\natural(\varphi_{h_j}^\natural, \overline{Q}_{e,h_j}^\natural) < \infty$, we have

$$W_{\text{mp}}(\overline{Q}_{e,h_j}^{\natural,T} \nabla_{\eta}^{h_j} \varphi_{h_j}^\natural [(\nabla_x \Theta)^\natural]^{-1}) = W_{\text{mp}}\left(\overline{Q}_{e,h_j}^{\natural,T} (\nabla_{(\eta_1, \eta_2)} \varphi_{h_j}^\natural | \frac{1}{h_j} \partial_{\eta_3} \varphi_{h_j}^\natural) [(\nabla_x \Theta)^\natural]^{-1}\right) \quad (7.10)$$

$$\geq W_{\text{mp}}^{\text{hom}, \natural}\left(\mathcal{E}_{\varphi_{h_j}^\natural, \overline{Q}_{e,h_j}^\natural}\right), \quad (7.11)$$

where we recall that $\mathcal{E}_{\varphi_{h_j}^\natural, \overline{Q}_{e,h_j}^\natural} := (\overline{Q}_{e,h_j}^{\natural,T} \nabla_{(\eta_1, \eta_2)} \varphi_{h_j}^\natural - (\nabla y_0)^\natural | 0) [(\nabla_x \Theta)^\natural]^{-1}$.

Then by taking the integral over Ω_1 on both sides and taking the \liminf for h_j , we obtain

$$\liminf_{h_j \rightarrow 0} \int_{\Omega_1} W_{\text{mp}}(\overline{Q}_{e,h_j}^{\natural,T} \nabla_{\eta}^{h_j} \varphi_{h_j}^\natural [(\nabla_x \Theta)^\natural]^{-1}) \det[\nabla_x \Theta]^\natural(\eta) dV_\eta \geq \liminf_{h_j \rightarrow 0} \int_{\Omega_1} W_{\text{mp}}^{\text{hom}, \natural}\left(\mathcal{E}_{\varphi_{h_j}^\natural, \overline{Q}_{e,h_j}^\natural}\right) \det[\nabla_x \Theta]^\natural(\eta) dV_\eta.$$

In the expression of $\mathcal{E}_{\varphi_{h_j}^\natural, \overline{Q}_{e,h_j}^\natural}$, the quantity $[(\nabla_x \Theta)^\natural]^{-1}$ is evaluated in $(x_1, x_2, x_3) = (\eta_1, \eta_2, h \eta_3)$. Therefore, we have to study its behaviour for $h_j \rightarrow 0$. In addition, we recall the convergence results [47, Lemma 1]:

$$\begin{aligned} \lim_{h_j \rightarrow 0} \det[\nabla_x \Theta]^\natural(\eta_1, \eta_2, \eta_3) &= \lim_{h_j \rightarrow 0} \det[\nabla_x \Theta]^\natural(x_1, x_2, \frac{1}{h_j} x_3) = \det[\nabla_x \Theta]^\natural(\eta_1, \eta_2, 0) \\ &= \det(\nabla y_0 | n_0) \quad \text{in } C^0(\overline{\Omega}), \\ \lim_{h_j \rightarrow 0} [(\nabla_x \Theta)^{-1}]^\natural(\eta_1, \eta_2, \eta_3) &= \lim_{h_j \rightarrow 0} [(\nabla_x \Theta)^{-1}]^\natural(x_1, x_2, \frac{1}{h_j} x_3) \\ &= [(\nabla_x \Theta)^{-1}]^\natural(\eta_1, \eta_2, 0) = (\nabla_x \Theta)^{-1}(0) \quad \text{in } C^0(\overline{\Omega}). \end{aligned} \quad (7.12)$$

Due to (7.12), the weak convergence of the sequence $\varphi_{h_j}^\natural$ and the strong convergence of the sequence $\overline{Q}_{e,h_j}^\natural$, we have the weak convergence

$$\begin{aligned} \mathcal{E}_{\varphi_{h_j}^\natural, \overline{Q}_{e,h_j}^\natural} &= (\overline{Q}_{e,h_j}^{\natural,T} \nabla_{(\eta_1, \eta_2)} \varphi_{h_j}^\natural - (\nabla y_0)^\natural | 0) [(\nabla_x \Theta)^\natural]^{-1} \\ &\rightharpoonup (\overline{Q}_{e,0}^{\natural,T} \nabla_{(\eta_1, \eta_2)} \varphi_0^\natural - (\nabla y_0)^\natural | 0) [(\nabla_x \Theta)^\natural]^{-1}(0) =: \mathcal{E}_{\varphi_0^\natural, \overline{Q}_{e,0}^\natural}. \end{aligned} \quad (7.13)$$

Using again (7.12), the convexity of the energy function $W_{\text{mp}}^{\text{hom}, \natural}$ with respect to $\mathcal{E}_{\varphi_{h_j}^\natural, \overline{Q}_{e,h_j}^\natural}$, the Fatou's Lemma, the characterization of \liminf and the weak convergence (7.13) we get

$$\liminf_{h_j \rightarrow 0} \int_{\Omega_1} W_{\text{mp}}^{\text{hom}, \natural}\left(\mathcal{E}_{\varphi_{h_j}^\natural, \overline{Q}_{e,h_j}^\natural}\right) \det[\nabla_x \Theta]^\natural(\eta) dV_\eta \geq \int_{\Omega_1} W_{\text{mp}}^{\text{hom}, \natural}\left(\mathcal{E}_{\varphi_0^\natural, \overline{Q}_{e,0}^\natural}\right) \det(\nabla y_0 | n_0) dV_\eta. \quad (7.14)$$

Since both φ_0^\natural and $\overline{Q}_{e,0}^\natural$ are independent of the transverse variable η_3 , we also obtain

$$\begin{aligned} \liminf_{h_j \rightarrow 0} \int_{\Omega_1} W_{\text{mp}}(\overline{Q}_{e,h_j}^{\natural,T} \nabla_{\eta}^{h_j} \varphi_{h_j}^\natural [(\nabla_x \Theta)^\natural]^{-1}) \det[\nabla_x \Theta]^\natural(\eta) dV_\eta &\geq \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\omega} W_{\text{mp}}^{\text{hom}, \natural}\left(\mathcal{E}_{\varphi_0^\natural, \overline{Q}_{e,0}^\natural}\right) \det(\nabla y_0 | n_0) dV_\eta \\ &= \int_{\omega} W_{\text{mp}}^{\text{hom}}\left(\mathcal{E}_{\varphi_0^\natural, \overline{Q}_{e,0}^\natural}\right) \det(\nabla y_0 | n_0) d\omega. \end{aligned} \quad (7.15)$$

We do the same process for the curvature energy, by using (6.21), the convexity of $\widetilde{W}_{\text{curv}}^{\text{hom}}$ in its argument and the weak convergence

$$\begin{aligned} &\left(\text{axl}(\overline{Q}_{e,h_j}^{\natural,T} \partial_{\eta_1} \overline{Q}_{e,h_j}^\natural) | \text{axl}(\overline{Q}_{e,h_j}^{\natural,T} \partial_{\eta_2} \overline{Q}_{e,h_j}^\natural) | 0\right) [(\nabla_x \Theta)^\natural(\eta)]^{-1} \\ &\rightharpoonup \left(\text{axl}(\overline{Q}_{e,0}^{\natural,T} \partial_{\eta_1} \overline{Q}_{e,0}^\natural) | \text{axl}(\overline{Q}_{e,0}^{\natural,T} \partial_{\eta_2} \overline{Q}_{e,0}^\natural) | 0\right) [\nabla_x \Theta(0)]^{-1}. \end{aligned} \quad (7.16)$$

Using also (7.12), we arrive at

$$\begin{aligned}
\liminf_{h_j} \int_{\Omega_1} \widetilde{W}_{\text{curv}}(\Gamma_h^\natural) \det[\nabla_x \Theta]^\natural(\eta) dV_\eta &\geq \liminf_{h_j} \int_{\Omega_1} \widetilde{W}_{\text{curv}}^{\text{hom}, \natural}(\mathcal{K}_e^\natural) \det[\nabla_x \Theta]^\natural(\eta) dV_\eta \\
&\geq \liminf_{h_j} \int_{\Omega_1} \widetilde{W}_{\text{curv}}^{\text{hom}}(\mathcal{K}_{e,s}) \det[\nabla_x \Theta]^\natural(\eta) dV_\eta \geq \int_{\Omega_1} \widetilde{W}_{\text{curv}}^{\text{hom}}(\mathcal{K}_{e,s}) \det(\nabla y_0|n_0) dV_\eta \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\omega} \widetilde{W}_{\text{curv}}^{\text{hom}}(\mathcal{K}_{e,s}) \det(\nabla y_0|n_0) dV_\eta = \int_{\omega} \widetilde{W}_{\text{curv}}^{\text{hom}}(\mathcal{K}_{e,s}) \det(\nabla y_0|n_0) d\omega.
\end{aligned} \tag{7.17}$$

Since, $W_{\text{mp}}(\overline{Q}_{e,h_j}^{\natural,T} \nabla_\eta^{h_j} \varphi_{h_j}^\natural [(\nabla_x \Theta)^\natural]^{-1}) > 0$ and $\widetilde{W}_{\text{curv}}(\Gamma_h^\natural) > 0$, by combining (7.15) and (7.17) we deduce

$$\begin{aligned}
\liminf_{h_j} \int_{\Omega_1} [W_{\text{mp}}(\overline{Q}_{e,h_j}^{\natural,T} \nabla_\eta^{h_j} \varphi_{h_j}^\natural [(\nabla_x \Theta)^\natural]^{-1}) + \widetilde{W}_{\text{curv}}(\Gamma_h^\natural)] \det[\nabla_x \Theta]^\natural(\eta) dV_\eta \\
\geq \int_{\omega} \left(W_{\text{mp}}^{\text{hom}}(\mathcal{E}_{m, \overline{Q}_{e,0}}) + \widetilde{W}_{\text{curv}}^{\text{hom}}(\mathcal{K}_{e,s}) \right) \det(\nabla y_0|n_0) d\omega = \mathcal{J}_0(m, \overline{Q}_{e,0}),
\end{aligned} \tag{7.18}$$

where we have used that $\overline{Q}_{e,0}^\natural \equiv \overline{Q}_{e,0}$ and $m = \varphi_0$. Hence, the lim-inf inequality (7.7) is proven. \blacksquare

7.2 Step 2 of the proof: The lim-sup condition - recovery sequence

Now we show the following lemma

Lemma 7.3. *In the hypothesis of Theorem 7.1, for all $(\varphi_0^\natural, \overline{Q}_{e,0}^\natural) \in L^2(\Omega_1) \times L^2(\Omega_1, \text{SO}(3))$ there exists $(\varphi_{h_j}^\natural, \overline{Q}_{e,h_j}^\natural) \in L^2(\Omega_1) \times L^2(\Omega_1, \text{SO}(3))$ with $(\varphi_{h_j}^\natural, \overline{Q}_{e,h_j}^\natural) \rightarrow (\varphi_0^\natural, \overline{Q}_{e,0}^\natural)$ such that*

$$\mathcal{J}_0(\varphi_0^\natural, \overline{Q}_{e,0}^\natural) \geq \limsup_{h_j \rightarrow 0} \mathcal{J}_{h_j}(\varphi_{h_j}^\natural, \overline{Q}_{e,h_j}^\natural). \tag{7.19}$$

Proof. Similar to the case of the lim-inf inequality, we can restrict our attention to sequences $(\varphi_{h_j}^\natural, \overline{Q}_{e,h_j}^\natural) \in X$ such that $\mathcal{J}_{h_j}(\varphi_{h_j}^\natural, \overline{Q}_{e,h_j}^\natural) < \infty$. Therefore, the sequence $(\varphi_{h_j}^\natural, \overline{Q}_{e,h_j}^\natural) \in X$ has a weakly convergent subsequence in X' , and we can focus on the space $H^1(\Omega_1, \mathbb{R}^3) \times H^1(\Omega_1, \text{SO}(3))$.

One of the requirements for Γ -convergence, is the existence of a recovery sequence. Thus, the idea is to define an expansion for the deformation and the microrotation through the thickness. In reality, the minimizers of the energy model can be a good candidate for constructing the recovery sequence. To do so, we look at the first order Taylor expansion of the nonlinear deformation $\varphi_{h_j}^\natural$ in thickness direction η_3

$$\varphi_{h_j}^\natural(\eta_1, \eta_2, \eta_3) = \varphi_{h_j}^\natural(\eta_1, \eta_2, 0) + \eta_3 \partial_{\eta_3} \varphi_{h_j}^\natural(\eta_1, \eta_2, 0). \tag{7.20}$$

With the formula

$$d^* = \left(1 - \frac{\lambda}{2\mu + \lambda} \langle \mathcal{E}_{m,s}, \mathbb{1}_3 \rangle \right) \overline{Q}_{e,0}^\natural n_0 + \frac{\mu_c - \mu}{\mu_c + \mu} \overline{Q}_{e,0}^\natural \mathcal{E}_{m,s}^T n_0, \tag{7.21}$$

and replacing $\frac{1}{h_j} \partial_{\eta_3} \varphi_{h_j}^\natural(\eta_1, \eta_2, 0)$ with $d^*(\eta_1, \eta_2)$, which means replacing $\partial_{\eta_3} \varphi_{h_j}^\natural(\eta_1, \eta_2, 0)$ by $h_j d^*(\eta_1, \eta_2)$, we make an ansatz for our recovery sequence as following

$$\varphi_{h_j}^\natural(\eta_1, \eta_2, \eta_3) := \varphi_0^\natural(\eta_1, \eta_2) + h_j \eta_3 d^*(\eta_1, \eta_2). \tag{7.22}$$

Since $\nabla_{(\eta_1, \eta_2)} \varphi^\natural \in L^2(\omega, \mathbb{R}^3)$ and $\overline{Q}_{e,0} \in \text{SO}(3)$, we obtain that d^* belongs to $L^2(\omega, \mathbb{R}^3)$ and by letting $h_j \rightarrow 0$, it can be seen that for this ansatz $\varphi_{h_j}^\natural \rightarrow \varphi_0^\natural$.

The reconstruction for the rotation $\overline{Q}_{e,0}$ is not obvious, since on the one hand we have to maintain the rotation constraint along the sequence and on the other hand we must approach the lower bound, which excludes the simple reconstruction $\overline{Q}_{e,h_j}^\natural(\eta_1, \eta_2, \eta_3) = \overline{Q}_{e,0}(\eta_1, \eta_2)$. In order to meet both requirements we consider therefore

$$\overline{Q}_{e,h_j}^\natural(\eta_1, \eta_2, \eta_3) := \overline{Q}_{e,0}(\eta_1, \eta_2) \cdot \exp(h_j \eta_3 A^*(\eta_1, \eta_2)), \tag{7.23}$$

where $A^* \in \mathfrak{so}(3)$ is the term obtained in (6.20), depending on the given $\overline{Q}_{e,0}$, and we note that $A^* \in L^2(\omega, \mathfrak{so}(3))$ by the coercivity of $\widetilde{W}_{\text{curv}}$. Since $\exp : \mathfrak{so}(3) \rightarrow \text{SO}(3)$, we obtain that $\overline{Q}_{e,h_j}^{\natural} \in \text{SO}(3)$ and for $h_j \rightarrow 0$, we have $\overline{Q}_{e,h_j}^{\natural} \rightarrow \overline{Q}_{e,0} \in L^2(\Omega_1, \text{SO}(3))$.

Since d^* need not to be differentiable, we should consider another modified recovery sequence. For fixed $\varepsilon > 0$, we select $d_\varepsilon \in H^1(\omega, \mathbb{R}^3)$ such that $\|d_\varepsilon - d^*\|_{L^2(\omega, \mathbb{R}^3)} < \varepsilon$. Therefore, accordingly we define the final recovery sequence for the deformation as following

$$\varphi_{h_j, \varepsilon}^{\natural}(\eta_1, \eta_2, \eta_3) := \varphi_0^{\natural}(\eta_1, \eta_2) + h_j \eta_3 d_\varepsilon(\eta_1, \eta_2). \quad (7.24)$$

The same argument holds for A^* , i.e., for fixed $\varepsilon > 0$ we may choose $A_\varepsilon \in H^1(\omega, \mathfrak{so}(3))$ such that $\|A_\varepsilon - A^*\|_{L^2(\omega, \mathfrak{so}(3))} < \varepsilon$. Hence, the final recovery sequence for the microrotation is

$$\overline{Q}_{e,h_j, \varepsilon}^{\natural}(\eta_1, \eta_2, \eta_3) := \overline{Q}_{e,0}(\eta_1, \eta_2) \cdot \exp(h_j \eta_3 A_\varepsilon(\eta_1, \eta_2)). \quad (7.25)$$

The gradient of the new recovery sequence of deformation is

$$\begin{aligned} \nabla_\eta \varphi_{h_j, \varepsilon}^{\natural}(\eta_1, \eta_2, \eta_3) &= (\nabla_{(\eta_1, \eta_2)} \varphi_0^{\natural}(\eta_1, \eta_2)|0) + h_j(0|d_\varepsilon(\eta_1, \eta_2)) + h_j \eta_3 (\nabla_{(\eta_1, \eta_2)} d_\varepsilon(\eta_1, \eta_2)|0) \\ &= (\nabla \varphi_0^{\natural}(\eta_1, \eta_2)|h_j d_\varepsilon(\eta_1, \eta_2)) + h_j \eta_3 (\nabla d_\varepsilon(\eta_1, \eta_2)|0), \end{aligned} \quad (7.26)$$

and the different terms in the curvature energy are

$$\begin{aligned} \overline{Q}_{e,h_j, \varepsilon}^{\natural, T} \partial_{\eta_1} \overline{Q}_{e,h_j, \varepsilon}^{\natural} &= \exp(h_j \eta_3 A_\varepsilon)^T \overline{Q}_{e,0}^T [\partial_{\eta_1} \overline{Q}_{e,0} \exp(h_j \eta_3 A_\varepsilon) + \overline{Q}_{e,0} D \exp(h_j \eta_3 A_\varepsilon) \cdot [h_j \eta_3 \partial_{\eta_1} A_\varepsilon]], \\ \overline{Q}_{e,h_j, \varepsilon}^{\natural, T} \partial_{\eta_2} \overline{Q}_{e,h_j, \varepsilon}^{\natural} &= \exp(h_j \eta_3 A_\varepsilon)^T \overline{Q}_{e,0}^T [\partial_{\eta_2} \overline{Q}_{e,0} \exp(h_j \eta_3 A_\varepsilon) + \overline{Q}_{e,0} D \exp(h_j \eta_3 A_\varepsilon) \cdot [h_j \eta_3 \partial_{\eta_2} A_\varepsilon]], \\ \overline{Q}_{e,h_j, \varepsilon}^{\natural, T} \partial_{\eta_3} \overline{Q}_{e,h_j, \varepsilon}^{\natural} &= \exp(h_j \eta_3 A_\varepsilon)^T \overline{Q}_{e,0}^T [\partial_{\eta_3} \overline{Q}_{e,0} \exp(h_j \eta_3 A_\varepsilon) + \overline{Q}_{e,0} D \exp(h_j \eta_3 A_\varepsilon) \cdot [h_j A_\varepsilon]] \\ &= h_j \exp(h_j \eta_3 A_\varepsilon(\eta_1, \eta_2))^T D \exp(h_j \eta_3 A_\varepsilon(\eta_1, \eta_2)) \cdot [A_\varepsilon], \end{aligned} \quad (7.27)$$

with $\partial_{\eta_i} A_\varepsilon \in \mathfrak{so}(3)$. Now we introduce the quantities

$$\begin{aligned} \tilde{U}_0 &= \overline{Q}_{e,0}^T (\nabla \varphi_0(\eta_1, \eta_2)|d^*(\eta_1, \eta_2))[(\nabla_x \Theta)(0)]^{-1}, \\ \tilde{U}_{h_j}^\varepsilon &= \overline{Q}_{e,h_j, \varepsilon}^{\natural, T} \left((\nabla \varphi_0(\eta_1, \eta_2)|d_\varepsilon(\eta_1, \eta_2)) + h_j \eta_3 (\nabla d_\varepsilon(\eta_1, \eta_2)|0) \right) [(\nabla_x \Theta)^\natural(\eta)]^{-1}, \\ \tilde{U}_0^\varepsilon &= \overline{Q}_{e,0}^T (\nabla \varphi_0(\eta_1, \eta_2)|d_\varepsilon(\eta_1, \eta_2))[(\nabla_x \Theta)(0)]^{-1}, \\ \Gamma_{h_j, \varepsilon}^{\natural} &:= \underbrace{\left(\text{axl} \left(\overline{Q}_{e,h_j, \varepsilon}^{\natural, T} \partial_{\eta_1} \overline{Q}_{e,h_j, \varepsilon}^{\natural} \right) \right)}_{:= \Gamma_{h_j, \varepsilon}^{1, \natural}} \underbrace{\left(\overline{Q}_{e,h_j, \varepsilon}^{\natural, T} \partial_{\eta_2} \overline{Q}_{e,h_j, \varepsilon}^{\natural} \right)}_{:= \Gamma_{h_j, \varepsilon}^{2, \natural}} \underbrace{\left(\frac{1}{h_j} \text{axl} \left(\overline{Q}_{e,h_j, \varepsilon}^{\natural, T} \partial_{\eta_3} \overline{Q}_{e,h_j, \varepsilon}^{\natural} \right) \right)}_{:= \Gamma_{h_j, \varepsilon}^{3, \natural}} [(\nabla_x \Theta)^\natural(\eta)]^{-1}, \\ \Gamma_0 &:= \underbrace{\left(\text{axl} \left(\overline{Q}_{e,0}^T \partial_{\eta_1} \overline{Q}_{e,0} \right) \right)}_{:= \mathcal{K}_{e,s}^1} \underbrace{\left(\overline{Q}_{e,0}^T \partial_{\eta_2} \overline{Q}_{e,0} \right)}_{:= \mathcal{K}_{e,s}^2} | 0 \left[(\nabla_x \Theta)(0) \right]^{-1}. \end{aligned} \quad (7.28)$$

Note that

$$\Gamma_{h_j, \varepsilon}^{3, \natural} := \text{axl} \left(\exp(h_j \eta_3 A_\varepsilon(\eta_1, \eta_2))^T D \exp(h_j \eta_3 A_\varepsilon(\eta_1, \eta_2)) \cdot [A_\varepsilon] \right). \quad (7.29)$$

It holds

$$\begin{aligned} \|\tilde{U}_{h_j}^\varepsilon - \tilde{U}_0^\varepsilon\| &\rightarrow 0, \quad \text{as } h_j \rightarrow 0, & \|\tilde{U}_{h_j}^\varepsilon - \tilde{U}_0\| &\rightarrow 0, \quad \text{as } h_j \rightarrow 0, \varepsilon \rightarrow 0, \\ \|\Gamma_{h_j, \varepsilon}^{i, \natural} - \mathcal{K}_{e,s}^i\| &\rightarrow 0, \quad \text{as } h_j \rightarrow 0, \varepsilon \rightarrow 0, \quad i = 1, 2, & \|\Gamma_{h_j, \varepsilon}^{3, \natural} - \text{axl } A_\varepsilon\| &\rightarrow 0, \quad \text{as } h_j \rightarrow 0. \end{aligned}$$

We also have

$$\begin{aligned}
\|\tilde{U}_0^\varepsilon - \tilde{U}_0\|^2 &= \|\bar{Q}_{e,0}^T(\nabla\varphi_0|d_\varepsilon)[\nabla_x\Theta(0)]^{-1} - \bar{Q}_{e,0}^T(\nabla\varphi_0|d^*)[\nabla_x\Theta(0)]^{-1}\|^2 \\
&= \|\bar{Q}_{e,0}^T(0|0|d_\varepsilon - d^*)[\nabla_x\Theta(0)]^{-1}\|^2 \\
&= \langle \bar{Q}_{e,0}^T(0|0|d_\varepsilon - d^*)[\nabla_x\Theta(0)]^{-1}, \bar{Q}_{e,0}^T(0|0|d_\varepsilon - d^*)[\nabla_x\Theta(0)]^{-1} \rangle \\
&= \langle \bar{Q}_{e,0} \bar{Q}_{e,0}^T(0|0|d_\varepsilon - d^*), (0|0|d_\varepsilon - d^*)[\nabla_x\Theta(0)]^{-1}[\nabla_x\Theta(0)]^{-T} \rangle \\
&= \langle (0|0|d_\varepsilon - d^*), (0|0|d_\varepsilon - d^*)(\hat{\Gamma}_{y_0})^{-1} \rangle = \langle (0|0|d_\varepsilon - d^*)^T(0|0|d_\varepsilon - d^*), (\hat{\Gamma}_{y_0})^{-1} \rangle \\
&= \langle (0|0|(d_\varepsilon - d^*)^T(d_\varepsilon - d^*)), (\hat{\Gamma}_{y_0})^{-1} \rangle = \langle d_\varepsilon - d^*, d_\varepsilon - d^* \rangle = \|d_\varepsilon - d^*\|^2 \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.
\end{aligned} \tag{7.30}$$

We may write

$$\begin{aligned}
\mathcal{J}_{h_j}^{\natural, \text{mp}}(\varphi_{h_j, \varepsilon}^{\natural}, \bar{Q}_{e, h_j, \varepsilon}^{\natural}) &:= \int_{\Omega_1} W_{\text{mp}}(\tilde{U}_{h_j}^\varepsilon) \det((\nabla_x\Theta)^{\natural})(\eta) dV_\eta \\
&= \int_{\Omega_1} \left[W_{\text{mp}}(\tilde{U}_{h_j}^\varepsilon) - W_{\text{mp}}(\tilde{U}_0) + W_{\text{mp}}(\tilde{U}_0) \right] \det((\nabla_x\Theta)^{\natural})(\eta) dV_\eta \\
&= \int_{\Omega_1} \left[W_{\text{mp}}(\tilde{U}_{h_j}^\varepsilon + \tilde{U}_0 - \tilde{U}_0) - W_{\text{mp}}(\tilde{U}_0) + W_{\text{mp}}(\tilde{U}_0) \right] \det((\nabla_x\Theta)^{\natural})(\eta) dV_\eta \\
&\leq \int_{\Omega_1} \left[|W_{\text{mp}}(\tilde{U}_{h_j}^\varepsilon + \tilde{U}_0 - \tilde{U}_0) - W_{\text{mp}}(\tilde{U}_0)| + W_{\text{mp}}(\tilde{U}_0) \right] \det((\nabla_x\Theta)^{\natural})(\eta) dV_\eta,
\end{aligned} \tag{7.31}$$

where we used that W_{mp} is positive. The exact quadratic expansion in the neighborhood of the point $\tilde{U}_{h_j}^\varepsilon = \tilde{U}_0 + \tilde{U}_{h_j}^\varepsilon - \tilde{U}_0$ for W_{mp} is given by

$$W_{\text{mp}}(\tilde{U}_0 + \tilde{U}_{h_j}^\varepsilon - \tilde{U}_0) = W_{\text{mp}}(\tilde{U}_0) + \langle DW_{\text{mp}}(\tilde{U}_0), \tilde{U}_{h_j}^\varepsilon - \tilde{U}_0 \rangle + \frac{1}{2} D^2 W_{\text{mp}}(\tilde{U}_0) \cdot (\tilde{U}_{h_j}^\varepsilon - \tilde{U}_0, \tilde{U}_{h_j}^\varepsilon - \tilde{U}_0).$$

Therefore, with the assumption that $\|\tilde{U}_{h_j}^\varepsilon - \tilde{U}_0\| \leq 1$, we have the following relations

$$\begin{aligned}
\mathcal{J}_{h_j}^{\natural, \text{mp}}(\varphi_{h_j, \varepsilon}^{\natural}, \bar{Q}_{e, h_j, \varepsilon}^{\natural}) &\leq \int_{\Omega_1} \left[W_{\text{mp}}(\tilde{U}_0) + \|DW_{\text{mp}}(\tilde{U}_0)\| \|\tilde{U}_{h_j}^\varepsilon - \tilde{U}_0\| + \frac{1}{4} \|D^2 W_{\text{mp}}(\tilde{U}_0)\| \|\tilde{U}_{h_j}^\varepsilon - \tilde{U}_0\|^2 \right] \det((\nabla_x\Theta)^{\natural})(\eta) dV_\eta \\
&\leq \int_{\Omega_1} \left[W_{\text{mp}}(\tilde{U}_0) + C \|\tilde{U}_0\| \|\tilde{U}_{h_j}^\varepsilon - \tilde{U}_0\| + C_1 \|\tilde{U}_{h_j}^\varepsilon - \tilde{U}_0\| \right] \det((\nabla_x\Theta)^{\natural})(\eta) dV_\eta \\
&\leq \int_{\Omega_1} \left[W_{\text{mp}}(\tilde{U}_0) + (C \|\tilde{U}_0\| + C_1) \|\tilde{U}_{h_j}^\varepsilon - \tilde{U}_0\| \right] \det((\nabla_x\Theta)^{\natural})(\eta) dV_\eta,
\end{aligned} \tag{7.32}$$

where C and C_1 are upper bounds for $\|DW_{\text{mp}}(\tilde{U}_0)\|$ and $\|D^2 W_{\text{mp}}(\tilde{U}_0)\|$, respectively. Now we consider the terms of \tilde{W}_{curv}

$$\begin{aligned}
\mathcal{J}_{h_j}^{\natural, \text{curv}}(\Gamma_{h_j, \varepsilon}^{\natural}) &:= \int_{\Omega_1} \tilde{W}_{\text{curv}} \left((\Gamma_{h_j, \varepsilon}^{1, \natural}, \Gamma_{h_j, \varepsilon}^{2, \natural}, \Gamma_{h_j, \varepsilon}^{3, \natural}) [(\nabla_x\Theta)^{\natural}(\eta)]^{-1} \right) \det((\nabla_x\Theta)^{\natural})(\eta) dV_\eta \\
&\leq \int_{\Omega_1} \left[\tilde{W}_{\text{curv}} \left((\Gamma_{h_j, \varepsilon}^{1, \natural}, \Gamma_{h_j, \varepsilon}^{2, \natural}, \Gamma_{h_j, \varepsilon}^{3, \natural}) [(\nabla_x\Theta)^{\natural}(\eta)]^{-1} \right) - \tilde{W}_{\text{curv}} \left((\Gamma_0^1, \Gamma_0^2, A_\varepsilon) [(\nabla_x\Theta)^{\natural}(\eta)]^{-1} \right) \right. \\
&\quad + \tilde{W}_{\text{curv}} \left((\Gamma_0^1, \Gamma_0^2, A_\varepsilon) [(\nabla_x\Theta)^{\natural}(\eta)]^{-1} \right) - \tilde{W}_{\text{curv}} \left((\Gamma_0^1, \Gamma_0^2, A^*) [(\nabla_x\Theta)^{\natural}(\eta)]^{-1} \right) \\
&\quad \left. + \tilde{W}_{\text{curv}} \left((\Gamma_0^1, \Gamma_0^2, A^*) [(\nabla_x\Theta)^{\natural}(\eta)]^{-1} \right) \right] \det((\nabla_x\Theta)^{\natural})(\eta) dV_\eta \\
&\leq \int_{\Omega_1} \left[\left| \tilde{W}_{\text{curv}} \left((\Gamma_{h_j, \varepsilon}^{1, \natural}, \Gamma_{h_j, \varepsilon}^{2, \natural}, \Gamma_{h_j, \varepsilon}^{3, \natural}) [(\nabla_x\Theta)^{\natural}(\eta)]^{-1} \right) - \tilde{W}_{\text{curv}} \left((\Gamma_0^1, \Gamma_0^2, A_\varepsilon) [(\nabla_x\Theta)^{\natural}(\eta)]^{-1} \right) \right| \right. \\
&\quad + \left| \tilde{W}_{\text{curv}} \left((\Gamma_0^1, \Gamma_0^2, A_\varepsilon) [(\nabla_x\Theta)^{\natural}(\eta)]^{-1} \right) - \tilde{W}_{\text{curv}} \left((\Gamma_0^1, \Gamma_0^2, A^*) [(\nabla_x\Theta)^{\natural}(\eta)]^{-1} \right) \right| \\
&\quad \left. + \tilde{W}_{\text{curv}} \left((\Gamma_0^1, \Gamma_0^2, A^*) [(\nabla_x\Theta)^{\natural}(\eta)]^{-1} \right) \right] \det((\nabla_x\Theta)^{\natural})(\eta) dV_\eta,
\end{aligned} \tag{7.33}$$

where we have used the triangle inequality.

Note that beside the boundedness of $\det[\nabla_x \Theta]^\sharp(0)$, due to the hypothesis that $\det[\nabla_x \Theta(0)] \geq a_0 > 0$, it follows that there exists a constant $C > 0$ such that

$$\forall x \in \bar{\omega}: \quad \|[\nabla_x \Theta(0)]^{-1}\| \leq C. \quad (7.34)$$

We notice that both energy parts are positive and $\det[\nabla_x \Theta](0)$ is bounded. Also $\widetilde{W}_{\text{curv}}$ is continuous and $\|A_\varepsilon - A^*\|_{L^2(\omega, \mathfrak{so}(3))} < \varepsilon$. By using (6.20) and (7.12), and applying $\limsup_{h_j \rightarrow 0}$ on both sides of (7.32) and (7.33) with $h_j \rightarrow 0$ and $\varepsilon \rightarrow 0$ we get

$$\begin{aligned} \limsup_{h_j \rightarrow 0} \mathcal{J}_{h_j}^\sharp(\varphi_{h_j, \varepsilon}^\sharp, \overline{Q}_{e, h_j, \varepsilon}^\sharp) &\leq \int_{\Omega_1} (W_{\text{mp}}(\widetilde{U}_0) + \widetilde{W}_{\text{curv}}((\Gamma_0^1, \Gamma_0^2, A^*)(\nabla_x \Theta)^\sharp(0))^{-1}) \det[\nabla_x \Theta](0) dV_\eta \\ &= \int_{\Omega_1} (W_{\text{mp}}(\widetilde{U}_0) + \widetilde{W}_{\text{curv}}((\Gamma_0^1, \Gamma_0^2, 0)[(\nabla_x \Theta)(0)]^{-1}) \det[\nabla_x \Theta](0) dV_\eta. \end{aligned} \quad (7.35)$$

However, $W_{\text{mp}}(\widetilde{U}_0)$ and $\widetilde{W}_{\text{curv}}((\Gamma_0^1, \Gamma_0^2, 0)[(\nabla_x \Theta)(0)]^{-1})$ are already independent of the third variable η_3 , hence we deduce

$$\limsup_{h_j \rightarrow 0} \mathcal{J}_{h_j}^\sharp(\varphi_{h_j, \varepsilon}^\sharp, \overline{Q}_{e, h_j, \varepsilon}^\sharp) \leq \mathcal{J}_0(m, \overline{Q}_{e, 0}), \quad \varphi^\sharp \equiv \varphi, \quad \overline{Q}_{e, 0}^\sharp \equiv \overline{Q}_{e, 0} \quad \text{and} \quad m = \varphi_0. \quad \blacksquare$$

8 The Gamma-limit including external loads

The main result of this paper is the following theorem

Theorem 8.1. *Assume that the initial configuration is defined by a continuous injective mapping $y_0 : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ which admits an extension to $\bar{\omega}$ into $C^2(\bar{\omega}; \mathbb{R}^3)$ such that $\det[\nabla_x \Theta(0)] \geq a_0 > 0$ on $\bar{\omega}$, where a_0 is a constant, and assume that the boundary data satisfy the conditions*

$$\varphi_d^\sharp = \varphi_d|_{\Gamma_1} \text{ (in the sense of traces) for } \varphi_d \in H^1(\Omega_1; \mathbb{R}^3). \quad (8.1)$$

Let the constitutive parameters satisfy

$$\mu > 0, \quad \kappa > 0, \quad \mu_c > 0, \quad a_1 > 0, \quad a_2 > 0, \quad a_3 > 0, \quad (8.2)$$

Then, for any sequence $(\varphi_{h_j}^\sharp, \overline{Q}_{e, h_j}^\sharp) \in X$ such that $(\varphi_{h_j}^\sharp, \overline{Q}_{e, h_j}^\sharp) \rightarrow (\varphi_0, \overline{Q}_{e, 0})$ as $h_j \rightarrow 0$, the sequence of functionals $\mathcal{I}_{h_j} : X \rightarrow \overline{\mathbb{R}}$

$$\mathcal{I}_{h_j}^\sharp(\varphi^\sharp, \nabla_\eta^{h_j} \varphi^\sharp, \overline{Q}_e^\sharp, \Gamma_{h_j}^\sharp) = \begin{cases} \frac{1}{h_j} J_{h_j}^\sharp(\varphi^\sharp, \nabla_\eta^{h_j} \varphi^\sharp, \overline{Q}_e^\sharp, \Gamma_{h_j}^\sharp) - \frac{1}{h} \Pi_{h_j}^\sharp(\varphi^\sharp, \overline{Q}_e^\sharp) & \text{if } (\varphi^\sharp, \overline{Q}_e^\sharp) \in \mathcal{S}', \\ +\infty & \text{else in } X, \end{cases} \quad (8.3)$$

Γ -converges to the limit energy functional $\mathcal{I}_0 : X \rightarrow \overline{\mathbb{R}}$ defined by

$$\mathcal{I}_0(m, \overline{Q}_{e, 0}) = \begin{cases} \mathcal{J}_0(m, \overline{Q}_{e, 0}) - \Pi(m, \overline{Q}_{e, 0}) & \text{if } (m, \overline{Q}_{e, 0}) \in \mathcal{S}'_\omega, \\ +\infty & \text{else in } X, \end{cases} \quad (8.4)$$

where

$$\mathcal{J}_0(m, \overline{Q}_{e, 0}) = \begin{cases} \int_\omega [W_{\text{mp}}^{\text{hom}}(\mathcal{E}_{m, \overline{Q}_{e, 0}}) + \widetilde{W}_{\text{curv}}^{\text{hom}}(\mathcal{K}_{e, s})] \det(\nabla y_0 | n_0) d\omega & \text{if } (m, \overline{Q}_{e, 0}) \in \mathcal{S}'_\omega, \\ +\infty & \text{else in } X, \end{cases} \quad (8.5)$$

and $\Pi(m, \overline{Q}_{e, 0}) = \Pi_{\tilde{f}, \omega}(\tilde{u}_0) + \Pi_{\tilde{c}, \gamma_1}(\overline{Q}_{e, 0})$ defined by the external loads.

Remark 1. Before proving the above theorem, we will give the expression of the external loads potential in Ω_1 . We have

$$\Pi_h^\natural(\varphi^\natural, \overline{Q}_e^\natural) = \Pi_f^\natural(\varphi^\natural) + \Pi_c^\natural(\overline{Q}_e^\natural), \quad \Pi_f^\natural(\varphi^\natural) = h \int_{\Omega_1} \langle \tilde{f}^\natural, \tilde{u}^\natural \rangle dV_\eta, \quad \Pi_c^\natural(\overline{Q}_e^\natural) = h \int_{\Gamma_1} \langle \tilde{c}^\natural, \overline{Q}_e^\natural \rangle dS_\eta, \quad (8.6)$$

with $\tilde{f}^\natural(\eta) = \tilde{f}(\zeta(\eta))$, $\tilde{u}^\natural(\eta) = \tilde{u}(\zeta(\eta))$, $\tilde{c}^\natural(\zeta) = \tilde{c}(\zeta(\eta))$, $\overline{Q}_e^\natural(\eta) = \overline{Q}_e(\zeta(\eta))$ and $\tilde{u}^\natural(\eta_i) = \varphi^\natural(\eta_i) - \Theta^\natural(\eta_i)$. We use the following expressions

$$\begin{aligned} \Theta^\natural(\eta) &= y_0^\natural(\eta_1, \eta_2) + h_j \eta_3 n_0(\eta_1, \eta_2), \quad \varphi_{h_j}^\natural(\eta) = \varphi_0^\natural(\eta_1, \eta_2) + h_j \eta_3 d^*(\eta_1, \eta_2), \\ \tilde{u}^\natural(\eta_i) &= \varphi^\natural(\eta_i) - \Theta^\natural(\eta_i) = \underbrace{(\varphi_0^\natural(\eta_1, \eta_2) - y_0^\natural(\eta_1, \eta_2))}_{\tilde{u}_0(\eta_1, \eta_2)} + h_j \eta_3 (d^*(\eta_1, \eta_2) - n_0(\eta_1, \eta_2)). \end{aligned} \quad (8.7)$$

We calculate the work due to the loads separately. We have

$$\begin{aligned} \Pi_f^\natural(\varphi_{h_j}^\natural) &= h_j \int_{\Omega_1} \langle \tilde{f}^\natural, \tilde{u}^\natural \rangle dV_\eta = h_j \int_{\Omega_1} \langle \tilde{f}^\natural, \tilde{u}_0(\eta_1, \eta_2) \rangle dV_\eta + h_j^2 \eta_3 \int_{\Omega_1} \langle \tilde{f}^\natural, (d^*(\eta_1, \eta_2) - n_0(\eta_1, \eta_2)) \rangle dV_\eta \\ &= h_j \int_{\omega} \int_{-\frac{1}{2}}^{\frac{1}{2}} \langle \tilde{f}^\natural, \tilde{u}_0(\eta_1, \eta_2) \rangle d\eta_3 d\omega + h_j^2 \int_{\omega} \int_{-\frac{1}{2}}^{\frac{1}{2}} \eta_3 \langle \tilde{f}^\natural, (d^*(\eta_1, \eta_2) - n_0(\eta_1, \eta_2)) \rangle d\eta_3 d\omega \\ &= h_j \int_{\omega} \langle \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{f}^\natural d\eta_3, \tilde{u}_0(\eta_1, \eta_2) \rangle d\omega + h_j^2 \int_{\omega} \langle \int_{-\frac{1}{2}}^{\frac{1}{2}} \eta_3 \tilde{f}^\natural d\eta_3, (d^* - n_0)(\eta_1, \eta_2) \rangle d\omega := \Pi_{\tilde{f}, \omega}(\tilde{u}_0). \end{aligned} \quad (8.8)$$

For applying the same method for the potential of external applied boundary surface couple, we need to have an approximation for the exponential function which is already used in the expression of the recovery sequence for the microrotation $\overline{Q}_{e, h_j}^\natural$, i.e., $\exp(X) = \mathbb{1} + X + \frac{1}{2!}X^2 + \dots$, which implies

$$\overline{Q}_{e, h_j}^\natural = \overline{Q}_{e, 0} \cdot \exp(h_j \eta_3 A^*(\eta_1, \eta_2)) = \overline{Q}_{e, 0} + \overline{Q}_{e, 0} h_j \eta_3 A^*(\eta_1, \eta_2) + \frac{1}{2} \overline{Q}_{e, 0} h_j^2 \eta_3^2 A^*(\eta_1, \eta_2)^2 + \dots \quad (8.9)$$

Hence,

$$\begin{aligned} \Pi_c^\natural(\overline{Q}_{e, h_j}^\natural) &= h_j \int_{\Gamma_1} \langle \tilde{c}^\natural, \overline{Q}_{e, 0} + \overline{Q}_{e, 0} h_j \eta_3 A^*(\eta_1, \eta_2) + \frac{1}{2} \overline{Q}_{e, 0} h_j^2 \eta_3^2 A^*(\eta_1, \eta_2)^2 + \dots \rangle dS_\eta \\ &= h_j \int_{\Gamma_1} \langle \tilde{c}^\natural, \overline{Q}_{e, 0} \rangle dS_\eta + h_j^2 \eta_3 \int_{\Gamma_1} \langle \tilde{c}^\natural, \overline{Q}_{e, 0} A^*(\eta_1, \eta_2) \rangle dS_\eta + \frac{1}{2} h_j^3 \eta_3^2 \int_{\Gamma_1} \langle \tilde{c}^\natural, \overline{Q}_{e, 0} A^*(\eta_1, \eta_2)^2 \rangle dS_\eta + \dots \\ &= h_j \int_{(\gamma_1 \times [-\frac{1}{2}, \frac{1}{2}])} \langle \tilde{c}^\natural, \overline{Q}_{e, 0} \rangle dS_\eta + h_j^2 \eta_3 \int_{(\gamma_1 \times [-\frac{1}{2}, \frac{1}{2}])} \langle \tilde{c}^\natural, \overline{Q}_{e, 0} A^*(\eta_1, \eta_2) \rangle dS_\eta + O(h_j^3) \\ &= h_j \int_{\gamma_1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \langle \tilde{c}^\natural, \overline{Q}_{e, 0} \rangle d\eta_3 ds + h_j^2 \eta_3 \int_{\gamma_1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \langle \tilde{c}^\natural, \overline{Q}_{e, 0} A^*(\eta_1, \eta_2) \rangle d\eta_3 ds + O(h_j^3) \\ &= h_j \underbrace{\int_{\gamma_1} \langle \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{c}^\natural d\eta_3, \overline{Q}_{e, 0} \rangle ds + h_j^2 \int_{\gamma_1} \langle \int_{-\frac{1}{2}}^{\frac{1}{2}} \eta_3 \tilde{c}^\natural d\eta_3, \overline{Q}_{e, 0} A^*(\eta_1, \eta_2) \rangle ds}_{:= \Pi_{\tilde{c}, \gamma_1}(\overline{Q}_{e, 0})} + O(h_j^3). \end{aligned} \quad (8.10)$$

Therefore,

$$\Pi_{h_j}^\natural(\varphi^\natural, \overline{Q}_e^\natural) = \Pi_{\tilde{f}, \omega}(\tilde{u}_0) + \Pi_{\tilde{c}, \gamma_1}(\overline{Q}_{e, 0}) + O(h_j^3) = \Pi(m, \overline{Q}_{e, 0}) + O(h_j^3), \quad \tilde{u}_0 = m - y_0, \quad (8.11)$$

which regularity condition confirm the boundedness and continuity of external loads.

Now we come back to the proof of Theorem 8.1.

Proof of Theorem 8.1. As a first step we have considered the functionals

$$\mathcal{J}_h^\natural(\varphi^\natural, \nabla_\eta^h \varphi^\natural, \overline{Q}_e^\natural, \Gamma_h^\natural) = \begin{cases} \frac{1}{h} J_h^\natural(\varphi^\natural, \nabla_\eta^h \varphi^\natural, \overline{Q}_e^\natural, \Gamma_h^\natural) & \text{if } (\varphi^\natural, \overline{Q}_e^\natural) \in \mathcal{S}', \\ +\infty & \text{else in } X. \end{cases} \quad (8.12)$$

In subsections 7.1 and 7.2, we have shown that the following inequality holds

$$\limsup_{h_j \rightarrow 0} \mathcal{J}_{h_j}^{\natural}(\varphi_{h_j, \varepsilon}^{\natural}, \overline{Q}_{e, h_j, \varepsilon}^{\natural}) \leq \mathcal{J}_0(\varphi_0, \overline{Q}_{e, 0}) \leq \liminf_{h_j \rightarrow 0} \mathcal{J}_{h_j}^{\natural}(\varphi_{h_j, \varepsilon}^{\natural}, \overline{Q}_{e, h_j, \varepsilon}^{\natural}), \quad (8.13)$$

which implies that $\mathcal{J}_0(\varphi_0, \overline{Q}_{e, 0})$ is the Γ -lim of the sequence $\mathcal{J}_{h_j}^{\natural}(\varphi_{h_j, \varepsilon}^{\natural}, \overline{Q}_{e, h_j, \varepsilon}^{\natural})$, i.e.,

$$\mathcal{J}_0(\varphi_0, \overline{Q}_{e, 0}) = \Gamma\text{-}\lim(\mathcal{J}_{h_j}^{\natural}(\varphi_{h_j, \varepsilon}^{\natural}, \overline{Q}_{e, h_j, \varepsilon}^{\natural})), \quad m \equiv \varphi_0. \quad (8.14)$$

Remark 1, shows that the family $(\mathcal{J}_{h_j}^{\natural}(\varphi^{\natural}, \overline{Q}_e^{\natural}) - \Pi_{h_j}^{\natural}(\varphi^{\natural}, \overline{Q}_e^{\natural}))_j$ is Γ -convergent (because the external load potential is continuous). This guarantees the existence of Γ -convergence for the family $(\mathcal{I}_{h_j}^{\natural})_j$. Therefore, we may write

$$\mathcal{I}_0(m, \overline{Q}_{e, 0}) = \Gamma\text{-}\lim \mathcal{I}_{h_j}^{\natural}(\varphi_{h_j, \varepsilon}^{\natural}, \overline{Q}_{e, h_j, \varepsilon}^{\natural}) = \mathcal{J}_0(m, \overline{Q}_{e, 0}) - \Pi(\varphi_0, \overline{Q}_{e, 0}), \quad m \equiv \varphi_0, \quad (8.15)$$

which is the desired formula. ■

9 Consistency with related shell and plate models

9.1 A comparison to the Cosserat flat shell Γ -limit

In this part we check whether our model is consistent with the Cosserat flat shell model obtained in [55]. In the case of the plate model (flat initial configuration) we can assume that $\Theta(x_1, x_2, x_3) = (x_1, x_2, x_3)$ which gives $\nabla_x \Theta = \mathbb{1}_3$ and $y_0(x_1, x_2) = (x_1, x_2) := \text{id}(x_1, x_2)$. Also $Q_0 = \mathbb{1}_3$, $n_0 = e_3$ and $\overline{Q}_{e, 0}(x_1, x_2) = \overline{R}(x_1, x_2)$.

The family of functionals [17, 18] coincide with that considered in the analysis of Γ -convergence for a flat referential configuration, while its descaled Γ -limit is

$$\mathcal{J}_0(m, \overline{R}) = \begin{cases} \int_{\omega} h [W_{\text{mp}}^{\text{hom}}(\mathcal{E}_{m, s}^{\text{plate}}) + \overline{W}_{\text{curv}}^{\text{hom}}(\mathcal{K}_{e, s}^{\text{plate}})] d\omega & \text{if } (m, \overline{R}) \in \mathcal{S}'_{\omega}, \\ +\infty & \text{else in } X, \end{cases} \quad (9.1)$$

where

$$\begin{aligned} \mathcal{E}_{m, s}^{\text{plate}} &= \overline{R}^T (\nabla m | 0) - \mathbb{1}_2^b = \overline{R}^T (\nabla m | 0) - \mathbb{1}_3 + e_3 \otimes e_3, \\ \mathcal{K}_{e, s}^{\text{plate}} &= \left(\text{axl}(\overline{Q}_{e, 0}^T \partial_{x_1} \overline{Q}_{e, 0}) \mid \text{axl}(\overline{Q}_{e, 0}^T \partial_{x_2} \overline{Q}_{e, 0}) \mid 0 \right) \notin \text{Sym}(3), \end{aligned} \quad (9.2)$$

and

$$\begin{aligned} W_{\text{mp}}^{\text{hom}}(\mathcal{E}_{m, s}^{\text{plate}}) &= \mu \|\text{sym} [\mathcal{E}_{m, s}^{\text{plate}}]\|^2 + \mu_c \|\text{skew} [\mathcal{E}_{m, s}^{\text{plate}}]\|^2 + \frac{\lambda \mu}{\lambda + 2\mu} [\text{tr}([\mathcal{E}_{m, s}^{\text{plate}}])]^2 + \frac{2\mu \mu_c}{\mu_c + \mu} \|[\mathcal{E}_{m, s}^{\text{plate}}]^T e_3\|^2 \\ &= W_{\text{shell}}([\mathcal{E}_{m, s}^{\text{plate}}]) + \frac{2\mu \mu_c}{\mu_c + \mu} \|[\mathcal{E}_{m, s}^{\text{plate}}]^{\perp}\|^2, \\ \overline{W}_{\text{curv}}^{\text{hom}}(\mathcal{K}_{e, s}^{\text{plate}}) &= \inf_{A \in \mathfrak{so}(3)} \overline{W}_{\text{curv}}^* \left(\text{axl}(\overline{R}^T \partial_{\eta_1} \overline{R}) \mid \text{axl}(\overline{R}^T \partial_{\eta_2} \overline{R}) \mid \text{axl}(A) \right), \end{aligned} \quad (9.3)$$

together with

$$[\mathcal{E}_{m, s}^{\text{plate}}]^{\parallel} := (\mathbb{1}_3 - e_3 \otimes e_3) [\mathcal{E}_{m, s}^{\text{plate}}], \quad [\mathcal{E}_{m, s}^{\text{plate}}]^{\perp} := (e_3 \otimes e_3) [\mathcal{E}_{m, s}^{\text{plate}}], \quad (9.4)$$

where $W_{\text{shell}}(X) = \mu \|\text{sym} X\|^2 + \mu_c \|\text{skew} X\|^2 + \frac{\lambda \mu}{\lambda + 2\mu} [\text{tr}(X)]^2$. Let us denote by \overline{R}_i the columns of the matrix \overline{R} , i.e., $\overline{R} = (\overline{R}_1 \mid \overline{R}_2 \mid \overline{R}_3)$, $\overline{R}_i = \overline{R} e_i$. Since $(\mathbb{1}_3 - e_3 \otimes e_3) \overline{R}^T = (\overline{R}_1 \mid \overline{R}_2 \mid 0)^T$, it follows that $[\mathcal{E}_{m, s}^{\text{plate}}]^{\parallel} = (\overline{R}_1 \mid \overline{R}_2 \mid 0)^T (\nabla m | 0) - \mathbb{1}_2^b = ((\overline{R}_1 \mid \overline{R}_2)^T \nabla m)^b - \mathbb{1}_2^b$, while

$$[\mathcal{E}_{m, s}^{\text{plate}}]^{\perp} = (0 \mid 0 \mid \overline{R}_3)^T (\nabla m | 0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \langle \overline{R}_3, \partial_{x_1} m \rangle & \langle \overline{R}_3, \partial_{x_2} m \rangle & 0 \end{pmatrix}. \quad (9.5)$$

Hence, in the Cosserat flat shell model we have

$$\begin{aligned} W_{\text{mp}}^{\text{hom}}(\mathcal{E}_{m,s}^{\text{plate}}) &= \mu \|\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla m - \mathbb{1}_2)\|^2 + \mu_c \|\text{skew}((\bar{R}_1 | \bar{R}_2)^T \nabla m - \mathbb{1}_2)\|^2 \\ &\quad + \frac{\lambda \mu}{\lambda + 2\mu} [\text{tr}((\bar{R}_1 | \bar{R}_2)^T \nabla m - \mathbb{1}_2)]^2 + \frac{2\mu \mu_c}{\mu_c + \mu} (\langle \bar{R}_3, \partial_{x_1} m \rangle^2 + \langle \bar{R}_3, \partial_{x_2} m \rangle^2), \end{aligned} \quad (9.6)$$

which agrees with the Γ -limit found in [55].

9.2 A comparison with the nonlinear derivation Cosserat shell model

In [35], under assumptions (3.3) upon the thickness by using the derivation approach, the authors have obtained the following two-dimensional minimization problem for the deformation of the midsurface $m : \omega \rightarrow \mathbb{R}^3$ and the microrotation of the shell $\bar{Q}_{e,s} : \omega \rightarrow \text{SO}(3)$ solving on $\omega \subset \mathbb{R}^2$: minimize with respect to $(m, \bar{Q}_{e,s})$ the functional

$$I(m, \bar{Q}_{e,s}) = \int_{\omega} \left[W_{\text{memb}}(\mathcal{E}_{m,s}) + W_{\text{memb,bend}}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) + W_{\text{bend,curv}}(\mathcal{K}_{e,s}) \right] \underbrace{\det(\nabla y_0 | n_0)}_{\det \nabla \Theta} d\omega, \quad (9.7)$$

where the membrane part $W_{\text{memb}}(\mathcal{E}_{m,s})$, the membrane–bending part $W_{\text{memb,bend}}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s})$ and the bending–curvature part $W_{\text{bend,curv}}(\mathcal{K}_{e,s})$ of the shell energy density are given by

$$\begin{aligned} W_{\text{memb}}(\mathcal{E}_{m,s}) &= \left(h + K \frac{h^3}{12} \right) W_{\text{shell}}(\mathcal{E}_{m,s}), \\ W_{\text{memb,bend}}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) &= \left(\frac{h^3}{12} - K \frac{h^5}{80} \right) W_{\text{shell}}(\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) \\ &\quad - \frac{h^3}{3} H W_{\text{shell}}(\mathcal{E}_{m,s}, \mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) + \frac{h^3}{6} \mathcal{W}_{\text{shell}}(\mathcal{E}_{m,s}, (\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) B_{y_0}) \\ &\quad + \frac{h^5}{80} W_{\text{mp}}((\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s}) B_{y_0}), \\ W_{\text{bend,curv}}(\mathcal{K}_{e,s}) &= \left(h - K \frac{h^3}{12} \right) W_{\text{curv}}(\mathcal{K}_{e,s}) + \left(\frac{h^3}{12} - K \frac{h^5}{80} \right) W_{\text{curv}}(\mathcal{K}_{e,s} B_{y_0}) + \frac{h^5}{80} W_{\text{curv}}(\mathcal{K}_{e,s} B_{y_0}^2), \end{aligned} \quad (9.8)$$

where

$$\begin{aligned} W_{\text{shell}}(X) &= \mu \|\text{sym } X\|^2 + \mu_c \|\text{skew } X\|^2 + \frac{\lambda \mu}{\lambda + 2\mu} [\text{tr}(X)]^2, \\ &= \mu \|\text{dev sym } X\|^2 + \mu_c \|\text{skew } X\|^2 + \frac{2\mu(2\lambda + \mu)}{3(\lambda + 2\mu)} [\text{tr}(X)]^2, \\ W_{\text{shell}}(X, Y) &= \mu \langle \text{sym } X, \text{sym } Y \rangle + \mu_c \langle \text{skew } X, \text{skew } Y \rangle + \frac{\lambda \mu}{\lambda + 2\mu} \text{tr}(X) \text{tr}(Y), \\ W_{\text{mp}}(X) &= \mu \|\text{sym } X\|^2 + \mu_c \|\text{skew } X\|^2 + \frac{\lambda}{2} [\text{tr}(X)]^2 = \mathcal{W}_{\text{shell}}(X) + \frac{\lambda^2}{2(\lambda + 2\mu)} [\text{tr}(X)]^2, \\ W_{\text{curv}}(X) &= \mu L_c^2 (b_1 \|\text{dev sym } X\|^2 + b_2 \|\text{skew } X\|^2 + 4b_3 [\text{tr}(X)]^2), \quad \forall X, Y \in \mathbb{R}^{3 \times 3}. \end{aligned} \quad (9.9)$$

In the formulation of the minimization problem, the *Weingarten map (or shape operator)* is defined by $L_{y_0} = I_{y_0}^{-1} \Pi_{y_0} \in \mathbb{R}^{2 \times 2}$, where $I_{y_0} := [\nabla y_0]^T \nabla y_0 \in \mathbb{R}^{2 \times 2}$ and $\Pi_{y_0} := -[\nabla y_0]^T \nabla n_0 \in \mathbb{R}^{2 \times 2}$ are the matrix representations of the *first fundamental form (metric)* and the *second fundamental form* of the surface, respectively. In that paper, the authors have also introduced the tensors defined by

$$A_{y_0} := (\nabla y_0 | 0) [\nabla \Theta_x(0)]^{-1} \in \mathbb{R}^{3 \times 3}, \quad B_{y_0} := -(\nabla n_0 | 0) [\nabla \Theta_x(0)]^{-1} \in \mathbb{R}^{3 \times 3}, \quad (9.10)$$

and the so-called *alternator tensor* C_{y_0} of the surface [62]

$$C_{y_0} := \det(\nabla \Theta_x(0)) [\nabla \Theta_x(0)]^{-T} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} [\nabla \Theta_x(0)]^{-1}. \quad (9.11)$$

Comparing with the Γ -limit obtained in the present paper, the internal energy density obtained via the derivation approach depends also on

$$\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \mathcal{K}_{e,s} = - [\nabla \Theta_x(0)]^{-T} \begin{pmatrix} \mathcal{R} - \mathcal{G} L_{y_0} & 0 \\ \mathcal{T} L_{y_0} & 0 \end{pmatrix} [\nabla \Theta_x(0)]^{-1}, \quad (9.12)$$

where the nonsymmetric quantity $\mathcal{R} - \mathcal{G} L_{y_0}$ represents *the change of curvature tensor*. The choice of this name is justified subsequently in the framework of the linearized theory, see [38, 39]. Let us notice that the elastic shell bending–curvature tensor $\mathcal{K}_{e,s}$ appearing in the Cosserat Γ -limit is not capable to measure the change of curvature, see [37, 38, 39, 41], and that sometimes a confusion is made between bending and change of curvature measures, see also [1, 6, 7, 10, 61]

If we ignore the effect of the change of curvature tensor (9.12) in the model obtained via the derivation approach, there exists no coupling terms in $\mathcal{E}_{m,s}$ and $\mathcal{K}_{e,s}$ and we obtain a particular form of the energy, i.e.,

$$W_{\text{our}}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) = \left(h + K \frac{h^3}{12}\right) W_{\text{shell}}(\mathcal{E}_{m,s}) + \left(h - K \frac{h^3}{12}\right) W_{\text{curv}}(\mathcal{K}_{e,s}), \quad (9.13)$$

where

$$\begin{aligned} W_{\text{shell}}(\mathcal{E}_{m,s}) &= \mu \|\text{sym } \mathcal{E}_{m,s}^\parallel\|^2 + \mu_c \|\text{skew } \mathcal{E}_{m,s}^\parallel\|^2 + \frac{\lambda \mu}{\lambda + 2\mu} [\text{tr}(\mathcal{E}_{m,s}^\parallel)]^2 + \frac{\mu + \mu_c}{2} \|\mathcal{E}_{m,s}^\perp\|^2 \\ &= \mu \|\text{sym } \mathcal{E}_{m,s}^\parallel\|^2 + \mu_c \|\text{skew } \mathcal{E}_{m,s}^\parallel\|^2 + \frac{\lambda \mu}{\lambda + 2\mu} [\text{tr}(\mathcal{E}_{m,s}^\parallel)]^2 + \frac{\mu + \mu_c}{2} \|\mathcal{E}_{m,s}^T n_0\|^2, \end{aligned} \quad (9.14)$$

and

$$W_{\text{curv}}(\mathcal{K}_{e,s}) = \mu L_c^2 \left(b_1 \|\text{sym } \mathcal{K}_{e,s}^\parallel\|^2 + b_2 \|\text{skew } \mathcal{K}_{e,s}^\parallel\|^2 + \frac{12b_3 - b_1}{3} [\text{tr}(\mathcal{K}_{e,s}^\parallel)]^2 + \frac{b_1 + b_2}{2} \|\mathcal{K}_{e,s}^\perp\|^2 \right). \quad (9.15)$$

Skipping now all bending related h^3 -terms we note that there is only one difference between the membrane energy obtained via the derivation approach and the membrane energy obtained via Γ -convergence, i.e., the weight of the energy term $\|\mathcal{E}_{m,s}^T n_0\|^2$:

- derivation approach: the algebraic mean of μ and μ_c , i.e., $\frac{\mu + \mu_c}{2}$;
- Γ -convergence: the harmonic mean of μ and μ_c , i.e., $\frac{2\mu\mu_c}{\mu + \mu_c}$.

This difference has already been observed for the Cosserat flat shell Γ -limit [56].

We recall again the obtained curvature energy in [40] as

$$W_{\text{curv}}^{\text{hom}}(\mathcal{K}_{e,s}) = \mu L_c^2 \left(b_1 \|\text{sym } \mathcal{K}_{e,s}^\parallel\|^2 + b_2 \|\text{skew } \mathcal{K}_{e,s}^\parallel\|^2 + \frac{b_1 b_3}{(b_1 + b_3)} \text{tr}(\mathcal{K}_{e,s}^\parallel)^2 + \frac{2b_1 b_2}{b_1 + b_2} \|\mathcal{K}_{e,s}^\perp\|^2 \right). \quad (9.16)$$

A comparison between (9.15) and (9.16) shows that, like in the case for the membrane part, the weight of the energy term $\|\mathcal{K}_{e,s}^\perp\|^2 = \|\mathcal{K}_{e,s}^T n_0\|^2$ are different as following

- derivation approach: the algebraic mean of b_1 and b_2 , i.e., $\frac{b_1 + b_2}{2}$;
- Γ -convergence: the harmonic mean of b_1 and b_2 , i.e., $\frac{2b_1 b_2}{b_1 + b_2}$.

In the model obtained via the derivation approach [35], the constitutive coefficients in the shell model depend on both the Gauß curvature K and the mean curvature H . In the approach presented in the current paper this does not occur. However, we will consider this aspect in forthcoming works, by considering the Γ -limit method in order to obtain higher order terms in terms of the thickness in the membrane energy, see [30, 31, 32, 33].

9.3 A comparison with the general 6-parameter shell model

In the resultant 6-parameter theory of shells, the strain energy density for isotropic shells has been presented in various forms. The simplest expression $W_P(\mathcal{E}_{m,s}, \mathcal{K}_{e,s})$ has been proposed in the papers [17, 18] in the form

$$2W_P(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) = C[\nu(\text{tr} \mathcal{E}_{m,s}^\parallel)^2 + (1-\nu)\text{tr}((\mathcal{E}_{m,s}^\parallel)^T \mathcal{E}_{m,s}^\parallel)] + \alpha_s C(1-\nu) \|\mathcal{E}_{m,s}^T n_0\|^2 \\ + D[\nu(\text{tr} \mathcal{K}_{e,s}^\parallel)^2 + (1-\nu)\text{tr}((\mathcal{K}_{e,s}^\parallel)^T \mathcal{K}_{e,s}^\parallel)] + \alpha_t D(1-\nu) \|\mathcal{K}_{e,s}^T n_0\|^2, \quad (9.17)$$

with the Poisson ratio $\nu = \frac{\lambda}{2(\mu+\lambda)}$.

In [28], Eremeyev and Pietraszkiewicz have proposed a more general form of the strain energy density, namely

$$2W_{EP}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) = \alpha_1 (\text{tr} \mathcal{E}_{m,s}^\parallel)^2 + \alpha_2 \text{tr}(\mathcal{E}_{m,s}^\parallel)^2 + \alpha_3 \text{tr}((\mathcal{E}_{m,s}^\parallel)^T \mathcal{E}_{m,s}^\parallel) + \alpha_4 \|\mathcal{E}_{m,s}^T n_0\|^2 \\ + \beta_1 (\text{tr} \mathcal{K}_{e,s}^\parallel)^2 + \beta_2 \text{tr}(\mathcal{K}_{e,s}^\parallel)^2 + \beta_3 \text{tr}((\mathcal{K}_{e,s}^\parallel)^T \mathcal{K}_{e,s}^\parallel) + \beta_4 \|\mathcal{K}_{e,s}^T n_0\|^2. \quad (9.18)$$

Already, note the absence of coupling terms involving $\mathcal{K}_{e,s}^\parallel$ and $\mathcal{E}_{m,s}^\parallel$. The eight coefficients α_k, β_k ($k = 1, 2, 3, 4$) can depend in general on the structure of the curvature tensor $\mathcal{K}^0 = Q_0(\text{axl}(Q_0^T \partial_{x_1} Q_0) | \text{axl}(Q_0^T \partial_{x_2} Q_0) | 0)[\nabla \Theta(0)]^{-1}$ of the curved reference configuration. We can decompose the strain energy density (9.18) in the in-plane part $W_{\text{plane-EP}}(\mathcal{E}_{m,s})$ and the curvature part $W_{\text{curv-EP}}(\mathcal{K}_{e,s})$ and write their expressions in the form

$$W_{EP}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) = W_{\text{plane-EP}}(\mathcal{E}_{m,s}) + W_{\text{curv-EP}}(\mathcal{K}_{e,s}), \quad (9.19) \\ 2W_{\text{plane-EP}}(\mathcal{E}_{m,s}) = (\alpha_2 + \alpha_3) \|\text{sym} \mathcal{E}_{m,s}^\parallel\|^2 + (\alpha_3 - \alpha_2) \|\text{skew} \mathcal{E}_{m,s}^\parallel\|^2 + \alpha_1 (\text{tr}(\mathcal{E}_{m,s}^\parallel))^2 + \alpha_4 \|\mathcal{E}_{m,s}^T n_0\|^2, \\ 2W_{\text{curv-EP}}(\mathcal{K}_{e,s}) = (\beta_2 + \beta_3) \|\text{sym} \mathcal{K}_{e,s}^\parallel\|^2 + (\beta_3 - \beta_2) \|\text{skew} \mathcal{K}_{e,s}^\parallel\|^2 + \beta_1 (\text{tr}(\mathcal{K}_{e,s}^\parallel))^2 + \beta_4 \|\mathcal{K}_{e,s}^T n_0\|^2.$$

By comparing our membrane energy

$$W_{\text{mp}}^{\text{hom}}(\mathcal{E}_{m,s}) = \mu \|\text{sym} \mathcal{E}_{m,s}^\parallel\|^2 + \mu_c \|\text{skew} \mathcal{E}_{m,s}^\parallel\|^2 + \frac{\lambda \mu}{\lambda + 2\mu} [\text{tr}(\mathcal{E}_{m,s}^\parallel)]^2 + \frac{2\mu \mu_c}{\mu_c + \mu} \|\mathcal{E}_{m,s}^T n_0\|^2 \quad (9.20) \\ = W_{\text{shell}}(\mathcal{E}_{m,s}) + \frac{4\mu \mu_c}{\mu_c + \mu} \|\mathcal{E}_{m,s}^\perp\|^2,$$

with $W_{EP}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s})$ we deduce the following identification of the constitutive coefficients $\alpha_1, \dots, \alpha_4$

$$\alpha_1 = h \frac{2\mu \lambda}{2\mu + \lambda}, \quad \alpha_2 = h(\mu - \mu_c), \quad \alpha_3 = h(\mu + \mu_c), \quad \alpha_4 = h \frac{2\mu \mu_c}{\mu + \mu_c}.$$

We observe that $\mu_c^{\text{drill}} := \alpha_3 - \alpha_2 = 2h\mu_c$, which means that the in-plane rotational couple modulus μ_c^{drill} of the Cosserat shell model is determined by the Cosserat couple modulus μ_c of the 3D Cosserat material. An analogous conclusion is given in [4] where linear deformations are considered.

Now a comparison between our curvature energy

$$W_{\text{curv}}^{\text{hom}}(\mathcal{K}_{e,s}) = \mu L_c^2 \left(b_1 \|\text{sym} \mathcal{K}_{e,s}^\parallel\|^2 + b_2 \|\text{skew} \mathcal{K}_{e,s}^\parallel\|^2 + \frac{b_1 b_3}{(b_1 + b_3)} \text{tr}(\mathcal{K}_{e,s}^\parallel)^2 + \frac{2b_1 b_2}{b_1 + b_2} \|\mathcal{K}_{e,s}^\perp\|^2 \right). \quad (9.21)$$

and $W_{\text{curv-EP}}(\mathcal{K}_{e,s})$, leads us to the identification of the constitutive coefficients β_1, \dots, β_4

$$\beta_1 = 2\mu L_c^2 \frac{b_1 b_3}{b_1 + b_3}, \quad \beta_2 = \mu L_c^2 b_1, \quad \beta_3 = \mu L_c^2 (b_1 + b_2), \quad \beta_4 = 4\mu L_c^2 \frac{b_1 b_2}{b_1 + b_2}.$$

9.4 A comparison to another $O(h^5)$ -Cosserat shell model

In [13], by using a method which extends the reduction procedure from classical elasticity to the case of Cosserat shells, Birsan has obtained a minimization problem, which for the particular case of a quadratic ansatz for the deformation map and skipping higher order terms is based on the following energy

$$I(m, \overline{Q}_{e,s}) = \int_{\omega} \left[W_{\text{memb,bend}}^{(\text{quad})}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) + W_{\text{bend,curv}}(\mathcal{K}_{e,s}) \right] \det(\nabla y_0 | n_0) d\omega, \quad (9.22)$$

with $W_{\text{memb,bend}}^{(\text{quad})}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) = h W_{\text{Coss}}(\mathcal{E}_{m,s})$ and $W_{\text{bend,curv}}(\mathcal{K}_{e,s}) = h W_{\text{curv}}(\mathcal{K}_{e,s})$, where

$$\begin{aligned} W_{\text{Coss}}(X) &= \mathcal{W}_{\text{Coss}}(X, X) = \mu \|\text{sym } X^\parallel\|^2 + \mu_c \|\text{skew } X^\parallel\|^2 + \frac{2\mu\mu_c}{\mu + \mu_c} \|X^\perp\|^2 + \frac{\lambda\mu}{\lambda + 2\mu} [\text{tr}(X)]^2, \\ \mathcal{W}_{\text{Coss}}(X, Y) &= \mu \langle \text{sym } X^\parallel, \text{sym } Y^\parallel \rangle + \mu_c \langle \text{skew } X^\parallel, \text{skew } Y^\parallel \rangle + \frac{2\mu\mu_c}{\mu + \mu_c} \langle X^\perp, Y^\perp \rangle + \frac{\lambda\mu}{\lambda + 2\mu} \text{tr}(X) \text{tr}(Y), \\ W_{\text{mp}}(X) &= \mu \|\text{sym } X\|^2 + \mu_c \|\text{skew } X\|^2 + \frac{\lambda}{2} [\text{tr}(X)]^2 = \mathcal{W}_{\text{shell}}(X, X) + \frac{\lambda^2}{2(\lambda + 2\mu)} [\text{tr}(X)]^2, \\ W_{\text{curv}}(X) &= \mu L_c^2 (b_1 \|\text{dev sym } X\|^2 + b_2 \|\text{skew } X\|^2 + 4b_3 [\text{tr}(X)]^2), \quad \forall X, Y \in \mathbb{R}^{3 \times 3}. \end{aligned}$$

As it can be seen, in the obtained model by Biršan, there are some coupled terms of stress tensor and bending-curvature tensor, too. This is not surprising, since Biršan has obtained the starting example from the model in [35]. The main difference, in comparison to the model obtained in [35] is that

$$W_{\text{Coss}}(X) = \mathcal{W}_{\text{Coss}}(X, X) = \mu \|\text{sym } X^\parallel\|^2 + \mu_c \|\text{skew } X^\parallel\|^2 + \frac{2\mu\mu_c}{\mu + \mu_c} \|X^\perp\|^2 + \frac{\lambda\mu}{\lambda + 2\mu} [\text{tr}(X)]^2,$$

from [35] is replaced by

$$\mathcal{W}_{\text{Coss}}(X, Y) := \mathcal{W}_{\text{shell}}(X^\parallel, Y^\parallel) + \frac{2\mu\mu_c}{\mu + \mu_c} \langle X^\perp, Y^\perp \rangle, \quad (9.23)$$

for all tensors $X, Y \in \mathbb{R}^{3 \times 3}$ of the form $(*|*|0) \cdot [\nabla_x \Theta(0)]^{-1}$. Note that

$$\mathcal{W}_{\text{shell}}(X, Y) := \mathcal{W}_{\text{shell}}(X^\parallel, Y^\parallel) + \frac{\mu + \mu_c}{2} \langle X^\perp, Y^\perp \rangle, \quad (9.24)$$

holds true for all tensors $X, Y \in \mathbb{R}^{3 \times 3}$ of the form $(*|*|0) \cdot [\nabla_x \Theta(0)]^{-1}$. Hence, for this type of tensors we have

$$\mathcal{W}_{\text{Coss}}(X, Y) := \mathcal{W}_{\text{shell}}(X, Y) - \frac{\mu + \mu_c}{2} \langle X^\perp, Y^\perp \rangle + \frac{2\mu\mu_c}{\mu + \mu_c} \langle X^\perp, Y^\perp \rangle. \quad (9.25)$$

The main point of the comparison presented in this subsection is that the membrane term of order $O(h)$ coincide with the homogenized membrane energy determined by us in the present paper, i.e.,

$$W_{\text{mp}}^{\text{hom}}(\mathcal{E}_{m,s}) \equiv W_{\text{Coss}}(\mathcal{E}_{m,s}). \quad (9.26)$$

With a small comparison between the obtained membrane energy via Γ -convergence and the one obtained via the derivation approach model by Biršan, obviously we see that for a $O(h)$ -Cosserat shell theory, there is no difference between the coefficients, i.e.,

- special derivation approach: the harmonic mean of μ and μ_c ; $\frac{2\mu\mu_c}{\mu + \mu_c}$,
- Γ -limit approach: the harmonic mean of μ and μ_c ; $\frac{2\mu\mu_c}{\mu + \mu_c}$.

10 Linearisation of the Γ -limit Cosserat membrane shell model

10.1 The linearised model

In this section we develop the linearization of the Γ -limit functional for the elastic Cosserat shell model, i.e., for situations of small midsurface deformations and small Cosserat-curvature change. Let us consider

$$m(x_1, x_2) = y_0(x_1, x_2) + v(x_1, x_2), \quad (10.1)$$

where $v : \omega \rightarrow \mathbb{R}^3$ is the infinitesimal shell-midsurface displacement. For the rotation tensor $\overline{Q}_{e,0} \in \text{SO}(3)$ there exists a skew-symmetric matrix

$$\overline{A}_\vartheta := \text{Anti}(\vartheta_1, \vartheta_2, \vartheta_3) := \begin{pmatrix} 0 & -\vartheta_3 & \vartheta_2 \\ \vartheta_3 & 0 & -\vartheta_1 \\ -\vartheta_2 & \vartheta_1 & 0 \end{pmatrix} \in \mathfrak{so}(3), \quad \text{Anti} : \mathbb{R}^3 \rightarrow \mathfrak{so}(3), \quad (10.2)$$

where $\vartheta = \text{axl}(\bar{A}_\vartheta)$ denotes the axial vector of \bar{A}_ϑ , such that $\bar{Q}_{e,0} := \exp(\bar{A}_\vartheta) = \sum_{k=0}^{\infty} \frac{1}{k!} \bar{A}_\vartheta^k = \mathbb{1}_3 + \bar{A}_\vartheta + \text{h.o.t.}$. The tensor field \bar{A}_ϑ is the infinitesimal microrotation. Here, “h.o.t.” stands for terms of higher order than linear with respect to u and \bar{A}_ϑ .

Using these linearisations of the kinematic variables, we find the linearisations of the strain tensors. Indeed, since

$$\bar{Q}_{e,0}^T \nabla m - \nabla y_0 = (\mathbb{1}_3 + \bar{A}_\vartheta^T + \text{h.o.t.})(\nabla v + \nabla y_0) - \nabla y_0 = \nabla v - \bar{A}_\vartheta \nabla y_0 + \text{h.o.t.}, \quad (10.3)$$

we get for the non-symmetric *shell strain tensor* (which characterises both the in-plane deformation and the transverse shear deformation)

$$\mathcal{E}_{m,s} = (\bar{Q}_{e,0}^T \nabla m - \nabla y_0 \mid 0) [\nabla \Theta]^{-1},$$

the linearization

$$\mathcal{E}_{m,s}^{\text{lin}} = (\nabla v - \bar{A}_\vartheta \nabla y_0 \mid 0) [\nabla \Theta]^{-1} = (\partial_{x_1} u - \vartheta \times a_1 \mid \partial_{x_2} u - \vartheta \times a_2 \mid 0) [\nabla \Theta]^{-1} \notin \text{Sym}(3).$$

And for the *shell bending-curvature tensor*

$$\mathcal{K}_{e,s} := \left(\text{axl}(\bar{Q}_{e,0}^T \partial_{x_1} \bar{Q}_{e,0}) \mid \text{axl}(\bar{Q}_{e,0}^T \partial_{x_2} \bar{Q}_{e,0}) \mid 0 \right) [\nabla \Theta]^{-1}, \quad (10.4)$$

we calculate

$$\begin{aligned} \bar{Q}_{e,0}^T \partial_{x_\alpha} \bar{Q}_{e,0} &= (\mathbb{1}_3 - \bar{A}_\vartheta) \partial_{x_\alpha} \bar{A}_\vartheta + \text{h.o.t.} = \partial_{x_\alpha} \bar{A}_\vartheta + \text{h.o.t.} = \underbrace{\bar{A}_{\partial_{x_\alpha} \vartheta}}_{\equiv \text{Anti } \partial_{x_\alpha} \vartheta = \partial_{x_\alpha} \text{Anti } \vartheta} + \text{h.o.t.}, \end{aligned} \quad (10.5)$$

i.e.,

$$\text{axl}(\bar{Q}_{e,0}^T \partial_{x_\alpha} \bar{Q}_{e,0}) = \partial_{x_\alpha} \vartheta + \text{h.o.t.}, \quad (10.6)$$

and we deduce

$$\mathcal{K}_{e,s}^{\text{lin}} = (\text{axl}(\partial_{x_1} \bar{A}_\vartheta) \mid \text{axl}(\partial_{x_2} \bar{A}_\vartheta) \mid 0) [\nabla \Theta]^{-1}, \quad (10.7)$$

together with

$$\mathcal{K}_{e,s}^{\text{lin}} = (\partial_{x_1} \vartheta \mid \partial_{x_2} \vartheta \mid 0) [\nabla \Theta]^{-1} = (\nabla \vartheta \mid 0) [\nabla \Theta]^{-1}. \quad (10.8)$$

The form of the energy density remains unchanged upon linearization, since the model is physically linear. Thus, the linearization of the Γ -limits reads: for a midsurface displacement vector field $v : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ and the micro-rotation vector field $\vartheta : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$:

$$\mathcal{J}_0(m, \bar{Q}_{e,0}) = \int_{\omega} h \left[\bar{W}_{\text{mp}}^{\text{hom}}(\mathcal{E}_{m,s}^{\text{lin}}) + \bar{W}_{\text{curv}}^{\text{hom}}(\mathcal{K}_{e,s}^{\text{lin}}) \right] \det(\nabla y_0 \mid n_0) d\omega - \bar{\Pi}^{\text{lin}}(u, \vartheta),$$

where

$$\begin{aligned} \bar{W}_{\text{mp}}^{\text{hom}}(\mathcal{E}_{m,s}^{\text{lin}}) &= \mu \|\text{sym } \mathcal{E}_{m,s}^{\text{lin},\parallel}\|^2 + \mu_c \|\text{skew } \mathcal{E}_{m,s}^{\text{lin},\parallel}\|^2 + \frac{\lambda \mu}{\lambda + 2\mu} [\text{tr}(\mathcal{E}_{m,s}^{\text{lin},\parallel})]^2 + \frac{2\mu \mu_c}{\mu_c + \mu} \|\mathcal{E}_{m,s}^{\text{lin},\text{T}} n_0\|^2 \\ &= W_{\text{shell}}(\mathcal{E}_{m,s}^{\text{lin},\parallel}) + \frac{2\mu \mu_c}{\mu_c + \mu} \|\mathcal{E}_{m,s}^{\text{lin},\perp}\|^2, \\ \bar{W}_{\text{curv}}^{\text{hom}}(\mathcal{K}_{e,s}^{\text{lin}}) &= \mu L_c^2 \left(b_1 \|\text{sym } \mathcal{K}_{e,s}^{\text{lin},\parallel}\|^2 + b_2 \|\text{skew } \mathcal{K}_{e,s}^{\text{lin},\parallel}\|^2 + \frac{b_1 b_3}{(b_1 + b_3)} \text{tr}(\mathcal{K}_{e,s}^{\text{lin},\parallel})^2 + \frac{2b_1 b_2}{b_1 + b_2} \|\mathcal{K}_{e,s}^{\text{lin},\perp}\|^2 \right), \end{aligned} \quad (10.9)$$

and $\bar{\Pi}^{\text{lin}}(u, \vartheta)$ is the linearization of the continuous external loading potential $\bar{\Pi}$.

10.2 Comparison with the linear Reissner-Mindlin membrane-bending model

The following model

$$\begin{aligned} \int_{\omega} h \left(\mu \|\text{sym } \nabla(v_1, v_2)\|^2 + \frac{\kappa \mu}{2} \|\nabla v_3 - \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}\|^2 + \frac{\mu \lambda}{2\mu + \lambda} \text{tr}(\text{sym } \nabla(v_1, v_2))^2 \right) \\ + \frac{h^3}{12} \left(\mu \|\text{sym } \nabla(\theta_1, \theta_2)\|^2 + \frac{\mu \lambda}{2\mu + \lambda} \text{tr}(\nabla(\theta_1, \theta_2))^2 \right) d\omega \rightarrow \min \text{ w.r.t. } (v, \theta), \\ v|_{\gamma_0} = u^d(x, y, 0), \quad -\theta|_{\gamma_0} = (u_{1,z}^d, u_{2,z}^d, 0)^T, \end{aligned} \quad (10.10)$$

is the linear Reissner-Mindlin membrane-bending model which has five degree of freedom, three from the midsurface displacement $v: \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ and the other two are from the out-of-plane rotation parameter $\theta: \omega \rightarrow \mathbb{R}^2$ that describes the infinitesimal increment of the director and $0 < \kappa \leq 1$ is the so called *shear correction factor*. In this model the drill rotations (rotations about the normal) are absent.

As derived in [54], the Reissner-Mindlin membrane-bending model can be obtained as Γ -limit of the linear Cosserat elasticity model. Neff et al. in [56] applied the nonlinear scaling for the displacement and linear scaling for the infinitesimal microrotation for the minimization problem with respect to (u, \bar{A}) :

$$I(u, \bar{A}) = \int_{\Omega_h} W_{\text{mp}}(\bar{\varepsilon}) + W_{\text{curv}}(\nabla \text{axl } \bar{A}) dV \mapsto \min \quad \text{w.r.t. } (u, \bar{A}), \quad (10.11)$$

where $\bar{\varepsilon} = \nabla u - \bar{A}$, and

$$\begin{aligned} W_{\text{mp}}(\bar{\varepsilon}) &= \mu \|\text{sym } \bar{\varepsilon}\|^2 + \mu_c \|\text{skew } \bar{\varepsilon}\|^2 + \frac{\lambda}{2} [\text{tr}(\bar{\varepsilon})]^2, \\ W_{\text{curv}}(\mathcal{A}) &= \mu \frac{\widehat{L}_c^2(h)}{2} \left(\alpha_1 \|\text{sym } \nabla \text{axl } \bar{A}\|^2 + \alpha_2 \|\text{skew } \nabla \text{axl } \bar{A}\|^2 + \frac{\alpha_3}{2} [\text{tr}(\nabla \text{axl } \bar{A})]^2 \right), \end{aligned} \quad (10.12)$$

for $\alpha_1, \alpha_2, \alpha_3 \geq 0$. Then, they obtained the following minimization problem:

$$I^{\text{hom}}(v, \bar{A}) = \int_{\omega} W_{\text{mp}}^{\text{hom}}(\nabla v, \text{axl } \bar{A}) + W_{\text{curv}}^{\text{hom}}(\nabla \text{axl } \bar{A}) d\omega, \quad (10.13)$$

with respect to (v, θ) , where $v: \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is the deformation of the midsurface and $\bar{A}: \omega \subset \mathbb{R}^2 \rightarrow \mathfrak{so}(3)$ as the infinitesimal microrotation of the plate on ω with the boundary condition $v|_{\gamma_0} = u_d(x, y, 0)$, $\gamma_0 \subset \partial\omega$ and

$$\begin{aligned} W_{\text{mp}}^{\text{hom}}(\nabla v, \theta) &:= \mu \|\text{sym } \nabla_{(\eta_1, \eta_2)}(v_1, v_2)\|^2 + 2 \frac{\mu \mu_c}{\mu + \mu_c} \|\nabla_{(\eta_1, \eta_2)} v_3 - \begin{pmatrix} -\theta_2 \\ \theta_1 \end{pmatrix}\|^2 + \frac{\mu \lambda}{2\mu + \lambda} \text{tr}[\nabla_{(\eta_1, \eta_2)}(v_1, v_2)]^2, \\ W_{\text{curv}}^{\text{hom}}(\nabla \theta) &:= \mu \frac{\widehat{L}_c^2(h)}{2} \left(\alpha_1 \|\text{sym } \nabla_{(\eta_1, \eta_2)}(\theta_1, \theta_2)\|^2 + \frac{\alpha_1 \alpha_3}{2\alpha_1 + \alpha_3} \text{tr}[\nabla_{(\eta_1, \eta_2)}(\theta_1, \theta_2)]^2 \right). \end{aligned} \quad (10.14)$$

Comparing the Reissner-Mindlin membrane-bending model with the linearisation of the Γ -model obtained in the present paper, it can be seen that the Reissner-Mindlin model is obtained by Γ -convergence, upon selecting $\alpha_1 = \mu$, $\alpha_3 = \lambda$ in our model and by neglecting the drilling (the third component of the director).

In this formula one can recognize the harmonic mean \mathcal{H}

$$\frac{1}{2} \mathcal{H}\left(\mu, \frac{\lambda}{2}\right) = \frac{\mu \lambda}{2\mu + \lambda}, \quad \mathcal{H}(\mu, \mu_c) = \frac{2\mu \mu_c}{\mu + \mu_c}, \quad \frac{1}{2} \mathcal{H}\left(\alpha_1, \frac{\alpha_3}{2}\right) = \frac{\alpha_1 \alpha_3}{2\alpha_1 + \alpha_3}. \quad (10.15)$$

In our paper we used the nonlinear scaling for both deformation and microrotation, while in [56], they applied linear scaling for microrotation and nonlinear scaling for deformation. The other comparison is regarding the th elastic shell strain tensor and elastic shell bending curvature tensor which in our model are not de-coupled, and in (10.14) the in-plane deflections v_1, v_2 are not decoupled from θ_3 as well.

10.3 Aganovic and Neff's flat shell model

Aganović et al.[2] proposed a linear Cosserat flat shell model based on asymptotic analysis of the linear isotropic micropolar Cosserat model. They used the nonlinear scaling for both the displacement and infinitesimal microrotations. Therefore, their minimization problem reads:

$$\begin{aligned} & \int_{\omega} h \left(\mu \|\text{sym } (\nabla(v_1, v_2) - \begin{pmatrix} 0 & -\theta_3 \\ \theta_3 & 0 \end{pmatrix})\|^2 + \mu_c \|\text{skew } (\nabla(v_1, v_2) - \begin{pmatrix} 0 & -\theta_3 \\ \theta_3 & 0 \end{pmatrix})\|^2 + \frac{2\mu \mu_c}{\mu + \mu_c} \|\nabla v_3 - \begin{pmatrix} -\theta_2 \\ \theta_1 \end{pmatrix}\|^2 \right. \\ & \quad \left. + \frac{\mu \lambda}{2\mu + \lambda} \text{tr}(\text{sym } (\nabla(v_1, v_2) - \begin{pmatrix} 0 & -\theta_3 \\ \theta_3 & 0 \end{pmatrix}))^2 \right) \\ & \quad + \mu \frac{h L_c^2}{2} \left(\alpha_1 \|\text{sym } \nabla(\theta_1, \theta_2)\|^2 + \alpha_2 \|\text{skew } \nabla(\theta_1, \theta_2)\|^2 + \frac{2\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} \|\nabla \theta_3\|^2 + \frac{\alpha_1 \alpha_3}{2\alpha_1 + \alpha_3} \text{tr}(\nabla(\theta_1, \theta_2))^2 \right) d\omega \\ & \rightarrow \min \text{ w.r.t. } (v, \theta), \end{aligned} \quad (10.16)$$

where it is assumed that $\alpha_2, \kappa > 0$, otherwise this model with the assumption $\alpha_2 = 0$ will give the Reissner-Mindlin model. This means that we can not ignore the in-plane drill component θ_3 here and in the case of $\alpha_2 > 0$ one does not obtain the Reissner-Mindlin model. The asymptotic model coincides with the assumptions of Neff et al. in [55], where their assumption was about scaling the nonlinear Cosserat plate model with nonlinear scaling for both deformation and microrotation. The membrane part of this energy coincides with the homogenized membrane energy of our model with the same coefficients.

11 Conclusion

In this paper we have considered the Γ -limit procedure in order to derive a Cosserat thin shell model having a curved reference configuration. The paper is based on the development in [55], where the Γ -limit was obtained for a flat reference configuration of the shell. Here, the major complication arises from the curvy shell reference configuration. By introducing suitable mappings, we can encode the "curvy" information on a fictitious flat reference configuration. There, we use the nonlinear scaling for both the nonlinear deformation and the microrotation. This leads to a Cosserat membrane model, in which the effect of Cosserat-curvature survive the Γ -limit procedure. The homogenized membrane and curvature energy expressions are made explicit after some lengthy technical calculations. This is only possible because we use a physically linear, isotropic Cosserat model. Since the limit equations are obtained by Γ -convergence, they are automatically well-posed. We finally compare the Cosserat membrane shell model with some other dimensionally reduced proposals and linearizations. The full regularity of weak solutions for this Cosserat shell model (for some choice of constitutive parameters) will be established in [34].

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A Appendix

A.1 An auxiliary optimization problem

In this section we solve the auxiliary optimization problem (6.5). We calculate the variation of the energy (6.5) at equilibrium to be minimized over $c \in \mathbb{R}^3$ in order to determine the minimizer d^* . For arbitrary increment $\delta d^* \in \mathbb{R}^3$, we have

$$\forall \delta d^* \in \mathbb{R}^3 : \quad \langle DW_{\text{mp}}(\bar{Q}_e^{\sharp, T}(\nabla_{(\eta_1, \eta_2)} \varphi^\sharp | d^*)[(\nabla_x \Theta)^\sharp]^{-1}), \bar{Q}_e^{\sharp, T}(0|0|\delta d^*)[(\nabla_x \Theta)^\sharp]^{-1} \rangle = 0. \quad (\text{A.1})$$

By applying DW_{mp} we obtain

$$\begin{aligned} & \langle 2\mu \left(\text{sym}(\bar{Q}_e^{\sharp, T}(\nabla_{(\eta_1, \eta_2)} \varphi^\sharp | d^*)[(\nabla_x \Theta)^\sharp]^{-1} - \mathbb{1}_3) \right), \bar{Q}_e^{\sharp, T}(0|0|\delta d^*)[(\nabla_x \Theta)^\sharp]^{-1} \rangle_{\mathbb{R}^{3 \times 3}} \\ & + \langle 2\mu_c \left(\text{skew}(\bar{Q}_e^{\sharp, T}(\nabla_{(\eta_1, \eta_2)} \varphi^\sharp | d^*)[(\nabla_x \Theta)^\sharp]^{-1}) \right), \bar{Q}_e^{\sharp, T}(0|0|\delta d^*)[(\nabla_x \Theta)^\sharp]^{-1} \rangle_{\mathbb{R}^{3 \times 3}} \\ & + \lambda \text{tr} \left(\text{sym}(\bar{Q}_e^{\sharp, T}(\nabla_{(\eta_1, \eta_2)} \varphi^\sharp | d^*)[(\nabla_x \Theta)^\sharp]^{-1} - \mathbb{1}_3) \right) \langle \mathbb{1}_3, \bar{Q}_e^{\sharp, T}(0|0|\delta d^*)[(\nabla_x \Theta)^\sharp]^{-1} \rangle_{\mathbb{R}^{3 \times 3}} = 0. \end{aligned} \quad (\text{A.2})$$

This is equivalent to

$$\begin{aligned} & \langle 2\mu \bar{Q}_e^{\sharp} \left(\text{sym}(\bar{Q}_e^{\sharp, T}(\nabla_{(\eta_1, \eta_2)} \varphi^\sharp | d^*)[(\nabla_x \Theta)^\sharp]^{-1} - \mathbb{1}_3) \right) [(\nabla_x \Theta)^\sharp]^{-T} e_3, \delta d^* \rangle_{\mathbb{R}^3} \\ & + \langle 2\mu_c \bar{Q}_e^{\sharp} \left(\text{skew}(\bar{Q}_e^{\sharp, T}(\nabla_{(\eta_1, \eta_2)} \varphi^\sharp | d^*)[(\nabla_x \Theta)^\sharp]^{-1}) \right) [(\nabla_x \Theta)^\sharp]^{-T} e_3, \delta d^* \rangle_{\mathbb{R}^3} \\ & + \lambda \text{tr} \left(\text{sym}(\bar{Q}_e^{\sharp, T}(\nabla_{(\eta_1, \eta_2)} \varphi^\sharp | d^*)[(\nabla_x \Theta)^\sharp]^{-1} - \mathbb{1}_3) \right) \langle \bar{Q}_e^{\sharp} [(\nabla_x \Theta)^\sharp]^{-T} e_3, \delta d^* \rangle_{\mathbb{R}^3} = 0, \end{aligned} \quad (\text{A.3})$$

and it gives

$$\begin{aligned} & \langle 2\mu \bar{Q}_e^{\natural,T} \left(\text{sym}(\bar{Q}_e^{\natural,T} (\nabla_{(\eta_1, \eta_2)} \varphi^{\natural} |d^*|) [(\nabla_x \Theta)^{\natural}]^{-1} - \mathbb{1}_3) \right) n_0, \delta d^* \rangle_{\mathbb{R}^3} \\ & + \left\langle 2\mu_c \bar{Q}_e^{\natural,T} \left(\text{skew}(\bar{Q}_e^{\natural,T} (\nabla_{(\eta_1, \eta_2)} \varphi^{\natural} |d^*|) [(\nabla_x \Theta)^{\natural}]^{-1}) \right) n_0, \delta d^* \right\rangle_{\mathbb{R}^3} \\ & + \lambda \text{tr} \left(\text{sym}(\bar{Q}_e^{\natural,T} (\nabla_{(\eta_1, \eta_2)} \varphi^{\natural} |d^*|) [(\nabla_x \Theta)^{\natural}]^{-1} - \mathbb{1}_3) \right) \langle \bar{Q}_e^{\natural} n_0, \delta d^* \rangle_{\mathbb{R}^3} = 0. \end{aligned} \quad (\text{A.4})$$

Recall that the *first Piola–Kirchhoff stress tensor* in the reference configuration Ω_ξ is given by $S_1(F_\xi, \bar{R}_\xi) := D_{F_\xi} W_{\text{mp}}(F_\xi, \bar{R}_\xi)$, while the *Biot-type stress tensor* is $T_{\text{Biot}}(\bar{U}_\xi) := D_{\bar{U}_\xi} W_{\text{mp}}(\bar{U}_\xi)$. Since $D_{F_\xi} \bar{U}_\xi \cdot X = \bar{R}_\xi^T X$ and

$$\langle D_{F_\xi} W_{\text{mp}}(F_\xi, \bar{R}_\xi), X \rangle = \langle D_{\bar{U}_\xi} W_{\text{mp}}(\bar{U}_\xi), D_{F_\xi} \bar{U}_\xi X \rangle, \quad \forall X \in \mathbb{R}^{3 \times 3},$$

we obtain

$$D_{F_\xi} W_{\text{mp}}(F_\xi, \bar{R}_\xi) = \bar{R}_\xi D_{\bar{U}_\xi} W_{\text{mp}}(\bar{U}_\xi). \quad (\text{A.5})$$

Therefore, $S_1(F_\xi, \bar{R}_\xi) = \bar{R}_\xi T_{\text{Biot}}(\bar{U}_\xi)$ and $T_{\text{Biot}}(\bar{U}_\xi) = \bar{R}_\xi^T S_1(F_\xi, \bar{R}_\xi)$. Here, we have

$$T_{\text{Biot}}(\bar{U}_\xi) = 2\mu \text{sym}(\bar{U}_\xi - \mathbb{1}_3) + 2\mu_c \text{skew}(\bar{U}_\xi - \mathbb{1}_3) + \lambda \text{tr}(\text{sym}(\bar{U}_\xi - \mathbb{1}_3)) \mathbb{1}_3, \quad (\text{A.6})$$

where $\bar{U}_\xi(\Theta(x_1, x_2, x_3)) = \bar{U}_e(x_1, x_2, x_3)$. Thus, we can express the first Piola Kirchhoff stress tensor

$$S_1(F_\xi, \bar{R}_\xi) = \bar{R}_\xi \left[2\mu \text{sym}(\bar{R}_\xi^T F_\xi - \mathbb{1}_3) + 2\mu_c \text{skew}(\bar{R}_\xi^T F_\xi - \mathbb{1}_3) + \lambda \text{tr}(\text{sym}(\bar{R}_\xi^T F_\xi - \mathbb{1}_3)) \mathbb{1}_3 \right], \quad (\text{A.7})$$

with $\bar{R}_\xi(\Theta(x_1, x_2, x_3)) = \bar{Q}_e(x_1, x_2, x_3)$ for the elastic microrotation $\bar{Q}_e: \Omega_h \rightarrow \text{SO}(3)$. Hence, we must have

$$\forall \delta d^* \in \mathbb{R}^3: \quad \langle S_1((\nabla_{(\eta_1, \eta_2)} \varphi^{\natural} |d^*|) [(\nabla_x \Theta)^{\natural}]^{-1}, \bar{Q}_e^{\natural}) n_0, \delta d^* \rangle_{\mathbb{R}^3} = 0, \quad (\text{A.8})$$

implying

$$S_1((\nabla_{(\eta_1, \eta_2)} \varphi^{\natural} |d^*|) [(\nabla_x \Theta)^{\natural}]^{-1}, \bar{Q}_e^{\natural}) n_0 = 0 \quad \forall \eta_3 \in \left[-\frac{1}{2}, \frac{1}{2} \right]. \quad (\text{A.9})$$

In shell theories, the usual assumption is that the normal stress on the transverse boundaries are vanishing, that is

$$S_1(F_\xi, \bar{R}_\xi)|_{\omega_\xi^\pm} (\pm n_0) = 0, \quad (\text{normal stress on lower and upper faces is zero}). \quad (\text{A.10})$$

We notice that the condition (A.9) is for all $\eta_3 \in [-\frac{1}{2}, \frac{1}{2}]$, while the condition (A.10) is only for $\eta_3 = \pm \frac{1}{2}$. Therefore, it is possible that the Cosserat-membrane type Γ -limit underestimates the real stresses (e.g., the transverse shear stresses). From the relation between the first Piola-Kirchhoff tensor and the Biot-stress tensor we obtain

$$T_{\text{Biot}}(\bar{Q}_e^{\natural,T} (\nabla_{(\eta_1, \eta_2)} \varphi^{\natural} |d^*|) [(\nabla_x \Theta)^{\natural}]^{-1}) n_0 = 0, \quad \forall \eta_3 \in [-\frac{1}{2}, \frac{1}{2}], \quad (\text{A.11})$$

or, equivalently,

$$T_{\text{Biot}}(\bar{U}_{\varphi^{\natural}, \bar{Q}_e^{\natural}, d^*}) n_0 = 0, \quad (\text{A.12})$$

where

$$T_{\text{Biot}}(\bar{U}_{\varphi^{\natural}, \bar{Q}_e^{\natural}, d^*}) = 2\mu \text{sym}(\bar{U}_{\varphi^{\natural}, \bar{Q}_e^{\natural}, d^*} - \mathbb{1}_3) + 2\mu_c \text{skew}(\bar{U}_{\varphi^{\natural}, \bar{Q}_e^{\natural}, d^*} - \mathbb{1}_3) + \lambda \text{tr}(\text{sym}(\bar{U}_{\varphi^{\natural}, \bar{Q}_e^{\natural}, d^*} - \mathbb{1}_3)) \mathbb{1}_3, \quad (\text{A.13})$$

and we have introduced the notation $\bar{U}_{\varphi^{\natural}, \bar{Q}_e^{\natural}, d^*} := \bar{Q}_e^{\natural,T} (\nabla_{(\eta_1, \eta_2)} \varphi^{\natural} |d^*|) [(\nabla_x \Theta)^{\natural}]^{-1}$. With the help of the following decomposition

$$\begin{aligned} \bar{U}_{\varphi^{\natural}, \bar{Q}_e^{\natural}, d^*} - \mathbb{1}_3 &= (\bar{Q}_e^{\natural,T} \nabla_{(\eta_1, \eta_2)} \varphi^{\natural} - (\nabla y_0)^{\natural} |0|) [(\nabla_x \Theta)^{\natural}]^{-1} + (0|0| \bar{Q}_e^{\natural,T} d^* - n_0) [(\nabla_x \Theta)^{\natural}]^{-1} \\ &= \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}} + (0|0| \bar{Q}_e^{\natural,T} d^* - n_0) [(\nabla_x \Theta)^{\natural}]^{-1}, \end{aligned} \quad (\text{A.14})$$

with $\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}} = (\bar{Q}_e^{\natural,T} \nabla_{(\eta_1, \eta_2)} \varphi^{\natural} - (\nabla y_0)^{\natural} |0|) [(\nabla_x \Theta)^{\natural}]^{-1}$, and relations (A.29)-(A.31), the relation (A.13) can be expressed as

$$\begin{aligned} T_{\text{Biot}}(\bar{U}_{\varphi^{\natural}, \bar{Q}_e^{\natural}, d^*}) n_0 &= \mu \left(\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0 + (\bar{Q}_e^{\natural,T} d^* - n_0) + [(\nabla_x \Theta)^{\natural}]^{-T} (0|0| \bar{Q}_e^{\natural,T} d^* - n_0)^T n_0 \right) \\ &+ \mu_c \left(-\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0 + (\bar{Q}_e^{\natural,T} d^* - n_0) - [(\nabla_x \Theta)^{\natural}]^{-T} (0|0| \bar{Q}_e^{\natural,T} d^* - n_0)^T n_0 \right) \\ &+ \lambda \left(\langle \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}, \mathbb{1}_3 \rangle n_0 + (\bar{Q}_e^{\natural,T} d^* - n_0) n_0 \otimes n_0 \right) \\ &= (\mu + \mu_c) (\bar{Q}_e^{\natural,T} d^* - n_0) + (\mu - \mu_c) \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0 + (\mu - \mu_c) ((0|0| \bar{Q}_e^{\natural,T} d^* - n_0) [(\nabla_x \Theta)^{\natural}]^{-1})^T n_0 \\ &+ \lambda \text{tr}(\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}) n_0 + \lambda (\bar{Q}_e^{\natural,T} d^* - n_0) n_0 \otimes n_0, \end{aligned} \quad (\text{A.15})$$

and the condition (A.12) on T_{Biot} reads

$$(\mu + \mu_c)(\bar{Q}_e^{\natural,T} d^* - n_0) + (\mu - \mu_c)(\bar{Q}_e^{\natural,T} d^* - n_0)n_0 \otimes n_0 + \lambda(\bar{Q}_e^{\natural,T} d^* - n_0)n_0 \otimes n_0 \\ = -\left[(\mu - \mu_c)\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 + \lambda \text{tr}(\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural})n_0\right], \quad (\text{A.16})$$

where $((0|0|\bar{Q}_e^{\natural,T} d^* - n_0)[(\nabla_x \Theta)^\natural]^{-1})^T n_0 = (\bar{Q}_e^{\natural,T} d^* - n_0)n_0 \otimes n_0$. Before continuing the calculations, we introduce the tensor

$$A_{y_0} := (\nabla y_0|0) [(\nabla_x \Theta)(0)]^{-1} = \mathbb{1}_3 - n_0 \otimes n_0 \in \text{Sym}(3), \quad (\text{A.17})$$

and we notice that, identically as in the proof of Lemma 4.3 in [35], we can show that

$$\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural} A_{y_0} = \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural} \iff \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural} n_0 \otimes n_0 = 0. \quad (\text{A.18})$$

Actually, for an arbitrary matrix $X = (*|*|0)[\nabla_x \Theta(0)]^{-1}$, since $A_{y_0}^2 = A_{y_0} \in \text{Sym}(3)$ and $XA_{y_0} = X$, we have

$$\langle (\mathbb{1}_3 - A_{y_0})X, A_{y_0}X \rangle = \langle (A_{y_0} - A_{y_0}^2)X, X \rangle = 0,$$

but also

$$(\mathbb{1}_3 - A_{y_0})X^T = (X(\mathbb{1}_3 - A_{y_0}))^T = (X - XA_{y_0})^T = 0, \quad (\text{A.19})$$

and consequently

$$\langle X^T(\mathbb{1}_3 - A_{y_0}), A_{y_0}X \rangle = 0 \quad \text{as well as} \quad \langle X^T(\mathbb{1}_3 - A_{y_0}), (\mathbb{1}_3 - A_{y_0})X \rangle = 0.$$

In addition, since $A_{y_0} = \mathbb{1}_3 - (0|0|n_0)(0|0|n_0)^T = \mathbb{1}_3 - n_0 \otimes n_0$, the following equalities holds

$$\|(\mathbb{1}_3 - A_{y_0})X\|^2 = \langle X, (\mathbb{1}_3 - A_{y_0})^2 X \rangle = \langle X, (\mathbb{1}_3 - A_{y_0})X \rangle = \langle X, (0|0|n_0)(0|0|n_0)^T X \rangle \\ = \langle (0|0|n_0)^T X, (0|0|n_0)^T X \rangle = \|X(0|0|n_0)^T\|^2 = \|X^T(0|0|n_0)\|^2 = \|X^T n_0\|^2. \quad (\text{A.20})$$

We have the following decomposition

$$(\bar{Q}_e^{\natural,T} d^* - n_0) = \mathbb{1}_3(\bar{Q}_e^{\natural,T} d^* - n_0) = (A_{y_0} + n_0 \otimes n_0)(\bar{Q}_e^{\natural,T} d^* - n_0) \\ = A_{y_0}(\bar{Q}_e^{\natural,T} d^* - n_0) + n_0 \otimes n_0(\bar{Q}_e^{\natural,T} d^* - n_0). \quad (\text{A.21})$$

By using that

$$n_0 \otimes n_0(\bar{Q}_e^{\natural,T} d^* - n_0) = n_0 \langle n_0, (\bar{Q}_e^{\natural,T} d^* - n_0) \rangle = \langle (\bar{Q}_e^{\natural,T} d^* - n_0), n_0 \rangle n_0 = (\bar{Q}_e^{\natural,T} d^* - n_0)n_0 \otimes n_0, \quad (\text{A.22})$$

and with (A.16), we get

$$(\mu + \mu_c)A_{y_0}(\bar{Q}_e^{\natural,T} d^* - n_0) + (\mu + \mu_c)n_0 \otimes n_0(\bar{Q}_e^{\natural,T} d^* - n_0) + (\mu - \mu_c)n_0 \otimes n_0(\bar{Q}_e^{\natural,T} d^* - n_0) \\ + \lambda n_0 \otimes n_0(\bar{Q}_e^{\natural,T} d^* - n_0) = -\left[(\mu - \mu_c)\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 + \lambda \text{tr}(\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural})n_0\right]. \quad (\text{A.23})$$

Therefore,

$$\left((\mu + \mu_c)A_{y_0} + (2\mu + \lambda)n_0 \otimes n_0\right)(\bar{Q}_e^{\natural,T} d^* - n_0) = -\left[(\mu - \mu_c)\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 + \lambda \text{tr}(\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural})n_0\right]. \quad (\text{A.24})$$

Direct calculation shows

$$\left((\mu + \mu_c)A_{y_0} + (2\mu + \lambda)n_0 \otimes n_0\right)^{-1} := \left(\frac{1}{\mu + \mu_c}A_{y_0} + \frac{1}{2\mu + \lambda}n_0 \otimes n_0\right). \quad (\text{A.25})$$

Next, by using

$$A_{y_0}n_0 = (\mathbb{1}_3 - n_0 \otimes n_0)n_0 = n_0 - n_0 \langle n_0, n_0 \rangle = n_0 - n_0 = 0, \\ n_0 \otimes n_0 \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 = (0|0|n_0)(0|0|n_0)^T \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 = (0|0|n_0) \left((\bar{Q}_e^{\natural,T} \nabla_{(\eta_1, \eta_2)} \varphi^\natural - (\nabla y_0)^\natural |0| [(\nabla_x \Theta)^\natural]^{-1} (0|0|n_0) \right)^T n_0 \\ = (0|0|n_0) \left((\bar{Q}_e^{\natural,T} \nabla_{(\eta_1, \eta_2)} \varphi^\natural - (\nabla y_0)^\natural |0| (0|0|e_3) \right)^T n_0 = 0, \quad (\text{A.26})$$

eq. (A.24) can be written as

$$\bar{Q}_e^{\natural,T} d^* - n_0 = -\left[\frac{1}{\mu + \mu_c}A_{y_0} + \frac{1}{2\mu + \lambda}n_0 \otimes n_0\right] \times \left[(\mu - \mu_c)\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 + \lambda \text{tr}(\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural})n_0\right] \\ = -\left[\frac{\mu - \mu_c}{\mu + \mu_c}A_{y_0} \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 + \frac{\mu - \mu_c}{2\mu + \lambda}n_0 \otimes n_0 \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 + \frac{\lambda}{\mu + \mu_c} \text{tr}(\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural})A_{y_0}n_0 \right. \\ \left. + \frac{\lambda}{2\mu + \lambda} \text{tr}(\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural})(n_0 \otimes n_0)n_0\right] = -\left[\frac{\mu - \mu_c}{\mu + \mu_c}A_{y_0} \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 + \frac{\lambda}{2\mu + \lambda} \text{tr}(\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural})n_0\right]. \quad (\text{A.27})$$

Simplifying (A.27) we obtain

$$d^* = \left(1 - \frac{\lambda}{2\mu + \lambda} \langle \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}, \mathbb{1}_3 \rangle\right) \bar{Q}_e^\natural n_0 + \frac{\mu_c - \mu}{\mu_c + \mu} \bar{Q}_e^\natural \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0.$$

In terms of $\bar{Q}_e^\natural = \bar{R}^\natural Q_0^{\natural,T}$ we obtain the following expression for d^*

$$d^* = \left(1 - \frac{\lambda}{2\mu + \lambda} \langle (Q_0^\natural \bar{R}^{\natural,T} \nabla_{(\eta_1, \eta_2)} \varphi^\natural - (\nabla y_0)^\natural |0| [(\nabla_x \Theta)^\natural]^{-1}, \mathbb{1}_3 \rangle\right) \bar{R}^\natural Q_0^{\natural,T} n_0 \\ + \frac{\mu_c - \mu}{\mu_c + \mu} \bar{R}^\natural Q_0^{\natural,T} \left((Q_0^\natural \bar{R}^{\natural,T} \nabla_{(\eta_1, \eta_2)} \varphi^\natural - (\nabla y_0)^\natural |0| [(\nabla_x \Theta)^\natural]^{-1} \right)^T n_0. \quad (\text{A.28})$$

A.2 Calculations for the T_{Biot} stress

Here we present the lengthy calculation related to the T_{Biot} stress tensor in expression (A.13). We have

$$\begin{aligned}
2 \text{sym}(\bar{U}_{\varphi^{\natural}, \bar{Q}_e^{\natural}, d^*} - \mathbb{1}_3) n_0 &= \left(2 \text{sym}(\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}) + 2 \text{sym}((0|0|\bar{Q}_e^{\natural, T} d^* - n_0)[(\nabla_x \Theta)^{\natural}]^{-1}) \right) n_0 \\
&= \left(\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}} + \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T \right) n_0 + \left((0|0|\bar{Q}_e^{\natural, T} d^* - n_0)[(\nabla_x \Theta)^{\natural}]^{-1} + [(\nabla_x \Theta)^{\natural}]^{-T} (0|0|\bar{Q}_e^{\natural, T} d^* - n_0)^T \right) n_0 \\
&= \underbrace{\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}} n_0 + \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0}_{=0} + (0|0|\bar{Q}_e^{\natural, T} d^* - n_0)[(\nabla_x \Theta)^{\natural}]^{-1} n_0 + [(\nabla_x \Theta)^{\natural}]^{-T} (0|0|\bar{Q}_e^{\natural, T} d^* - n_0)^T n_0 \\
&= \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0 + (0|0|\bar{Q}_e^{\natural, T} d^* - n_0) e_3 + [(\nabla_x \Theta)^{\natural}]^{-T} (0|0|\bar{Q}_e^{\natural, T} d^* - n_0)^T n_0 \\
&= \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0 + (\bar{Q}_e^{\natural, T} d^* - n_0) + [(\nabla_x \Theta)^{\natural}]^{-T} (0|0|\bar{Q}_e^{\natural, T} d^* - n_0)^T n_0,
\end{aligned} \tag{A.29}$$

and

$$\begin{aligned}
2 \text{skew}(\bar{U}_{\varphi^{\natural}, \bar{Q}_e^{\natural}, d^*} - \mathbb{1}_3) n_0 &= \left(2 \text{skew}(\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}) + 2 \text{skew}((0|0|\bar{Q}_e^{\natural, T} d^* - n_0)[(\nabla_x \Theta)^{\natural}]^{-1}) \right) n_0 \\
&= \left(\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}} - \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T \right) n_0 + \left((0|0|\bar{Q}_e^{\natural, T} d^* - n_0)[(\nabla_x \Theta)^{\natural}]^{-1} - [(\nabla_x \Theta)^{\natural}]^{-T} (0|0|\bar{Q}_e^{\natural, T} d^* - n_0)^T \right) n_0 \\
&= -\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0 + (\bar{Q}_e^{\natural, T} d^* - n_0) - [(\nabla_x \Theta)^{\natural}]^{-T} (0|0|\bar{Q}_e^{\natural, T} d^* - n_0)^T n_0.
\end{aligned} \tag{A.30}$$

Calculating the trace of T_{Biot} gives

$$\begin{aligned}
\text{tr}(\text{sym}(\bar{U}_{\varphi^{\natural}, \bar{Q}_e^{\natural}, d^*} - \mathbb{1}_3)) n_0 &= \langle \text{sym}(\bar{U}_{\varphi^{\natural}, \bar{Q}_e^{\natural}, d^*} - \mathbb{1}_3), \mathbb{1}_3 \rangle n_0 = \left(\langle \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}, \mathbb{1}_3 \rangle + \langle (0|0|\bar{Q}_e^{\natural, T} d^* - n_0)[(\nabla_x \Theta)^{\natural}]^{-1}, \mathbb{1}_3 \rangle \right) n_0 \\
&= \langle \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}, \mathbb{1}_3 \rangle n_0 + (\bar{Q}_e^{\natural, T} d^* - n_0) n_0 \otimes n_0,
\end{aligned} \tag{A.31}$$

where we have used that $\langle (0|0|\bar{Q}_e^{\natural, T} d^* - n_0)[(\nabla_x \Theta)^{\natural}]^{-1}, \mathbb{1}_3 \rangle_{\mathbb{R}^3 \times 3} n_0 = \langle (\bar{Q}_e^{\natural, T} d^* - n_0), n_0 \rangle_{\mathbb{R}^3} n_0 = (\bar{Q}_e^{\natural, T} d^* - n_0) n_0 \otimes n_0$.

A.3 Calculations for the homogenized membrane energy

In this part we do the calculations for obtaining the minimizer separately. By inserting d^* in the membrane part of the relation (4.10), we have

$$\begin{aligned}
\|\text{sym}(\bar{U}_h^{\natural} - \mathbb{1}_3)\|^2 &= \|\text{sym}(\bar{Q}_e^{\natural, T} (\nabla_{(\eta_1, \eta_2)} \varphi^{\natural} | d^*) [(\nabla_x \Theta)^{\natural}]^{-1} - \mathbb{1}_3)\|^2 \\
&= \|\text{sym} \left(\underbrace{\bar{Q}_e^{\natural, T} (\nabla_{(\eta_1, \eta_2)} \varphi^{\natural} - [\nabla y_0]^{\natural} | 0) [(\nabla_x \Theta)^{\natural}]^{-1}}_{=\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}} + (0|0|\bar{Q}_e^{\natural, T} d^* - n_0)[(\nabla_x \Theta)^{\natural}]^{-1} \right)\|^2 \\
&= \|\text{sym} \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}\|^2 + \|\text{sym}((0|0|\bar{Q}_e^{\natural, T} d^* - n_0)[(\nabla_x \Theta)^{\natural}]^{-1})\|^2 \\
&\quad + 2 \left\langle \text{sym} \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}, \text{sym}((0|0|\bar{Q}_e^{\natural, T} d^* - n_0)[(\nabla_x \Theta)^{\natural}]^{-1}) \right\rangle \\
&= \|\text{sym} \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}\|^2 + \|\text{sym} \left(\frac{\mu_c - \mu}{\mu_c + \mu} \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0 \otimes n_0 - \frac{\lambda}{2\mu + \lambda} \text{tr}(\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}) n_0 \otimes n_0 \right)\|^2 \\
&\quad + 2 \left\langle \text{sym} \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}, \text{sym} \left(\frac{\mu_c - \mu}{\mu_c + \mu} \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0 \otimes n_0 - \frac{\lambda}{2\mu + \lambda} \text{tr}(\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}) n_0 \otimes n_0 \right) \right\rangle.
\end{aligned} \tag{A.32}$$

We have

$$\begin{aligned}
&\|\text{sym} \left(\frac{\mu_c - \mu}{\mu_c + \mu} \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0 \otimes n_0 - \frac{\lambda}{2\mu + \lambda} \text{tr}(\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}) n_0 \otimes n_0 \right)\|^2 \\
&= \frac{(\mu_c - \mu)^2}{(\mu_c + \mu)^2} \|\text{sym}(\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0 \otimes n_0)\|^2 + \frac{\lambda^2}{(2\mu + \lambda)^2} \text{tr}(\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}})^2 \|n_0 \otimes n_0\|^2 \\
&\quad - 2 \frac{\mu_c - \mu}{\mu_c + \mu} \frac{\lambda}{2\mu + \lambda} \text{tr}(\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}) \left\langle \text{sym}(\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0 \otimes n_0), n_0 \otimes n_0 \right\rangle \\
&= \frac{(\mu_c - \mu)^2}{(\mu_c + \mu)^2} \left\langle \text{sym}(\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0 \otimes n_0), \text{sym}(\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0 \otimes n_0) \right\rangle + \frac{\lambda^2}{(2\mu + \lambda)^2} \text{tr}(\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}})^2 \\
&\quad - \frac{\mu_c - \mu}{\mu_c + \mu} \frac{\lambda}{2\mu + \lambda} \text{tr}(\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}) \left\langle \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0 \otimes n_0, n_0 \otimes n_0 \right\rangle \\
&\quad - \frac{\mu_c - \mu}{\mu_c + \mu} \frac{\lambda}{2\mu + \lambda} \text{tr}(\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}) \left\langle n_0 \otimes n_0 \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}, n_0 \otimes n_0 \right\rangle \\
&= \frac{(\mu_c - \mu)^2}{4(\mu_c + \mu)^2} \langle \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0 \otimes n_0, \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0 \otimes n_0 \rangle + \frac{(\mu_c - \mu)^2}{4(\mu_c + \mu)^2} \langle \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0 \otimes n_0, n_0 \otimes n_0 \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}} \rangle
\end{aligned} \tag{A.33}$$

$$\begin{aligned}
& + \frac{(\mu_c - \mu)^2}{4(\mu_c + \mu)^2} \langle n_0 \otimes n_0 \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}, \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 \otimes n_0 \rangle + \frac{(\mu_c - \mu)^2}{4(\mu_c + \mu)^2} \langle n_0 \otimes n_0 \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}, n_0 \otimes n_0 \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural} \rangle \\
& + \frac{\lambda^2}{(2\mu + \lambda)^2} \text{tr}(\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural})^2 = \frac{(\mu_c - \mu)^2}{2(\mu_c + \mu)^2} \|\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0\|^2 + \frac{\lambda^2}{(2\mu + \lambda)^2} \text{tr}(\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural})^2.
\end{aligned}$$

Since, using (A.18) we have $\langle \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 \otimes n_0, n_0 \otimes n_0 \rangle = \langle n_0 \otimes n_0, \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural} n_0 \otimes n_0 \rangle = 0$, and since we have used the matrix expression $\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural} = (*|*|0)[(\nabla_x \Theta)^\natural]^{-1}$ and $n_0 \otimes n_0 = (0|0|n_0)[(\nabla_x \Theta)^\natural(0)]^{-1}$, we deduce

$$\begin{aligned}
\langle \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 \otimes n_0, \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 \otimes n_0 \rangle & = \langle \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T (0|0|n_0)[(\nabla_x \Theta)^\natural(0)]^{-1}, \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T (0|0|n_0)[(\nabla_x \Theta)^\natural(0)]^{-1} \rangle \\
& = \langle (0|0|\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0)^T (0|0|\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0), [(\nabla_x \Theta)^\natural(0)]^{-1} [(\nabla_x \Theta)^\natural(0)]^{-T} \rangle \\
& = \langle (0|0|\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0)^T (0|0|\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0), (\hat{\Gamma}_{y_0})^{-1} \rangle \\
& = \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 & & \end{pmatrix} (0|0|\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0), \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle = \langle \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0, \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 \rangle = \|\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0\|^2.
\end{aligned} \tag{A.34}$$

On the other hand,

$$\begin{aligned}
& 2 \left\langle \text{sym} \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}, \text{sym} \left(\frac{\mu_c - \mu}{\mu_c + \mu} \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 \otimes n_0 - \frac{\lambda}{2\mu + \lambda} \text{tr}(\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}) n_0 \otimes n_0 \right) \right\rangle \\
& = \frac{1}{2} \left\langle \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural} + \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T, \frac{\mu_c - \mu}{\mu_c + \mu} \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 \otimes n_0 + \frac{\mu_c - \mu}{\mu_c + \mu} n_0 \otimes n_0 \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural} - \frac{2\lambda}{2\mu + \lambda} \text{tr}(\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}) n_0 \otimes n_0 \right\rangle \\
& = \frac{\mu_c - \mu}{2(\mu_c + \mu)} \left\langle \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}, \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 \otimes n_0 \right\rangle + \frac{\mu_c - \mu}{2(\mu_c + \mu)} \left\langle \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}, n_0 \otimes n_0 \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural} \right\rangle \\
& \quad - \frac{\lambda}{(2\mu + \lambda)} \text{tr}(\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}) \langle \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}, n_0 \otimes n_0 \rangle + \frac{\mu_c - \mu}{2(\mu_c + \mu)} \left\langle \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T, \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 \otimes n_0 \right\rangle \\
& \quad + \frac{\mu_c - \mu}{2(\mu_c + \mu)} \left\langle \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T, n_0 \otimes n_0 \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural} \right\rangle - \frac{\lambda}{(2\mu + \lambda)} \text{tr}(\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}) \langle \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T, n_0 \otimes n_0 \rangle = \frac{\mu_c - \mu}{\mu_c + \mu} \|\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0\|^2,
\end{aligned} \tag{A.35}$$

due to (A.20). Therefore, (A.32) can be reduced to

$$\begin{aligned}
& \|\text{sym}(\bar{Q}_e^{\natural, T} (\nabla_{(\eta_1, \eta_2)} \varphi^\natural | c) [(\nabla_x \Theta)^\natural]^{-1} - \mathbb{1}_3)\|^2 \\
& = \|\text{sym} \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}\|^2 + \frac{(\mu_c - \mu)^2}{2(\mu_c + \mu)^2} \|\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0\|^2 + \frac{\lambda^2}{(2\mu + \lambda)^2} \text{tr}(\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural})^2 + \frac{\mu_c - \mu}{(\mu_c + \mu)} \|\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0\|^2.
\end{aligned} \tag{A.36}$$

Now we continue the calculations for the skew symmetric part,

$$\begin{aligned}
\|\text{skew}(\bar{Q}_e^{\natural, T} (\nabla_{(\eta_1, \eta_2)} \varphi^\natural | d^*) [(\nabla_x \Theta)^\natural]^{-1})\|^2 & = \|\text{skew}(\bar{Q}_e^{\natural, T} (\nabla_{(\eta_1, \eta_2)} \varphi^\natural | 0) [(\nabla_x \Theta)^\natural]^{-1})\|^2 + \|\text{skew}((0|0|\bar{Q}_e^{\natural, T} d^*) [(\nabla_x \Theta)^\natural]^{-1})\|^2 \\
& \quad + 2 \langle \text{skew}(\bar{Q}_e^{\natural, T} (\nabla_{(\eta_1, \eta_2)} \varphi^\natural | 0) [(\nabla_x \Theta)^\natural]^{-1}), \text{skew}((0|0|\bar{Q}_e^{\natural, T} d^*) [(\nabla_x \Theta)^\natural]^{-1}) \rangle.
\end{aligned} \tag{A.37}$$

In a similar manner, we calculate the terms separately. Since $n_0 \otimes n_0$ is symmetric, we obtain

$$\begin{aligned}
\|\text{skew}((0|0|\bar{Q}_e^{\natural, T} d^*) [(\nabla_x \Theta)^\natural]^{-1})\|^2 & = \|\text{skew}(n_0 \otimes n_0 + \frac{\mu_c - \mu}{\mu_c + \mu} \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 \otimes n_0 - \frac{\lambda}{2\mu + \lambda} \text{tr}(\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}) n_0 \otimes n_0)\|^2 \\
& = \frac{(\mu_c - \mu)^2}{(\mu_c + \mu)^2} \|\text{skew}(\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 \otimes n_0)\|^2.
\end{aligned}$$

But, we have

$$\begin{aligned}
\|\text{skew}(\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 \otimes n_0)\|^2 & = \frac{1}{4} \left\langle \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 \otimes n_0, \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 \otimes n_0 \right\rangle - \frac{1}{4} \left\langle \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 \otimes n_0, n_0 \otimes n_0 \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural} \right\rangle \\
& \quad - \frac{1}{4} \left\langle n_0 \otimes n_0 \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}, \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 \otimes n_0 \right\rangle + \frac{1}{4} \left\langle n_0 \otimes n_0 \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}, n_0 \otimes n_0 \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural} \right\rangle = \frac{1}{2} \|\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0\|^2,
\end{aligned} \tag{A.38}$$

where we used the fact that $(n_0 \otimes n_0)^2 = (n_0 \otimes n_0)$. The difficulty in the skew symmetric part of (A.37) is solved in the following calculation

$$\begin{aligned}
& 2 \langle \text{skew}(\bar{Q}_e^{\natural, T} (\nabla_{(\eta_1, \eta_2)} \varphi^\natural | 0) [(\nabla_x \Theta)^\natural]^{-1}), \text{skew}((0|0|\bar{Q}_e^{\natural, T} d^*) [(\nabla_x \Theta)^\natural]^{-1}) \rangle \\
& = 2 \frac{(\mu_c - \mu)}{(\mu_c + \mu)} \left\langle \text{skew}(\bar{Q}_e^{\natural, T} (\nabla_{(\eta_1, \eta_2)} \varphi^\natural | 0) [(\nabla_x \Theta)^\natural]^{-1}), \text{skew}(\mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 \otimes n_0) \right\rangle \\
& = \frac{(\mu_c - \mu)}{2(\mu_c + \mu)} \langle \bar{Q}_e^{\natural, T} (\nabla_{(\eta_1, \eta_2)} \varphi^\natural | 0) [(\nabla_x \Theta)^\natural]^{-1}, \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural}^T n_0 \otimes n_0 \rangle \\
& \quad - \frac{(\mu_c - \mu)}{2(\mu_c + \mu)} \langle \bar{Q}_e^{\natural, T} (\nabla_{(\eta_1, \eta_2)} \varphi^\natural | 0) [(\nabla_x \Theta)^\natural]^{-1}, n_0 \otimes n_0 \mathcal{E}_{\varphi^\natural, \bar{Q}_e^\natural} \rangle
\end{aligned} \tag{A.39}$$

$$\begin{aligned}
& -\frac{(\mu_c - \mu)}{2(\mu_c + \mu)} \langle (\bar{Q}_e^{\natural, T} (\nabla_{(\eta_1, \eta_2)} \varphi^{\natural} | 0) [(\nabla_x \Theta)^{\natural}]^{-1})^T, \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0 \otimes n_0 \rangle \\
& + \frac{(\mu_c - \mu)}{2(\mu_c + \mu)} \langle (\bar{Q}_e^{\natural, T} (\nabla_{(\eta_1, \eta_2)} \varphi^{\natural} | 0) [(\nabla_x \Theta)^{\natural}]^{-1})^T, n_0 \otimes n_0 \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T \rangle = -\frac{(\mu_c - \mu)}{(\mu_c + \mu)} \|\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0\|^2.
\end{aligned}$$

Therefore,

$$2 \langle \text{skew}(\bar{Q}_e^{\natural, T} (\nabla_{(\eta_1, \eta_2)} \varphi^{\natural} | 0) [(\nabla_x \Theta)^{\natural}]^{-1}), \text{skew}((0 | 0 | \bar{Q}_e^{\natural, T} d^*) [(\nabla_x \Theta)^{\natural}]^{-1}) \rangle = -\frac{(\mu_c - \mu)}{(\mu_c + \mu)} \|\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0\|^2, \quad (\text{A.40})$$

and we obtain

$$\begin{aligned}
\|\text{skew}(\bar{Q}_e^{\natural, T} (\nabla_{(\eta_1, \eta_2)} \varphi^{\natural} | d^*) [(\nabla_x \Theta)^{\natural}]^{-1})\|^2 &= \|\text{skew}(\bar{Q}_e^{\natural, T} (\nabla_{(\eta_1, \eta_2)} \varphi^{\natural} | 0) [(\nabla_x \Theta)^{\natural}]^{-1})\|^2 + \frac{(\mu_c - \mu)^2}{2(\mu_c + \mu)^2} \|\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0\|^2 \\
&\quad - \frac{(\mu_c - \mu)}{(\mu_c + \mu)} \|\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0\|^2.
\end{aligned} \quad (\text{A.41})$$

The last requirement for our calculations, is

$$\begin{aligned}
& \left[\text{tr} \left(\text{sym}(\bar{Q}_e^{\natural, T} (\nabla_{(\eta_1, \eta_2)} \varphi^{\natural} | d^*) [(\nabla_x \Theta)^{\natural}]^{-1} - \mathbb{1}_3) \right) \right]^2 \\
&= \left(\text{tr} \left(\text{sym}(\bar{Q}_e^{\natural, T} \nabla_{(\eta_1, \eta_2)} \varphi^{\natural} - [\nabla y_0]^{\natural} | 0) [(\nabla_x \Theta)^{\natural}]^{-1} \right) + \text{tr} \left(\text{sym}((0 | 0 | \bar{Q}_e^{\natural, T} d^* - n_0) [(\nabla_x \Theta)^{\natural}]^{-1}) \right) \right)^2 \\
&= \left(\text{tr}(\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}) + \frac{(\mu_c - \mu)}{2(\mu_c + \mu)} (\langle \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T n_0 \otimes n_0, \mathbb{1}_3 \rangle + \langle n_0 \otimes n_0 \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T, \mathbb{1}_3 \rangle) - \frac{\lambda}{2\mu + \lambda} \text{tr}(\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}) \underbrace{\langle n_0 \otimes n_0, \mathbb{1}_3 \rangle}_{=1} \right)^2 \\
&= \left(\frac{2\mu}{2\mu + \lambda} \text{tr}(\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}) + \frac{(\mu_c - \mu)}{2(\mu_c + \mu)} (\langle \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T, n_0 \otimes n_0 \rangle + \langle \mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}}^T, n_0 \otimes n_0 \rangle) \right)^2 = \frac{4\mu^2}{(2\mu + \lambda)^2} \text{tr}(\mathcal{E}_{\varphi^{\natural}, \bar{Q}_e^{\natural}})^2.
\end{aligned} \quad (\text{A.42})$$

A.4 Homogenized quadratic curvature energy

In [40], the authors obtained the homogenized curvature energy for the following curvature energy

$$W_{\text{curv}}(\Gamma^{\natural}[(\nabla_x \Theta)^{\natural}]^{-1}) = \mu L_c^2 \left(b_1 \|\text{sym} \Gamma^{\natural}[(\nabla_x \Theta)^{\natural}]^{-1}\|^2 + b_2 \|\text{skew} \Gamma^{\natural}[(\nabla_x \Theta)^{\natural}]^{-1}\|^2 + b_3 \text{tr}(\Gamma^{\natural}[(\nabla_x \Theta)^{\natural}]^{-1})^2 \right), \quad (\text{A.43})$$

as

$$\begin{aligned}
W_{\text{curv}}^{\text{hom}}(\mathcal{K}_{e,s}) &= \mu L_c^2 \left(b_1 \|\text{sym} \mathcal{K}_{e,s}\|^2 + b_2 \|\text{skew} \mathcal{K}_{e,s}\|^2 - \frac{(b_1 - b_2)^2}{2(b_1 + b_2)} \|\mathcal{K}_{e,s}^T n_0\|^2 + \frac{b_1 b_3}{(b_1 + b_3)} \text{tr}(\mathcal{K}_{e,s})^2 \right) \\
&= \mu L_c^2 \left(b_1 \|\text{sym} \mathcal{K}_{e,s}^{\parallel}\|^2 + b_2 \|\text{skew} \mathcal{K}_{e,s}^{\parallel}\|^2 - \frac{(b_1 - b_2)^2}{2(b_1 + b_2)} \|\mathcal{K}_{e,s}^T n_0\|^2 + \frac{b_1 b_3}{(b_1 + b_3)} \text{tr}(\mathcal{K}_{e,s}^{\parallel})^2 + \frac{b_1 + b_2}{2} \|\mathcal{K}_{e,s}^T n_0\| \right) \\
&= \mu L_c^2 \left(b_1 \|\text{sym} \mathcal{K}_{e,s}^{\parallel}\|^2 + b_2 \|\text{skew} \mathcal{K}_{e,s}^{\parallel}\|^2 + \frac{b_1 b_3}{(b_1 + b_3)} \text{tr}(\mathcal{K}_{e,s}^{\parallel})^2 + \frac{2b_1 b_2}{b_1 + b_2} \|\mathcal{K}_{e,s}^{\perp}\| \right),
\end{aligned} \quad (\text{A.44})$$

where $\mathcal{K}_{e,s} = (\Gamma_1 | \Gamma_2 | 0) [(\nabla_x \Theta)^{\natural}]^{-1}$ with the decomposition

$$X = X^{\parallel} + X^{\perp}, \quad X^{\parallel} := A_{y_0} X, \quad X^{\perp} := (\mathbb{1}_3 - A_{y_0}) X, \quad (\text{A.45})$$

for every matrix X .