1D MODEL FOR THE 3D MAGNETOHYDRODYNAMICS

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ABSTRACT. We propose a one-dimensional (1D) model for the three-dimensional (3D) incompressible ideal magnetohydrodynamics. We establish a regularity criterion of the Beale-Kato-Majda type for this 1D model. Without the stretching effect, the model with only transport effect equipped with a proper sign is shown to have global in time strong solution. Some numerical simulations suggest that solutions of this model with smooth periodic initial data do not tend to develop singularities at finite time.

KEY WORDS: magnetohydrodynamics; regularity; Hilbert transform; 1D model.

CLASSIFICATION CODE: 35Q35, 65M70, 76D03, 76W05, 76B03.

1. INTRODUCTION

The ideal incompressible magnetohydrodynamics (MHD) governed by the set of partial differential equations

$$
\begin{aligned}
u_t + (u \cdot \nabla)u - (B \cdot \nabla)B + \nabla \Pi &= 0, \\
B_t + (u \cdot \nabla)B - (B \cdot \nabla)u &= 0, \\
\nabla \cdot u = 0, \quad \nabla \cdot B &= 0,
\end{aligned}
\tag{1.1}
$$

is an important model in geophysics and astrophysics. In the system, the vector fields u and B denote the fluid velocity and magnetic field respectively; the scalar function Π is the pressure. We notice that (1.1) reduces to the incompressible Euler equation if $B \equiv 0$,

$$
\begin{aligned}
u_t + (u \cdot \nabla)u + \nabla \Pi &= 0, \\
\nabla \cdot u &= 0.
\end{aligned}
\tag{1.2}
$$

The mathematical question of whether or not a solution of the 3D Euler (1.2) develops singularity at finite time remains open. So does it for the 3D MHD (1.1).

Denote the vorticity by $\omega = \nabla \times u$. Taking a curl on (1.2) gives

$$
\omega_t + (u \cdot \nabla)\omega + (\omega \cdot \nabla)u = 0,
\tag{1.3a}
$$

$$
u = \nabla \times (-\Delta)^{-1}\omega.
\tag{1.3b}
$$

We note that u can be recovered from ω through the Biot-Savart law (1.3b) which involves a nonlocal operator. In (1.3a), the quadratic term $(u \cdot \nabla)\omega$ is regarded as the transport term, while $(\omega \cdot \nabla)u$ represents the stretching effect. The general belief is that the stretching effect is responsible for dramatic wild behaviours of solutions, for instance, the appearance of finite-time singularity.

1.1. **1D models for Euler equation and related equations.** To gain insights towards understanding the properties of solutions to the Euler equation (1.2), approximating models and toy models have been proposed and studied in the literature. One type of 1D models for the vorticity form of Euler equation has attracted a great deal of attention, which can be traced back to the work of Constantin, Lax and Majda [1]. The authors of [1] proposed the following 1D model for system (1.3a)-(1.3b),

$$\omega_t = \omega H \omega, \tag{1.4a}$$

$$u_x = H \omega, \tag{1.4b}$$

with $\omega = \omega(t, x)$ and $u = u(t, x)$ for $t \geq 0$ and $x \in \mathbb{R}$. In the system, H denotes the Hilbert transform defined by

$$H f = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(y)}{x - y} \, dy. \tag{1.5}$$

We note that equation (1.4b) is a 1D analogue of the Biot-Savart law (1.3b). With only stretching effect in equation (1.4a), the authors solved system (1.4a)-(1.4b) exactly and showed the formation of finite-time singularities for a class of initial data. Since then, various generalisations of (1.4a)-(1.4b) have been studied both analytically and numerically. The De Gregorio model [4, 5]

$$\omega_t + u\omega_x - \omega H \omega = 0, \tag{1.6a}$$

$$u_x = H \omega, \tag{1.6b}$$

includes both transport and stretching effects. Numerical results of [4, 5] provide evidence that finite-time blow-up may not occur for system (1.6a)-(1.6b). It indicates that the convection (transport) term has a regularization effect and it dominates the stretching term. Later on, in order to understand the competing effects of convection and stretching terms, Okamoto, Sakajo and Wunsch [13] suggested to study the following family of models

$$\omega_t + au\omega_x - \omega H \omega = 0, \tag{1.7a}$$

$$u_x = H \omega, \tag{1.7b}$$

with a parameter $a \in \mathbb{R}$. The authors also conjectured global in time existence of solutions to (1.7a)-(1.7b) with $a = 1$ which is the De Gregorio model (1.6a)-(1.6b). Indeed, Jia, Stewart and Šverák [11] proved that solutions of (1.6a)-(1.6b) with initial data near a steady state are global and converge to this steady state. In contrast, Elgindi and Jeong [7] showed singularity formation for (1.6a)-(1.6b) in classes of Hölder continuous solutions. Moreover, the authors of [7] established that, there exists smooth initial data such that solution of the Okamoto-Sakajo-Wunsch model (1.7a)-(1.7b) with small $|a|$ develops self-similar type of blow-up at finite time. Later on, Elgindi, Ghoul and Masmoudi [6] further showed that such self-similar blow-up is stable.

When $a = -1$, (1.7a)-(1.7b) is the Cordoba-Cordoba-Fontelos model introduced in [2] for the 2D quasi-geostrophic equation. Cordoba, Cordoba and Fontelos [2, 3] showed finite-time singularity formation for this model with a general class of initial data.

For axisymmetric 3D incompressible Navier-Stokes equation with swirl, Hou, Li, Shi, Wang and Yu [9] proposed a 1D nonlocal model for a simplified 3D nonlocal

system [10]. For this 1D model, the authors proved finite-time singularity formation rigorously and showed numerical evidences.

1.2. **1D models for MHD.** Inspired by the works discussed above, we will propose a family of nonlocal nonlinear models for the MHD system (1.1) as an attempt to understand the intricate structures involved in this system. In the context of MHD, besides the convection and stretching effects, the coupling and interaction between the fluid velocity and magnetic field also play crucial roles, which naturally introduce additional challenges.

Denote the Elsässer variables by

$$p = u + B, \quad m = u - B.$$

Equivalent to (1.1), (p, m) satisfies the system

$$
\begin{aligned}
p_t + (m \cdot \nabla)p + \nabla \Pi &= 0, \\
m_t + (p \cdot \nabla)m + \nabla \Pi &= 0, \\
\nabla \cdot p = 0, \quad \nabla \cdot m &= 0.
\end{aligned}
\tag{1.8}
$$

The structure of system (1.8) indicates that p and m are transported by each other. We also note that (1.8) appears in a rather symmetric form. Denote the vorticity of p and m by

$$\Omega = \nabla \times p, \quad \omega = \nabla \times m.$$

It follows from the Biot-Savart law that

$$p = \nabla \times (-\Delta)^{-1}\Omega, \quad m = \nabla \times (-\Delta)^{-1}\omega.$$

Taking the curl $\nabla \times$ on the equations of (1.8) gives

$$
\begin{aligned}
\Omega_t + (m \cdot \nabla)\Omega - (\Omega \cdot \nabla)m + \nabla \times (m\nabla p) &= 0, \\
\omega_t + (p \cdot \nabla)\omega - (\omega \cdot \nabla)p + \nabla \times (p\nabla m) &= 0, \\
p &= \nabla \times (-\Delta)^{-1}\Omega, \\
m &= \nabla \times (-\Delta)^{-1}\omega,
\end{aligned}
\tag{1.9}
$$

where $m\nabla p = (m_i \partial_j p_i)_j$ and $p\nabla m = (p_i \partial_j m_i)_j$. We propose the following 1D model to mimic system (1.9),

$$
\begin{aligned}
\Omega_t + m\Omega_x - \Omega m_x - \omega p_x + m\Omega_x &= 0, \\
\omega_t + p\omega_x - \omega p_x - \Omega m_x + p\omega_x &= 0, \\
p_x = H\Omega, \quad m_x &= H\omega.
\end{aligned}
\tag{1.10}
$$

In this paper, we will work with a simplified version of (1.10) by dropping the stretching effects Ωm_x and ωp_x and focusing on the transport effects, namely

$$
\begin{aligned}
\Omega_t + am\Omega_x - \omega p_x &= 0, \\
\omega_t + ap\omega_x - \Omega m_x &= 0, \\
p_x = H\Omega, \quad m_x &= H\omega,
\end{aligned}
\tag{1.11}
$$

with a parameter $a \in \mathbb{R}$. We will investigate (1.11) on the periodic interval $S^1 = [-\pi, \pi]$. Correspondingly, the Hilbert transform for periodic functions on S^1 can be defined as

$$Hf(x) = \frac{1}{2\pi} P.V. \int_{-\pi}^{\pi} f(y) \cot\left(\frac{x-y}{2}\right) dy.
\tag{1.12}$$

Indeed, the Cauchy kernel $\frac{1}{x}$ in definition (1.5) can be made periodic using the following identity

$$\frac{1}{2}\cot\left(\frac{x-y}{2}\right) = \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{x+2n\pi} + \frac{1}{x-2n\pi}\right).$$

To uniquely determine p from Ω and m from ω, we make the choice of Gauge by taking zero-mean value

$$\int_{-\pi}^{\pi} p(t,x)\,dx = \int_{-\pi}^{\pi} m(t,x)\,dx = 0. \tag{1.13}$$

We note that the mean value of Ω and ω is invariant for system (1.11) with $a = 1$. Indeed, we have for a smooth solution (Ω, ω) that

$$\frac{d}{dt}\int_{-\pi}^{\pi}\Omega(t,x)\,dx = \int_{-\pi}^{\pi}(-am\Omega_x + \omega p_x)\,dx$$

$$= \int_{-\pi}^{\pi}(am_x\Omega + \omega p_x)\,dx$$

$$= \int_{-\pi}^{\pi}(a\Omega H\omega + \omega H\Omega)\,dx$$

$$= (1-a)\int_{-\pi}^{\pi}\omega H\Omega\,dx$$

where we have used integration by parts and the skew symmetry property of the Hilbert transform. Similarly, we have

$$\frac{d}{dt}\int_{-\pi}^{\pi}\omega(t,x)\,dx = \int_{-\pi}^{\pi}(-ap\omega_x + \Omega m_x)\,dx = (1-a)\int_{-\pi}^{\pi}\Omega H\omega\,dx.$$

Obviously when $a = 1$, it follows

$$\frac{d}{dt}\int_{-\pi}^{\pi}\Omega(t,x)\,dx = \frac{d}{dt}\int_{-\pi}^{\pi}\omega(t,x)\,dx = 0,$$

and this is not true in general for $a \neq 1$. Hence, it is not appropriate to consider solutions of (1.11) in spaces of functions with zero mean for general value of a.

Consider the rescaled variables

$$\tilde{\Omega} = a\Omega, \quad \tilde{\omega} = a\omega$$

with corresponding $\tilde{p}$ and $\tilde{m}$ such that

$$\tilde{p}_x = H\tilde{\Omega}, \quad \tilde{m}_x = H\tilde{\omega}.$$

We can verify that $\tilde{p} = ap$ and $\tilde{m} = am$. In view of (1.11), $(\tilde{\Omega}, \tilde{\omega})$ satisfies the system

$$\tilde{\Omega}_t + \tilde{m}\tilde{\Omega}_x - a^{-1}\tilde{\omega}\tilde{p}_x = 0,$$
$$\tilde{\omega}_t + \tilde{p}\tilde{\omega}_x - a^{-1}\tilde{\Omega}\tilde{m}_x = 0. \tag{1.14}$$

Formally, taking $a \to \infty$, (1.14) turns to the system with only convection effect (with the tilde sign suppressed),

$$\Omega_t + m\Omega_x = 0,$$
$$\omega_t + p\omega_x = 0, \tag{1.15}$$
$$p_x = H\Omega, \quad m_x = H\omega.$$

We will investigate both systems (1.11) and (1.15) in the paper. We point out that formulating the problem in Elsässer variables does not give us essential advantage; rather it has the benefit of dealing with less nonlinear terms.

1.3. **Main results.** For general $a \in \mathbb{R}$, we show the existence of local in time solutions to (1.11) in the space $\mathcal{H}^1(S^1)$.

Theorem 1.1. *Let $a \in \mathbb{R}$ and $\Omega_0, \omega_0 \in \mathcal{H}^1(S^1)$. There exists a time T which depends on $\|\Omega_{0,x}\|_{L^2}$ and $\|\omega_{0,x}\|_{L^2}$ such that there exists a unique solution $(\Omega(t,x), \omega(t,x))$ to (1.11) with initial data $\Omega(0,x) = \Omega_0$ and $\omega(0,x) = \omega_0$ on $[0,T)$, which satisfies*

$$\Omega, \omega \in C^0\left([0,T); \mathcal{H}^1(S^1)\right) \cap C^1\left([0,T); L^2(S^1)\right).$$

The following theorem provides a Beale-Kato-Majda type of regularity criterion.

Theorem 1.2. *Let $(\Omega(t,x), \omega(t,x))$ be the solution of (1.11) on $[0,T)$ obtained in Theorem 1.1. If*

$$\int_0^T \left(\|H\Omega(t)\|_{L^\infty} + \|H\omega(t)\|_{L^\infty}\right) dt < \infty, \qquad (1.16)$$

the solution can be extended beyond T in the space $\mathcal{H}^1(S^1) \times \mathcal{H}^1(S^1)$.

Furthermore, if the initial data is in a space with higher regularity, the solution obtained in Theorem 1.1 also has higher regularity. Specifically, we will show:

Theorem 1.3. *Assume $\Omega_0, \omega_0 \in \mathcal{H}^n(S^1)$ with $n \geq 2$. Let (Ω, ω) be a solution of (1.11) with initial data (Ω_0, ω_0) on $[0,T)$, satisfying $\Omega, \omega \in C([0,T); \mathcal{H}^1)$. Then, we have*

$$\sup_{0 \leq t < T} \left(\|\Omega(t)\|_{\mathcal{H}^n} + \|\omega(t)\|_{\mathcal{H}^n}\right) < \infty.$$

With the absence of stretching effect, the solution of (1.15) can be shown to exist in the space $H^1(S^1)$ for all the time. Namely, we have

Theorem 1.4. *Assume $\Omega_0, \omega_0 \in \mathcal{H}^1(S^1)$. Then there exists a unique solution $(\Omega(t), \omega(t))$ of (1.15) with initial data (Ω_0, ω_0) on $[0,\infty)$.*

Some numerical simulations will be provided in Section 6. The numerical results suggest that starting from smooth periodic initial data, solutions of the model (1.11) with $a = 1$ or $a = -1$ are unlikely to develop singularities at finite time. This observation agrees with the numerical results done by De Gregorio [4, 5] and Okamoto, Sakajo and Wunsch [13] for the De Gregorio model (1.6a)-(1.6b).

2. NOTATIONS AND PRELIMINARIES

2.1. **Functional setting.** Denote

$$L^2(S^1) = \left\{ f \mid f \in L^2(-\pi, \pi), f \text{ is periodic on} [-\pi, \pi] \right\},$$

$$\mathcal{H}^k(S^1) = \left\{ f \mid f^{(s)} \in L^2(-\pi, \pi), f^{(s)} \text{ is periodic on} [-\pi, \pi], \text{ for all } 0 \leq s \leq k \right\}.$$

In particular, we consider the triplet of spaces

$$\mathcal{V} = \left\{ f \,\middle|\, f \in \mathcal{H}^2(S^1), \int_{-\pi}^{\pi} f(x)\, dx = 0 \right\}$$

$$\mathcal{W} = \mathcal{H}^1(S^1), \quad \mathcal{X} = L^2(S^1),$$

with the obvious embedding $\mathcal{V} \subset \mathcal{W} \subset \mathcal{X}$.

We denote $(,)$ by

$$(f,g) = \int_{-\pi}^{\pi} fg \, dx.$$

The space $\mathcal{H}^k(S^1)$ is a Hilbert space endowed with the natural inner product

$$(f,g)_{\mathcal{H}^k} = \sum_{s=0}^{k} \left(f^{(s)}, g^{(s)} \right) \quad \text{for functions} \ \ f, g \in \mathcal{H}^k(S^1),$$

and norm $(f,f)_{\mathcal{H}^k}^{\frac{1}{2}}$.

A bilinear form $\langle, \rangle : \mathcal{V} \times \mathcal{X} \to \mathbb{R}$ is defined as

$$\langle f, g \rangle = - \int_{-\pi}^{\pi} f_{xx} g \, dx.$$

Applying the integration by parts, we have for all $f \in \mathcal{V}$ and $g \in \mathcal{W}$

$$\langle f, g \rangle = (f_x, g_x).$$

For a space $\mathcal{Z}$, we denote $\mathcal{Z}^2 = \mathcal{Z} \times \mathcal{Z}$ by convention. In the context of a coupled system, for instance (1.11), it is convenient to introduce the triplet $\{\mathcal{V}^2, \mathcal{W}^2, \mathcal{X}^2\}$. Naturally, the Hilbert space $\mathcal{W}^2$ is endowed with the inner product

$$(f,g)_{\mathcal{W}^2} = (f_1, g_1)_{\mathcal{W}} + (f_2, g_2)_{\mathcal{W}} \quad \forall \ \ f = (f_1, f_2) \in \mathcal{W}^2, \ \ g = (g_1, g_2) \in \mathcal{W}^2.$$

In an analogous way, inner product can be defined for $\mathcal{V}^2$ and $\mathcal{X}^2$. A bilinear form $\langle, \rangle : \mathcal{V}^2 \times \mathcal{X}^2 \to \mathbb{R}$ is defined as

$$\langle f, g \rangle = - \int_{-\pi}^{\pi} f_{1,xx} g_1 \, dx - \int_{-\pi}^{\pi} f_{2,xx} g_2 \, dx. \tag{2.1}$$

For all $f = (f_1, f_2) \in \mathcal{V}^2$ and $g = (g_1, g_2) \in \mathcal{W}^2$, we also have

$$\langle f, g \rangle = (f_{1,x}, g_{1,x}) + (f_{2,x}, g_{2,x}).$$

Definition 2.1. A family $\{\mathcal{Z}, \mathcal{H}, \mathcal{Y}\}$ of three real separable Banach spaces is called an admissible triplet if the following conditions hold:

(i) The inclusions $\mathcal{Z} \subset \mathcal{H} \subset \mathcal{Y}$ are continuous and dense.

(ii) H is a Hilbert space endowed with inner product $(,)_{\mathcal{H}}$ and norm $\| \| \|_{\mathcal{H}} = (,)_{\mathcal{H}}^{\frac{1}{2}}$.

(iii) There is a continuous non-degenerate bilinear form on $\mathcal{Z} \times \mathcal{Y}$, denoted by $\langle, \rangle$, such that

$$\langle v, u \rangle = (v, u)_{\mathcal{H}}, \ \ \text{for} \ \ v \in \mathcal{V} \ \ \text{and} \ \ u \in \mathcal{H}. \tag{2.2}$$

Denote C_w by the space of functions with weak continuity and C_w^1 the space of functions with weak differentiability.

An abstract theorem of existence of Kato-Lai [12] is stated as follows.

Theorem 2.2. *Let $\{\mathcal{Z}, \mathcal{H}, \mathcal{Y}\}$ be an admissible triplet. Let $A : \mathcal{H} \to \mathcal{Y}$ be a weakly continuous map such that*

$$\langle v, A(v) \rangle \geq -\beta(\|v\|_{\mathcal{H}}^2), \ \ \forall \ \ v \in \mathcal{Z} \tag{2.3}$$

where $\beta(r) \geq 0$ is a monotone increasing function of $r \geq 0$. Then for any $u_0 \in \mathcal{H}$, there exists a time $T > 0$ such that the Cauchy problem

$$u_t + A(u) = 0, \quad u(0,x) = u_0$$

has a solution $u(t, x)$ on $[0, T]$ satisfying

$$u \in C_w([0, T]; \mathcal{H}) \cap C_w^1([0, T]; \mathcal{Y}).$$

Moreover, $\sup_{0 < t < T} \|u(t)\|_H$ *depends only on T, β and $\|u_0\|_{\mathcal{H}}$.*

In order to prove the existence part of Theorem 1.1, the Kato-Lai theorem will be applied to system (1.11) with the admissible triplet $\{\mathcal{V}^2, \mathcal{W}^2, \mathcal{X}^2\}$.

2.2. **Properties of Hilbert transform.**

The Hilbert transform has the following simple properties

$$H(cf) = cHf, \quad \text{for a constant} \quad c,$$
$$H \sin(kx) = -\cos(kx), \quad H \cos(kx) = \sin(kx).$$

And more generally, we have

$$H \sin(kx + \theta) = -\cos(kx + \theta), \quad H \cos(kx + \theta) = \sin(kx + \theta).$$

For any periodic function f, the mean value of its Hilbert transform is zero, that is

$$\int_{-\pi}^{\pi} Hf \, dx = 0. \tag{2.4}$$

Lemma 2.3. [14] *The Hilbert transform H is a bounded linear operator from space L^p to L^p with $1 < p < \infty$ and*

$$\|Hf\|_{L^p} \le C_p \|f\|_{L^p} \tag{2.5}$$

for a constant $C_p > 0$ depending on p.

3. Local existence

This section is devoted to a proof of Theorem 1.1. The proof includes three steps: (i) establishing the local existence of a solution by employing Theorem 2.2; (ii) showing the uniqueness of solution by a rather standard argument; (iii) justifying the strong continuity which is a consequence of the uniqueness and the time-reversible property of system (1.11).

Proof of Theorem 1.1: Denote $u = (\Omega, \omega)$, $q = (p, m)$, and naturally $q_x = Hu = (H\Omega, H\omega)$. Denote $A(u) = (A_1(u), A_2(u))$ with

$$A_1(u) = am\Omega_x - \omega p_x, \quad A_2(u) = ap\omega_x - \Omega m_x.$$

Thus, system (1.11) can be written as

$$u_t + A(u) = 0.$$

It is obvious that the family $\{\mathcal{V}^2, \mathcal{W}^2, \mathcal{X}^2\}$ is an admissible triplet associated with the bilinear form $\langle, \rangle$ defined in (2.1). To apply Theorem 2.2, we will need to show

that the operator A maps $\mathcal{W}^2$ into $\mathcal{X}^2$ continuously and it satisfies (2.3). Indeed, for any $u = (\Omega, \omega) \in \mathcal{W}^2$ with $q = (p, m) \in \mathcal{V}^2$, we have

$$
\begin{aligned}
\|A(u)\|_{\mathcal{X}^2} &= \left(\|am\Omega_x - \omega p_x\|_{L^2}^2 + \|ap\omega_x - \Omega m_x\|_{L^2}^2\right)^{\frac{1}{2}} \\
&\leq \|am\Omega_x - \omega p_x\|_{L^2} + \|ap\omega_x - \Omega m_x\|_{L^2} \\
&\leq |a|\|m\|_{L^\infty}\|\Omega_x\|_{L^2} + \|\omega\|_{L^\infty}\|p_x\|_{L^2} \\
&\quad + |a|\|p\|_{L^\infty}\|\omega_x\|_{L^2} + \|\Omega\|_{L^\infty}\|m_x\|_{L^2} \\
&\leq c_0\left(|a|\|m_x\|_{L^2}\|\Omega_x\|_{L^2} + \|\omega\|_{\mathcal{H}^1}\|p_x\|_{L^2}\right. \\
&\quad \left. + |a|\|p_x\|_{L^2}\|\omega_x\|_{L^2} + \|\Omega\|_{\mathcal{H}^1}\|m_x\|_{L^2}\right) \\
&\leq c_0\left(|a| + 1\right)\left(\|H\omega\|_{L^2}\|\Omega\|_{\mathcal{H}^1} + \|\omega\|_{\mathcal{H}^1}\|H\Omega\|_{L^2}\right) \\
&\leq c_0\left(|a| + 1\right)\left(\|\omega\|_{L^2}\|\Omega\|_{\mathcal{H}^1} + \|\omega\|_{\mathcal{H}^1}\|\Omega\|_{L^2}\right)
\end{aligned}
$$

where we have used the Hölder inequality, Sobolev inequality, the fact that p and m have zero mean, and the property (2.5). It follows that A maps $\mathcal{W}^2$ into $\mathcal{X}^2$. On the other hand, for any $u_1 = (\Omega_1, \omega_1) \in \mathcal{W}^2$ with $q_1 = (p_1, m_1) \in \mathcal{V}^2$ and $u_2 = (\Omega_2, \omega_2) \in \mathcal{W}^2$ with $q_2 = (p_2, m_2) \in \mathcal{V}^2$, we deduce

$$
\begin{aligned}
\|A(u_1) - A(u_2)\|_{\mathcal{X}^2} &= \left(\|(am_1\Omega_{1,x} - \omega_1 p_{1,x}) - (am_2\Omega_{2,x} - \omega_2 p_{2,x})\|_{L^2}^2\right. \\
&\quad \left. + \|(ap_1\omega_{1,x} - \Omega_1 m_{1,x}) - (ap_2\omega_{2,x} - \Omega_2 m_{2,x})\|_{L^2}^2\right)^{\frac{1}{2}} \\
&\leq \|(am_1\Omega_{1,x} - \omega_1 p_{1,x}) - (am_2\Omega_{2,x} - \omega_2 p_{2,x})\|_{L^2} \\
&\quad + \|(ap_1\omega_{1,x} - \Omega_1 m_{1,x}) - (ap_2\omega_{2,x} - \Omega_2 m_{2,x})\|_{L^2}.
\end{aligned}
\tag{3.1}
$$

Applying the Hölder inequality, Sobolev inequality, and (2.5) leads to

$$
\begin{aligned}
&\|(am_1\Omega_{1,x} - \omega_1 p_{1,x}) - (am_2\Omega_{2,x} - \omega_2 p_{2,x})\|_{L^2} \\
&\leq |a|\|\Omega_{1,x}\|_{L^2}\|m_1 - m_2\|_{L^\infty} + |a|\|\Omega_{1,x} - \Omega_{2,x}\|_{L^2}\|m_2\|_{L^\infty} \\
&\quad + \|\omega_2\|_{L^\infty}\|p_{2,x} - p_{1,x}\|_{L^2} + |a|\|\omega_2 - \omega_1\|_{L^\infty}\|p_{1,x}\|_{L^2} \\
&\leq c_0|a|\|\Omega_{1,x}\|_{L^2}\|m_{1,x} - m_{2,x}\|_{L^2} + c_0|a|\|\Omega_{1,x} - \Omega_{2,x}\|_{L^2}\|m_{2,x}\|_{L^2} \\
&\quad + c_0\|\omega_2\|_{\mathcal{H}^1}\|p_{2,x} - p_{1,x}\|_{L^2} + c_0|a|\|\omega_2 - \omega_1\|_{\mathcal{H}^1}\|p_{1,x}\|_{L^2} \\
&\leq c_0(|a| + 1)\left(\|\Omega_1\|_{\mathcal{H}^1} + \|\omega_2\|_{\mathcal{H}^1}\right)\left(\|\Omega_1 - \Omega_2\|_{\mathcal{H}^1} + \|\omega_1 - \omega_2\|_{\mathcal{H}^1}\right),
\end{aligned}
\tag{3.2}
$$

and similarly

$$
\begin{aligned}
&\|(ap_1\omega_{1,x} - \Omega_1 m_{1,x}) - (ap_2\omega_{2,x} - \Omega_2 m_{2,x})\|_{L^2} \\
&\leq c_0(|a| + 1)\left(\|\Omega_1\|_{\mathcal{H}^1} + \|\omega_2\|_{\mathcal{H}^1}\right)\left(\|\Omega_1 - \Omega_2\|_{\mathcal{H}^1} + \|\omega_1 - \omega_2\|_{\mathcal{H}^1}\right).
\end{aligned}
\tag{3.3}
$$

The estimates (3.1)-(3.3) together indicate that $A : \mathcal{W}^2 \to \mathcal{X}^2$ is strongly continuous.

By the definition of the bilinear form in (2.1), we have for any $u = (\Omega, \omega) \in \mathcal{V}^2$

$$
\begin{aligned}
\langle u, A(u)\rangle &= -(u_{xx}, A(u)) = (u_x, (A(u))_x) \\
&= (\Omega_x, (am\Omega_x - \omega p_x)_x) + (\omega_x, (ap\omega_x - \Omega m_x)_x) \\
&= \int_{-\pi}^{\pi} \Omega_x(am_x\Omega_x + am\Omega_{xx} - \omega_x p_x - \omega p_{xx})\,dx \\
&\quad + \int_{-\pi}^{\pi} \omega_x(ap_x\omega_x + ap\omega_{xx} - \Omega_x m_x - \Omega m_{xx})\,dx.
\end{aligned}
\tag{3.4}
$$

Note that $A(u) \in \mathcal{X}^2$ and (3.4) can be made rigorous through a standard approximating procedure. Applying integration by parts to the right hand side of (3.4), it has

$$a \int_{-\pi}^{\pi} m\Omega_x\Omega_{xx}\, dx = -a \int_{-\pi}^{\pi} m_x\Omega_x\Omega_x\, dx - a \int_{-\pi}^{\pi} m\Omega_{xx}\Omega_x\, dx.$$

Hence we conclude

$$a \int_{-\pi}^{\pi} m\Omega_x\Omega_{xx}\, dx = -\frac{a}{2} \int_{-\pi}^{\pi} m_x\Omega_x^2\, dx, \tag{3.5}$$

and similarly

$$a \int_{-\pi}^{\pi} p\omega_x\omega_{xx}\, dx = -\frac{a}{2} \int_{-\pi}^{\pi} p_x\omega_x^2\, dx. \tag{3.6}$$

Since $p_x = H\Omega$ and $m_x = H\omega$, combining (3.4)-(3.6) gives

$$\langle u, A(u) \rangle = (u_x, (A(u))_x)$$

$$= \frac{a}{2} \int_{-\pi}^{\pi} (H\omega)\Omega_x^2\, dx + \frac{a}{2} \int_{-\pi}^{\pi} (H\Omega)\omega_x^2\, dx$$

$$- \int_{-\pi}^{\pi} \Omega_x\omega_x H\Omega\, dx - \int_{-\pi}^{\pi} \omega\Omega_x H\Omega_x\, dx \tag{3.7}$$

$$- \int_{-\pi}^{\pi} \Omega_x\omega_x H\omega\, dx - \int_{-\pi}^{\pi} \Omega\omega_x H\omega_x\, dx.$$

Applying Hölder's inequality, Sobolev's inequality, (2.4) and (2.5), we have

$$\left| \int_{-\pi}^{\pi} H\omega\Omega_x^2\, dx \right| \leq \|H\omega\|_{L^\infty}\|\Omega_x\|_{L^2}^2$$

$$\leq c_0\|H\omega_x\|_{L^2}\|\Omega_x\|_{L^2}^2 \tag{3.8}$$

$$\leq c_0\|\omega_x\|_{L^2}\|\Omega_x\|_{L^2}^2,$$

and similarly

$$\left| \int_{-\pi}^{\pi} \Omega_x\omega_x H\Omega\, dx \right| + \left| \int_{-\pi}^{\pi} \omega\Omega_x H\Omega_x\, dx \right| \leq c_0\|\omega\|_{\mathcal{H}^1}\|\Omega_x\|_{L^2}^2,$$

$$\left| \int_{-\pi}^{\pi} (H\Omega)\omega_x^2\, dx \right| \leq c_0\|\omega_x\|_{L^2}^2\|\Omega_x\|_{L^2}, \tag{3.9}$$

$$\left| \int_{-\pi}^{\pi} \omega_x\Omega_x H\omega\, dx \right| + \left| \int_{-\pi}^{\pi} \omega_x\Omega H\omega_x\, dx \right| \leq c_0\|\omega_x\|_{L^2}^2\|\Omega\|_{\mathcal{H}^1}.$$

Therefore, putting together (3.7)-(3.9), we deduce

$$|\langle u, A(u) \rangle| \leq c_0(|a| + 1)\left(\|\omega\|_{\mathcal{H}^1}\|\Omega_x\|_{L^2}^2 + \|\omega_x\|_{L^2}^2\|\Omega\|_{\mathcal{H}^1} \right)$$

$$\leq c_0(|a| + 1)\left(\|\omega\|_{\mathcal{H}^1} + \|\Omega\|_{\mathcal{H}^1} \right)^3. \tag{3.10}$$

Hence, the operator A satisfies (2.3) with $\beta(r) = c_0(|a| + 1)r^{\frac{3}{2}}$. As a consequence, applying Theorem 2.2, we conclude that there exists a time $T > 0$ such that system (1.11) has a solution $(\Omega(t, x), \omega(t, x))$ on $[0, T]$ satisfying

$$\Omega, \omega \in C_w([0, T]; \mathcal{W}) \cap C_w^1([0, T]; \mathcal{X}).$$

Next we show the uniqueness of solution to (1.11). Let $u_1 = (\Omega_1, \omega_1)$ be a solution to (1.11) with initial data $u_0 = (\Omega_0, \omega_0)$. Let $q_1 = (p_1, m_1)$ such that $p_{1,x} = H\Omega_1$ and $m_{1,x} = H\omega_1$. Let $u_2 = (\Omega_2, \omega_2)$ be another solution to (1.11)

with the same initial data (Ω_0, ω_0) and associated with $q_2 = (p_2, m_2)$. Since both (Ω_1, ω_1) and (Ω_2, ω_2) satisfy (1.11), we are able to show that (details omitted)

$$\frac{1}{2}\frac{d}{dt}\left(\|\Omega_1(t) - \Omega_2(t)\|_{L^2}^2 + \|\omega_1(t) - \omega_2(t)\|_{L^2}^2\right)$$
$$\leq c_0(|a| + 1)\max_{0 \leq t \leq T}\left(\|\Omega_1\|_{\mathcal{H}^1} + \|\Omega_2\|_{\mathcal{H}^1} + \|\omega_1\|_{\mathcal{H}^1} + \|\omega_2\|_{\mathcal{H}^1}\right) \qquad (3.11)$$
$$\cdot\left(\|\Omega_1(t) - \Omega_2(t)\|_{L^2}^2 + \|\omega_1(t) - \omega_2(t)\|_{L^2}^2\right).$$

Thus, uniqueness follows from (3.11) and Grönwall's inequality.

Strong continuity in time follows from the uniqueness and the fact that system (1.11) is time-reversible. Indeed, it follows from (3.10) that

$$\|\Omega_x(t)\|_{L^2} + \|\omega_x(t)\|_{L^2} \to \|\Omega_{0,x}\|_{L^2} + \|\omega_{0,x}\|_{L^2} \text{ as } t \to 0.$$

Hence, we know

$$\Omega(t) \to \Omega_0, \quad \omega(t) \to \omega_0 \text{ strongly in } \mathcal{H}^1 \text{ as } t \to 0.$$

As a consequence of uniqueness, Ω and ω are strongly right-continuous. In addition, the property of time-reversibility implies that Ω and ω are strongly left-continuous as well.

$\square$

4. Regularity criterion

In this section, we prove Theorem 1.2 and the higher regularity result in Theorem 1.3.

Proof of Theorem 1.2: In view of the local existence theorem, we just need to show that the $\mathcal{H}^1$ norm of $\Omega(t)$ and $\omega(t)$ remains bounded as $t \to T$ under condition (1.16).

Assume $u = (\Omega, \omega)$ is a solution of (1.11) on $[0, T]$. We note that

$$\frac{1}{2}\frac{d}{dt}\left(\|\Omega_x\|_{L^2}^2 + \|\omega_x\|_{L^2}^2\right)$$
$$= (\Omega_x, \Omega_{tx}) + (\omega_x, \omega_{tx})$$
$$= (\Omega_x, -(am\Omega_x - \omega p_x)_x) + (\omega_x, -(ap\omega_x - \Omega m_x)_x)$$
$$= -\frac{a}{2}\int_{-\pi}^{\pi}(H\omega)\Omega_x^2\,dx - \frac{a}{2}\int_{-\pi}^{\pi}(H\Omega)\omega_x^2\,dx \qquad (4.1)$$
$$+ \int_{-\pi}^{\pi}\Omega_x\omega_x H\Omega\,dx + \int_{-\pi}^{\pi}\omega\Omega_x H\Omega_x\,dx$$
$$+ \int_{-\pi}^{\pi}\Omega_x\omega_x H\omega\,dx + \int_{-\pi}^{\pi}\Omega\omega_x H\omega_x\,dx$$

where we used (3.7) in the last step. Applying the identities

$$(v, u) = (Hv, Hu), \quad H(vHv) = \frac{1}{2}\left((Hv)^2 - v^2\right),$$

we infer

$$\int_{-\pi}^{\pi}\omega\Omega_x H\Omega_x\,dx = \int_{-\pi}^{\pi}(H\omega)H(\Omega_x H\Omega_x)\,dx$$
$$= \frac{1}{2}\int_{-\pi}^{\pi}(H\omega)\left((H\Omega_x)^2 - (\Omega_x)^2\right)\,dx, \qquad (4.2)$$

$$\int_{-\pi}^{\pi} \Omega \omega_x H \omega_x \, dx = \int_{-\pi}^{\pi} (H\Omega) H(\omega_x H \omega_x) \, dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} (H\Omega) \left((H\omega_x)^2 - (\omega_x)^2 \right) \, dx. \tag{4.3}$$

Combining (4.1)-(4.3), we have

$$\frac{1}{2} \frac{d}{dt} \left(\|\Omega_x\|_{L^2}^2 + \|\omega_x\|_{L^2}^2 \right)$$

$$= - \frac{a+1}{2} \int_{-\pi}^{\pi} (H\omega) \Omega_x^2 \, dx - \frac{a+1}{2} \int_{-\pi}^{\pi} (H\Omega) \omega_x^2 \, dx$$

$$+ \frac{1}{2} \int_{-\pi}^{\pi} (H\omega)(H\Omega_x)^2 \, dx + \frac{1}{2} \int_{-\pi}^{\pi} (H\Omega)(H\omega_x)^2 \, dx$$

$$+ \int_{-\pi}^{\pi} \Omega_x \omega_x \left(H\Omega + H\omega \right) \, dx \tag{4.4}$$

$$\leq \frac{|a+1|}{2} \|H\omega\|_{L^\infty} \|\Omega_x\|_{L^2}^2 + \frac{|a+1|}{2} \|H\Omega\|_{L^\infty} \|\omega_x\|_{L^2}^2$$

$$+ \frac{1}{2} \|H\omega\|_{L^\infty} \|\Omega_x\|_{L^2}^2 + \frac{1}{2} \|H\Omega\|_{L^\infty} \|\omega_x\|_{L^2}^2$$

$$+ \|H\Omega + H\omega\|_{L^\infty} \|\Omega_x\|_{L^2} \|\omega_x\|_{L^2}$$

$$\leq c_0(|a|+1) \left(\|H\Omega\|_{L^\infty} + \|H\omega\|_{L^\infty} \right) \left(\|\Omega_x\|_{L^2}^2 + \|\omega_x\|_{L^2}^2 \right)$$

for a constant $c_0 > 0$. It follows from Grönwall's inequality that

$$\left(\|\Omega_x(t)\|_{L^2}^2 + \|\omega_x(t)\|_{L^2}^2 \right)$$

$$\leq \left(\|\Omega_x(0)\|_{L^2}^2 + \|\omega_x(0)\|_{L^2}^2 \right) \exp \left\{ 2c_0(|a|+1) \int_0^t \left(\|H\Omega(\tau)\|_{L^\infty} + \|H\omega(\tau)\|_{L^\infty} \right) \, d\tau \right\}.$$

Thus, the statement of the theorem is justified.

$\square$

Proof of Theorem 1.3: The statement can be established through standard energy method. We only deal with the case of $n = 2$ and obtain the a priori estimate for $\|\Omega(t)\|_{\mathcal{H}^2}$ and $\|\omega(t)\|_{\mathcal{H}^2}$. Formally, differentiating the equations of (1.11) twice in space yields

$$\begin{aligned}
\Omega_{txx} = & - 2am_x \Omega_{xx} + 2\omega_x p_{xx} - am_{xx} \Omega_x \\
& + \omega_{xx} p_x - am\Omega_{xxx} + \omega p_{xxx}, \\
\omega_{txx} = & - 2ap_x \omega_{xx} + 2\Omega_x m_{xx} - ap_{xx} \omega_x \\
& + \Omega_{xx} m_x - ap\omega_{xxx} + \Omega m_{xxx}.
\end{aligned} \tag{4.5}$$

Taking the inner product of the first equation with Ω_{xx} and the second one with ω_{xx}, we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\|\Omega_{xx}\|_{L^2}^2 + \|\omega_{xx}\|_{L^2}^2 \right) \\
= & - 2a(\Omega_{xx}, m_x \Omega_{xx}) + 2(\Omega_{xx}, \omega_x p_{xx}) - a(\Omega_{xx}, m_{xx} \Omega_x) \\
& + (\Omega_{xx}, \omega_{xx} p_x) - a(\Omega_{xx}, m\Omega_{xxx}) + (\Omega_{xx}, \omega p_{xxx}) \\
& - 2a(\omega_{xx}, p_x \omega_{xx}) + (\omega_{xx}, \Omega_x m_{xx}) - a(\omega_{xx}, p_{xx} \omega_x) \\
& + (\omega_{xx}, \Omega_{xx} m_x) - a(\omega_{xx}, p\omega_{xxx}) + (\omega_{xx}, \Omega m_{xxx}).
\end{aligned} \tag{4.6}$$

Notice that, by integration by parts,

$$- a(\Omega_{xx}, m\Omega_{xxx}) = a(\Omega_{xxx}, m\Omega_{xx}) + a(\Omega_{xx}, m_x\Omega_{xx})$$

which implies

$$- a(\Omega_{xx}, m\Omega_{xxx}) = \frac{a}{2}(\Omega_{xx}, m_x\Omega_{xx}).$$

Similarly, we have

$$- a(\omega_{xx}, p\omega_{xxx}) = \frac{a}{2}(\omega_{xx}, p_x\omega_{xx}).$$

Applying Hölder's inequality, the Hilbert transform boundedness on L^p, it follows

$$
\begin{aligned}
|(\Omega_{xx}, m_x\Omega_{xx})| &= |(\Omega_{xx}, (H\omega)\Omega_{xx})| \\
&\leq c_0 \|H\omega\|_{L^\infty} \|\Omega_{xx}\|_{L^2}^2 \\
&\leq c_0 \|\omega_x\|_{L^2} \|\Omega_{xx}\|_{L^2}^2,
\end{aligned}
$$

and similarly

$$|(\omega_{xx}, p_x\omega_{xx})| = |(\Omega_{xx}, (H\omega)\Omega_{xx})| \leq c_0 \|\Omega_x\|_{L^2} \|\omega_{xx}\|_{L^2}^2.$$

We estimate $(\Omega_{xx}, \omega_x p_{xx})$ as

$$
\begin{aligned}
|(\Omega_{xx}, \omega_x p_{xx})| &= |(\Omega_{xx}, \omega_x H\Omega_x)| \\
&\leq c_0 \|\Omega_{xx}\|_{L^2} \|\omega_x\|_{L^4} \|H\Omega_x\|_{L^4} \\
&\leq c_0 \|\Omega_{xx}\|_{L^2} \|\omega_x\|_{L^2}^{\frac{3}{4}} \|\omega_{xx}\|_{L^2}^{\frac{1}{4}} \|H\Omega_x\|_{L^2}^{\frac{3}{4}} \|H\Omega_{xx}\|_{L^2}^{\frac{1}{4}} \\
&\leq c_0 \|\Omega_{xx}\|_{L^2}^{\frac{5}{4}} \|H\Omega_x\|_{L^2}^{\frac{3}{4}} \|\omega_x\|_{L^2}^{\frac{3}{4}} \|\omega_{xx}\|_{L^2}^{\frac{1}{4}} \\
&\leq c_0 \|\Omega_{xx}\|_{L^2}^{\frac{15}{8}} \|H\Omega_x\|_{L^2}^{\frac{9}{8}} + c_0 \|\omega_x\|_{L^2}^{\frac{9}{4}} \|\omega_{xx}\|_{L^2}^{\frac{3}{4}} \\
&\leq c_0 \|\Omega_x\|_{L^2} \|\Omega_{xx}\|_{L^2}^2 + c_0 \|\omega_x\|_{L^2} \|\omega_{xx}\|_{L^2}^2,
\end{aligned}
$$

where we used the inequalities of Hölder, Galiardo-Nirenberg and Young, and the facts that $\|\omega_x\|_{L^2} \leq \|\omega_{xx}\|_{L^2}$ and $\|\Omega_x\|_{L^2} \leq \|\Omega_{xx}\|_{L^2}$. Other terms on the right hand side of (4.6) can be handled similarly as above. We conclude

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\Omega_{xx}\|_{L^2}^2 + \|\omega_{xx}\|_{L^2}^2\right) \\
&\leq c_0(1 + |a|)\left(\|\Omega_x\|_{L^2} + \|\omega_x\|_{L^2}\right)\left(\|\Omega_{xx}\|_{L^2}^2 + \|\omega_{xx}\|_{L^2}^2\right),
\end{aligned}
$$

which immediately gives, by Grönwall's inequality

$$
\begin{aligned}
&\left(\|\Omega_{xx}(t)\|_{L^2}^2 + \|\omega_{xx}(t)\|_{L^2}^2\right) \\
&\leq \left(\|\Omega_{xx}(0)\|_{L^2}^2 + \|\omega_{xx}(0)\|_{L^2}^2\right) e^{\int_0^t 2c_0(1+|a|)\left(\|\Omega_x(\tau)\|_{L^2} + \|\omega_x(\tau)\|_{L^2}\right)d\tau}.
\end{aligned}
\tag{4.7}
$$

Combining (4.7) with the assumption that $\Omega, \omega \in C([0,T]; H^1)$, it follows that

$$\sup_{0 \leq t \leq T}\left(\|\Omega(t)\|_{H^2} + \|\omega(t)\|_{H^2}\right) < \infty.$$

$\square$

5. Pure transport case

In this section we prove Theorem 1.4. According to Theorem 1.1, there exists a unique solution $(\Omega(t), \omega(t))$ of (1.15) on $[0, T]$ for some $T > 0$. In view of Theorem 1.2, in order to show the global existence, it is sufficient to prove

$$\int_0^T \left(\|H\Omega(t)\|_{L^\infty} + \|H\omega(t)\|_{L^\infty} \right) dt < \infty \quad \text{for all } T > 0.$$

On the other hand, due to the boundedness of Hilbert transform, we have

$$\|H\Omega(t)\|_{L^\infty} \le c_0 \|H\Omega(t)\|_{C^\beta} \le c_0 \|\Omega(t)\|_{C^\beta},$$
$$\|H\omega(t)\|_{L^\infty} \le c_0 \|H\omega(t)\|_{C^\beta} \le c_0 \|\omega(t)\|_{C^\beta}$$

for $\beta \in (0, 1)$. As a consequence, we only need to prove:

Proposition 5.1. *Assume $\Omega_0, \omega_0 \in H^1(S^1)$. Let $(\Omega(t), \omega(t))$ be the solution of (1.15) with initial data (Ω_0, ω_0) on $[0, T]$. Then there exists $\beta_1, \beta_2 \in (0, 1)$ such that*

$$\sup_{0 \le t \le T} \left(\|\omega(t)\|_{C^{\beta_1}} + \|\Omega(t)\|_{C^{\beta_2}} \right) < \infty. \tag{5.1}$$

Proof: Recall the equations satisfied by (Ω, ω),

$$\Omega_t + m\Omega_x = 0,$$
$$\omega_t + p\omega_x = 0.$$

Consider the characteristics $X_t(x)$ and $Y_t(x)$ satisfying

$$\frac{d}{dt} X_t = p(t, X_t(\xi)), \quad X_0(\xi) = \xi, \tag{5.2}$$

$$\frac{d}{dt} Y_t = m(t, Y_t(\xi)), \quad Y_0(\xi) = \xi, \tag{5.3}$$

such that

$$\Omega(t, Y_t(x)) = \Omega_0(x), \quad \omega(t, X_t(x)) = \omega_0(x). \tag{5.4}$$

We notice that there exists a unique solution $X_t(x)$ to the Cauchy problem (5.2) and a unique solution $Y_t(x)$ to (5.3). Indeed, since $\Omega(t), \omega(t) \in H^1(S^1) \subset C^{\frac{1}{2}}(S^1)$ and the Hilbert transform is bounded on C^β, we have

$$\|p(t)\|_{C^{1,\frac{1}{2}}} \le c_0 \|\Omega(t)\|_{C^{1,\frac{1}{2}}} < \infty,$$
$$\|m(t)\|_{C^{1,\frac{1}{2}}} \le c_0 \|\omega(t)\|_{C^{1,\frac{1}{2}}} < \infty.$$

Hence, p and m are Lipschitz in time. Thus, the standard ordinary differential equation theory implies existence and uniqueness of solution to (5.2) and (5.3).

Denote the inverse (backward) trajectory of $X_t(x)$ and $Y_t(x)$ by $q_1(t, x) = X_t^{-1}(x)$ and $q_2(t, x) = Y_t^{-1}(x)$, respectively. Note that $q_1(t, x)$ and $q_2(t, x)$ satisfy respectively,

$$\partial_t q_1 = -p(t, q_1(t, x)), \quad q_1(0, x) = x, \tag{5.5}$$

$$\partial_t q_2 = -m(t, q_2(t, x)), \quad q_2(0, x) = x. \tag{5.6}$$

We claim that p and m satisfy the estimate

$$|p(t, x) - p(t, y)| \le F(|x - y|), \quad x, y \in [-\pi, \pi],$$
$$|m(t, x) - m(t, y)| \le G(|x - y|), \quad x, y \in [-\pi, \pi], \tag{5.7}$$

with

$$F(s) = \begin{cases} c_0\|\Omega_0\|_{L^\infty} s(1 - \log s), & 0 \le s \le 1, \\ c_0\|\Omega_0\|_{L^\infty}, & s > 1, \end{cases} \tag{5.8}$$

and

$$G(s) = \begin{cases} c_0\|\omega_0\|_{L^\infty} s(1 - \log s), & 0 \le s \le 1, \\ c_0\|\omega_0\|_{L^\infty}, & s > 1, \end{cases} \tag{5.9}$$

for a universal constant $c_0 > 0$. We only need to show one of them, for instance, the estimate for p. Recall that, by (1.12)

$$p_x(t, x) = H\Omega = \frac{1}{2\pi} P.V. \int_{-\pi}^{\pi} \Omega(t, y) \cot\left(\frac{x - y}{2}\right) dy.$$

Hence, we have

$$p(t, x) = \frac{1}{\pi} P.V. \int_{-\pi}^{\pi} \Omega(t, y) \log\left|\sin\left(\frac{x - y}{2}\right)\right| dy.$$

Without loss of generality, we take $x, y \in (-\pi, \pi)$ such that $-\pi < x < y < \pi$ and $\delta = y - x$. We split the interval $[-\pi, \pi]$ into subintervals

$$I_1 = [-\pi, x - \frac{\delta}{2}), \quad I_2 = [x - \frac{\delta}{2}, x + \frac{\delta}{2}), \quad I_3 = [x + \frac{\delta}{2}, y + \frac{\delta}{2}), \quad I_4 = [y + \frac{\delta}{2}, \pi].$$

In the case of $x - \frac{\delta}{2} \le -\pi$ or $y + \frac{\delta}{2} > \pi$, we treat I_1 or I_4 as an empty set. In order to prove the estimate on p in (5.7), we proceed as

$$|p(t, x) - p(t, y)| = \left| \frac{1}{\pi} P.V. \int_{-\pi}^{\pi} \Omega(t, z) \left(\log\left|\sin\left(\frac{x - z}{2}\right)\right| - \log\left|\sin\left(\frac{y - z}{2}\right)\right| \right) dz \right|$$

$$\le \left| \frac{1}{\pi} P.V. \int_{I_1} \Omega(t, z) \left(\log\left|\sin\left(\frac{x - z}{2}\right)\right| - \log\left|\sin\left(\frac{y - z}{2}\right)\right| \right) dz \right|$$

$$+ \left| \frac{1}{\pi} P.V. \int_{I_2} \cdots dz \right| + \left| \frac{1}{\pi} P.V. \int_{I_3} \cdots dz \right| + \left| \frac{1}{\pi} P.V. \int_{I_4} \cdots dz \right|.$$

The second term on the right hand side can be estimated as

$$\left| \frac{1}{\pi} P.V. \int_{I_2} \Omega(t, z) \left(\log\left|\sin\left(\frac{x - z}{2}\right)\right| - \log\left|\sin\left(\frac{y - z}{2}\right)\right| \right) dz \right|$$

$$\le c_0\|\Omega(t)\|_{L^\infty} P.V. \int_{x - \frac{\delta}{2}}^{x - \frac{\delta}{2}} |\log|x - z|| + |\log|y - z|| \, dz$$

$$\le c_0\|\Omega(t)\|_{L^\infty} \delta (1 + |\log \delta|)$$

$$\le \begin{cases} c_0\|\Omega_0\|_{L^\infty} \delta (1 - \log \delta), & 0 < \delta < 1 \\ c_0\|\Omega_0\|_{L^\infty}, & \delta \ge 1. \end{cases}$$

The integrals on I_1, I_3 and I_4 can be estimated similarly. The estimate for m in (5.7) can be established in an analogous way.

In view of (5.2)-(5.3) and (5.7), we have

$$\partial_t |q_1(t, x) - q_1(t, y)| \le F(|q_1(t, x) - q_1(t, y)|),$$
$$\partial_t |q_2(t, x) - q_2(t, y)| \le G(|q_2(t, x) - q_2(t, y)|). \tag{5.10}$$

Denote $\beta_1(t) = e^{-c_0\|\Omega_0\|_{L^\infty}t}$ and $\beta_2(t) = e^{-c_0\|\omega_0\|_{L^\infty}t}$. For fixed x and y with $|x - y| < 1$, define

$$z_1(t) = \begin{cases} |x - y|^{\beta_1(t)} e^{1-\beta_1(t)}, & 0 \le t < t_0, \\ 1 + c_0\|\Omega_0\|_{L^\infty}(t - t_0), & t \ge t_0, \end{cases}$$

where t_0 is such that $|x - y|^{\beta_1(t_0)} e^{1-\beta_1(t_0)} = 1$. Note that $\beta_1(0) = 1$ and $z_1(0) = |x - y| < 1$. Hence, $z_1(t)$ is well-defined on $[0, \infty)$. One can verify that $z_1(t)$ is the solution of the differential equation

$$\partial_t z = F(z), \quad z(0) = |x - y|.$$

Combining with the first inequality of (5.10), we conclude

$$|q_1(t, x) - q_1(t, y)| \le z_1(t). \tag{5.11}$$

Similarly, we define

$$z_2(t) = \begin{cases} |x - y|^{\beta_2(t)} e^{1-\beta_2(t)}, & 0 \le t < t_0, \\ 1 + c_0\|\omega_0\|_{L^\infty}(t - t_0), & t \ge t_0, \end{cases}$$

with t_0 such that $|x-y|^{\beta_2(t_0)} e^{1-\beta_2(t_0)} = 1$. Analogously, using the second inequality of (5.10), we infer

$$|q_2(t, x) - q_2(t, y)| \le z_2(t). \tag{5.12}$$

We are ready to show (5.1). Noticing that $\omega(t, x) = \omega(0, X_t^{-1}(x))$, we deduce

$$\begin{aligned} |\omega(t, x) - \omega(t, y)| &= \left|\omega(0, X_t^{-1}(x)) - \omega(0, X_t^{-1}(y))\right| \\ &= \left|\int_{X_t^{-1}(y)}^{X_t^{-1}(x)} \omega_{0,x}(\zeta)\, d\zeta\right| \\ &\le c_0\|\omega_{0,x}\|_{L^2} \left|X_t^{-1}(x) - X_t^{-1}(y)\right|^{\frac{1}{2}} \\ &\le c_0\|\omega_{0,x}\|_{L^2} \left|q_1(t, x) - q_1(t, y)\right|^{\frac{1}{2}} \end{aligned}$$

where mean value theorem and Hölder's inequality were applied. As a consequence, we conclude

$$\sup_{0 \le t \le T} \|\omega(t)\|_{C^{\beta_1}} < \infty.$$

thanks to (5.11). Analogously, we can show

$$\sup_{0 \le t \le T} \|\Omega(t)\|_{C^{\beta_2}} < \infty.$$

It completes the proof of the proposition.

$\square$

6. NUMERICAL SIMULATIONS

In this section, we perform some numerical study for the 1D model (1.11) of MHD. For convenience, we recall (1.11) here,

$$\begin{aligned} \Omega_t + am\Omega_x - \omega p_x &= 0, \\ \omega_t + ap\omega_x - \Omega m_x &= 0, \qquad x \in [-\pi, \pi] \\ p_x &= H\Omega, \quad m_x = H\omega, \end{aligned} \tag{6.1}$$

and the Hilbert transform for a periodic function

$$Hf = \frac{1}{2\pi} P.V. \int_{-\pi}^{\pi} f(y) \cot\left(\frac{x-y}{2}\right) \, dy.$$

As mentioned earlier, in order for p and m to be uniquely defined, we can choose the gauge and set them to have either zero mean over the interval $[-\pi, \pi]$ or zero point value at a fixed point, e.g., $p(x_0, t) = m(x_0, t) = 0$ for some x_0, see [11].

We use a Fourier-collocation spectral method for the spatial approximation and a five stage fourth order low storage Runge-Kutta method for time discretization. An exponential type filter is used for stabilization of the spectral method, see [8]. For a periodic function $f(x)$, its Hilbert transform can be approximated in spectral method via the following formula:

$$\widehat{Hf}(k) = -i\,\text{sgn}(k)\hat{f}(k),$$

where $\hat{f}(k)$ are coefficients in Fourier series of $f(x)$, see [9, 13]. Similarly, for periodic functions $p(x)$ and $\Omega(x)$, the equation $p_x = H\Omega$ can be approximated in spectral method through the relation

$$ik\hat{p}(k) = -i\,\text{sgn}(k)\hat{\Omega}(k).$$

6.1. **Numerical results for the 1D model of MHD.** One can check that, for arbitrary constants A_1, A_2, θ_1, θ_2 and k

$$\Omega(x) = A_1 \sin(kx + \theta_1), \quad \omega(x) = A_2 \sin(kx + \theta_2)$$

and

$$\Omega(x) = A_1 \cos(kx + \theta_1), \quad \omega(x) = A_2 \cos(kx + \theta_2)$$

are steady states of system (6.1). Thus, we choose to consider the initial condition

$$\Omega_0 = \sin(x) + \cos(4x) + 5, \quad \omega_0 = \sin(2x) + 2, \tag{6.2}$$

such that there are two non-zero modes in the Fourier representation of Ω_0.

We conduct simulations for (6.1) with initial data (6.2) under the following two settings: (i) $a = 1$ and (ii) $a = -1$. In the computation, we take $N = 12800$ points in the Fourier-collocation spectral method.

Figure 1 shows the numerical results for case (i). The time evolution of $\Omega(t, x)$ and $\omega(t, x)$ are plotted in Figure 1(a) and Figure 1(b), respectively. One can see that $\Omega(t, x)$ and $\omega(t, x)$ are rather smooth. The first order derivative Ω_x shown in Figure 1(c) seems smooth as well, while ω_x illustrated in Figure 1 (d) develops some mild spines at time $t = 4$. However, we observe spines for the second derivatives Ω_{xx} and ω_{xx} at larger time in Figure 1(e) and (f). In particular, there is a notable spine near $x = 0$ at $t = 4$. Notice that

$$u = \frac{1}{2}(p + m), \quad B = \frac{1}{2}(p - m),$$

and hence

$$u_x = \frac{1}{2}(p_x + m_x) = \frac{1}{2}H(\Omega + \omega), B_x = \frac{1}{2}(p_x - m_x) = \frac{1}{2}H(\Omega - \omega),$$
$$H\Omega = u_x + B_x, \quad H\omega = u_x - B_x. \tag{6.3}$$

Figure 1(g) shows the time evolution of $\|H\Omega\|_{L^\infty} + \|H\omega\|_{L^\infty}$, while Figure 1(h) shows $\|u_x(t)\|_{L^\infty}$ and $\|B_x(t)\|_{L^\infty}$. We observe oscillations in these graphs and the amplitudes grow slowly in a linear manner. Combined with the regularity

criterion (1.16), it seems that the solution starting with data (6.2) may not develop singularities at finite time.

For case (ii) with $a = -1$ and $p(t, 0) = m(t, 0) = 0$, the results are illustrated in Figure 2. One can see from Figure 2(a) and (b) that the solution is less smooth compared to the solution in case (i) shown in Figure 1(a) and (b). This suggests that the convection term with a negative sign causes the solution to behave more singularly. Nevertheless, 2(c) and (d) show that the amplitudes of $\|H\Omega\|_{L^\infty} + \|H\omega\|_{L^\infty}$, $\|u_x(t)\|_{L^\infty}$ and $\|B_x(t)\|_{L^\infty}$ grow faster than that of case (i), but remain in a linear growth. Thus one may speculate that solutions of system (6.1) with $a = -1$ starting from smooth initial data do not develop singularities in finite time.

6.2. **Numerical results for the De Gregorio model revisited.** Numerical simulations for the De Gregorio model (1.6a)-(1.6b) have been performed in [4, 5, 13] among others. The main information is that singularity formation for this model with smooth initial data is unlikely to happen.

We apply our numerical scheme to (1.6a)-(1.6b) with the initial data

$$\omega_0(x) = \sin x + 0.1\sin(2x)$$

by taking $N = 12800$ points in the Fourier-collocation spectral method. The obtained simulations are shown in Figure 3, which recover the numerical results done by Okamoto, Sakajo, and Wunsch [13].

We note that $u_x = H\omega$ for the De Gregorio model (1.6a)-(1.6b) and $u_x = \frac{1}{2}H(\Omega + \omega)$ for our 1D MHD model (6.1), see (6.3). Comparing Figure 1(h) and Figure 3(e), we observe oscillations of $\|u_x\|_{L^\infty}$ for the 1D MHD model and absence of such oscillations for the pure fluid model. It is reasonable to infer that the interactions between fluid velocity and magnetic field cause such oscillations and more complicated dynamics.

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Figure 1. $a = 1$

Figure 2. $a = -1$

(a)

(b)

(c)

(d)

(e)

FIGURE 3. The De Gregorio model.