Fractional Order Periodic Maps: Stability Analysis and Application to the Periodic-2 Limit Cycles in the Nonlinear Systems

Sachin Bhalekar[a], Prashant M. Gade[b]

[a]*School of Mathematics and Statistics, University of Hyderabad, Hyderabad, 500046 India*
[b]*Department of Physics, Rashtrasant Tukadoji Maharaj Nagpur University, Nagpur*

Abstract

We consider the stability of periodic map with period-2 in linear fractional difference equations where the function is $f(x) = ax$ at even times and $f(x) = bx$ at odd times. The stability of such a map for an integer order map depends on product ab. The conditions are much complex for fractional maps and depend on ab as well as $a + b$. There are no superstable period-2 orbits. These conditions are useful in obtaining stability conditions of asymptotically periodic orbits with period-2 in the nonlinear case. The stability conditions are demonstrated numerically. The formalism can be generalized to higher periods.

1. Introduction

The stability analysis of the fixed point is extremely helpful in the study of nonlinear maps of integer order. The stability of periodic orbits is no less important. In fact, one of the definitions of chaos is as follows: if V is a set and $F : V \to V$ is chaotic if it has a) sensitive dependence on initial conditions, b) is topologically transitive and c) the periodic points of F are dense on V[1]. It has been shown that conditions b) and c) imply a). Thus periodic points are crucial in the theory of chaos. Several invariant properties of the chaotic attractor can be computed using unstable periodic orbits[2].

Email addresses: `sachin.math@yahoo.co.in, sachinbhalekar@uohyd.ac.in (Corresponding Author)` (Sachin Bhalekar), `prashant.m.gade@gmail.com` (Prashant M. Gade)

The multifractal spectrum of the attractor can be computed using unstable periodic orbits[3]. In a very striking result, it has been shown that the statistical properties of turbulence can be computed using only one unstable periodic orbit[4, 5]. We can compute Lyapunov exponent of the system using unstable periodic orbit[6]. Several dynamic quantities can be computed using eigenvalues of a few short fundamental cycles because they are structured hierarchically. Longer cycles only offer higher order corrections to these quantities[2]. Apart from invariant density, fractal dimension and Lyapunov exponents, we can compute the topological and metric entropy of the attractor[7]. Periodic orbits form a skeleton of chaotic attractors and control schemes such as the Ott-Grebogi-Yorke scheme have been used in controlling chaos and stabilizing a particular periodic orbit[8]. Several bifurcations in the system, such as crisis, can be explained by understanding the periodic orbits and their stable and unstable manifold[9]. (We note that unstable periodic orbits are an important theoretical tool in studying quantum chaos as well[10].) They have important applications, including control of cardiac chaos[11]. Methods based on the detection of unstable periodic orbits have been used to establish low dimensional chaos in crayfish caudal photoreceptor [12]. They are used in the characterization, control and prediction of experimental systems [13, 14]. In short, the importance of periodic orbit in the theory of nonlinear dynamics and chaos cannot be overemphasized.

In fractional order systems, the chaos theory is not as well developed as in integer order systems. However, specific results about the stability of the fixed point are obtained. We linearize around the fixed point and ensure stability if the eigenvalues are inside the unit circle. In integer order maps, the chain rule is applicable. Thus, we can study the stability of fixed points of the function $f^n(x)$ for n-period orbits. We again linearize and the stability of the n-period orbit $(x_1, x_2, \ldots, x_n)$ of 1-d map is dictated by the condition that $|f'(x_n)f'(x_{n-1})\ldots f'(x_1)| < 1$.

Unfortunately, these conditions do not work for fractional order maps even for fixed points and the stability is ensured if the eigenvalues are inside the cardioid-shaped stability region in the complex plane[**?**]. This work shows that the conditions are even more complicated for periodic points. (In fractional maps, we have only asymptotically periodic points, not strictly periodic ones). While it is indeed true that the fixed points are given by $f(x) = x$ even for fractional order map, the period-n orbit is not given by roots of equation $f^n(x) = x$. The fixed points of the twice iterated nonlinear map $f^2(x)$ do not give the 2-period orbit for the fractional order map, which is reached asymptotically. We will explicitly solve the system for an asymptotically 2-period orbit. Besides, the stability of orbit (x_1, x_2) is not given by product $f'(x_1)f'(x_2)$. It is also dependent in $f'(x_1) + f'(x_2)$. Thus the stability analysis of higher period orbits for fractional maps is much more complicated than integer order maps. The bifurcation diagrams can be complicated as well. In integer order maps, we observe a period doubling cascade. When certain period becomes unstable, we obtain the stable solution with twice the period. In fractional order maps, the fixed point and period two orbit can both be stable at same parameter value.

In this work, we first derive the analytic conditions for the stability of the periodic map. The map is linear. However, it is different for odd and even times. We find that the same conditions work for linearized asymptotically period-two orbits of fractional nonlinear maps.

2. Preliminaries

In this section, we present some basic definitions and results. Let $h > 0$, $a \in \mathbb{R}$, $(h\mathbb{N})_a = \{a, a+h, a+2h, \ldots\}$ and $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$.

Definition 2.1. *(see [15, 16, 17]). For a function $x : (h\mathbb{N})_a \to \mathbb{R}$, the forward h-difference operator if defined as*

$$(\Delta_h x)(t) = \frac{x(t+h) - x(t)}{h},$$

where $t \in (h\mathbb{N})_a$.

Throughout this paper, we take $a = 0$ and $h = 1$.

Definition 2.2. *[17] For a function $x : \mathbb{N}_\circ \to \mathbb{R}$ the fractional sum of order $\alpha > 0$ is given by*

$$(\Delta^{-\alpha} x)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{n} \frac{\Gamma(\alpha + n - s)}{\Gamma(n - s + 1)} x(s), \tag{1}$$

where, $t = \alpha + n$, $n \in \mathbb{N}_\circ$.

Definition 2.3. *[17, 18] Let $\mu > 0$ and $m - 1 < \mu < m$, where $m \in \mathbb{N}$, $m = \lceil \mu \rceil$. The μth fractional Caputo-like difference is defined as*

$$\Delta^\mu x(t) = \Delta^{-(m-\mu)} \left(\Delta^m x(t)\right), \tag{2}$$

where $t \in \mathbb{N}_{m-\mu}$ *and*

$$\Delta^m x(t) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} x(t+k). \tag{3}$$

Note: The discrete dynamical system $x(t+1) = f(x(t))$ can be written equivalently as a difference equation $\Delta x(t) = f(x(t)) - x(t)$ by subtracting the term $x(t)$ from both sides. Further, we can generalize this difference equation by replacing the operator Δ with the operator Δ^α, where $0 < \alpha < 1$. If we consider the difference equations in the higher dimensions and if the function f is linear, then we get the systems of the form Equation (**??**). This is the motivation behind the term $A - I$. Of course, we can write the matrix $A' = A - I$ and use the results discussed in the literature (e.g., [19]) to analyze this system.

Definition 2.4. *[17] The Z-transform of a sequence* $\{y(n)\}_{n=0}^{\infty}$ *is a complex function given by* $Y(z) = Z[y](z) = \sum_{k=0}^{\infty} y(k) z^{-k}$ *where* $z \in \mathbb{C}$ *is a complex number for which the series converges absolutely.*

Definition 2.5. *[17] Let* $\tilde{\phi}_\alpha(n)$ *be a family of binomial functions defined on* $\mathbb{Z}$*, parametrized by* α *defined by*

$$\begin{aligned} \tilde{\phi}_\alpha(n) &= \frac{\Gamma(n+\alpha-1)}{\Gamma(\alpha)\Gamma(n)} \\ &= \begin{pmatrix} n+\alpha-1 \\ n \end{pmatrix} = (-1)^n \begin{pmatrix} -\alpha \\ n \end{pmatrix}. \end{aligned} \tag{4}$$

Then

$$Z(\tilde{\phi}_\alpha(t)) = \frac{1}{(1-z^{-1})^\alpha}, \quad |z| > 1.$$

Definition 2.6. *[17] The convolution* $\phi * x$ *of the functions* ϕ *and* x *defined on* $\mathbb{N}$ *is defined as*

$$(\phi * x)(n) = \sum_{s=0}^{n} \phi(n-s) x(s) = \sum_{s=0}^{n} \phi(s) x(n-s).$$

Then the Z-transform of this convolution is

$$Z(\phi * x)(n) = (Z(\phi)(n))(Z(x)(n)). \tag{5}$$

3. The Model and Characteristic Equation

Let $x:\mathbb{N}_\circ \to \mathbb{R}$ and f be a map defined by

$$f(x(t)) = \begin{cases} ax(t), & \text{if } t \text{ is even} \\ bx(t), & \text{if } t \text{ is odd,} \end{cases} \tag{6}$$

where a and b are real numbers.
We define the fractional order discrete dynamical system using this map as

$$x(t+1) = x(0) + \sum_{j=0}^{t} \frac{\Gamma(t-j+\alpha)}{\Gamma(\alpha)\Gamma(t-j+1)} \left[f(x(j)) - x(j)\right]. \tag{7}$$

The traditional way to obtain the characteristic equation of the systems of the form (7) is to take Z-transform and equate the coefficient of $Z[x(t)]$ to zero. The map f defined in (3) can also be written as

$$f(x(t)) = \frac{a+b+(-1)^t(a-b)}{2} x(t).$$

Applying Z-transform to (7), we get

$$X(z) - zx(0) = \frac{x(0)}{1-z^{-1}} + \frac{1}{(1-z^{-1})^\alpha}\left(\frac{a+b}{2} - 1\right) X(z) + \frac{1}{(1-z^{-1})^\alpha}\left(\frac{a-b}{2} - 1\right) X(-z). \tag{8}$$

Note that the equation (8) cannot be used to find the characteristic equation because of $X(-z)$ in the last term.
An elegant way to get the solution to this problem is to separate the terms $x(t)$ with even t from odd values of t. Let us define $p(t) = x(2t)$ and $q(t) = x(2t+1)$. The system (7) can now be written in an equivalent form as

$$\begin{aligned} p(t+1) &= x(0) + \sum_{k=0}^{t} \frac{\Gamma(2t+1-2k+\alpha)}{\Gamma(\alpha)\Gamma(2t-2k+2)} \left[(a-1)p(k)\right] \\ &\quad + \sum_{k=0}^{t} \frac{\Gamma(2t-2k+\alpha)}{\Gamma(\alpha)\Gamma(2t-2k+1)} \left[(b-1)q(k)\right], \\ q(t+1) &= x(0) + \sum_{k=0}^{t} \frac{\Gamma(2t+2-2k+\alpha)}{\Gamma(\alpha)\Gamma(2t-2k+3)} \left[(a-1)p(k)\right] \\ &\quad + (a-1)p(t+1) + \sum_{k=0}^{t} \frac{\Gamma(2t+1-2k+\alpha)}{\Gamma(\alpha)\Gamma(2t-2k+2)} \left[(b-1)q(k)\right]. \end{aligned} \tag{9}$$

If we define $\phi_1(t) = \tilde{\phi}_\alpha(2t)$, $\phi_2(t) = \tilde{\phi}_\alpha(2t+1)$ and $\phi_3(t) = \tilde{\phi}_\alpha(2t+2)$ then the system (9) can be written as

$$\begin{aligned} p(t+1) &= x(0) + (a-1)\left(\phi_2 * p\right)(t) + (b-1)\left(\phi_1 * q\right)(t), \\ q(t+1) &= x(0) + (a-1)\left(\phi_3 * p\right)(t) + (a-1)p(t+1) + (b-1)\left(\phi_2 * q\right)(t). \end{aligned} \tag{10}$$

With a few computations, we get the Z-transforms as

$$\begin{aligned} Z\left(\phi_1(t)\right) &= \frac{\left(\sqrt{z}-1\right)^{-\alpha} + \left(\sqrt{z}+1\right)^{-\alpha}}{2z^{-\alpha/2}}, \\ Z\left(\phi_2(t)\right) &= \frac{\left(\sqrt{z}-1\right)^{-\alpha} - \left(\sqrt{z}+1\right)^{-\alpha}}{2z^{(-\alpha-1)/2}}, \\ Z\left(\phi_3(t)\right) &= -z + \frac{\left(\sqrt{z}-1\right)^{-\alpha} + \left(\sqrt{z}+1\right)^{-\alpha}}{2z^{-1-\alpha/2}}, \\ Z\left(p(t+1)\right) &= zP(z) - zp(0), \; Z\left(q(t+1)\right) = zQ(z) - zq(0), \end{aligned} \tag{11}$$

where $Z(p(t)) = P(z)$, $Z(q(t)) = Q(z)$, $p(0) = x(0)$ and $q(0) = x(1) = ax(0)$. Applying Z-transform to the system (10) and using (11), we get

$$\begin{aligned} &\left[z - (a-1)\frac{\left(\sqrt{z}-1\right)^{-\alpha} - \left(\sqrt{z}+1\right)^{-\alpha}}{2z^{(-\alpha-1)/2}}\right] P(z) \\ &\quad -(b-1)\frac{\left(\sqrt{z}-1\right)^{-\alpha} + \left(\sqrt{z}+1\right)^{-\alpha}}{2z^{-\alpha/2}} Q(z) = \frac{-z^2}{1-z}x(0), \\ &(a-1)\frac{\left(\sqrt{z}-1\right)^{-\alpha} + \left(\sqrt{z}+1\right)^{-\alpha}}{2z^{-1-\alpha/2}} P(z) \\ &\quad + \left[(b-1)\frac{\left(\sqrt{z}-1\right)^{-\alpha} - \left(\sqrt{z}+1\right)^{-\alpha}}{2z^{(-\alpha-1)/2}} - z\right] Q(z) = \frac{z^2}{1-z}x(0). \end{aligned} \tag{12}$$

The characteristic equation of the system (9) (and hence of the system (7)) can now be obtained by equating the determinant of coefficients of the terms $P(z)$ and $Q(z)$ in the system (12) to zero as below:

$$-z(z-1)^\alpha - \frac{1}{2}(a+b-2)z^{\frac{1+\alpha}{2}}\left[\left(\sqrt{z}-1\right)^{-\alpha} - \left(\sqrt{z}+1\right)^{-\alpha}\right] + (a-1)(b-1)z^\alpha = 0. \tag{13}$$

4. Stable Region

The zero solution of system (7) is locally asymptotically stable if and only if all the roots z of the characteristic equation (13) satisfy $|z| < 1$. Therefore, the boundary of the stable region of the system (7) can be obtained by substituting $z = e^{\iota t}$ in the characteristic equation (13). We have,

$$
\begin{aligned}
z - 1 &= e^{\iota t} - 1 = 2\sin(t/2)e^{\iota(\pi+t)/2}, \\
\sqrt{z} - 1 &= e^{\iota t/2} - 1 = 2\sin(t/4)e^{\iota(2\pi+t)/4}, \\
\sqrt{z} + 1 &= e^{\iota t/2} + 1 = 2\cos(t/4)e^{\iota t/4}. \qquad (14)
\end{aligned}
$$

Using (14), we can rewrite the characteristic equation (13) as

$$
\begin{aligned}
-2^{\alpha}\left(\sin(t/2)\right)^{\alpha} e^{\iota[t+\alpha(\pi+t)/2]} + (a-1)(b-1)e^{\iota\alpha t} \qquad (15) \\
-\frac{1}{2}(a+b-2)e^{\iota t(1+\alpha)/2}2^{\alpha}\left[\left(\sin(t/4)\right)^{\alpha} e^{\iota\alpha(2\pi+t)/4} - \left(\cos(t/4)\right)^{\alpha} e^{\iota\alpha t/4}\right] &= 0.
\end{aligned}
$$

Separating real and imaginary parts in (15), we get

$$
\begin{aligned}
-2^{\alpha}\left(\sin(t/2)\right)^{\alpha}\cos\left(\frac{\alpha(\pi+t)}{2}+t\right) + (a-1)(b-1)\cos(\alpha t) \\
-2^{\alpha-1}(a+b-2)\left(\sin(t/4)\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}+t\left(\frac{1}{2}+\frac{3\alpha}{4}\right)\right) \\
+2^{\alpha-1}(a+b-2)\left(\cos(t/4)\right)^{\alpha}\cos\left(t\left(\frac{1}{2}+\frac{3\alpha}{4}\right)\right) &= 0, \quad (16) \\
-2^{\alpha}\left(\sin(t/2)\right)^{\alpha}\sin\left(\frac{\alpha(\pi+t)}{2}+t\right) + (a-1)(b-1)\sin(\alpha t) \\
-2^{\alpha-1}(a+b-2)\left(\sin(t/4)\right)^{\alpha}\sin\left(\frac{\alpha\pi}{2}+t\left(\frac{1}{2}+\frac{3\alpha}{4}\right)\right) \\
+2^{\alpha-1}(a+b-2)\left(\cos(t/4)\right)^{\alpha}\sin\left(t\left(\frac{1}{2}+\frac{3\alpha}{4}\right)\right) &= 0. \quad (17)
\end{aligned}
$$

Equation (17) is identically satisfied for $t = 0$. For this value of t, the equation (16) gives

$$b = \frac{2\left(2^{\alpha}-1\right) + 2a\left(1-2^{\alpha-1}\right)}{2\left(a-1+2^{\alpha-1}\right)}. \qquad (18)$$

This boundary curve (18) can also be written as

$$\left(a - \left[1-2^{\alpha-1}\right]\right)\left(b - \left[1-2^{\alpha-1}\right]\right) = 4^{\alpha-1}. \qquad (19)$$

Let us call this boundary curve as Γ_1. The lines $a = 1-2^{\alpha-1}$ and $b = 1-2^{\alpha-1}$ are asymptotes for Γ_1.

If we substitute $t = \pi$, then the equations (16) and (17) generate the following common boundary curve Γ_2

$$b = -\frac{1 + 2^{\alpha} - a + 2^{\alpha/2}(a-2)\sin(\alpha\pi/4)}{a - 1 + 2^{\alpha/2}\sin(\alpha\pi/4)}. \tag{20}$$

Equivalently,

$$\left(a - \left[1 - 2^{\alpha/2}\sin(\alpha\pi/4)\right]\right)\left(b - \left[1 - 2^{\alpha/2}\sin(\alpha\pi/4)\right]\right) = -2^{\alpha}\left(\cos(\alpha\pi/4)\right)^2. \tag{21}$$

The lines $a = 1 - 2^{\alpha/2}\sin(\alpha\pi/4)$ and $b = 1 - 2^{\alpha/2}\sin(\alpha\pi/4)$ are asymptotes for Γ_2.

Furthermore, the system (16)-(17) can be solved for a and b as parametric functions of $t \in [0, 2\pi]$. We proceed as below:

Let us define $s_1 = \sin^{\alpha}(t/2)$, $s_2 = \sin(t + \alpha(\pi - t)/2)$, $s_3 = \sin^{\alpha}(t/4)$, $s_4 = \sin(t(\alpha-2)/4)$, $s_5 = \cos^{\alpha}(t/4)$, $s_6 = \sin(((\alpha-2)t - 2\alpha\pi)/4)$. Then

$$\begin{aligned} a(t) &= 1 + \frac{-s_1 s_2 + \sqrt{s_1\left(2^{\alpha}(-s_3 s_4 + s_5 s_6)(s_5 s_4 - s_3 s_6) + s_1 s_2^2\right)}}{s_5 s_4 - s_3 s_6}, \\ b(t) &= 1 - \frac{s_1 s_2 + \sqrt{s_1\left(2^{\alpha}(-s_3 s_4 + s_5 s_6)(s_5 s_4 - s_3 s_6) + s_1 s_2^2\right)}}{s_5 s_4 - s_3 s_6} \end{aligned} \tag{22}$$

is required parametric representation of the boundary curve, which we call Γ_3.

Theorem 4.1. *For $0 < \alpha < 1$, the region inside the boundary curves Γ_1, Γ_2 and Γ_3 is bounded in the ab-plane. For any pair (a, b) in this bounded region, the zero solution of system (7) is locally asymptotically stable.*

Proof: We assume that $0 < \alpha < 1$. The intersection points between the curves Γ_1 and Γ_2 are obtained by equating the right sides of the equations

(18) and (20). This gives the following two points:

$$
\begin{aligned}
(a_1, b_1) &= \left(\frac{2^{1+\alpha/2} - 2\sin(\alpha\pi/4) - \sqrt{2^\alpha + 4^\alpha - 2^{1+3\alpha/2}\sin(\alpha\pi/4)}}{2^{\alpha/2} - 2\sin(\alpha\pi/4)}, \right. && (23)\\
&\left. \frac{2^{1+\alpha/2} - 2^{1+\alpha}\sin(\alpha\pi/4) - (2-2^\alpha)\sqrt{2^\alpha + 4^\alpha - 2^{1+3\alpha/2}\sin(\alpha\pi/4)}}{2^{1+\alpha/2} + 2^{3\alpha/2} - 2^{1+\alpha}\sin(\alpha\pi/4) - 2\sqrt{2^\alpha + 4^\alpha - 2^{1+3\alpha/2}\sin(\alpha\pi/4)}} \right),\\
(a_2, b_2) &= \left(\frac{2^{1+\alpha/2} - 2\sin(\alpha\pi/4) + \sqrt{2^\alpha + 4^\alpha - 2^{1+3\alpha/2}\sin(\alpha\pi/4)}}{2^{\alpha/2} - 2\sin(\alpha\pi/4)}, \right. && (24)\\
&\left. \frac{2^{1+\alpha/2} - 2^{1+\alpha}\sin(\alpha\pi/4) + (2-2^\alpha)\sqrt{2^\alpha + 4^\alpha - 2^{1+3\alpha/2}\sin(\alpha\pi/4)}}{2^{1+\alpha/2} + 2^{3\alpha/2} - 2^{1+\alpha}\sin(\alpha\pi/4) + 2\sqrt{2^\alpha + 4^\alpha - 2^{1+3\alpha/2}\sin(\alpha\pi/4)}} \right).
\end{aligned}
$$

As the curves Γ_j are symmetric about the line $a = b$, so are these intersection points. The point (a_1, b_1) is above whereas (a_2, b_2) is below the line $a = b$ and both are in the first quadrant. This also shows that there is no intersection between the curves Γ_1 and Γ_2 for the negative values of a or b.
The curves Γ_2 and Γ_3 intersects each other at the points (a_3, b_3) and (b_3, a_3), where

$$
\begin{aligned}
a_3 &= 1 + 2^{\alpha/2}\frac{2 - \alpha + \sqrt{2(2 - 2\alpha + \alpha^2)\cos^2(\alpha\pi/4)}}{\alpha\cos(\alpha\pi/4) + (\alpha - 2)\sin(\alpha\pi/4)},\\
b_3 &= 1 + 2^{\alpha/2}\frac{2 - \alpha - \sqrt{2(2 - 2\alpha + \alpha^2)\cos^2(\alpha\pi/4)}}{\alpha\cos(\alpha\pi/4) + (\alpha - 2)\sin(\alpha\pi/4)}. && (25)
\end{aligned}
$$

Note that $a_3 < 0$ and $b_3 > 0$.
Furthermore, for any negative values of a or b, the curve Γ_3 lies between the corresponding branches of the curves Γ_1 and Γ_2. This shows that the region inside the boundary curves Γ_1, Γ_2 and Γ_3 is bounded in the ab-plane. Since the change in stability can occur only at these boundary curves and the system (7) is stable at the origin, the bounded region mentioned above is the stable region for the system (7). This proves the result.

The curves Γ_1 (blue color), Γ_2 (red color), Γ_3 (black color) and the stable region for $\alpha = 0.5$ is shown in Figure 1. The stable orbit of the system (7) with $\alpha = 0.5$ and $(a, b) = (0.6, 0.7)$ inside the stable region in Figure 1 is shown in Figure 2. On the other hand, the unstable orbit of this system with $(a, b) = (-2.5, 3.6)$ outside the stable region in Figure 1 is shown in Figure 3.

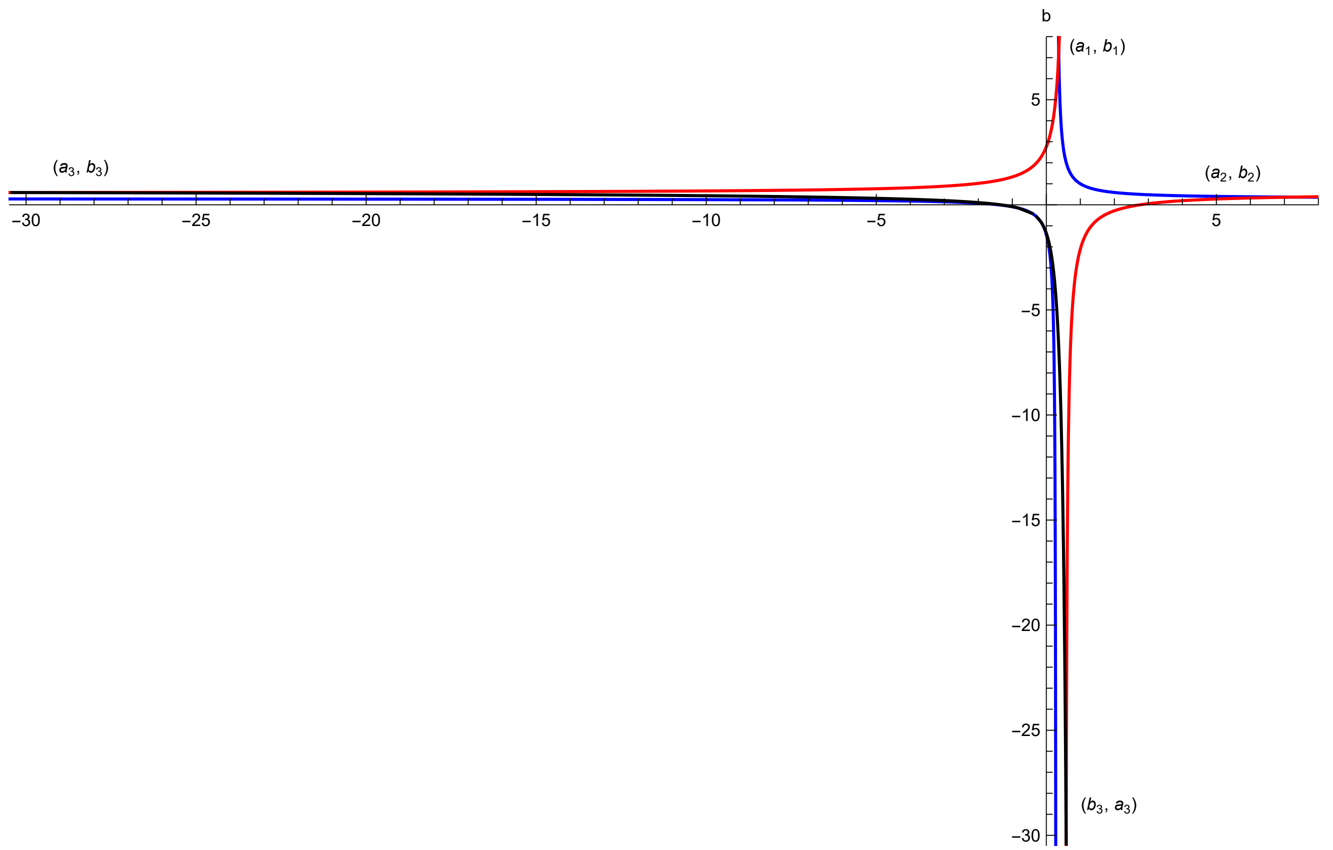

Figure 1: The stable region of system (7) with $\alpha = 0.5$

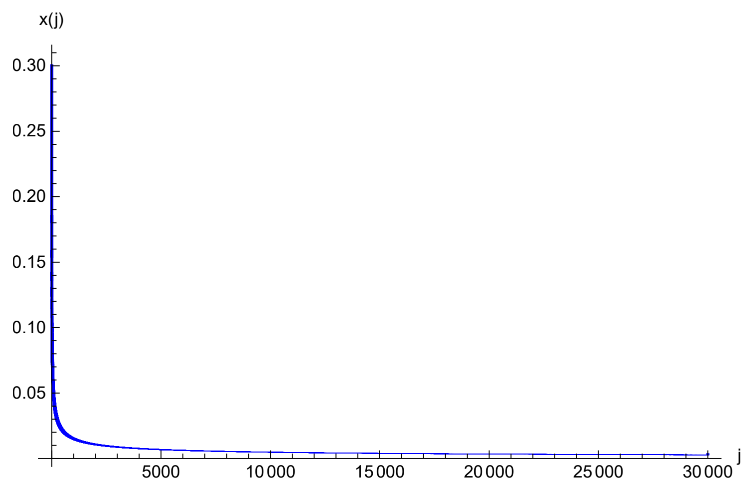

Figure 2: The stable orbit of system (7) with $\alpha = 0.5$ and $(a, b) = (0.6, 0.7)$

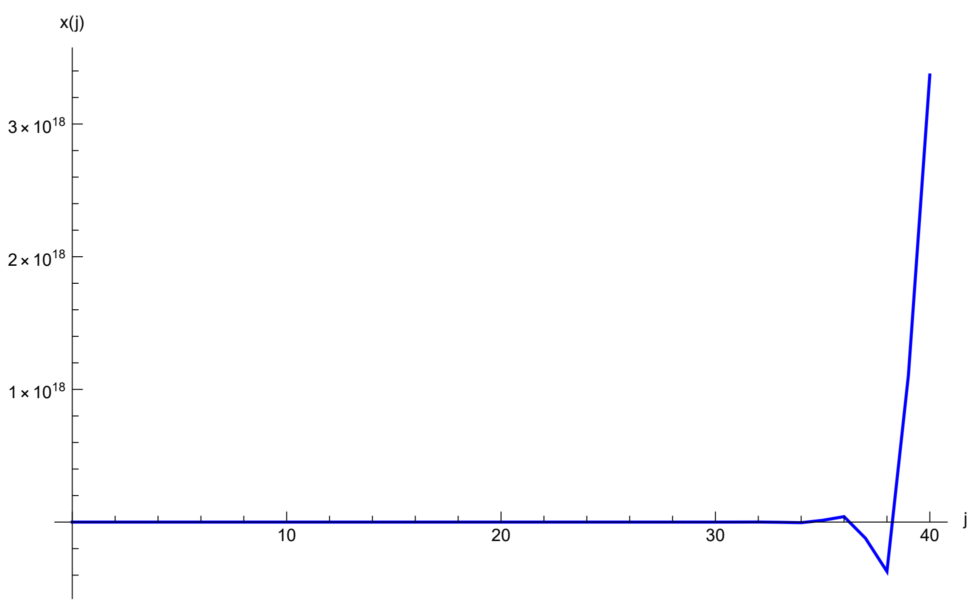

Figure 3: The unstable orbit of system (7) with $\alpha = 0.5$ and $(a, b) = (-2.5, 3.6)$

Note: If $\alpha = 1$, then the curve Γ_3 merges with the branch of Γ_1 in the third quadrant of ab-plane. Furthermore, the curves Γ_1 and Γ_2 do not intersect each other in this case and they get reduced to the curve $|ab| = 1$. Note that $|ab| = 1$ is the boundary of the stable region for the classical map i.e., the system $x(t+1) = f(x(t))$, where f is defined by (3). This shows that our system and stability analysis are the continuous generalization to the classical map and the corresponding stability.

5. Application to the nonlinear systems with period-2 limit cycles

It is proved that [20] the continuous-time fractional order autonomous systems of differential equations cannot have periodic solutions. However, such systems can have "asymptotic" periodic solutions or a limit cycle [20, 21]. In this section, we show that the discrete-time fractional order systems

$$x(t+1) = x(0) + \sum_{j=0}^{t} \frac{\Gamma(t-j+\alpha)}{\Gamma(\alpha)\Gamma(t-j+1)} \left[f\left(x(j)\right) - x(j)\right] \tag{26}$$

also have the same property. Further, we propose a necessary and sufficient condition for the existence of a period-2 limit cycle in the system (26).

Theorem 5.1. *The system (26) cannot have a period-2 orbit.*

Proof: If there exists the points u and v such that u, v is a period-2 orbit of system (26) then $x(2k) = u, x(2k+1) = v$ for $k = 0, 1, 2, \cdots$. Therefore, for $t = 0, 1$ and 2, the system (26) gives

$$f(u) = v, \tag{27}$$
$$f(v) = (1-\alpha)v + \alpha u, \tag{28}$$
$$v = u + \left(\frac{\alpha(\alpha+1)}{2} + 1\right)(f(u) - u) + \alpha\left(f(v) - v\right), \tag{29}$$

respectively. Using (27) and (28) in (29), we get

$$\left(\alpha - \frac{\alpha+1}{2}\right)(u - v) = 0. \tag{30}$$

This implies either $u = v$ or $\alpha = 1$. This contradiction shows that there cannot be a period-2 orbit of the system (26).

5.1. The necessary and sufficient condition for the period-2 limit cycle u, v in the system (26)

As in (9), we can split the system (26) as

$$\begin{aligned} p(t+1) &= x(0) + \sum_{k=0}^{t} \frac{\Gamma(2t+1-2k+\alpha)}{\Gamma(\alpha)\Gamma(2t-2k+2)} \left[f(p(k)) - p(k)\right] \\ &\quad + \sum_{k=0}^{t} \frac{\Gamma(2t-2k+\alpha)}{\Gamma(\alpha)\Gamma(2t-2k+1)} \left[f(q(k)) - q(k)\right], \end{aligned} \tag{31}$$

$$\begin{aligned} q(t+1) &= x(0) + \sum_{k=0}^{t} \frac{\Gamma(2t+2-2k+\alpha)}{\Gamma(\alpha)\Gamma(2t-2k+3)} \left[f(p(k)) - p(k)\right] \\ &\quad + (a-1)p(t+1) + \sum_{k=0}^{t} \frac{\Gamma(2t+1-2k+\alpha)}{\Gamma(\alpha)\Gamma(2t-2k+2)} \left[f(q(k)) - q(k)\right], \end{aligned} \tag{32}$$

where $p(t) = x(2t)$ and $q(t) = x(2t+1)$. If there exists a period-2 limit cycle u, v in the system (26) then

$$\lim_{t\to\infty} p(t) = u$$

and

$$\lim_{t\to\infty} q(t) = v$$

.

If t is very large and k is very small, then the ratios of Gamma functions in (31) and (32) become zero. On the other hand, if k is very large in such cases, then $p(k) \approx u$ and $q(k) \approx v$. Therefore, taking limit as $t \to \infty$ and subtracting (31) from (32), we get

$$\begin{aligned} u - v &= (f(u)-u) \lim_{t\to\infty} \left(\sum_{k=0}^{t-1} \frac{\Gamma(2t-2k-1+\alpha)}{\Gamma(2t-2k)\Gamma(\alpha)} - \sum_{k=0}^{t} \frac{\Gamma(2t-2k+\alpha)}{\Gamma(2t-2k+1)\Gamma(\alpha)} \right) \\ &\quad + (f(v)-v) \lim_{t\to\infty} \left(\sum_{k=0}^{t-1} \frac{\Gamma(2t-2k-2+\alpha)}{\Gamma(2t-2k-1)\Gamma(\alpha)} - \sum_{k=0}^{t-1} \frac{\Gamma(2t-2k-1+\alpha)}{\Gamma(2t-2k)\Gamma(\alpha)} \right) \\ &= -[(f(u)-u) - (f(v)-v)]2^{-\alpha}. \end{aligned} \tag{33}$$

Similarly, taking limit as $t \to \infty$ and adding (31) and (32), we get

$$u+v = 2x(0) + 2[(f(u)-u) + (f(v)-v)] \times [\lim_{t\to\infty} \sum_{k=0}^{2t} \frac{\Gamma(2t-k+\alpha)}{\Gamma(2t-k+1)\Gamma(\alpha)}]. \tag{34}$$

Since the limit in the equation (34) tends to infinity, and all other terms are finite, we must have

$$u + v = f(u) + f(v). \tag{35}$$

Solving equations (33) and (35), we get

$$\begin{aligned} f(u) &= u + 2^{\alpha-1}(v - u) && (36)\\ f(v) &= v + 2^{\alpha-1}(u - v). && (37) \end{aligned}$$

Note that for $\alpha = 1$, the conditions (36)–(37) get reduced $f(u) = v$, $f(v) = u$, the conditions for classical map $x(t+1) = f(x(t))$ to have a period-2 orbit. Recall that the period-2 orbit u, v of the classical map is stable if $|ab| < 1$, where $a = f'(u)$, $b = f'(v)$. Furthermore, there are only an asymptotic period-2 orbits in the fractional order system (26) and $f(p(k)) \approx f(u)+ap(k)$, $f(q(k)) \approx f(v) + bq(k)$. Therefore, heuristically we can use the linearized stability analysis and expect that the point $(a, b) = (f'(u), f'(v))$ should lie inside the stable region of the system (9).

Thus the necessary and sufficient condition for the period-2 limit cycle u, v in the system (26) is
(i) $f(u) = u + 2^{\alpha-1}(v - u)$, $f(v) = v + 2^{\alpha-1}(u - v)$, and
(ii) The point $(a, b) = (f'(u), f'(v))$ lies inside the stable region bounded by the curves Γ_1, Γ_2 and Γ_3 defined in the Section 4. We verify this result with various well-known systems in the subsection below.

5.2. Examples

Example 5.1. *Consider the fractional order logistic map. In this case, we take the equation (26) with $f(x) = \lambda x(1 - x)$, λ is a real parameter.*

The conditions (36) and (37) give

$$\begin{aligned} (u, v) &= \Bigg(\frac{(2^\alpha - 1) + \lambda + \sqrt{(\lambda - (1 - 2^\alpha))(\lambda - (1 + 2^\alpha))}}{2\lambda},\\ &\qquad \frac{(2^\alpha - 1) + \lambda - \sqrt{(\lambda - (1 - 2^\alpha))(\lambda - (1 + 2^\alpha))}}{2\lambda}\Bigg). \end{aligned}$$

These points are real if $\lambda > 1 + 2^\alpha$.

Now, the point

$$\begin{aligned} (a, b) &= (f'(u), f'(v))\\ &= \left(1 - 2^\alpha + \sqrt{(\lambda - (1 - 2^\alpha))(\lambda - (1 + 2^\alpha))}, 1 - 2^\alpha - \sqrt{(\lambda - (1 - 2^\alpha))(\lambda - (1 + 2^\alpha))}\right) \end{aligned}$$

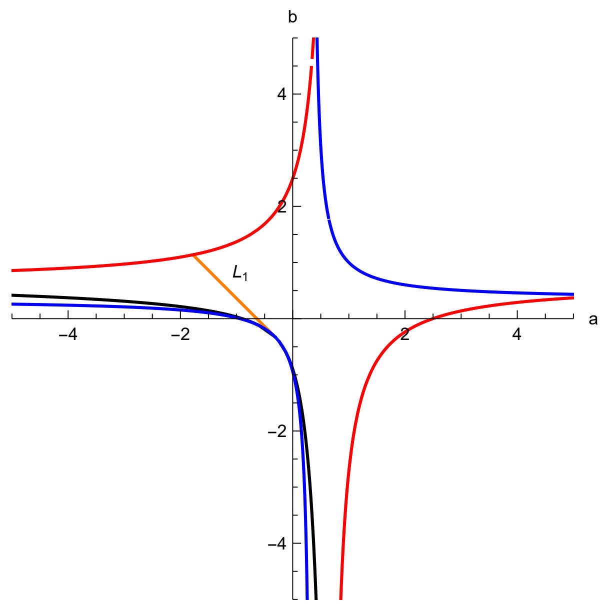

Figure 4: The points $(a, b) = (f'(u), f'(v))$ on the line L_1 indicates the period-2 limit cycle in the fractional order logistic map with $\alpha = 0.4$.

forms a straight line L_1, a parametric curve in λ that intersects the curve Γ_1 at $\lambda = 1 + 2^\alpha$ and Γ_2 at $\lambda = 1 + \sqrt{2^\alpha + 2^{1+2\alpha} - 2^{1+3\alpha/2}\sin(\alpha\pi/4)}$ (cf. Figure 4 for $\alpha = 0.4$). This shows that the fractional order logistic map has period-2 limit cycle if and only if $1 + 2^\alpha < \lambda < 1 + \sqrt{2^\alpha + 2^{1+2\alpha} - 2^{1+3\alpha/2}\sin(\alpha\pi/4)}$. If $\alpha = 0.4$ then we need $\lambda \in (2.31951, 2.96595)$ for period-2 limit cycle. For $\lambda < 2.31951$, the trajectory settles down to an equilibrium point. We can observe period-2 limit cycles when $\lambda \in (2.31951, 2.96595)$ as expected (cf. Figure 5). Period-doubling is observed for $\lambda > 2.96595$.

Example 5.2. *Now, we consider the fractional order cubic map. We take the equation (26) with $f(x) = \beta x(6 - x^2)$, where $\beta < 0$ is a real parameter.*

We get the three expressions (u_0, v_0), (u_1, v_1) and $(-u_1, -v_1)$ for the points

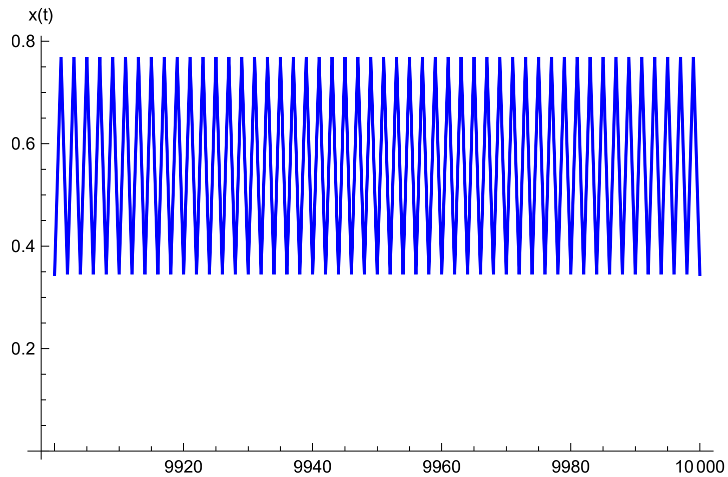

Figure 5: Period-2 limit cycle in the fractional order logistic map with $\alpha = 0.4$ and $\lambda = 2.8$.

(u, v) by using the conditions (36) and (37), where

$$
\begin{aligned}
u_0 &= \sqrt{(6\beta + 2^\alpha - 1)/\beta},\ v_0 = -u_0, \\
u_1 &= \frac{1}{2}\sqrt{12 + \frac{2^\alpha - 2}{\beta} - z}, \\
v_1 &= \frac{2^{-1-\alpha} u_1 \left((2^\alpha - 2)\beta + (12 + z)\beta^2\right)}{\beta}, \\
z &= \frac{-\sqrt{4 - 2^{2+\alpha} - 3 \times 4^\alpha + 24(2^\alpha - 2)\beta + 144\beta^2}}{\beta}.
\end{aligned}
$$

Note that, the points $(a_0, b_0) = (f'(u_0), f'(v_0))$ form a straight-line L_2 in the $ab-$plane, where $a_0 = b_0 = -12\beta - 3(2^\alpha - 1)$. This line L_2 intersects both the branches of boundary curve Γ_1 at the parameter values $\beta_0 = (2 - 3 \times 2^\alpha)/12$ and $\beta_1 = (1 - 2^\alpha)/6$. For $\alpha = 0.7$, we have $\beta_0 = -0.23946$ and $\beta_1 = -0.104084$ respectively. If $\beta > \beta_1$ then the numbers a_0 and b_0 are not real; whereas if $\beta < \beta_0$ then the points (a_0, b_0) are outside the stable region and hence the corresponding points (u_0, v_0) don't form period-2 limit cycle for this system.

The points $(a_1, b_1) = (f'(u_1), f'(v_1))$ form a curve L_3 defined by $4(a^2 + b^2) + (9 \times 2^\alpha - 18)(a + b) + 10ab + 18 - 9 \times 2^{1+\alpha} = 0$ in the $ab-$plane. This curve L_3 intersects the boundary curves Γ_1 and Γ_2 at the parameter values β_0 and

β_2, where

$$\begin{aligned}
\beta_2 &= \frac{1}{384}\Big(64 - 33\times 2^{\alpha} + 2^{1+\alpha/2}\sin(\alpha\pi/4) + 2^{\alpha/2}\nu_1 - 3\sqrt{2}\nu_2 \\
&\quad -3\times 2^{(2+\alpha)/4}\big(179\times 2^{\alpha/2} + 189\times 2^{1+3\alpha/2} - 19\times 2^{\alpha/2}\cos(\alpha\pi/2) \\
&\quad -25\times 3^{\alpha+3}\sin(\alpha\pi/4) + 2^{1+\alpha/2}\sin^2(\alpha\pi/4) + 5(2^{2+\alpha/2} - 2^{1+\alpha})\nu_1 \\
&\quad +3\times 2^{(1+\alpha)/2}\nu_2 - 3\sqrt{2}\nu_1\nu_2 - 6\sqrt{2}\sin(\alpha\pi/4)\nu_2\big)^{1/2}\Big), \\
\nu_1 &= \big(34 + 81\times 2^{\alpha} - 2\cos(\alpha\pi/2) - 9\times 2^{2+\alpha/2}\sin(\alpha\pi/4)\big)^{1/4}, \\
\nu_2 &= 2^{\alpha/2}\big(18 + 9\times 2^{\alpha} - 2\cos(\alpha\pi/2) - 2^{\alpha/2}\nu_1 + 2(-5\times 2^{1+\alpha/2} + \nu_1)\sin(\alpha\pi/4)\big)^{1/2}.
\end{aligned}$$

For $\alpha = 0.7$, $\beta_2 = -0.277584$.

Thus, the condition for the existence of period-2 limit cycle in the fractional order cubic map is $\beta \in (\beta_2, \beta_1)$. Figure 6 shows the curves L_2 and L_3 in the stable region for $\alpha = 0.7$ and $-0.277584 < \beta < -0.104084$.

For $\alpha = 0.7$, the trajectory of fractional order cubic map converges to the period-2 point $(u_0, v_0) = (1.6963, -1.6963)$ when $\beta = -0.20 \in (\beta_0, \beta_1)$ (cf. Figure 7) and to the period-2 point $(u_1, v_1) = (1.45647, -2.14495)$ when $\beta = -0.26 \in (\beta_2, \beta_0)$ (cf. Figure 8). Note that the point $(-u_1, -v_1)$ indicates the existence of "coexisting" asymptotic period-2 orbits as shown in Figure 9.

Example 5.3. *In this example, we discuss the conditions for asymptotic period-2 orbits in the fractional order Gauss map (26), where $f(x) = e^{-7.5x^2} + \beta$ and β is a real parameter.*

Due to the transcendental nature of the function f, we cannot have the exact expressions for the points (u, v) and (a, b), unlike the previous examples. Therefore, we verify the results using numerical approximations. It is observed that, for $-0.05 \leq \beta \leq 0.56$ and $\alpha = 0.6$, the points $(a, b) = (f'(u), f'(v))$ form a curve L_4 that remains inside the stable region as shown in Figure 10. The system shows asymptotic period-2 orbits for all these parameter values, as expected. We did not observe period-2 limit cycles outside this range.

6. Discussion

We have obtained the analytic conditions for the stability of periodic linear map in fractional difference equations. We show that the same condi-

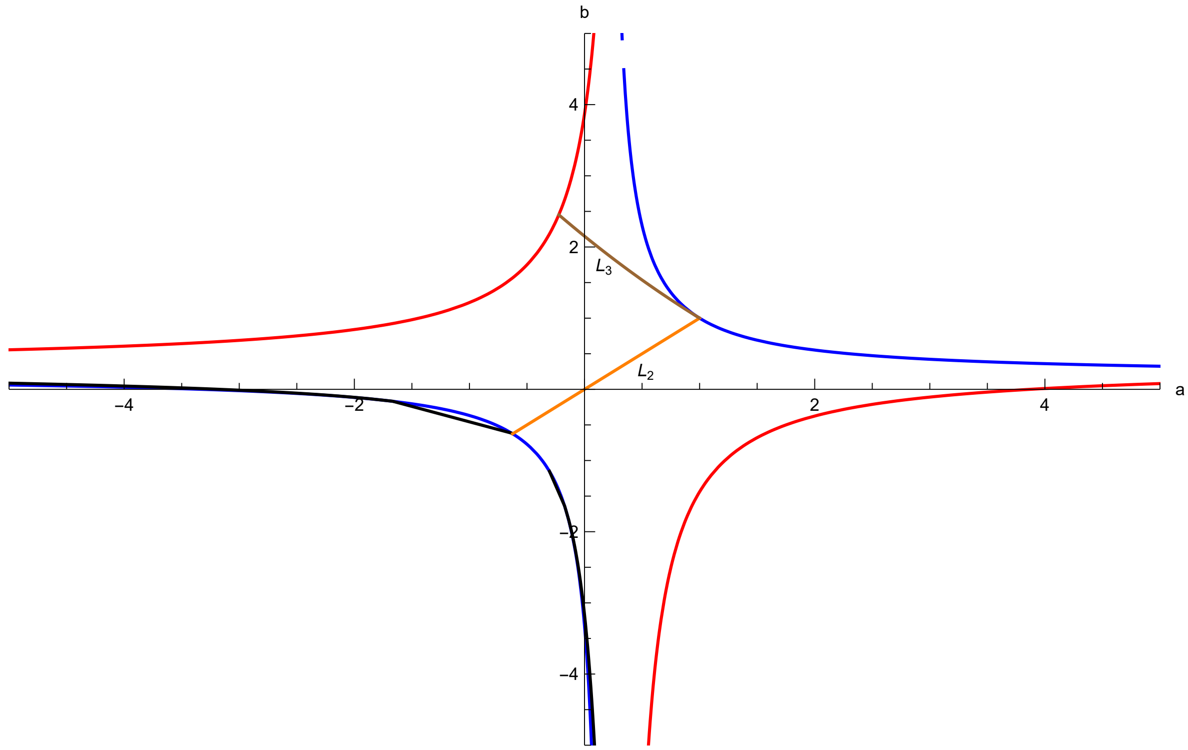

Figure 6: The curves L_2 and L_3 in the stable region for $\alpha = 0.7$ and $-0.277584 < \beta < -0.104084$.

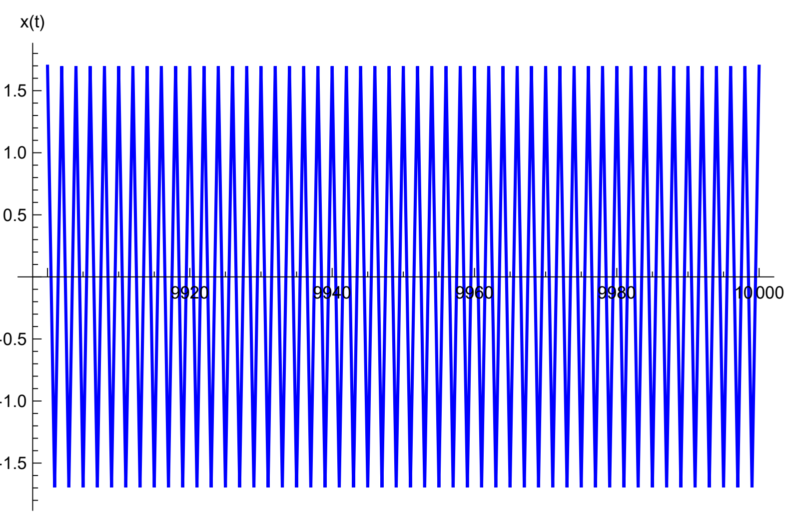

Figure 7: Asymptotic period-2 orbit in the fractional order cubic map for $\alpha = 0.7$ and $\beta = -0.20$.

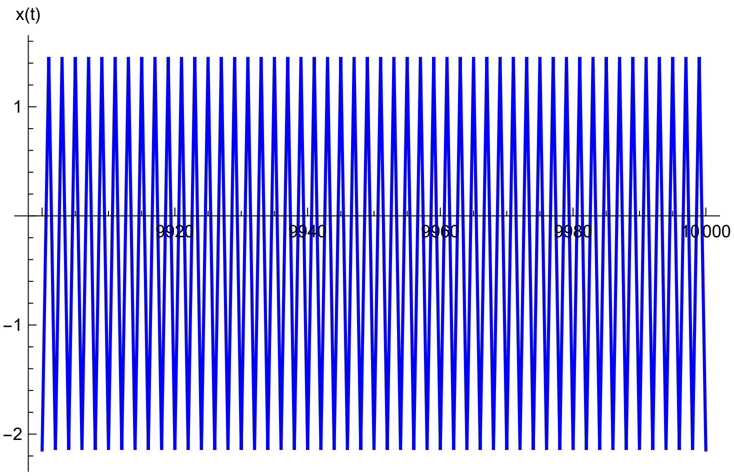

Figure 8: Asymptotic period-2 orbit in the fractional order cubic map for $\alpha = 0.7$ and $\beta = -0.26$.

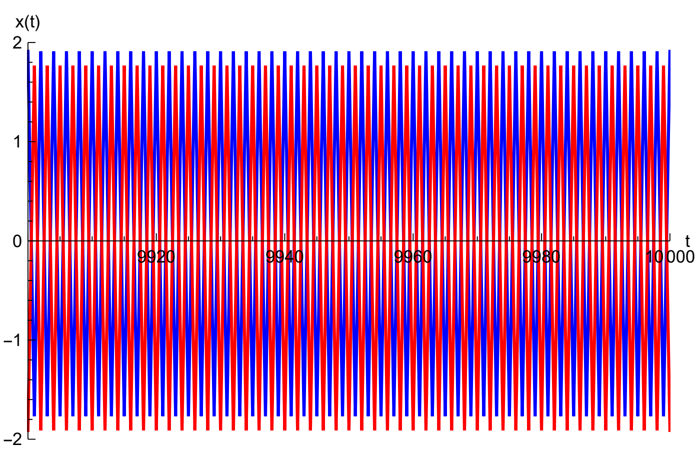

Figure 9: Coexisting asymptotic period-2 orbits in the fractional order cubic map for $\alpha = 0.7$ and $\beta = -0.24$.

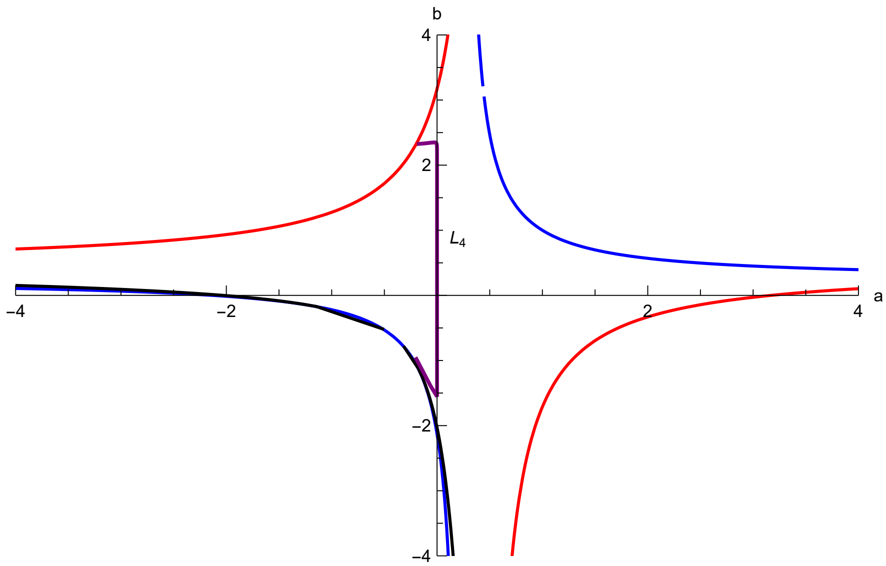

Figure 10: The points on the curve L_4 for $-0.05 \leq \beta \leq 0.56$ show asymptotic period-2 orbits in the fractional order Gauss map with $\alpha = 0.6$.

tions help us infer the stability of asymptotically periodic orbits of period-2 in nonlinear fractional difference equations. This formalism can be potentially generalized to higher periods.

Unstable periodic orbits form the skeleton of chaotic attractors in integer order systems. They are useful in characterization, prediction and control. Analysis of stable and unstable manifolds of periodic orbits is an indispensable tool in the theory of dynamical systems. The presence of chaos or the presence of stable or unstable manifolds of periodic orbits are open questions in fractional order systems. However, finding basic stability conditions for periodic orbit can be a useful step in formulating an analogous theory for fractional systems.

Acknowledgement

P. M. Gade thanks DST-SERB for financial assistance (Ref. CRG/2020/003993).

References

[1] R. L. Devaney, Chaotic dynamical systems: An introduction, Addison-Wesley, 1989.

[2] P. Cvitanović, Invariant measurement of strange sets in terms of cycles, Phys. Rev. Lett. 61 (1988) 2729–2732. doi:10.1103/PhysRevLett.61.2729.
URL https://link.aps.org/doi/10.1103/PhysRevLett.61.2729

[3] C. Grebogi, E. Ott, J. A. Yorke, Unstable periodic orbits and the dimension of chaotic attractors, Phys. Rev. A 36 (1987) 3522–3524. doi:10.1103/PhysRevA.36.3522.
URL https://link.aps.org/doi/10.1103/PhysRevA.36.3522

[4] G. KAWAHARA, S. KIDA, Periodic motion embedded in plane couette turbulence: regeneration cycle and burst, Journal of Fluid Mechanics 449 (2001) 291–300. doi:10.1017/S0022112001006243.

[5] M. Kawasaki, S.-i. Sasa, Statistics of unstable periodic orbits of a chaotic dynamical system with a large number of degrees of freedom, Phys. Rev. E 72 (2005) 037202. doi:10.1103/PhysRevE.72.037202.
URL https://link.aps.org/doi/10.1103/PhysRevE.72.037202

[6] R. Franzosi, P. Poggi, M. Cerruti-Sola, Lyapunov exponents from unstable periodic orbits, Phys. Rev. E 71 (2005) 036218. doi:10.1103/PhysRevE.71.036218.
URL https://link.aps.org/doi/10.1103/PhysRevE.71.036218

[7] H. Fujisaka, Statistical dynamics generated by fluctuations of local lyapunov exponents, Progress of theoretical physics 70 (5) (1983) 1264–1275.

[8] E. Ott, C. Grebogi, J. A. Yorke, Controlling chaos, Phys. Rev. Lett. 64 (1990) 1196–1199. doi:10.1103/PhysRevLett.64.1196.
URL https://link.aps.org/doi/10.1103/PhysRevLett.64.1196

[9] C. Grebogi, E. Ott, J. A. Yorke, Chaotic attractors in crisis, Phys. Rev. Lett. 48 (1982) 1507–1510. doi:10.1103/PhysRevLett.48.1507.
URL https://link.aps.org/doi/10.1103/PhysRevLett.48.1507

[10] H.-J. Stöckmann, Quantum Chaos: An Introduction, Cambridge University Press, 1999. doi:10.1017/CBO9780511524622.

[11] A. Garfinkel, M. L. Spano, W. L. Ditto, J. N. Weiss, Controlling cardiac chaos, Science 257 (5074) (1992) 1230–1235.

[12] X. Pei, F. Moss, Characterization of low-dimensional dynamics in the crayfish caudal photoreceptor, Nature 379 (1996) 618–621.

[13] R. Roy, T. W. Murphy, T. D. Maier, Z. Gills, E. R. Hunt, Dynamical control of a chaotic laser: Experimental stabilization of a globally coupled system, Phys. Rev. Lett. 68 (1992) 1259–1262. doi:10.1103/PhysRevLett.68.1259.
URL https://link.aps.org/doi/10.1103/PhysRevLett.68.1259

[14] L. Flepp, R. Holzner, E. Brun, M. Finardi, R. Badii, Model identification by periodic-orbit analysis for nmr-laser chaos, Phys. Rev. Lett. 67 (1991) 2244–2247. doi:10.1103/PhysRevLett.67.2244.
URL https://link.aps.org/doi/10.1103/PhysRevLett.67.2244

[15] R. A. Ferreira, D. F. Torres, Fractional h-difference equations arising from the calculus of variations, Applicable Analysis and Discrete Mathematics (2011) 110–121.

[16] N. R. Bastos, R. A. Ferreira, D. F. Torres, Discrete-time fractional variational problems, Signal Processing 91 (3) (2011) 513–524.

[17] D. Mozyrska, M. Wyrwas, The z-transform method and delta type fractional difference operators, Discrete Dynamics in Nature and Society 2015 (2015).

[18] C. Fulai, L. Xiannan, Z. Yong, Existence results for nonlinear fractional difference equation, Advances in Difference Equations (2011).

[19] R. Abu-Saris, Q. Al-Mdallal, On the asymptotic stability of linear system of fractional-order difference equations, Fractional Calculus and Applied Analysis 16 (3) (2013) 613–629.

[20] E. Kaslik, S. Sivasundaram, Non-existence of periodic solutions in fractional-order dynamical systems and a remarkable difference between integer and fractional-order derivatives of periodic functions, Nonlinear Analysis: Real World Applications 13 (3) (2012) 1489–1497.

[21] S. Bhalekar, M. Patil, Singular points in the solution trajectories of fractional order dynamical systems, Chaos: An Interdisciplinary Journal of Nonlinear Science 28 (11) (2018) 113123.