On the Poincare-Bendixson formula for planar piecewise smooth vector fields

Shimin Li (lishimin@hznu.edu.cn)
Hangzhou Normal University

Changjian Liu
Sun Yat-sen University

Jaume Llibre
Universitat Autonoma de Barcelona

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ON THE POINCARÉ-BENDIXSON FORMULA FOR PLANAR PIECEWISE SMOOTH VECTOR FIELDS

SHIMIN LI[1], CHANGJIAN LIU[2], JAUME LLIBRE[3]

Abstract. The topological index, or simply the index, of an equilibrium point of a differential system is an integer which saves important information about the local phase portrait of the equilibrium.

There are mainly two ways to calculate the index of an isolated equilibrium point of a smooth vector field. First Poincaré and Bendixson proved that the index of an equilibrium point can be obtained from the number of hyperbolic and elliptic sectors that there are in a neighborhood of the equilibrium point, which is known as Poincaré-Bendixson formula for the topological index of an equilibrium point. Second several works contributed to the algebraic method of Cauchy's index for computing the index of an equilibrium point.

In this paper we extend the Poincaré-Bendixson formula to planar piecewise smooth vector fields. Applying this formula we define the index of generic codimension-one equilibria for piecewise smooth vector fields, including boundary equilibria, pseudo-equilibria and tangency points.

1. Introduction and statement of the main results

A planar smooth differential system is defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = P(x,y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = Q(x,y), \tag{1}$$

where P and Q are C^r with $1 \leqslant r \leqslant \infty$. Let $Z(p) = (P(p), Q(p))$ be the smooth vector field at the point $p \in \mathbb{R}^2$ associated to system (1).

Suppose that $q \in \mathbb{R}^2$ is an isolated equilibrium point of the vector field (1), that is, $P(q) = Q(q) = 0$. One of the fundamental problems in the qualitative theory of planar differential systems is to provide a characterization of the local phase portraits in the vicinity of the equilibrium point q. The Jacobian matrix of system (1) at q is

$$DZ(q) := \begin{pmatrix} \dfrac{\partial P}{\partial x}(q) & \dfrac{\partial P}{\partial y}(q) \\ \dfrac{\partial Q}{\partial x}(q) & \dfrac{\partial Q}{\partial y}(q) \end{pmatrix}. \tag{2}$$

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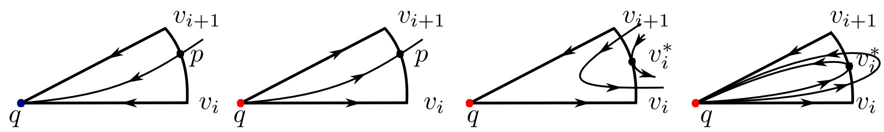

Fig.1.1 Fig.1.2 Fig.1.3 Fig.1.4

FIGURE 1. Illustration of the elliptic sector, hyperbolic sector and parabolic sector. Fig.1.1 is an attracting parabolic sector. Fig.1.2 is a repelling parabolic sector. Fig.1.3 is a hyperbolic sector. Fig.1.4 is an elliptic sector.

The local phase portrait at an equilibrium point q having the real part of its two eigenvalues of the Jacobian matrix of the differential system at q different from zero is determined by the Hartman-Grobman theorem (see for instance Theorem 2.15 of [9]). The semi-hyperbolic equilibria (where one of the eigenvalues equals to zero) are also classified (see Theorem 2.19 of [9]).

Concerning the degenerate equilibria (i.e. when both eigenvalues of the Jacobian matrix at an equilibrium point are equal to zero), the situation is much more difficult. The Andreev theorem classifies the nilpotent equilibrium point, whose associated Jacobian matrix is not identically zero, except the monodromic case (i.e. when the equilibrium point is a center or a focus), see for instance Theorem 3.5 of [9]. In general the study of the local dynamics of the orbits near an equilibrium point with linear part identically zero is quite complicated. In this case the only possibility is studying this equilibrium point doing the changes of variables called blow ups, see [1].

A *characteristic orbit* $\gamma(t)$ of vector field (1) at the equilibrium point q is an orbit tending to q in positive time (resp. in negative time) with a well defined direction θ_0, that is, $\gamma(t) \to q$ for $t \to +\infty$ (resp. $t \to -\infty$) and $\lim\limits_{t\to+\infty} \dfrac{\gamma(t)-q}{\parallel \gamma(t)-q \parallel} = (\cos\theta_0, \sin\theta_0)$ (resp. $\lim\limits_{t\to-\infty} \dfrac{\gamma(t)-q}{\parallel \gamma(t)-q \parallel} = (\cos\theta_0, \sin\theta_0)$) exists. Following Frommer [11] the direction of θ_0 is called a characteristic direction at q. There are several methods to investigate the number of orbits tending to an equilibrium point in a characteristic direction, see [21] for Z-sectors, see [29] for normal sectors, see [28] for generalized normal sectors, and see [22] for quasi normal sectors.

Definition 1. *Let V be a compact neighborhood of an isolated equilibrium point q of vector field* (1)*, and ∂V is the boundary of V. Suppose that there exist a finite number of characteristic orbits $c_1, c_2, \cdots, c_n$, each cutting ∂V transversely at one point v_i, then V can be divided into several sectors $S_i := int\{c_i \cup c_{i+1} \cup (v_i, v_{i+1})\}$ for $i = 1, 2, \cdots, n-1$ and $S_n := int\{c_n \cup c_1 \cup (v_n, v_1)\}$, where (v_i, v_{i+1}) denote the piece of ∂V between v_i and v_{i+1}. We denote by $\gamma(p)$ the orbit of the vector field* (1) *through the point p, and by $\gamma^+(p)$ (resp. $\gamma^-(p)$) the positive semi-orbit (resp. negative semi-orbit) of the vector field* (1) *at the point p. Taking into account the characteristic orbits around the equilibrium point q, we can divide the neighborhood V of q into several sectors as follows.*

- *Attracting parabolic sector. For any points of $[v_i, v_{i+1}] \subset \partial V$ the vector field* (1) *points inward, and for any $p \in S_i \setminus \{q\}$, $\omega(p) = \{q\}$ and $\gamma^-(p) \cap \partial V \neq \emptyset$, see* Fig.1.1.

• *Repelling parabolic sector. For any points of* $[v_i, v_{i+1}] \subset \partial V$ *the vector field* (1) *points outward, and for any* $p \in S_i \setminus \{q\}$, $\alpha(p) = \{q\}$ *and* $\gamma^+(p) \cap \partial V \neq \emptyset$ *see* Fig.1.2.

• *Hyperbolic sector. There exists a point* $v_i^* \in (v_i, v_{i+1}) \subset \partial V$ *such that: for any points of* $[v_i, v_i^*)$ *the vector field points outward (resp. inward) while at all points of* $(v_i^*, v_{i+1}]$ *the vector field points inward (resp. outward); at* v_i^* *the vector field is tangent at* ∂V *and the tangency is external in the sense that the orbit through the point of* v_i^* *stay outside* V*; and for any* $q \in S_i \setminus \{c_i \cup c_{i+1} \cup v_i^*\}$ *we have* $\gamma^+(p) \cap \partial V \neq \emptyset$ *and* $\gamma^-(p) \cap \partial V \neq \emptyset$*, see* Fig.1.3.

• *Elliptic sector. There exists a point* $v_i^* \in (v_i, v_{i+1}) \subset \partial V$ *such that* $\gamma(v_i^*) \subset V$ *with* $\omega(v_i^*) = \alpha(v_i^*) = \{q\}$*; at all points of* $[v_i, v_i^*)$ *the vector field points outward,* $\gamma^-(p) \in V$ *and* $\alpha(q) = q$*; at all points of* $p \in (v_i^*, v_{i+1}]$ *the vector field points inward,* $\gamma^+(p) \subset V$ *and* $\omega(p) = q$*. We denote by* $S_{[v_i, v_i^*]} = \bigcup_{p \in [v_i, v_i^*]} \gamma^-(p)$ *and* $S_{[v_i^*, v_{i+1}]} = \bigcup_{p \in [v_i^*, v_{i+1}]} \gamma^+(p)$*; at all points* p *of* $S \setminus \{S_{[v_i, v_i^*]} \cup S_{[v_i^*, v_{i+1}]} \cup \{q\}\}$ *we have* $\gamma(p) \subset V$ *and* $\omega(p) = \alpha(p) = q$*. The same definition also works interchanging* $[v_i, v_i^*]$ *and* $[v_i^*, v_{i+1}]$*, see* Fig.1.4.

A path in $\mathbb{R}^2$ is a continuous map σ from $I = [0, 1]$ to $\mathbb{R}^2$ ($\sigma : I \to \mathbb{R}^2$), such that $\sigma(t) = (\sigma_1(t), \sigma_2(t)) \in \mathbb{R}^2$ for every $t \in I$, where $\sigma_i : I \to \mathbb{R}$ are continuous maps. We say that the path σ is closed if $\sigma(0) = \sigma(1)$.

Assume that $q \in \mathbb{R}^2$ does not belong to $\sigma(I)$ and let r be a ray with origin at q. We denote by $\overrightarrow{q\sigma(t)}$ the ray from q to $\sigma(t)$. For every point $\sigma(t)$ we denote by $\bar{\varphi}(t)$ the angle between the rays r and $\overrightarrow{q\sigma(t)}$. The angle $\bar{\varphi}(t)$ is an element of the circle $\mathbb{R}/2\pi\mathbb{Z}$. The function $\bar{\varphi} : I \to \mathbb{R}/2\pi\mathbb{Z}$ is continuous with respect to the parameter t; see Figure 2.

Given an isolated equilibrium point q of a vector field $Z(p)$ in $\mathbb{R}^2$, there is a neighborhood V of q on which there is no other equilibria of $Z(p)$. Consider now a closed path $\sigma : I \to V \setminus \{q\}$ such that $\sigma(I)$ is a small circle surrounding q. We define the (topological) index of q equal to $i(Z \circ \sigma, q)$; the number of turns of the closed path $Z \circ \sigma$ around the equilibrium point of coordinates of $\mathbb{R}^2$ when the closed path is run in counter-clockwise sense. The index of q is independent of the chosen closed path σ, see for more details Chapter 6 of [9]. In order to obtained the index of the equilibrium point q, we can to compute the following integral

$$i_q = \frac{1}{2\pi} \oint_\sigma d \arctan \frac{Q(x, y)}{P(x, y)} = \frac{\bar{\varphi}(1) - \bar{\varphi}(0)}{2\pi}. \tag{3}$$

We say that an isolated equilibrium point q of a vector field (1) has the finite sectorial decomposition property if q is not a center, a focus or anode, and q has a finite number of characteristic orbits.

The well known Poincaré-Bendixson formula can be stated as follows:

Poincaré-Bendixson formula: Assume that q is an isolated equilibrium point of a vector field (1) having the finite sectorial decomposition property. Let e, h and p denote the number of elliptic, hyperbolic and parabolic sectors of the local phase portrait at q respectively. Then $i_q = 1 + \dfrac{e - h}{2}$.

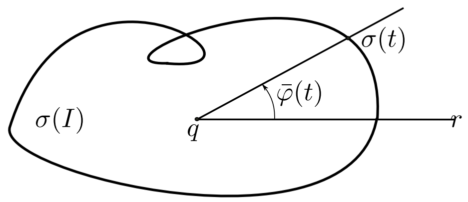

FIGURE 2. Definition of $\bar{\varphi}(t)$.

Note that the Poincaré-Bendixson formula can be extended to the case that the equilibrium point q is a focus, a center or a node.

Up to now piecewise smooth differential systems have appeared in control theory [3], impact and friction mechanics [5, 18], nonlinear oscillations [2], economics [14, 16], and biology [17], ..., see for more details the book [4] and the references therein. One of the most important goals concerning the theory of piecewise smooth differential systems is to look over the validity of the results coming from the classical theory of smooth differential systems into the piecewise smooth differential systems. On one hand, it is obvious that the existence and uniqueness theorem is not true in the piecewise smooth context [12]. While on the other hand, under suitable assumptions, Poincaré-Bendixson theorem [6], Bendixson-Dulac theorem [8], Peixoto theorem [26] and Poincaré recurrence theorem [10] have been generalized to piecewise smooth differential systems. This paper focuses on the Poincaré index formula and its generalization to piecewise smooth differential systems. In order to state our main result we need some preliminary definitions.

A planar piecewise smooth (PWS) differential system is defined by

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= P^{\pm}(x,y), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= Q^{\pm}(x,y), \end{aligned} \qquad \text{for } (x,y) \in \Sigma^{\pm}. \tag{4}$$

where the whole plane $\mathbb{R}^2$ is partitioned into two open zones $\Sigma^{\pm} := \{(x,y) | \pm h(x,y) > 0\}$ by the *discontinuous boundary* $\Sigma = h^{-1}(0)$, which is an one-dimensional smooth manifold and h has 0 as a regular value. Let $Z^{\pm}(p) = (P^{\pm}(p), Q^{\pm}(p))$ be the piecewise smooth vector field associated to the piecewise differential system (4).

Definition 2. *Let $Z^+h(p) = \langle Z^+(p), \nabla h(p)\rangle$ and $Z^-h(p) = \langle Z^-(p), \nabla h(p)\rangle$ be defined in Σ^+ and Σ^- respectively. The discontinuous boundary Σ can be divided as follows:*

- *Crossing region* $\Sigma^c = \{p \in \Sigma| \quad Z^+h(p)Z^-h(p) > 0\}$, *see* Fig.3.1.
- *Attracting region* $\Sigma^a = \{p \in \Sigma| \quad Z^+h(p) < 0, Z^-h(p) > 0\}$, *see* Fig.3.2.
- *Repelling region* $\Sigma^r = \{p \in \Sigma| \quad Z^+h(p) > 0, Z^-h(p) < 0\}$, *see* Fig.3.3.
- *Tangency points* $\Sigma^t = \{p \in \Sigma| \quad Z^+h(p)Z^-h(p) = 0\}$.

In the crossing region Σ^c the trajectories of Z^+ and Z^- can be concatenated naturally. However in the attracting region Σ^a (resp. repelling region Σ^r), the

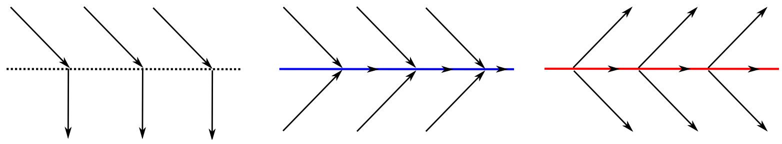

Fig.3.1 Fig.3.2 Fig.3.3

Figure 3. Illustration of the orbits in the crossing, attracting and repelling regions. Fig.3.1 is the crossing region Σ^c. Fig.3.2 is the attracting region Σ^a. Fig.3.3 is the repelling region Σ^r

trajectories cannot be continued through Σ^a (resp. Σ^r) and they slid in Σ^a (resp. Σ^r) in forward (resp. backward) time. Thus both attracting and repelling regions are called *sliding region*, that is $\Sigma^s := \Sigma^a \cup \Sigma^r$. Following the Filippov's convex method we construct the sliding vector field in the form

$$Z^s(p) = \lambda Z^+(p) + (1-\lambda)Z^-(p), \tag{5}$$

where $\lambda \in (0,1)$ is such that Z^s is tangent to Σ^s. In this case

$$\lambda = \frac{Z^-h(p)}{Z^-h(p) - Z^+h(p)}. \tag{6}$$

In the following we give the definition of pseudo-equilibrium of sliding vector field (5).

Definition 3. *The points $p \in \Sigma^s$ which satisfy $Z^s(p) = 0$ will be called pseudo-equilibria of sliding vector field* (5)*.*

Now we can define $\Sigma-$regular points and $\Sigma-$equilibria for piecewise smooth vector field (4), see [13].

Definition 4. *$p \in \Sigma$ will be called $\Sigma-$regular point of piecewise smooth vector field* (4) *if one of the following conditions hold.*

- *$p \in \Sigma^c$ is a crossing point, that is, $Z^+h(p)Z^-h(p) > 0$.*
- *$p \in \Sigma^s$ is not an equilibrium point of $Z^s(p)$, that is, $Z^+h(p)Z^-h(p) < 0$ and $Z^s(p) \neq 0$.*

Any other points $p \in \Sigma$ which are not $\Sigma-$ regular points will be called $\Sigma-$equilibria. More precisely, we have

Definition 5. *The $\Sigma-$equilibria of piecewise smooth vector field* (4) *are:*

- *$p \in \Sigma$ is a boundary equilibrium point, that is, either $Z^+(p) = 0$ or $Z^-(p) = 0$.*
- *$p \in \Sigma^s$ is a pseudo-equilibrium, that is, $Z^s(p) = 0$. Moreover, we will call stable pseudo-node to any point $p \in \Sigma^a$ such that $Z^s(p) = 0$ and $(Z^s)^{'}(p) < 0$, see* Fig.17.1*; unstable pseudo-node to any point $p \in \Sigma^r$ such that $Z^s(p) = 0$ and $(Z^s)^{'}(p) > 0$, see* Fig.17.2*; pseudo-saddle to any point $p \in \Sigma^a$ such that $Z^s(p) = 0$ and $(Z^s)^{'}(p) > 0$ (see* Fig.16.1*) or $q \in \Sigma^r$ such that $Z^s(p) = 0$ and $(Z^s)^{'}(p) < 0$ (see* Fig.16.2*).*
- *$p \in \Sigma^t$ is a tangency point, that is, $Z^+h(p)Z^-h(p) = 0$. A point $p \in \Sigma^t$ is a fold of Z^+ if $Z^+h(p) = 0$ and $(Z^+)^2h(p) := Z^+(Z^+h(p)) \neq 0$. Moreover, the fold*

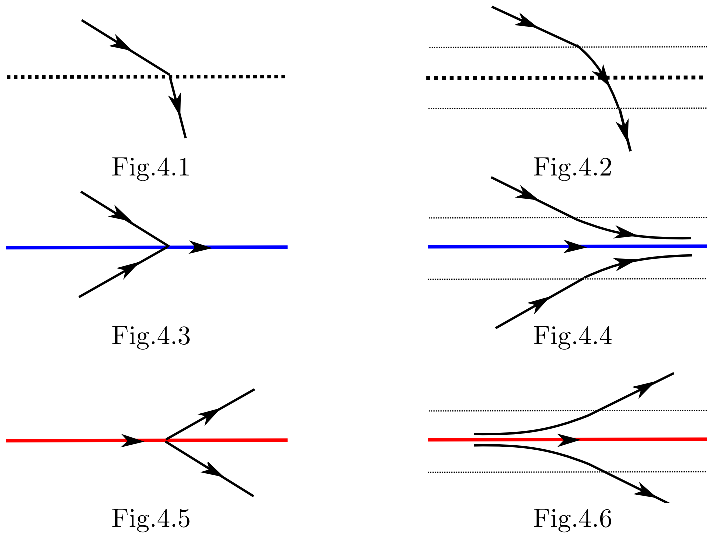

FIGURE 4. Illustration of a crossing, attracting and repelling region. Fig.4.1 is a crossing region. Fig.4.2 is the regularization of a crossing region. Fig.4.3 is an attracting region. Fig.4.4 is the regularization of an attracting region. Fig.4.5 is a repelling region. Fig.4.6 is the regularization of a repelling region.

$p \in Z^+$ is visible if $(Z^+)^2 h(p) > 0$, and it is invisible if $(Z^+)^2 h(p) < 0$. Similarly, a point $p \in \Sigma^t$ is a fold of Z^- if $Z^- h(p) = 0$ and $(Z^-)^2 h(p) \neq 0$. Furthermore, the fold $p \in Z^-$ is visible if $(Z^-)^2 h(p) < 0$, and it is invisible if $(Z^-)^2 h(p) > 0$.

Definition 6. *A point $p \in \Sigma$ is a two-fold if it is a fold for both Z^+ and Z^-. There are three types of two-fold, visible-visible two-fold see* Figure 13*; visible-invisible two-fold see* Figure 14*; invisible-invisible two-fold see* Figure 15.

In order to study the dynamics of a piecewise smooth vector field (4) we describe its regularization process as follows:

Definition 7. *A φ_ε-regularization of a piecewise smooth vector field* (4) *is an one-parameter family of vector fields $Z_\varepsilon(p)$ defined by*

$$Z^\varepsilon(p) := (P_\varepsilon(p), Q_\varepsilon(p)) = (1 - \varphi_\varepsilon(h(p)))\, Z^-(p) + \varphi_\varepsilon(h(p)) Z^+(p), \tag{7}$$

where $\varphi_\varepsilon(h(p)) = \varphi\left(\frac{h(p)}{\varepsilon}\right)$, and $\varphi : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying $\varphi(t) = 0$ if $t \leqslant -1$, $\varphi(t) = 1$ if $t \geqslant 1$ and $\varphi^{'}(t) > 0$ if $t \in (-1, 1)$.

We define the index of an isolated equilibrium point $q \in \Sigma$ of a piecewise smooth vector field (4) using its regularization vector field (7) as follows

$$i_q = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \oint_\sigma d \arctan \frac{Q_\varepsilon(x, y)}{P_\varepsilon(x, y)}. \tag{8}$$

In the paper [27] the authors regularized the regular points $q \in \Sigma$, see Figure 4. It is obvious that if the discontinuous boundary Σ is sliding, then the trajectories of

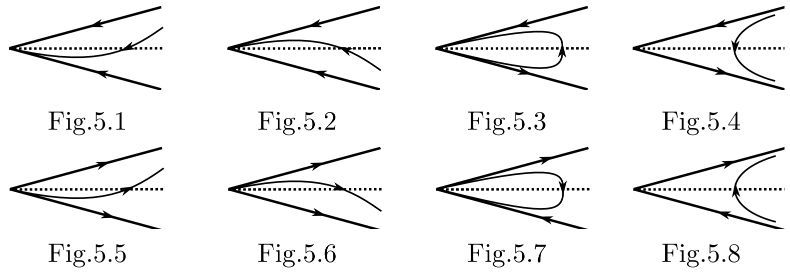

FIGURE 5. Illustration of an elliptic, a hyperbolic and a parabolic sector in a crossing region. Fig.5.1, Fig.5.2, Fig.5.5 and Fig.5.6 are parabolic sectors. Fig.5.3 and Fig.5.7 are elliptic sectors. Fig.5.4 and Fig.5.8 are hyperbolic sectors.

the piecewise smooth system (4) staring at Σ^+ (resp. Σ^-) cannot enter Σ^- (resp. Σ^+). However in the crossing region Σ^c the trajectories of the piecewise smooth system (4) will be affected by both Z^+ and Z^-. Thus in the crossing region it can exist parabolic, elliptic and hyperbolic sectors. According to the direction of the vector field in both Σ^+ and Σ^- there are eight situations in the regularized crossing regions, see Figure 5.

Now we can state our main result as follows.

Theorem 1. *Suppose that $Z^\pm$ is a piecewise smooth vector field* (4) *defined in a neighborhood V of q an isolated $\Sigma-$equilibrium point having the finite sectorial decomposition property. Let e^+, h^+ and p^+ denote the number of elliptic, hyperbolic, and parabolic sectors of q for vector field Z^+ in the region Σ^+, respectively. e^-, h^- and p^- denote the number of elliptic, hyperbolic, and parabolic sectors of q for vector field Z^- in the region Σ^-, respectively. Let $\tilde{e}$, $\tilde{h}$ and $\tilde{p}$ denote the number of elliptic, hyperbolic, and parabolic sectors of q in the regularized crossing regions. Then*

$$i_q = 1 + \frac{(e^+ + e^- + \tilde{e}) - (h^+ + h^- + \tilde{h})}{2}. \tag{9}$$

Remark 2. *According to Theorem 1 the index i_q of a piecewise smooth vector field $Z^\pm$ depends on the number of elliptic and hyperbolic sectors which belong not only to the regions in $\Sigma^\pm$ but also to the regularized crossing regions around Σ, see for example Figure 11. If q is in the crossing region of Σ and this case will be considered in section 3.3, then $\tilde{e} + \tilde{p} + \tilde{h} = 2$. If q is in Σ, either in hte sliding region or in the crossing region, we investigate it in section 3.2 and we shall see that $\tilde{e}+\tilde{p}+\tilde{h} = 1$. If q is in the sliding region of Σ it will be analyzed in section 3.1, and we shall see that $\tilde{e} = \tilde{p} = \tilde{h} = 0$ and hence $i_q = 1 + \dfrac{(e^+ + e^-) - (h^+ + h^-)}{2}$, see for instance Figure 16. In this last case the formula of Theorem 1 coincides with the Poincaré-Bendixson formula of a smooth vector field.*

Remark 3. *The authors of* [7] *also studied the extension of the Poincaré index to piecewise smooth vector fields, but they do not provide the explicit formula of Theorem 1.*

In [23], the authors consider the following piecewise smooth Liénard differential system

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= F(x) - y,\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= g(x),\end{aligned} \tag{10}$$

where $F(x) = \int_0^x f(x)\mathrm{d}x$ with

$$f(x) = \begin{cases} f_1(x), & x > 0,\\ f_2(x), & x < 0,\end{cases}$$

and

$$g(x) = \begin{cases} g_1(x), & x > 0,\\ g_2(x), & x < 0.\end{cases}$$

With the assumptions $xg(x) > 0$ and $xf(x) > 0$ for $x \neq 0$, then system (10) has a unique equilibrium point $q(0,0)$, which is a visible-invisible two-fold (VI1, see Fig.14.1) when $g_1(0)g_2(0) > 0$; visible-visible two-fold (VV2, see Fig.13.2) when $g_1(0) > 0, g_2(0) < 0$; invisible-invisible two-fold (II2, see Fig.15.2) when $g_1(0) < 0, g_2(0) > 0$.

Remark 4. *The authors of* [23] *analyzed the non-existence and uniqueness of limit cycles of piecewise smooth Liénard differential system* (10) *for the case* $g_1(0) < 0, g_2(0) > 0$.

According to our Theorem 1, it is obvious that the index of equilibrium point $i_q = 0$ *for the case* $g_1(0)g_2(0) > 0$ *(see section 3.6), and the the index of equilibrium point* $i_q = -1$ *for the case* $g_1(0) > 0, g_2(0) < 0$ *(see section 3.5). According to Corollary 11 of* [7], *we can conclude that system* (10) *cannot have limit cycles for the above two cases.*

This paper is organized as follows. In section 2 we give the proof of Theorem 1. We apply Theorem 1 for investigating the indices of all generic codimension-1 equilibria in the planar piecewise smooth vector fields, such as: boundary equilibria, pseudo-equilibria and tangency points.

2. Proof of Theorem 1

Without loss of generality we assume that $q = (0,0)$ and $h(x,y) = y$ in the piecewise smooth vector field (4). Thus the discontinuous boundary $\Sigma = \{(x,y)|y = 0\} = \{(x,0)|x > 0\} \cup o \cup \{(x,0)|x < 0\}$ divides the plane $\mathbb{R}^2$ into $\Sigma^+ = \{(x,y)|y > 0\}$ and $\Sigma^- = \{(x,y)|y < 0\}$.

Step 1. We regularized the piecewise smooth vector field (4) by an one-parameter family of continuous vector fields

$$Z_\varepsilon(p) := (P_\varepsilon(p), Q_\varepsilon(p)) = (1 - \varphi_\varepsilon(y))\, Z^-(p) + \varphi_\varepsilon(y) Z^+(p). \tag{11}$$

Step 2. Now we shall deduce the Poincaré-Bendixson formula for the continuous vector field Z_ε using similar ideas to the proof of Proposition 6.32 of [9]. Since we

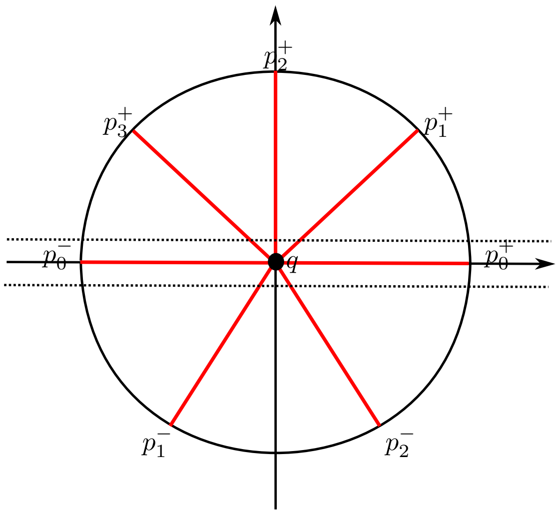

FIGURE 6. Illustration of a permissible parametrization ρ for both discontinuous boundaries $\{(x,0)|0 < x < x_0\}$ and $\{(x,0)| - x_0 < x < 0\}$ in the sliding region. Here $n^+ = 4$ and $n^- = 3$.

focus on the local dynamics at $q(0,0)$, then we can choose a suitable $x_0 > 0$ such that both intervals $\{(x,0)| - x_0 < x < 0\}$ and $\{(x,0)|0 < x < x_0\}$ are either in a regularized crossing region or in a regularized sliding region. Let V be a small disc centered at q and ∂V be the boundary of V. Thus $\partial V = \partial V^+ \cup \partial V^-$ where $\partial V^\pm := \partial V \cap \Sigma^\pm$.

2.1. **The discontinuous boundary has two sliding regions.** We consider that the discontinuous boundaries $\{(x,0)|0 < x < x_0\}$ and $\{(x,0)| - x_0 < x < 0\}$ are in the sliding region, see Figure 6.

Suppose that the curve ∂V^+ (resp. ∂V^-) has s_i^+ for $i = 1, 2, \cdots, e^+$ (resp. s_i^- for $i = 1, 2, \cdots, e^-$) points having an internal tangency, and r_j^+ for $j = 1, 2, \cdots, h^+$ (resp. r_j^- for $j = 1, 2, \cdots, h^-$) points having an external tangency. We can rearrange the $n^+ = e^+ + h^+$ contact points by q_k^+ for $k = 1, 2, \cdots, n^+$, and $n^- = e^- + h^-$ contact points by q_k^- for $k = 1, 2, \cdots, n^-$. Let $p_0^+ = \partial V \cap \{(x,0)|0 < x < x_0\}$ and $p_0^- = \partial V \cap \{(x,0)| - x_0 < x < 0\}$. Now we can choose intermediate points $p_k^+ \in (q_k^+, q_{k+1}^+)$ for $k = 1, \cdots, n^+ - 1$, and $p_k^- \in (q_k^-, q_{k+1}^-)$ for $k = 1, \cdots, n^- - 1$. From Definition 1 it is obvious that we can obtain an elliptic sector for each internal tangency, thus the number of elliptic sectors is equivalent to the number of internal tangency. Similarly we can get a hyperbolic sector for each external tangency and hence the number of hyperbolic sectors is equivalent to the number of external tangency.

We choose a permissible parametrization $\rho : \mathbb{S}^1 \to \partial V$ which satisfy $\rho(e^{ik\pi/n^+}) = p_k^+$ for $k = 0, 1, \cdots, n^+ - 1$ and $\rho(e^{i(\pi + k\pi/n^-)}) = p_k^-$ for $k = 0, 1, \cdots, n^- - 1$. See Figure 6 for the case $n^+ = 4$ and $n^- = 3$.

By means of a continuous transformation the index of an isolated equilibrium point with $(e^+ + e^-, h^+ + h^-, p^+ + p^-)$ sectors is equal to the index of an isolated equilibrium point with the same number of equal triangular sectors of the same kind and order than the sectors $(e^+ + e^-, h^+ + h^-, p^+ + p^-)$.

In the region Σ^+ each sector of (e^+, h^+, p^+) is a triangular sector of angle π/m^+ where $m^+ = e^+ + h^+ + p^+$. Now we can see the contribution to the index of each sector in Σ^+ as follows.

(i) A triangular parabolic sector which starts with the angle $\varphi(0) = \alpha$ and ends with the angle $\varphi(\pi/m^+) = \alpha + \pi/m^+$, with a net gain π/m^+. See Fig.1.1 and Fig.1.2.
(ii) A triangular hyperbolic sector which starts with the angle $\varphi(0) = \alpha$ and ends with the angle $\varphi(\pi/m^+) = \alpha + \pi/m^+ - \pi$, with a net contribution $\pi/m^+ - \pi$. See Fig.1.3.
(iii) A triangular elliptic sector which starts with the angle $\varphi(0) = \alpha$ and ends with the angle $\varphi(\pi/m^+) = \alpha+\pi/m^+ +\pi$, with a net contribution $\pi/m^+ +\pi$. See Fig.1.4.

Similarly in the region Σ^- each sector of (e^-, h^-, p^-) is a triangular sector of angle π/m^- where $m^- = e^- + h^- + p^-$. In the following we can deduce the contribution to the index of each sector in Σ^-.

(i) A triangular parabolic sector which starts with the angle $\varphi(0) = \alpha$ and ends with the angle $\varphi(\pi/m^-) = \alpha + \pi/m^-$, with a net contribution π/m^-.
(ii) A triangular hyperbolic sector which starts with the angle $\varphi(0) = \alpha$ and ends with the angle $\varphi(\pi/m^-) = \alpha + \pi/m^- - \pi$, with a net contribution $\pi/m^- - \pi$.
(iii) A triangular elliptic sector which starts with the angle $\varphi(0) = \alpha$ and ends with the angle $\varphi(\pi/m^-) = \alpha+\pi/m^- +\pi$, with a net contribution $\pi/m^- +\pi$.

Let $\sigma := \{(r, \theta)|\theta \in [0, 2\pi)\}$ be the circle ∂V of radius r surrounding the origin with r small enough. Going through the whole closed curve $\sigma = \sigma^+ \cup \sigma^-$ with $\sigma^+ = \{(r, \theta)|\theta \in [0, \pi)\}$ and $\sigma^- = \{(r, \theta)|\theta \in [\pi, 2\pi)\}$, we get

$$
\begin{aligned}
i_q \quad &= i(q, \sigma) = i(q, \sigma^+ \cup \sigma^-) \\
&= \frac{\sum_{i=1}^{p^+} \frac{\pi}{m^+} + \sum_{i=1}^{e^+} \left(\frac{\pi}{m^+} + \pi\right) + \sum_{i=1}^{h^+} \left(\frac{\pi}{m^+} - \pi\right)}{2\pi} \\
&\quad + \frac{\sum_{i=1}^{p^-} \frac{\pi}{m^-} + \sum_{i=1}^{e^-} \left(\frac{\pi}{m^-} + \pi\right) + \sum_{i=1}^{h^-} \left(\frac{\pi}{m^-} - \pi\right)}{2\pi} \\
&= \frac{\frac{e^+ + h^+ + p^+}{m^+}\pi + \frac{e^- + h^- + p^-}{m^-}\pi + (e^+ - h^+ + e^- - h^-)\pi}{2\pi} \\
&= 1 + \frac{(e^+ + e^-) - (h^+ + h^-)}{2}.
\end{aligned}
$$

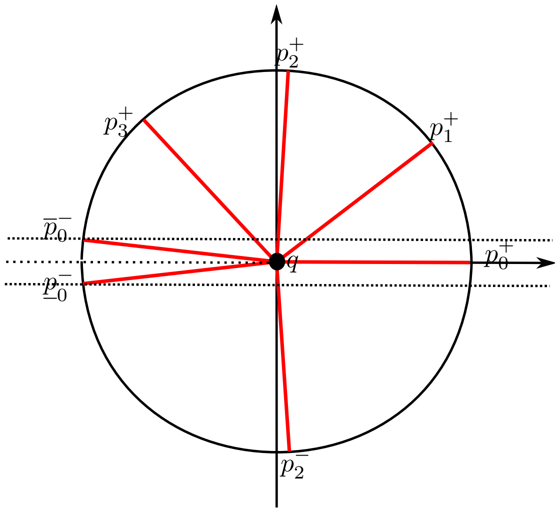

FIGURE 7. Illustration of a permissible parametrization ρ for discontinuous boundaries $\{(x,0)|0<x<x_0\}$ is in the sliding region and $\{(x,0)|-x_0<x<0\}$ is in the crossing region, where $m^+=4$ and $m^-=2$.

We have obtained the formula for the index stated in Theorem 1 because in this case $\tilde{e}=\tilde{h}=\tilde{p}=0$.

2.2. **The discontinuous boundary has a crossing region and a sliding region.** We consider that the discontinuous boundary has both crossing region and sliding region. More precisely, without loss of generality we assume that $\{(x,0)|-x_0<x<0\}$ is a crossing region and $\{(x,0)|0<x<x_0\}$ is a sliding region, see Figure 7.

Suppose that the curve ∂V^+ (resp. ∂V^-) has s_i^+ for $i=1,2,\cdots,e^+$ (resp. s_i^- for $i=1,2,\cdots,e^-$) points having an internal tangency, and r_j^+ for $j=1,2,\cdots,h^+$ (resp. r_j^- for $j=1,2,\cdots,h^-$) points having an external tangency.

We can rearrange the $n^+=e^++h^+$ contact points as q_k^+ for $k=1,2,\cdots,n^+$, and $n^-=e^-+h^-$ contact points as q_k^- for $k=1,2,\cdots,n^-$. Let $p_0^+=\partial V\cap\{(x,0)|0<x<x_0\}$, $\overline{p}_0^-=\partial V\cap\{(x,\varepsilon)|-x_0<x<0\}$ and $\underline{p}_0^-=\partial V\cap\{(x,-\varepsilon)|-x_0<x<0\}$ with $\varepsilon>0$ sufficiently small, see Figure 7. Now we can choose intermediate points $p_k^+\in(q_k^+,q_{k+1}^+)$ for $k=1,\cdots,n^+-1$ and $p_k^-\in(q_k^-,q_{k+1}^-)$ for $k=1,\cdots,n^--1$. Since $\{(x,0)|-x_0<x<0\}$ is a crossing region, we define the triangular sector $int(\overrightarrow{q\overline{p}_0^-}\cup\overrightarrow{q\underline{p}_0^-}\cup(\overline{p}_0^-,\underline{p}_0^-))$ in the regularized crossing region. We choose a permissible parametrization $\rho:\mathbb{S}^1\to\partial V$ such that $\rho(e^{ik(\pi-\arcsin\varepsilon)/m^+})=p_k^+$, for $k=0,1,\cdots,n^+-1$ and $\rho(e^{i(\pi+k(\pi-\arcsin\varepsilon)/m^-)})=p_k^-$ for $k=1,\cdots,n^--1$, $\rho(e^{i(\pi-\arcsin\varepsilon)})=\overline{p}_0^-$ and $\rho(e^{i(\pi+\arcsin\varepsilon)})=\underline{p}_0^-$, see Figure 7 again.

In short, by means of a continuous transformation the index of an isolated equilibrium point with $(e^++e^-+\tilde{e},h^++h^-+\tilde{h},p^++p^-+\tilde{p})$ sectors is equal to the index of an isolated equilibrium point with the same number and order of equal triangular sectors $(e^+ + e^- + \tilde{e}, h^+ + h^- + \tilde{h}, p^+ + p^- + \tilde{p})$.

In the region Σ^+ each sector of (e^+, h^+, p^+) is a triangular sector of angle $(\pi - \arcsin\varepsilon)/m^+$ where $m^+ = e^+ + h^+ + p^+$. Now we can investigate the contribution to the index of each sector in Σ^+ as follows.

(i) A triangular parabolic sector which starts with the angle $\varphi(0) = \alpha$ and ends with the angle $\varphi((\pi - \arcsin\varepsilon)/m^+) = \alpha + (\pi - \arcsin\varepsilon)/m^+$, with a net contribution $(\pi - \arcsin\varepsilon)/m^+$.
(ii) A triangular hyperbolic sector which starts with the angle $\varphi(0) = \alpha$ and ends with the angle $\varphi(\pi - \arcsin\varepsilon) = \alpha + (\pi - \arcsin\varepsilon)/m^+ - \pi$, with a net contribution $(\pi - \arcsin\varepsilon)/m^+ - \pi$.
(iii) A triangular elliptic sector which starts with the angle $\varphi(0) = \alpha$ and ends with the angle $\varphi((\pi - \arcsin\varepsilon)/m^+) = \alpha + (\pi - \arcsin\varepsilon)/m^+ + \pi$, with a net contribution $(\pi - \arcsin\varepsilon)/m^+ + \pi$.

Similarly in the region Σ^- each sector of (e^-, h^-, p^-) is a triangular sector of angle $(\pi - \arcsin\varepsilon)/m^-$ where $m^- = e^- + h^- + p^-$. In the following we can consider the contribution to the index of each sector in Σ^-.

(i) A triangular parabolic sector which starts with the angle $\varphi(0) = \alpha$ and will end with the angle $\varphi((\pi - \arcsin\varepsilon)/m^-) = \alpha + (\pi - \arcsin\varepsilon)/m^-$, with a net contribution $(\pi - \arcsin\varepsilon)/m^-$.
(ii) A triangular hyperbolic sector which starts with the angle $\varphi(0) = \alpha$ and will end with the angle $\varphi((\pi - \arcsin\varepsilon)/m^-) = \alpha + (\pi - \arcsin\varepsilon)/m^- - \pi$, with a net contribution $(\pi - \arcsin\varepsilon)/m^- - \pi$.
(iii) A triangular elliptic sector which starts with the angle $\varphi(0) = \alpha$ and will end with the angle $\varphi((\pi - \arcsin\varepsilon)/m^-) = \alpha + (\pi - \arcsin\varepsilon)/m^- + \pi$, with a net contribution $(\pi - \arcsin\varepsilon)/m^- + \pi$.

While in the regularized crossing region each sector of $(\tilde{e}, \tilde{h}, \tilde{p})$ is a triangular sector of angle $2\arcsin\varepsilon$. We can now consider the contribution to the index of each sector.

(i) A triangular parabolic sector which starts with the angle $\varphi(0) = \alpha$ and ends with the angle $\varphi(2\arcsin\varepsilon) = \alpha + 2\arcsin\varepsilon$, with a net contribution $2\arcsin\varepsilon$. See Fig.5.1.
(ii) A triangular hyperbolic sector which starts with the angle $\varphi(0) = \alpha$ and ends with the angle $\varphi(2\arcsin\varepsilon) = \alpha + 2\arcsin\varepsilon - \pi$, with a net contribution $2\arcsin\varepsilon - \pi$. See Fig.5.3.
(iii) A triangular elliptic sector which starts with the angle $\varphi(0) = \alpha$ and ends with the angle $\varphi(2\arcsin\varepsilon) = \alpha + 2\arcsin\varepsilon + \pi$, with a net contribution $2\arcsin\varepsilon + \pi$. See Fig.5.4.

Going through the whole closed curve $\partial V = \sigma = \sigma^+ \cup \sigma^- \cup \tilde{\sigma}$ with $\sigma^+ = \{(r, \theta) | \theta \in [0, \pi - \arcsin\varepsilon)\}$, $\sigma^- = \{(r, \theta) | \theta \in (\pi + \arcsin\varepsilon, 2\pi)\}$ and $\tilde{\sigma} = \{(r, \theta) | \theta \in$

$[\pi - \arcsin\varepsilon, \pi + \arcsin\varepsilon]\}$, we get

$$
\begin{aligned}
i_q &= i(q,\sigma) = i(q, \sigma^+ \cup \sigma^- \cup \tilde{\sigma}) \\
&= \frac{\sum_{i=1}^{p^+} \frac{\pi - \arcsin\varepsilon}{m^+} + \sum_{i=1}^{e^+} \left(\frac{\pi - \arcsin\varepsilon}{m^+} + \pi \right) + \sum_{i=1}^{h^+} \left(\frac{\pi - \arcsin\varepsilon}{m^+} - \pi \right)}{2\pi} \\
&+ \frac{\sum_{i=1}^{p^-} \frac{\pi - \arcsin\varepsilon}{m^-} + \sum_{i=1}^{e^-} \left(\frac{\pi - \arcsin\varepsilon}{m^-} + \pi \right) + \sum_{i=1}^{h^-} \left(\frac{\pi - \arcsin\varepsilon}{m^-} - \pi \right)}{2\pi} \\
&+ \frac{\sum_{i=1}^{\tilde{p}} 2\arcsin\varepsilon + \sum_{i=1}^{\tilde{e}} (2\arcsin\varepsilon + \pi) + \sum_{i=1}^{\tilde{h}} (2\arcsin\varepsilon - \pi)}{2\pi} \\
&= \frac{\frac{e^+ + h^+ + p^+}{m^+}\pi + \frac{e^- + h^- + p^-}{m^-}\pi + (e^+ - h^+ + e^- - h^- + \tilde{e} - \tilde{h})\pi}{2\pi} \\
&+ \frac{(\tilde{e} + \tilde{h} + \tilde{p} - 1)\arcsin\varepsilon}{\pi} \\
&= 1 + \frac{(e^+ + e^- + \tilde{e}) - (h^+ + h^- + \tilde{h})}{2} + \frac{(\tilde{e} + \tilde{h} + \tilde{p} - 1)\arcsin\varepsilon}{\pi}.
\end{aligned}
$$

So when $\varepsilon \to 0$ we obtain for the index the formula of the statement of Theorem 1.

2.3. **The discontinuous boundary has two crossing regions.** We consider that the discontinuous boundaries $\{(x,0)|0 < x < x_0\}$ and $\{(x,0)| - x_0 < x < 0\}$ are in the crossing region, see Figure 8.

Suppose that the curve ∂V^+ (resp. ∂V^-) has s_i^+ for $i = 1, 2, \cdots, e^+$ (resp. s_i^- for $i = 1, 2, \cdots, e^-$) points having an internal tangency, and r_j^+ for $j = 1, 2, \cdots, h^+$ (resp. r_j^- for $j = 1, 2, \cdots, h^-$) points having an external tangency.

We can rearrange the $n^+ = e^+ + h^+$ contact points as q_k^+ for $k = 1, 2, \cdots, n^+$, and $n^- = e^- + h^-$ contact points as q_k^- for $k = 1, 2, \cdots, n^-$. Let $\bar{p}_0^+ = \partial V \cap \{(x,\varepsilon)|0 < x < x_0\}$ and $\underline{p}_0^+ = \partial V \cap \{(x,-\varepsilon)|0 < x < x_0\}$, $\bar{p}_0^- = \partial V \cap \{(x,\varepsilon)| - x_0 < x < 0\}$ and $\underline{p}_0^- = \partial V \cap \{(x,-\varepsilon)| - x_0 < x < 0\}$, see Figure 8. Now we can choose intermediate points $p_k^+ \in (q_k^+, q_{k+1}^+)$ for $k = 1, \cdots, n^+ - 1$ and $p_k^- \in (q_k^-, q_{k+1}^-)$ for $k = 1, \cdots, n^- - 1$, see Figure 8 for the case $n^+ = 4$ and $n^- = 2$. Since $\{(x,0)|0 < x < x_0\}$ and $\{(x,0)|-x_0 < x < 0\}$ are in the crossing region, we define the triangular sectors $int(\overrightarrow{q\bar{p}_0^+} \cup \overrightarrow{q\underline{p}_0^+} \cup (\bar{p}_0^+, \underline{p}_0^+))$ and $int(\overrightarrow{q\bar{p}_0^-} \cup \overrightarrow{q\underline{p}_0^-} \cup (\bar{p}_0^-, \underline{p}_0^-))$ in the regularized crossing region. We choose a permissible parametrization $\rho : \mathbb{S}^1 \to \partial V$ such that $\rho(e^{ik(\pi - 2\arcsin\varepsilon)/n^+}) = p_k^+$ for $k = 1, \cdots, n^+ - 1$ and $\rho(e^{i(\pi + k(\pi - 2\arcsin\varepsilon)/n^-)}) = p_k^-$ for $k = 1, \cdots, n^- - 1$, $\rho(e^{i(\pi - \arcsin\varepsilon)}) = \bar{p}_0^-$, $\rho(e^{i(\pi + \arcsin\varepsilon)}) = \underline{p}_0^-$, $\rho(e^{i(\arcsin\varepsilon)}) = \bar{p}_0^+$ and $\rho(e^{i(-\arcsin\varepsilon)}) = \underline{p}_0^+$. See Figure 8 again.

In short, by means of a continuous transformation the index of a equilibrium point with $(e^+ + e^- + \tilde{e}, h^+ + h^- + \tilde{h}, p^+ + p^- + \tilde{p})$ sectors is equal to the index of a equilibrium point with the same number and order of equal triangular sectors $(e^+ + e^- + \tilde{e}, h^+ + h^- + \tilde{h}, p^+ + p^- + \tilde{p})$.

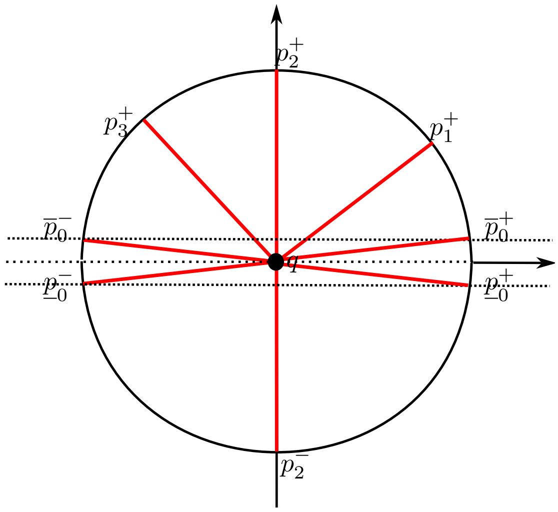

FIGURE 8. Illustration of a permissible parametrization ρ for both discontinuous boundaries $\{(x,0)|0 < x < x_0\}$ and $\{(x,0)| - x_0 < x < 0\}$ are in the crossing region, where $m^+ = 4$ and $m^- = 2$.

In the region Σ^+ each sector of (e^+, h^+, p^+) is a triangular sector of angle $(\pi - 2\arcsin\varepsilon)/m^+$ where $m^+ = e^+ + h^+ + p^+$. Now we can consider the contribution to the index of each sector in Σ^+ as follows.

(i) A triangular parabolic sector which starts with the angle $\varphi(0) = \alpha$ and ends with the angle $\varphi((\pi - 2\arcsin\varepsilon)/m^+) = \alpha + (\pi - 2\arcsin\varepsilon)/m^+$, with a net contribution $(\pi - 2\arcsin\varepsilon)/m^+$.
(ii) A triangular hyperbolic sector which starts with the angle $\varphi(0) = \alpha$ and ends with the angle $\varphi(\pi - 2\arcsin\varepsilon) = \alpha + (\pi - 2\arcsin\varepsilon)/m^+ - \pi$, with a net contribution $(\pi - 2\arcsin\varepsilon)/m^+ - \pi$.
(iii) A triangular elliptic sector which starts with the angle $\varphi(0) = \alpha$ and ends with the angle $\varphi((\pi - 2\arcsin\varepsilon)/m^+) = \alpha + (\pi - 2\arcsin\varepsilon)/m^+ + \pi$, with a net contribution $(\pi - 2\arcsin\varepsilon)/m^+ + \pi$.

Similarly in the region Σ^- each sector of (e^-, h^-, p^-) is a triangular sector of angle $(\pi - 2\arcsin\varepsilon)/m^-$ where $m^- = e^- + h^- + p^-$. In the following we can reveal the contribution to the index of each sector.

(i) A triangular parabolic sector which starts with the angle $\varphi(0) = \alpha$ and will end with the angle $\varphi((\pi - 2\arcsin\varepsilon)/m^-) = \alpha + (\pi - 2\arcsin\varepsilon)/m^-$, with a net contribution $(\pi - 2\arcsin\varepsilon)/m^-$.
(ii) A triangular hyperbolic sector which starts with the angle $\varphi(0) = \alpha$ and will end with the angle $\varphi((\pi - 2\arcsin\varepsilon)/m^-) = \alpha + (\pi - 2\arcsin\varepsilon)/m^- - \pi$, with a net contribution $(\pi - 2\arcsin\varepsilon)/m^- - \pi$.
(iii) A triangular elliptic sector which starts with the angle $\varphi(0) = \alpha$ and will end with the angle $\varphi((\pi - 2\arcsin\varepsilon)/m^-) = \alpha + (\pi - 2\arcsin\varepsilon)/m^- + \pi$, with a net contribution $(\pi - 2\arcsin\varepsilon)/m^- + \pi$.

While in the regularized crossing region each sector of $(\tilde{e}, \tilde{h}, \tilde{p})$ is a triangular sector of angle $2\arcsin\varepsilon$. Finally we can deduce the contribution to the index of each sector.

(i) A triangular parabolic sector which starts with the angle $\varphi(0) = \alpha$ and ends with the angle $\varphi(2\arcsin\varepsilon) = \alpha + 2\arcsin\varepsilon$, with a net contribution $2\arcsin\varepsilon$.
(ii) A triangular hyperbolic sector which starts with the angle $\varphi(0) = \alpha$ and ends with the angle $\varphi(2\arcsin\varepsilon) = \alpha + 2\arcsin\varepsilon - \pi$, with a net contribution $2\arcsin\varepsilon - \pi$.
(iii) A triangular elliptic sector which starts with the angle $\varphi(0) = \alpha$ and ends with the angle $\varphi(2\arcsin\varepsilon) = \alpha + 2\arcsin\varepsilon + \pi$, with a net contribution $2\arcsin\varepsilon + \pi$.

Going through the whole closed curve $\sigma = \sigma^+ \cup \sigma^- \cup \tilde{\sigma}$ with $\sigma^+ = \{(r,\theta)|\theta \in (\arcsin\varepsilon, \pi - \arcsin\varepsilon)\}$, $\sigma^- = \{(r,\theta)|\theta \in (\pi + \arcsin\varepsilon, 2\pi - \arcsin\varepsilon)\}$ and $\tilde{\sigma} = \{(r,\theta)|\theta \in [\pi - \arcsin\varepsilon, \pi + \arcsin\varepsilon]\} \cup \{(r,\theta)|\theta \in [-\arcsin\varepsilon, \arcsin\varepsilon]\}$, we get

$$
\begin{aligned}
i_q &= i(q,\sigma) = i(q, \sigma^+ \cup \sigma^- \cup \tilde{\sigma}) \\
&= \frac{\sum_{i=1}^{p^+} \frac{\pi - 2\arcsin\varepsilon}{m^+} + \sum_{i=1}^{e^+} \left(\frac{\pi - 2\arcsin\varepsilon}{m^+} + \pi \right) + \sum_{i=1}^{h^+} \left(\frac{\pi - 2\arcsin\varepsilon}{m^+} - \pi \right)}{2\pi} \\
&\quad + \frac{\sum_{i=1}^{p^-} \frac{\pi - 2\arcsin\varepsilon}{m^-} + \sum_{i=1}^{e^-} \left(\frac{\pi - 2\arcsin\varepsilon}{m^-} + \pi \right) + \sum_{i=1}^{h^-} \left(\frac{\pi - 2\arcsin\varepsilon}{m^-} - \pi \right)}{2\pi} \\
&\quad + \frac{\sum_{i=1}^{\tilde{p}} 2\arcsin\varepsilon + \sum_{i=1}^{\tilde{e}} (2\arcsin\varepsilon + \pi) + \sum_{i=1}^{\tilde{h}} (2\arcsin\varepsilon - \pi)}{2\pi} \\
&= \frac{\frac{e^+ + h^+ + p^+}{m^+}\pi + \frac{e^- + h^- + p^-}{m^-}\pi + (e^+ - h^+ + e^- - h^- + \tilde{e} - \tilde{h})\pi}{2\pi} \\
&\quad + \frac{\arcsin\varepsilon(\tilde{e} + \tilde{h} + \tilde{p} - 2)}{\pi} \\
&= 1 + \frac{(e^+ + e^- + \tilde{e}) - (h^+ + h^- + \tilde{h})}{2} + \frac{\arcsin\varepsilon(\tilde{e} + \tilde{h} + \tilde{p} - 2)}{\pi}.
\end{aligned}
$$

So when $\varepsilon \to 0$ we obtain for the index the formula of the statement of Theorem 1.

Step 3. We extend the index of the equilibria of the continuous vector field Z_ε to the piecewise smooth vector field $Z^\pm$. The following results explain the relationship between $i(Z_\varepsilon, \sigma)$ and $i(Z^\pm, \sigma)$, for more details see Proposition 8 of [7].

Lemma 1. *If σ is a closed continuous simple curve, and there is no equilibria of $Z^\pm$ on σ, then $i(Z_\varepsilon, \sigma) = i(Z^\pm, \sigma)$ for $\varepsilon > 0$ sufficiently small.*

According to Lemma 1 we have

$$
i_q = i(Z^\pm, \sigma) = \lim_{\varepsilon \to 0} i(Z_\varepsilon, \sigma) = 1 + \frac{(e^+ + e^- + \tilde{e}) - (h^+ + h^- + \tilde{h})}{2}. \tag{12}
$$

Table 1. The index of the boundary focus of system (4).

	e^+	h^+	e^-	h^-	$\tilde{e}$	$\tilde{h}$	i_q
$BF_{1,2}$	0	1	0	0	0	1	0
BF_3	0	0	0	0	0	0	1
BF_4	0	0	0	0	0	0	1
BF_5	0	1	0	0	0	1	0

This complete the proof of Theorem 1.

3. Applications

The equilibria in the disocntinuous boundary Σ of a piecewise-smooth vector field (4) have been investigated in [12, 15, 20, 25]. Using the regularization technique the works [19, 24] provided the bifurcations of a two-fold of the vector field (4). According to [20] the authors gave an overview of all generic codimension-1 bifurcations of a piecewise smooth vector field (4). In this section, we will define the index of these generic codimension-1 equilibria of (4) case by case by Poincaré-Bendixson formula which given in (9).

3.1. **Boundary-focus.** There are five generic critical cases: BF_i for $i = 1, 2, 3, 4, 5$, see Figure 9. It is worth to note that the cases BF_1 and BF_2 have the same local phase portraits before bifurcation.

For the case BF_i for $i = 1, 2$, see Fig.9.1. It is obvious that in the region Σ^+ there is a hyperbolic sector and no elliptic sectors. Neither hyperbolic sectors nor elliptic sectors in the region Σ^-. While in the crossing region there is a hyperbolic sector and no elliptic sectors. Thus we have $i_q = 1 + \dfrac{(0+0+0)-(1+0+1)}{2} = 0$. See the first row of Table 1.

For the case BF_3 see Fig.9.2. There is neither hyperbolic sectors nor elliptic sectors in both the regions $\Sigma^\pm$ and the regularized crossing regions, thus we have $i_q = 1 + \dfrac{(0+0+0)-(0+0+0)}{2} = 1$, see the second row of Table 1.

For the case BF_4 see Fig.9.3. The proof is similar with the case BF_3, thus we have $i_q = 1 + \dfrac{(0+0+0)-(0+0+0)}{2} = 1$, see the third row of Table 1.

For the case BF_5 see Fig.9.4. Similar with the case BF_1, we get $i_q = 1 + \dfrac{(0+0+0)-(1+0+1)}{2} = 0$, see the forth row of Table 1.

3.2. **Boundary-node.** There are two generic critical cases: BN_i for $i = 1, 2$, see Figure 10.

For the case BN_1 see Fig.10.1. There is neither hyperbolic sectors nor elliptic sectors in both the regions $\Sigma^\pm$ and in the regularized crossing region, hence we have $i_q = 1 + \dfrac{(0+0+0)-(0+0+0)}{2} = 1$, see the first row of Table 2.

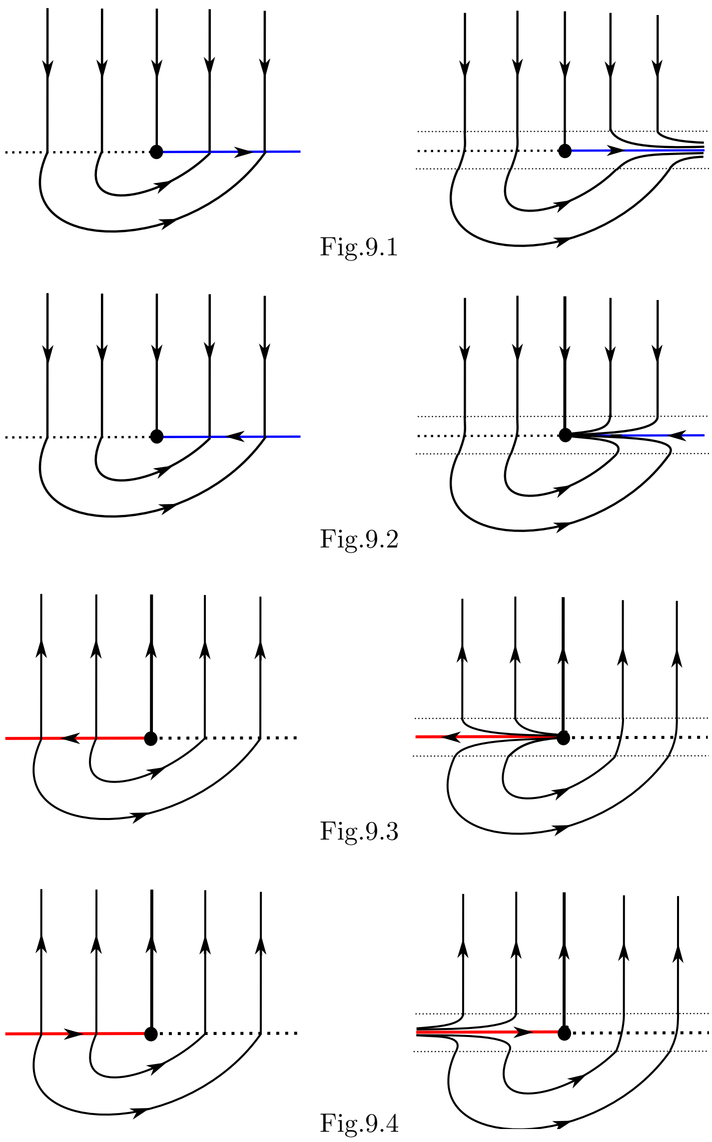

FIGURE 9. Boundary-focus and its regularisation. Fig.9.1, $BF_{1,2}$ and its regularization, the index of $BF_{1,2}$ is 0. Fig.9.2, BF_3 and its regularization, the index of BF_3 is 1. Fig.9.3, BF_4 and its regularization, the index of BF_4 is 1. Fig.9.4, BF_5 and its regularization, the index of BF_5 is 0.

For the case BN_2 see Fig.10.2. It is obvious that in the region Σ^+ there is one hyperbolic sector and no elliptic sectors. Neither hyperbolic sectors nor elliptic sectors in the regions Σ^-. While in the regularized crossing region there is one hyperbolic

TABLE 2. The index of the boundary-node of system (4).

	e^+	h^+	e^-	h^-	$\tilde{e}$	$\tilde{h}$	i_q
BN_1	0	0	0	0	0	0	1
BN_2	0	1	0	0	0	1	0

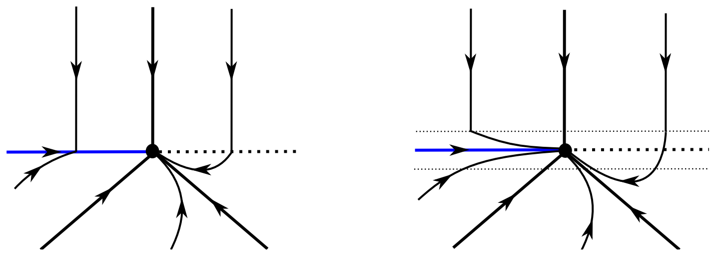

Fig.10.1

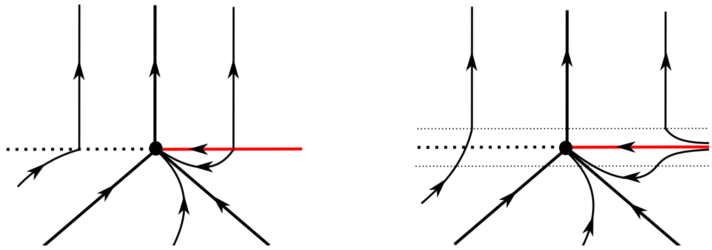

Fig.10.2

FIGURE 10. Boundary-node and its regularisation. Fig.10.1, BN_1 and its regularization, the index of BN_1 is 1. Fig.10.2, BN_2 and its regularization, the index of BN_1 is 0.

sector and no elliptic sectors. Thus we have $i_q = 1 + \dfrac{(0+0+0)-(1+0+1)}{2} = 0$, see the second row of Table 2.

3.3. **Boundary-saddle.** There are three generic critical cases: BS_i for $i = 1, 2, 3$, see Figure 11. It is worth note that the cases BS_1 and BS_2 have the same local phase portraits before bifurcation.

For the case BS_i for $i = 1, 2$ see Fig.11.1. It is obvious that in the region Σ^+ there is neither hyperbolic sectors nor elliptic sectors. While in both the region Σ^- and in the regularized crossing region, there is one hyperbolic sector and no elliptic sectors. Thus we obtain $i_q = 1 + \dfrac{(0+0+0)-(0+1+1)}{2} = 0$, see the first row of Table 3.

For the case BS_3 see Fig.11.2. There is one hyperbolic sector and no elliptic sectors in Σ^+. Two hyperbolic sectors and no elliptic sectors in the region Σ^-. In the regularized crossing region there is one hyperbolic sector and no elliptic sectors. Thus we get $i_q = 1 + \dfrac{(0+0+0)-(1+2+1)}{2} = -1$, see the second row of Table 3.

TABLE 3. The index of the boundary-saddle of system (4).

	e^+	h^+	e^-	h^-	$\tilde{e}$	$\tilde{h}$	i_q
$BS_{1,2}$	0	0	0	1	0	1	0
BS_1	0	1	0	2	0	1	-1

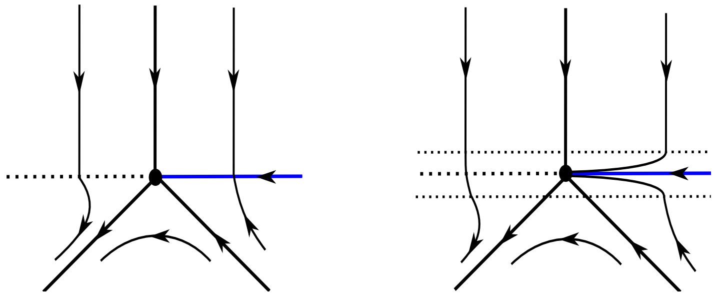

Fig.11.1

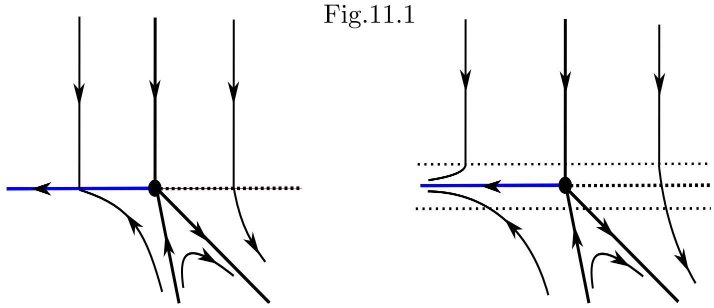

Fig.11.2

FIGURE 11. Boundary-saddle and its regularisation. Fig.11.1, $BS_{1,2}$ and its regularization, the index of $BS_{1,2}$ is 0. Fig.11.2, BS_3 and its regularization, the index of BS_3 is -1.

3.4. **Double tangency.** There are two generic critical cases: DT_i for $i = 1, 2$, see Figure 12.

For the case DT_1 see Fig.12.1. It is obvious that there is neither hyperbolic sectors nor elliptic sectors in both Σ^+ and Σ^-. While in the crossing region there are two hyperbolic sectors and no elliptic sectors. Thus we have $i_q = 1 + \dfrac{(0+0+0)-(0+0+2)}{2} = 0$, see the first row of Table 4.

For the case DT_2 see Fig.12.2. There is one hyperbolic sector and no elliptic sectors in both Σ^+ and Σ^-. Hence we obtain $i_q = 1 + \dfrac{(0+0+0)-(1+1+0)}{2} = 0$, see the second row of Table 4.

3.5. **Visible-visible two-fold.** There are two generic critical cases: VV_i for $i = 1, 2$, see Figure 13.

Table 4. The index of the double tangency of system (4).

	e^+	h^+	e^-	h^-	$\tilde{e}$	$\tilde{h}$	i_q
DT_1	0	0	0	0	0	2	0
DT_2	0	1	0	1	0	0	0

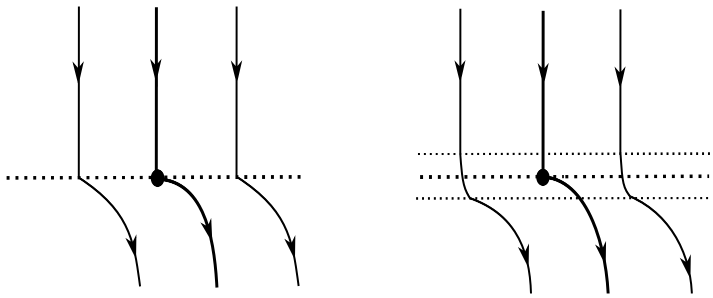

Fig.12.1

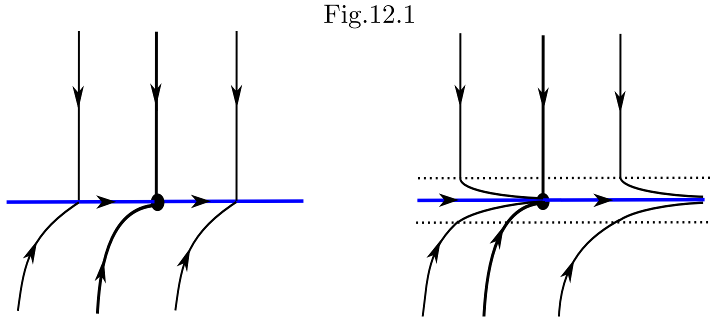

Fig.12.2

Figure 12. Double tangency and its regularisation. Fig.12.1, DT_1 and its regularization, the index of DT_1 is 0. Fig.12.2, DT_2 and its regularization, the index of DT_2 is 0.

Table 5. The index of the visible-visible two-fold of system (4).

	e^+	h^+	e^-	h^-	$\tilde{e}$	$\tilde{h}$	i_q
VV_1	0	1	0	1	0	0	0
VV_2	0	1	0	1	0	2	-1

For the case VV_1 see Fig.13.1. It is obvious that in the region $\Sigma^\pm$ there is one hyperbolic sector and no elliptic sectors. Thus we have $i_q = 1 + \dfrac{(0+0+0)-(1+1+0)}{2} = 0$, see the first row of Table 5.

For the case VV_2 see Fig.13.2. There is one hyperbolic sector and no elliptic sectors in both Σ^+ and Σ^-. Two hyperbolic sectors and no elliptic sectors in the regularized crossing regions. Thus we have $i_q = 1 + \dfrac{(0+0+0)-(1+1+2)}{2} = -1$, see the second row of Table 5.

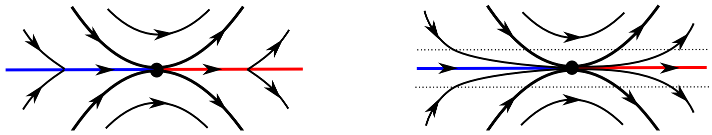

Fig.13.1

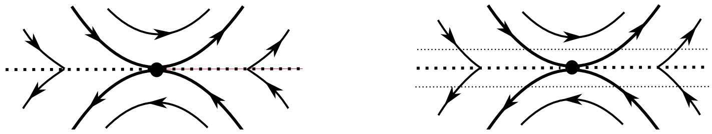

Fig.13.2

FIGURE 13. Visible-visible two-fold and its regularisation. Fig.13.1, VV_1 and its regularization, the index of VV_1 is 0. Fig.13.2, VV_2 and its regularization, the index of VV_2 is -1.

TABLE 6. The index of the visible-invisible two-fold of system (4).

	e^+	h^+	e^-	h^-	$\tilde{e}$	$\tilde{h}$	i_q
VI_1	0	1	0	0	0	1	0
VI_2	0	3	0	1	0	0	-1
VI_3	0	1	1	0	0	0	1

3.6. **Visible-invisible two-fold.** There are three generic critical cases: VI_i for $i = 1, 2, 3$, see Figure 14.

For the case VI_1 see Fig.14.1. It is obvious that in the region Σ^+ there is a hyperbolic sector and no elliptic sectors. Neither hyperbolic sectors nor elliptic sectors in the region Σ^-. While in the crossing region there is a hyperbolic sector and no elliptic sectors. Thus we have $i_q = 1 + \dfrac{(0+0+0)-(1+0+1)}{2} = 0$, see the first row of Table 6.

For the case VI_2 see Fig.14.2. There is three hyperbolic sectors and no elliptic sectors in Σ^+, one hyperbolic sector and no elliptic sectors in Σ^-. Hence we have $i_q = 1 + \dfrac{(0+0+0)-(3+1+0)}{2} = -1$, see the second row of Table 6.

For the case VI_3 see Fig.14.3. There is one hyperbolic sector and no elliptic sectors in Σ^+, one elliptic sector and no hyperbolic sectors in Σ^-. Then we have $i_q = 1 + \dfrac{(0+1+0)-(1+0+0)}{2} = 1$, see the third row of Table 6.

3.7. **Invisible-invisible two-fold.** There are two generic critical cases: II_i for $i = 1, 2$, see Figure 15.

For the case II_1 see Fig.15.1. It is obvious that there are one hyperbolic sector and no elliptic sectors in both Σ^+ and Σ^-. Thus we have $i_q = 1 + \dfrac{(0+0+0)-(1+1+0)}{2} = 0$, see the first row of Table 7.

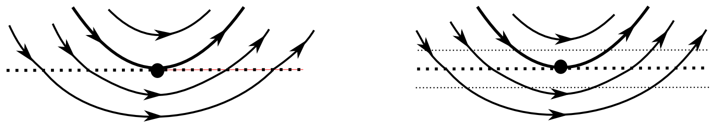

Fig.14.1

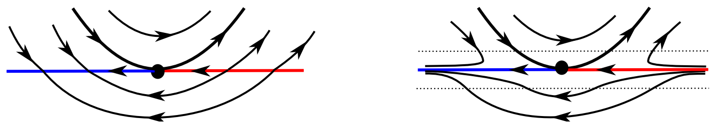

Fig.14.2

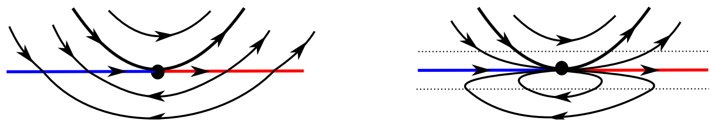

Fig.14.3

FIGURE 14. Visible-invisible two-fold VI and its regularisation. Fig.14.1, VI_1 and its regularization, the index of VI_1 is 0. Fig.14.2, VI_2 and its regularization, the index of VI_2 is -1. Fig.14.3, VI_3 and its regularization, the index of VI_3 is 1.

TABLE 7. The index of the invisible-invisible two-fold of system (4).

	e^+	h^+	e^-	h^-	$\tilde{e}$	$\tilde{h}$	i_q
II_1	0	1	0	1	0	0	0
II_2	0	0	0	0	0	0	1

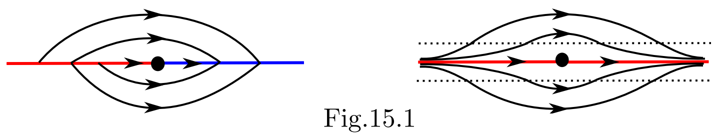

Fig.15.1

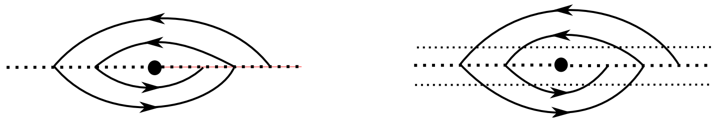

Fig.15.2

FIGURE 15. Invisible-invisible two-fold and its regularisation. Fig.15.1, II_1 and its regularization, the index of II_1 is 0. Fig.15.2, II_2 and its regularization, the index of II_2 is 1.

For the case II_2 see Fig.15.2. There is neither hyperbolic sectors nor elliptic sectors in both $\Sigma^{\pm}$ regions and in the regularized crossing region. Hence we have $i_q = 1 + \dfrac{(0+0+0)-(0+0+0)}{2} = 1$, see the second row of Table 7.

Table 8. The index of the pseudo-saddle of system (4).

	e^+	h^+	e^-	h^-	$\tilde{e}$	$\tilde{h}$	i_q
PS_1	0	2	0	2	0	0	-1
PS_2	0	2	0	2	0	0	-1

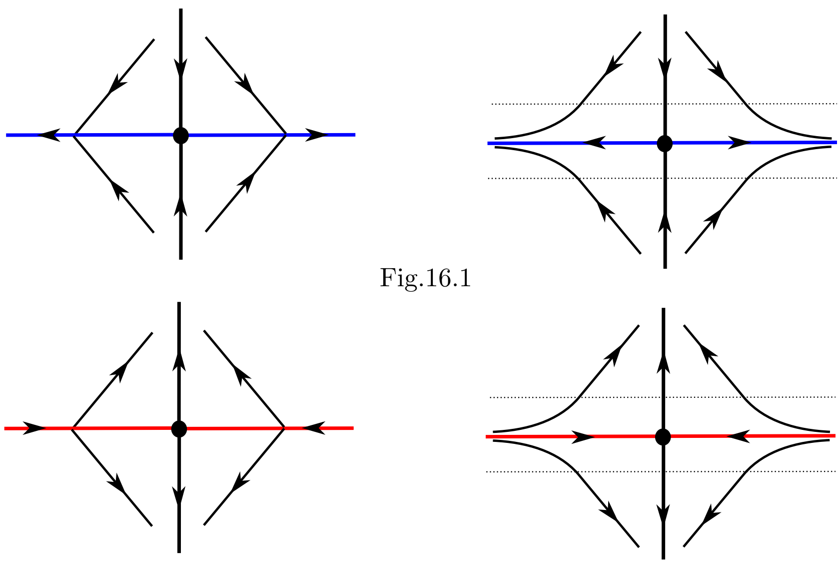

Fig.16.1

Fig.16.2

Figure 16. Pseudo-saddle and its regularisation. Fig.16.1, PS_1 and its regularization, the index of PS_1 is -1. Fig.16.2, PS_2 and its regularization, the index of PS_2 is -1.

3.8. **Pseudo-saddle.** There are two generic critical cases: PS_i for $i = 1, 2$, see Figure 16. For the case PS_1 it is obvious that there are two hyperbolic sectors and no elliptic sectors in both Σ^+ and Σ^-. Thus we have $i_q = 1 + \frac{(0+0+0)-(2+2+0)}{2} = -1$, see the first row of Table 8.

The proof of PS_2 is similar with PS_1.

3.9. **Pseudo-node.** There are two generic critical cases: PN_i for $i = 1, 2$, see Figure 17. For the case PN_1 there are neither hyperbolic sectors nor elliptic sectors in both Σ^+ and Σ^-. Thus we have $i_q = 1 + \frac{(0+0+0)-(0+0+0)}{2} = 1$. See the first row of Table 9.

The case PN_2 is similar with PN_1 and we omit its proof.

3.10. **Pseudo-saddle-node.** The pseudo-saddle node see Figure 18. It is obvious that there are one hyperbolic sector and no elliptic sectors in both Σ^+ and Σ^-. Thus we have $i_q = 1 + \frac{(0+0+0)-(1+1+0)}{2} = 0$. See Table 10.

TABLE 9. The index of the pseudo-node of system (4).

	e^+	h^+	e^-	h^-	$\tilde{e}$	$\tilde{h}$	i_q
PN_1	0	0	0	0	0	0	1
PN_2	0	0	0	0	0	0	1

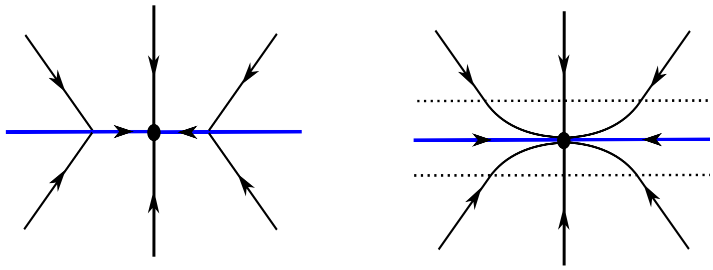

Fig.17.1

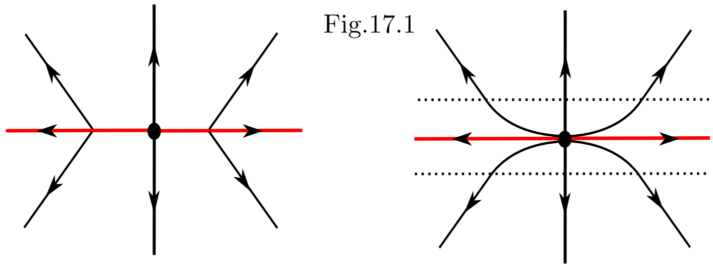

Fig.17.2

FIGURE 17. Pseudo-node and its regularisation. Fig.17.1, PN_1 and its regularization, the index of PN_1 is 1. Fig.17.2, PN_2 and its regularization, the index of PN_2 is 1.

TABLE 10. The index of the pseudo-saddle-node of system (4).

	e^+	h^+	e^-	h^-	$\tilde{e}$	$\tilde{h}$	i_q
PSN	0	1	0	1	0	0	0

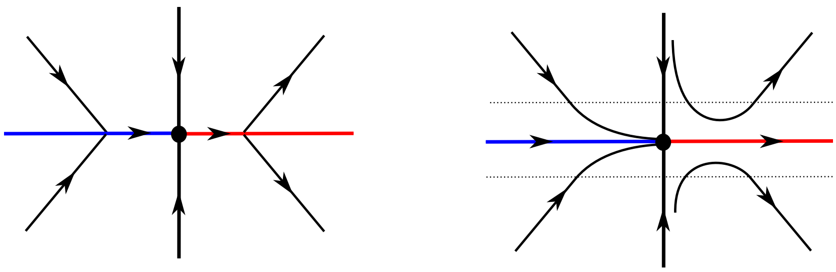

FIGURE 18. Pseudo-saddle-node and its regularisation. The index of PSN is 0.

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[1] School of Mathematics, Hangzhou Normal University, Hangzhou, 311121, P.R.China

E-mail address: lishimin@hznu.edu.cn

[2] School of Mathematics (Zhuhai), Sun Yat-sen University, Zhuhai, 519082, P.R.China

E-mail address: liuchangj@mail.sysu.edu.cn

[3] Departament de Matemàtiques, Universitat Autònoma de Barcelona,Bellaterra, Barcelona, Catalonia, 08193, Spain

E-mail address: jllibre@mat.uab.cat