Geometry preserving numerical methods

for physical systems with finite-dimensional Lie algebras

L. Blanco^{*}, F. Jiménez[§], J. de Lucas[†], C. Sardón^{*}

[†]Centre de Recherches Mathématiques, Université de Montréal,

Pavillon André-Aisenstadt, Chemin de la Tour 2920,

Montréal (Québec) H3T 1J4, Canada.

[†]Department of Mathematical Methods in Physics, University of Warsaw,

ul. Pasteura 5, 02-093, Warsaw, Poland.

*Departamento de Matemáticas Aplicadas a la Ingeniería Industrial,

Universidad Politécnica de Madrid (UPM)

c. José Gutiérrez Abascal 2, 28006, Madrid, Spain.

[§]Departamento de Matemática Aplicada I, ETSII,

Universidad Nacional de Educación a Distancia (UNED)

c. Juan del Rosal 12, 28040, Madrid, Spain.

Abstract

We propose a geometric integrator to numerically approximate the flow of Lie systems. The key is a novel procedure that integrates the Lie system on a Lie group intrinsically associated with a Lie system on a general manifold via a Lie group action, and then generates the discrete solution of the Lie system on the manifold via a solution of the Lie system on the Lie group.

One major result from the integration of a Lie system on a Lie group is that one is able to solve all associated Lie systems on manifolds at the same time, and that Lie systems on Lie groups can be described through first-order systems of linear homogeneous ordinary differential equations (ODEs) in normal form. This brings a lot of advantages, since solving a linear system of ODEs involves less numerical cost. Specifically, we use two families of numerical schemes on the Lie group, which are designed to preserve its geometrical structure: the first one based on the Magnus expansion, whereas the second is based on Runge-Kutta-Munthe-Kaas (RKMK) methods. Moreover, since the aforementioned action relates the Lie group and the manifold where the Lie system evolves, the resulting integrator preserves any geometric structure of the latter. We compare both methods for Lie systems with geometric invariants, particularly a class on Lie systems on curved spaces. We also illustrate the superiority of our method for describing long-term behavior and for differential equations admitting solutions whose geometric features depends heavily on initial conditions.

As already mentioned, our milestone is to show that the method we propose preserves all the geometric invariants very faithfully, in comparison with nongeometric numerical methods.

MSC 2020 classes: 34A26; 53A70 (primary) 37M15; 49M25 (secondary)

1 Introduction

The history of numerical methods on Lie groups is intertwined with the development of computational mathematics and the study of Lie theory. The foundations of Lie theory were settled by the Norwegian mathematician Sophus Lie in the late 19th century. However, it was not until the 20th century that the application of Lie groups to practical problems and the development of numerical methods gained momentum. In the 1970s, mathematicians and physicists began to explore numerical integration methods for Lie group equations of motion. Afterwards, pioneering work by Blanes, Casas, Oteo, and Ros provided explicit symplectic integrators for specific Lie groups, such as the rotation group SO(3) and the special Euclidean group SE(3). These methods preserved important geometric properties of Lie groups, such as the energy and/or the symplecticity [5, 6].

The computation of geodesics on Lie groups became a topic of interest in the 1980s. Researchers like Murray, Arimoto, and Sastry developed numerical methods to compute geodesics on Lie groups such as SO(3) and SE(3) [58, 70]. These methods relied on various techniques, including the exponential map, interpolation, and numerical optimization algorithms. The optimization of functions defined on Lie groups gained prominence in the 1990s. Researchers such as Absil, Mahony, and Mallick developed numerical optimization algorithms specifically tailored to the geometric properties of Lie groups [1, 49, 50]. These methods allowed for efficient optimization of functions over Lie groups, which found applications in robotics, computer vision, and control theory. The interpolation of motions on Lie groups received significant attention in the early 2000s. Researchers like Sola, Kuffner, and Agrawal proposed interpolation algorithms for Lie group elements, enabling smooth and visually appealing motion planning in applications such as robotics and computer graphics [72].

In recent years, there has been continued progress in numerical methods on Lie groups, fueled by advancements in computational power and the increasing demand for efficient algorithms in applications. Research continues to focus on refining existing methods, developing new techniques, and exploring applications in areas like machine learning, motion planning, and optimization.

The Runge-Kutta methods are a family of numerical integration techniques commonly used to solve ordinary differential equations. They involve evaluating the derivative of the function at multiple points within a time step and using a weighted sum of these derivatives to update the solution. A comprehensive survey on modern geometric Lie group methods, including new ideas and techniques, can be found in [37].

The RKMK method combines these two concepts by using the Munthe-Kaas rule to select the sampling points in the Runge-Kutta integration scheme. RKMK method is also the term we use to refer to the usual Runge-Kutta method (RK) applied on Lie groups. By considering the distribution of the highest derivative of the function being integrated, the RKMK method aims to improve the accuracy and efficiency of the integration process [55, 56, 57].

The specific details of the RKMK method, including the choice of sampling points and the weights assigned to the derivatives, can vary depending on the implementation and the problem at hand. Researchers have proposed different variants of the RKMK method with varying degrees of accuracy and computational complexity. Since the properties of a RKMK methods are the same of a classical RK, the symplecticity is preserved for certain orders: for example, the second-order Störmer-Verlet method, also known as the leapfrog method, is a well-known second-order symplectic integrator [75]. There are several fourth-order symplectic integrators, such as the Forest-Ruth method and the Yoshida method [79]. Higher-order symplectic integrators have also been developed, such as the sixth-order McLachlan integrator [54] and the eighth-order Blanes-Moan integrator [7]. These symplectic Runge-Kutta methods are designed to preserve the symplectic structure of Hamiltonian systems and offer improved accuracy and long-term stability compared to non-symplectic methods.

It is important to note that the choice of a specific symplectic Runge-Kutta method depends on the requirements of the problem at hand, including the desired accuracy, computational efficiency, and preservation of particular properties. In our case, we will work with a fourth-order RKMK.

The **RKMK** Methods

The basic idea behind applying the fourth-order RKMK method is to update the group elements

using Lie group operations while approximating the derivatives of the group elements at multiple intermediate points within a time step. The following steps outline a typical approach.

- Initialization: Start with an initial group element.
- Time Step Selection: Choose an appropriate time step size for the integration process.
- Derivative Evaluation: Evaluate the derivative of the group element at the initial time.
- State Update: Use the fourth-order RK method to update the group element by integrating the derivative. This involves evaluating the derivative at multiple intermediate points within the time step and combining them with weighted sums to update the state.
- Group Operation: Apply appropriate Lie group operations (e.g., matrix multiplication, exponentiation) to ensure the updated state remains on the Lie group manifold.
- Repeat: Repeat steps 3-5 until the desired integration time is reached.

By incorporating the Lie group operations in the state update step and properly handling the derivatives, the fourth-order RK method can be applied to approximate solutions on Lie groups.

Magnus Method and its Interpretation

To solve the initial-value problem for the linear system of ordinary differential equations on \mathbb{R}^n of the form

$$\frac{dY}{dt}(t) = A(t)Y(t), \quad Y(t_0) = Y_0, \tag{1.1}$$

where Y(t) is an unknown *n*-dimensional vector *t*-dependent function and A(t) is an $n \times n$ matrix with *t*-dependent entries, the Magnus approach was introduced. The solution for n = 1 is

$$Y(t) = \exp\left(\int_{t_0}^t A(s) \, ds\right) Y_0.$$

This solution also holds for n > 1 provided $A(t_1)A(t_2) = A(t_2)A(t_1)$ for any pair of values t_1 and t_2 , especially when A is independent of t. However, for the general case, the aforementioned expression is not a valid solution.

Wilhelm Magnus devised a method to solve the matrix initial-value problem (1.1) by introducing the exponential of a specific $n \times n$ matrix function $\Omega(t, t_0)$ as follows

$$Y(t) = \exp(\Omega(t, t_0)) Y_0,$$
(1.2)

where $\Omega(t)$ is constructed as a series expansion

$$\Omega(t) = \sum_{k=1}^{\infty} \Omega_k(t),$$

with $\Omega(t)$ representing $\Omega(t, t_0)$ for simplicity and taking $t_0 = 0$.

Using $\frac{d}{dt}(e^{\Omega})e^{-\Omega} = A(t)$ and the Poincaré-Hausdorff matrix identity, Magnus related the time derivative of Ω to the generating function of Bernoulli numbers and the adjoint endomorphism of Ω , namely $ad(\Omega) = [\Omega, \cdot]$, as follows

$$\frac{d\Omega}{dt} = \frac{\operatorname{ad}(\Omega)}{\exp(\operatorname{ad}(\Omega)) - \operatorname{Id}_n} A \tag{1.3}$$

to solve Ω in a recursive manner in terms of A as a continuous analog of the Baker-Campbell-Hausdorff (BCH) expansion [25, 74]. Note that Id_n stands for the $n \times n$ identity matrix.

Equation (1.3) is called the *Magnus expansion*, or *Magnus series*, for the solution of (1.1). The first four terms of this series read

$$\begin{aligned} \Omega_1(t) &= \int_0^t A(t_1) \, dt_1, \\ \Omega_2(t) &= \frac{1}{2} \int_0^t dt_1 \int_0^{t_1} [A(t_1), A(t_2)] dt_2, \\ \Omega_3(t) &= \frac{1}{6} \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} \left([A(t_1), [A(t_2), A(t_3)]] + [A(t_3), [A(t_2), A(t_1)]] \right) dt_3, \\ \Omega_4(t) &= \frac{1}{12} \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} \left([[[A(t_1), A(t_2)], A(t_3)], A(t_4)] + \ldots \right) dt_4. \end{aligned}$$

By expressing the solution in terms of the exponential of a matrix function (1.2), the Magnus series offers a systematic way to approximate the solution. The Magnus approach very often preserves important qualitative properties of the exact solution, such as the symplectic or unitary character, even in truncated forms. This method has found applications in various fields, including classical mechanics and quantum mechanics, where it offers an alternative to conventional perturbation theories. The Magnus expansion method stands as a valuable tool for analyzing and approximating solutions to linear differential equations [6, 48], and, naturally, has strong applications when Y(t)belongs to a matrix Lie group.

These two numerical integration techniques play a crucial role in the theory of Lie systems. These methods aim to preserve the geometric structures and qualitative properties of the underlying system, such as symplecticity, conservation laws and, in the first place, the Lie group structure itself. The Magnus expansion and RKMK methods are particularly useful for preserving the long-term behavior of Hamiltonian systems, and, in particular, the so called Lie-Hamilton systems [47], which are a special class of Lie systems.

Hamiltonian systems relative to a symplectic structure admit natural constants of motion, and numerical methods that preserve such symplectic structures and related invariants ensure that obtained approximate solutions will remain at every moment in certain regions containing the searched exact solution, e.g., in submanifolds given by a constant of motion. In the short term, this may not have a big impact in the accuracy of the chosen numerical solutions. But in the long term, this may ensure that the obtained numerical solution will always remain in a specific region containing the exact solution. In particular, symplectic preserving numerical methods may ensure that a numerical solution around a stable point of a Hamiltonian system given by the minimum of its Hamiltonian function remains close to the equilibrium point. Other numerical methods may not be able to reproduce such a behavior.

Lie systems occurred for the first time in the study of Riccati equations [39] as a consequence of the generalisation to a nonlinear realm of the known superposition rules for nonautonomous linear systems of first-order ordinary differential equations. Among other reasons, superposition rules are interesting to solve numerically systems of differential equations whose general solutions cannot be exactly found [76]. Although most differential equations cannot be studied via Lie systems, Lie systems have many relevant applications in physics, control theory, and other fields [13, 47]. In particular, Lie-Hamilton systems occur in the study of Smorodinsky-Winternitz oscillators, Milne– Pinney equations, dissipative harmonic oscillators, trigonometric oscillators, and so on (see [47] and references therein). Certain quantum mechanical systems, like quantum mechanical oscillators with time-dependent frequency and other time-dependent parameters, can also be studied via Lie systems on Lie groups [13]. Particular cases of matrix Riccati equations, which are also Lie systems, are associated with Painlevé trascendents, Sawada-Kotera equations, Kaup–Kupershmidt equations, etcetera [46]. For all that reasons, the study of Lie systems is fully motivated from the point of view of applications.

Our approach to the numerical integration of Lie systems in this manuscript is the Lie group approach, which is relevant since there exists a Lie group action establishing a relationship between a certain Lie group intrinsically associated to the Lie system and the manifold where the Lie system itself evolves [9]. The two Lie group integrators that we have introduced exploit the algebraic structure of the Lie group associated with the Lie system to construct accurate and efficient numerical schemes.

Our aim in this work is to depart from this preexisting technology on Lie group integrators and take advantage of it when numerically integrating Lie systems. These can evolve on manifolds with an additional particular compatible geometric structure, such as a group structure or curvature, and, therefore, a geometric integrator for them is in order. The Lie group action relating the Lie group underlying a Lie system and the manifold where it evolves represents a perfect tool to achieve this goal, generating a discrete sequence of points in the manifold (which naturally inherit its geometry) from a discrete sequence of points in the Lie group, which can be obtained from the Lie group integrators. In this way we establish a novel geometric integrator [23], which we will test on a particular class of Lie systems on curved spaces and other Lie systems with interesting properties. Our focus in this article, as shall be noticed during its reading, is the dynamical (convergence) and geometrical behavior of the discrete solutions of our integrators. Therefore, we have not paid much attention to the computation cost and time, which is a relevant issue to address in the future. However, as in other geometric integrators [23], it is to be expected a higher computation cost, due to the their particular design accomodating the intrinsic nature of the systems they approximate. On the other hand this special design shall redound on the long-term behavior, as pointed out in the next paragraph and throughout the paper. Note also that ordinary differential equations in normal form around equilibrium points may be approximated by linear systems of differential equations, which can be studied via Lie systems [47]. It may happen that higher-order approximations may be described through Lie systems. This also justifies the potential applicability of our techniques.

The previous approach has natural advantages to describe the long-term behavior of obtained numerical solutions. Indeed, Lie systems have an evolution that is determined by a family of vector fields that span an integrable generalized distribution. Then, the Lie system has a related stratification of the manifold and the solutions of the Lie system always evolve within them [9]. By using numerical methods in the Lie group associated with the Lie system, the numerical solutions obtained will always remain within the stratification related to the Lie system. In a natural manner, this ensures that obtained numerical solutions will always be well behaved not escaping away from the strata containing the exact solution to be obtained. In general, this may not have a deep impact in the short-term behavior. Nevertheless, it may be relevant in solving problems with a strong dependence on initial conditions, as shown in the examples of this work, which have some kind of "chaotic/bifurcation" behavior. Moreover, our methods ensure that numerical solutions will always satisfy certain geometric properties of the searched exact solutions in the long-term. In contrast, general numerical methods may provide numerical solutions whose features in the long-term have nothing to do with the searched solution.

The outline of the paper goes as follows. In §2 we introduce the fundamentals on Lie groups and Lie algebras needed hereafter. Moreover, we describe automorphic Lie systems and how to solve them in its underlying Lie group. The definition of the action of a Lie group on the manifold where the Lie system evolves is also presented. These two elements allow for the definition of the 7-step method for the reduction procedure of automorphic Lie systems in Definition 2.2. §3 depicts some basics on numerical schemes and the Lie group methods employed afterwards. In §4 we combine all the previous elements to propose our geometric method to numerically integrate automorphic Lie systems in Definition 4.1. In §5 we pick a class of Lie systems on curved spaces, we apply with high detail the 7-step method to analytically solve them and, afterwards, we employ our integrator, showing its geometric properties. The advantages of our methods in the description of problems with a strong dependence on the initial conditions and with respect to long-term analysis of solutions is analyzed in Section §6. The conclusions and outlook of our work are described in Section §7.

2 Geometric fundamentals

2.1 Lie groups and matrix Lie groups

Let G be a Lie group and let e be its neutral element. Every $g \in G$ defines a right-translation $R_g : h \in G \mapsto hg \in G$ and a left-translation $L_g : h \in G \mapsto gh \in G$ on G. A vector field, $X^{\mathbb{R}}$, on G is right-invariant if $X^{\mathbb{R}}(hg) = R_{g*,h}X^{\mathbb{R}}(h)$ for every $h, g \in G$, where $R_{g*,h}$ is the tangent map to R_g at $h \in G$. The value of a right-invariant vector field, $X^{\mathbb{R}}$, at every point of G is determined by its value at e, since, by definition, $X^{\mathbb{R}}(g) = R_{g*,e}X^{\mathbb{R}}(e)$ for every $g \in G$. Hence, each right-invariant vector field X^R on G gives rise to a unique $X^R(e) \in T_eG$ and vice versa. Then, the space of right-invariant vector fields on G is finite-dimensional, and it can be proved to be also a Lie algebra. Similarly, one may define left-invariant vector fields on G, establish a Lie algebra structure on the space of left-invariant vector fields and set an isomorphism between the space \mathfrak{g} of left-invariant vector fields on G and T_eG . The Lie algebra of left-invariant vector fields on G, with the Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ given by the commutator of vector fields, induces in T_eG a Lie algebra via the identification of left-invariant vector fields with their values at e. Note that we will frequently identify \mathfrak{g} with T_eG to simplify the notation.

There is a natural mapping from \mathfrak{g} to G, the so-called exponential map, of the form $\exp : a \in \mathfrak{g} \mapsto \gamma_a(1) \in G$, where $\gamma_a : \mathbb{R} \to G$ is the integral curve of the right-invariant vector field $X_a^{\mathbb{R}}$ on G satisfying $X_a^{\mathbb{R}}(e) = a$ and $\gamma(0) = e$. It is worth noting that the exponential map could be defined using left-invariant vector fields to give exactly the same mapping (cf. [25]). If $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{K})$, where $\mathfrak{gl}(n, \mathbb{K})$ is the Lie algebra of $n \times n$ square matrices with entries in a field \mathbb{K} relative to the Lie bracket given by the commutator of matrices, then $\mathfrak{gl}(n, \mathbb{K})$ can be considered as the Lie algebra of the Lie group $\operatorname{GL}(n, \mathbb{K})$ of $n \times n$ invertible matrices with entries in \mathbb{K} . It can be proved that in this case $\exp : A \in \mathfrak{gl}(n, \mathbb{K}) \mapsto \exp(A) \in \operatorname{GL}(n, \mathbb{K})$ is given by the standard expression of the exponential of a matrix [42], namely

$$\exp(A) = I_n + A + \frac{A^2}{2} + \frac{A^3}{6} + \dots = \sum_{k=0}^{\infty} \frac{A^k}{k!},$$

where we recall that I_n stands for the $n \times n$ identity matrix.

From the definition of the exponential map $\exp: T_e G \to G$, it follows that $\exp(sa) = \gamma_a(s)$ for each $s \in \mathbb{R}$ and $a \in T_e G$. Let us show this. Given the right-invariant vector field X_{sa}^{R} , with $sa \in T_e G$, one has

$$X_{sa}^{\mathrm{R}}(g) = R_{g*,e} X_{sa}^{\mathrm{R}}(e) = R_{g*,e}(sa) = s R_{g*,e}(a), \quad \forall g \in G.$$

In particular, for s = 1, it follows that $X_a^{\mathrm{R}}(g) = R_{g*,e}(a)$ and, for a general s, one has that $X_{sa}^{\mathrm{R}} = sX_a^{\mathrm{R}}$. Hence, if $\gamma_a, \gamma_{sa} : \mathbb{R} \to G$ are the integral curves of X_a^{R} and X_{sa}^{R} with initial condition e, then it can be proved that, for u = ts, one has that

$$\frac{d}{dt}\gamma_a(ts) = s\frac{d}{du}\gamma_a(u) = sX_a^{\mathrm{R}}(\gamma_a(ts)) = X_{sa}^{\mathrm{R}}(\gamma_a(ts)).$$

and $t \mapsto \gamma_a(st)$ is the integral curve of X_{sa}^{R} with initial condition e. Hence, $\gamma_a(st) = \gamma_{sa}(t)$. Therefore, $\exp(sa) = \gamma_{sa}(1) = \gamma_a(s)$. It is worth stressing that it is a consequence of Ado's theorem [2] that every Lie group can be written as a matrix Lie group on some open neighborhood of its neutral element.

The exponential map establishes a diffeomorphism from an open neighborhood $U_{\mathfrak{g}}$ of 0 in T_eG and $\exp(U_{\mathfrak{g}})$. More in detail, every basis $\mathcal{V} = \{v_1, \ldots, v_r\}$ of T_eG gives rise to the so-called canonical coordinates of the second-kind related to \mathcal{V} defined by the local diffeomorphism

$$U_{\mathfrak{g}} \subset T_e G \longrightarrow \exp(U_{\mathfrak{g}}) \subset G (\lambda_1, \dots, \lambda_r) \mapsto \prod_{\alpha=1}^r \exp(\lambda_\alpha v_\alpha),$$

for an appropriate open neighborhood $U_{\mathfrak{g}}$ of 0 in $T_e G \simeq \mathfrak{g}$.

In matrix Lie groups right-invariant vector fields take a simple useful form. In fact, let G be a matrix Lie group. It can be then considered as a Lie subgroup of $GL(n, \mathbb{K})$. Moreover, it can be proved that T_AG , for any $A \in G$, can be identified with a certain subspace of the space $\mathcal{M}_n(\mathbb{K})$ of $n \times n$ square matrices with coefficients in \mathbb{K} .

Since $R_A : B \in G \mapsto BA \in G$, then $R_{A*,e}(M) = MA \in T_AG$, for all $M \in T_eG$ and $A \in GL(n, \mathbb{K})$. As a consequence, if $X^{\mathbb{R}}(e) = M$ at the neutral element e of G, namely the identity I of the matrix Lie group G, then $X^{\mathbb{R}}(A) = R_{A*,I}(X^{\mathbb{R}}(I)) = R_{A*,I}(M) = MA$. It follows that, at any $A \in G$, every tangent vector $B \in T_AG$ can be written as B = CA for a unique $C \in T_IG$ [17, 25].

Let us describe some basic facts on Lie group actions on manifolds induced by Lie algebras of vector fields. It is known that every finite-dimensional Lie algebra, V, of vector fields on a manifold N gives rise to a (local) Lie group action

$$\varphi: G \times N \to N,\tag{2.1}$$

whose fundamental vector fields are given by the elements of V and G is a connected and simply connected Lie group whose Lie algebra is isomorphic to V (see [60]). If the vector fields of V are complete, then the Lie group action (2.1) is globally defined. Moreover, G does not always need to be simply connected. The Lie group action φ will be crucial in the definition of our integrators, since, as can be seen, relates the Lie group G and the manifold N, i.e., the manifold where we are going to define the time-evolution of our Lie systems. In fact, Lie group actions like φ are employed to reduce the integration of a Lie system on N to obtaining a particular solution of a Lie system on a Lie group [9, 13]. Let us show how to obtain φ from V, which will be of crucial importance in this work.

Let us restrict ourselves to an open neighborhood U_G of the neutral element of G, where we can use canonical coordinates of the second-kind related to a basis $\{v_1, \ldots, v_r\}$ of \mathfrak{g} with opposite structure constants than X_1, \ldots, X_r (see [9, 13]). Then, each $g \in U_G$ can be expressed as

$$g = \prod_{\alpha=1}^{r} \exp(\lambda_{\alpha} v_{\alpha}), \qquad (2.2)$$

for certain uniquely defined parameters $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$. To determine φ , we first calculate the curves

$$\gamma_x^{\alpha} : \mathbb{R} \to N : t \mapsto \varphi(\exp(tv_{\alpha}), x), \qquad \alpha = 1, \dots, r, \qquad \forall x \in N,$$
(2.3)

where γ_x^{α} must be the integral curve of X_{α} starting from the neutral element of G for $\alpha = 1, \ldots, r$. Indeed, for any element $g \in U_G \subset G$ expressed as in (2.2), using the intrinsic properties of a Lie group action,

$$\varphi(g,x) = \varphi\left(\prod_{\alpha=1}^{r} \exp(\lambda_{\alpha} v_{\alpha}), x\right) = \varphi(\exp(\lambda_{1} v_{1}), \varphi(\exp(\lambda_{2} v_{2}), \dots, \varphi(\exp(\lambda_{r} v_{r}), x) \dots)), \quad \forall x \in N,$$

and the Lie group action is completely defined for every $g \in U_G \subset G$.

In this work we will deal with some particular matrix Lie groups, starting from the general linear matrix group $GL(n, \mathbb{K})$, where we stress that \mathbb{K} may be \mathbb{R} or \mathbb{C} . As well known, any closed subgroup of $GL(n, \mathbb{K})$ is also a matrix Lie group [42, Theorem 15.29, pg. 392].

2.2 Generalities on Lie systems

Since we are hereafter dealing with non-autonomous systems of differential equations on a manifold N which are locally of the form

$$\frac{dx^{i}}{dt} = X^{i}(t, x), \qquad x \in N, \quad i = 1, \dots, n = \dim N, \qquad t \in \mathbb{R},$$

we can uniquely relate it to a t-dependent vector field, i.e., a t-parametric family of standard vector fields on N, given by

$$X(t,x) = \sum_{i=1}^{n} X^{i}(t,x) \frac{\partial}{\partial x^{i}}, \qquad x \in N, \quad t \in \mathbb{R},$$

and conversely. We will hereafter write $\mathfrak{X}_t(N)$ for the space of t-dependent vector fields on N [13]. Moreover, $X_t : x \in N \mapsto X(t, x) \in TN$ shall hereafter stand for the vector field induced by X at a particular value of $t \in \mathbb{R}$. The smallest Lie algebra of vector fields (in the sense of inclusion) containing all the vector fields $\{X_t\}_{t\in\mathbb{R}}$ will be denoted by $\operatorname{Lie}(\{X_t\}_{t\in\mathbb{R}})$. We call $\operatorname{Lie}(\{X_t\}_{t\in\mathbb{R}})$ the minimal Lie algebra of X.

A Lie system is a nonautonomous first-order system of ODEs that admits a superposition rule. A superposition rule for a system X on N (the manifold where X evolves) is a map $\Phi : N^m \times N \to N$ such that the general solution x(t) of X can be written as $x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)}(t); \rho)$, where $x_{(1)}(t), \ldots, x_{(m)}(t)$ is a generic family of particular solutions and ρ is a point in N related to the initial conditions of X.

A classic example of Lie system is given by Riccati equations [47, Example 3.3], that is,

$$\frac{dx}{dt} = b_1(t) + b_2(t)x + b_{12}(t)x^2, \qquad x \in \mathbb{R}, \qquad t \in \mathbb{R},$$
(2.4)

with $b_1(t), b_2(t), b_{12}(t)$ being arbitrary functions of t. It is known that the general solution, x(t), of a Riccati equation can be written as

$$x(t) = \frac{x_{(2)}(t)(x_{(3)}(t) - x_{(1)}(t)) + \rho x_{(3)}(t)(x_{(1)}(t) - x_{(2)}(t))}{(x_{(3)}(t) - x_{(1)}(t)) + \rho(x_{(1)}(t) - x_{(2)}(t))},$$
(2.5)

where $x_{(1)}(t), x_{(2)}(t), x_{(3)}(t)$ are three different particular solutions of (2.4) and $\rho \in \mathbb{R}$ is an arbitrary constant. This implies that every Riccati equation admits a superposition rule $\Phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ of the form

$$\Phi(x_{(1)}, x_{(2)}, x_{(3)}, \rho) = \frac{x_{(2)}(x_{(3)} - x_{(1)}) + \rho x_{(3)}(x_{(1)} - x_{(2)})}{(x_{(3)} - x_{(1)}) + \rho(x_{(1)} - x_{(2)})}$$

The conditions that guarantee the existence of a superposition rule are gathered in the Lie theorem [45, Theorem 44], which also provides a description of the underlying geometry of a Lie system. This theorem asserts that a first-order system X on N,

$$\frac{dx}{dt} = X(t, x), \qquad x \in N, \qquad t \in \mathbb{R}, \qquad X \in \mathfrak{X}_t(N), \tag{2.6}$$

admits a superposition rule if and only if X can be written as

$$X(t,x) = \sum_{\alpha=1}^{r} b_{\alpha}(t) X_{\alpha}(x), \qquad x \in N, \qquad t \in \mathbb{R},$$
(2.7)

for a certain family $b_1(t), \ldots, b_r(t)$ of t-dependent functions and a family of vector fields X_1, \ldots, X_r on N that generate an r-dimensional Lie algebra of vector fields. Moreover, Lie proved that $\dim V \leq m \dim N$, where m is the number of particular solutions of the associated superposition rule. The relation $\dim V \leq m \dim N$ is called the Lie's condition. The Lie algebra $\langle X_1, \ldots, X_r \rangle$ is called a Vessiot-Guldberg (VG) Lie algebra of the Lie system X.

The *t*-dependent vector field on the real line associated with (2.4) is $X_R = b_1(t)Y_0 + b_2(t)Y_1 + b_{12}(t)Y_2$, where Y_0, Y_1, Y_2 are vector fields on \mathbb{R} given by

$$Y_0 = \frac{\partial}{\partial x}, \qquad Y_1 = x \frac{\partial}{\partial x}, \qquad Y_2 = x^2 \frac{\partial}{\partial x}.$$

Since their commutation relations are

$$[Y_0, Y_1] = Y_0, \quad [Y_0, Y_2] = 2Y_1, \quad [Y_1, Y_2] = Y_2, \tag{2.8}$$

the vector fields Y_0, Y_1, Y_2 generate a VG Lie algebra V_R isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Then, Lie theorem guarantees that (2.4) admits a superposition rule, which is precisely the one shown in (2.5). Note that Lie's condition is satisfied since dim $V_R \leq 3 \cdot \dim \mathbb{R}$, where we have used that the superposition rule (2.4) depends on three particular solutions.

In practice, the process of determining if X is a Lie system or not relies on taking different vector fields X_t for different values of t and trying to determine the smallest Lie algebra (in the sense of inclusion) containing them by working out all their successive Lie brackets and checking whether all the X_t , with $t \in \mathbb{R}$, belong to the obtained family or not. Frequently, one finds two possible outcomes:

• a) After some calculations, the successive Lie brackets of some induced vector fields give rise to an infinite family of linearly independent vector fields. This happens in Abel equations of the form (see [13] and references therein)

$$\frac{dx}{dt} = x^2 + a(t)x^3, \qquad x \in \mathbb{R}, \qquad t \in \mathbb{R},$$

for any non-constant t-dependent function a(t). In this case, the associated t-dependent vector field reads $X_A = X_1 + a(t)X_2$, where the vector fields $X_1 = x^2 \partial/\partial x$, $X_2 = x^3 \partial/\partial x$ are such that

$$\left[x^2\frac{\partial}{\partial x}, x^k\frac{\partial}{\partial x}\right] = (k-2)x^{k+1}\frac{\partial}{\partial x}, \qquad k = 3, 4, \dots$$

Then, X_1, X_2 generate, along with their successive Lie brackets, an infinite family of linearly independent vector fields (over the reals). This shows that the minimal Lie algebra associated with X_A is infinite-dimensional and X_A is not a Lie system.

The finite-dimensional Lie algebras of analytic vector fields are also classified on \mathbb{R} and \mathbb{R}^2 under quite general conditions [20, 44, 47]. If one associates a differential equation (2.6) with its *t*dependent vector field X and it turns out that the vector fields $\{X_t\}_{t\in\mathbb{R}}$ do not generate a Lie algebra in one of the known types on N, then X is not a Lie system.

• b) In another particular case, one finds that all the $\{X_t\}_{t\in\mathbb{R}}$ are contained in a finite-dimensional Lie algebra of vector fields and X becomes a Lie system.

The Lie theorem yields that every Lie system X is related to (at least) one Vessiot-Guldberg (VG) Lie algebra, V, that satisfies that $\text{Lie}(\{X_t\}_{t\in\mathbb{R}}) \subset V$. This implies that the minimal Lie algebra has to be finite-dimensional [13]. A Lie system may have different VG Lie algebras. But the minimal Lie algebra is unique.

Let us give a particular illustrative example. Consider the Bernoulli equation

$$\frac{dx}{dt} = a_1(t)x + a_2(t)x^2, \qquad x \in \mathbb{R}, \qquad t \in \mathbb{R}$$

where $a_1(t), a_2(t)$ are arbitrary t-dependent functions, which is a Lie system as it is related to the t-dependent vector field $X_B = a_1(t)Y_1 + a_2(t)Y_2$ with $Y_1 = x\partial/\partial x$, $Y_2 = x^2\partial/\partial x$, and $[Y_1, Y_2] = Y_2$.

If we consider the particular case $a_1(t) = \cos t$ and $a_2(t) = \sin t$, then the minimal Lie algebra of X_B is the Lie algebra containing all $(X_B)_t$ with $t \in \mathbb{R}$. In particular, for t = 0 and $t = \pi/2$, one has $X_0 = Y_1$ and $X_{\pi/2} = Y_2$, respectively. Hence, the minimal Lie algebra of X must contain Y_1, Y_2 . But Y_1, Y_2 span a finite-dimensional Lie algebra and $(X_B)_t$, for every $t \in \mathbb{R}$, is a linear combination of Y_1, Y_2 . Hence, $V_B = \langle Y_1, Y_2 \rangle$ is the smallest Lie algebra of X_B . There exists another VG Lie algebra of X_B given by

$$Y_0 = \frac{\partial}{\partial x}, \qquad Y_1 = x \frac{\partial}{\partial x}, \qquad Y_2 = x^2 \frac{\partial}{\partial x}$$

The systems related to $\sum_{\alpha=0}^{2} a_{\alpha}(t)Y_{\alpha}$, for arbitrary *t*-dependent functions $a_{0}(t), a_{1}(t), a_{2}(t)$, are the Riccati equations (2.4). In other words, using the VG Lie algebra $V_{R} = \langle Y_{0}, Y_{1}, Y_{2} \rangle$, one has that Bernoulli equations with a quadratic term are particular cases of Riccati equations. Riccati equations have then a superposition rule depending on three solutions. Indeed, Lie's condition reads dim $V_{R} = 1 \cdot 3$. Meanwhile, for Bernoulli equations, dim $V_{B} = 2 \cdot 1$, as they are known to have a superposition rule depending on two solutions [13], but they also accept the superposition rule for Riccati equations, which depends on three particular solutions.

Let us analyze the relevance of using the minimal or a VG Lie algebra of a Lie system. It is known that $f \in C^{\infty}(N)$ is a first integral of a Lie system X on N if and only if f is a common first-integral of all the elements of the minimal Lie algebra of the Lie system [47]. Relevantly, it many happen that f is a constant of motion of X, but it is not a first integral of all the vector fields of a certain VG Lie algebra associated with X. Then, one may prefer to know the minimal Lie algebra: it characterizes autonomous constants of motion of Lie systems [4, 47]. Moreover, recall that Lie's condition shows that, if dim V is larger for a VG Lie algebra V of a Lie system X, the superposition rule associated with X depends on more particular solutions, which is not desirable in general. Moreover, many methods to determine superposition rules depend on solving a system of partial differential equations that becomes larger when the VG Lie algebra has larger dimension. That is why, again, one may prefer to determine the minimal Lie algebra for Lie systems.

On the contrary, the use of larger VG Lie algebras permits the analysis of properties that are common to more Lie systems. Additionally, some methods to derive superposition rules depend on the nature of VG Lie algebras. For instance, the coalgebra method [4, 47] is easier to implement for semi-simple VG Lie algebras, which have the so-called Casimir functions. This implies that it may be simpler to deal with larger VG Lie algebras if they are semi-simple and have Casimir functions, than with smaller ones that do not have them.

Let us illustrate the ideas of the previous paragraph. Consider the system of differential equations

$$\frac{dx_1}{dt} = a(t)x_1 + b(t)x_1^2,
\frac{dx_2}{dt} = a(t)x_2 + b(t)x_2^2,
\frac{dx_3}{dt} = a(t)x_3 + b(t)x_3^2,$$

appearing in the calculus of the superposition rule for Bernoulli equations with a square term (cf. [13]). In this case, one can relate the above system to the *t*-dependent vector field

$$Z = a(t)Z_2 + b(t)Z_3, \qquad Z_2 = \sum_{i=1}^3 x_i \frac{\partial}{\partial x_i}, \qquad Z_3 = \sum_{i=1}^3 x_i^2 \frac{\partial}{\partial x_i}, \qquad [Z_2, Z_3] = Z_3.$$

Hence, Z is a Lie system. A superposition rule for Bernoulli equations depending on two particular solutions can be obtained by a non-constant first integral for Z_2, Z_3 (see [13] for details), which amounts to solving directly a system of PDEs. If one considers Bernoulli equations as particular cases of Riccati equations, the associated superposition rule can be obtained by solving a system of PDEs on \mathbb{R}^4 and it depends on three particular solutions [13]. Fortunately, the superposition rule for Riccati equations can be obtained in a simpler manner than for Bernoulli equations via a Casimir function for $\mathfrak{sl}(2,\mathbb{R})$ [47].

2.2.1 Automorphic Lie systems

It is worth noting that the general solution of a Lie system on N with a VG Lie algebra V, can be obtained from a single particular solution of a Lie system on a Lie group G whose Lie algebra is isomorphic to V. These are the so-called automophic Lie systems [13, §1.4]. As the automorphic Lie system notion is going to be central in our paper, let us study it in some detail (see [13] for details).

Definition 2.1 An automorphic Lie system is a t-dependent system of first-order differential equations on a Lie group G of the form

$$\frac{dg}{dt} = \sum_{\alpha=1}^{r} b_{\alpha}(t) X_{\alpha}^{R}(g), \qquad g \in G, \quad t \in \mathbb{R},$$
(2.9)

where $\{X_1^R, \ldots, X_r^R\}$ is a basis of the space of right-invariant vector fields on G and $b_1(t), \ldots, b_r(t)$ are arbitrary t-dependent functions. Furthermore, we shall refer to the right-hand side of equation (2.9) as $\widehat{X}_R^G(t,g)$, i.e., $\widehat{X}_R^G(t,g) = \sum_{\alpha=1}^r b_\alpha(t) X_\alpha^R(g)$ (see [13, §1.3] for details).

Because of right-invariant vector fields, systems in the form of \widehat{X}_R^G have the following important property.

Proposition 2.1 Given a Lie group G and a particular solution g(t) of the Lie system defined on G, as

$$\frac{dg}{dt} = \sum_{\alpha=1}^{r} b_{\alpha}(t) X_{\alpha}^{\mathrm{R}}(g) = \widehat{X}_{R}^{G}(t,g), \qquad (2.10)$$

where $b_1(t), \ldots, b_r(t)$ are arbitrary t-dependent functions and $X_1^{\mathrm{R}}, \ldots, X_r^{\mathrm{R}}$ are right-invariant vector fields, we have that g(t)h is also a solution of (2.10) for each $h \in G$.

An immediate consequence of Proposition 2.1 is that, once we know a particular solution g(t) of \hat{X}_R^G , any other solution can be obtained simply by multiplying g(t) on the right by any element in G. More concretely, the solution h(t) of (2.10) with initial condition $h(0) = g(0)h_0$ can be expressed as $h(t) = g(t)h_0$. This justifies that henceforth we only worry about finding one particular solution g(t) of \hat{X}_R^G , e.g. the one that fulfills g(0) = e. The previous result can be understood in terms of the Lie theorem or via superposition rules. In fact, since (2.10) admits a superposition rule $\Phi : (g,h) \in G \times G \mapsto gh \in G$, the system (2.1) must be a Lie system. Alternatively, the same result follows from the Lie Theorem and the fact that the right-invariant vector fields on G span a finite-dimensional Lie algebra of vector fields.

There are several reasons to study automorphic Lie systems. One is that they can be locally written around the neutral element of its Lie group in the form

$$\frac{dA}{dt} = B(t)A, \qquad A \in \mathrm{GL}(n, \mathbb{K}), \quad B(t) \in \mathcal{M}_n(\mathbb{K}),$$

where we recall that $\mathcal{M}_n(\mathbb{K})$ is the set of $n \times n$ matrices with coefficients in \mathbb{K} .

The main reason to study automorphic Lie systems is given by the following results, which show how they can be used to solve any Lie system on a manifold. Let us start with a Lie system Xdefined on N. Hence, X can be written as

$$\frac{dx}{dt} = \sum_{\alpha=1}^{r} b_{\alpha}(t) X_{\alpha}(x), \qquad (2.11)$$

for certain t-dependent functions $b_1(t), \ldots, b_r(t)$ and vector fields $X_1, \ldots, X_r \in \mathfrak{X}(N)$ that generate an r-dimensional dimensional VG Lie algebra V. The VG Lie algebra V is always isomorphic to the Lie algebra \mathfrak{g} of a certain Lie group G. The VG Lie algebra V gives rise to a (local) Lie group action $\varphi: G \times N \to N$ whose fundamental vector fields are those of V. In particular, there exists a basis $\{v_1, \ldots, v_r\}$ of \mathfrak{g} so that

$$\frac{d}{dt}\Big|_{t=0}\varphi(\exp(tv_{\alpha}), x) = X_{\alpha}(x), \qquad \alpha = 1, \dots, r, \qquad \forall x \in N.$$

In other words, $\varphi_{\alpha} : (t, x) \in \mathbb{R} \times N \mapsto \varphi(\exp(tv_{\alpha}), x) \in N$ is the flow of the vector field X_{α} for $\alpha = 1, \ldots, r$. Note that if $[X_{\alpha}, X_{\beta}] = \sum_{\gamma=1}^{r} c_{\alpha\beta}^{\gamma} X_{\gamma}$ for $\alpha, \beta = 1, \ldots, r$ and certain constants $c_{\alpha\beta}^{\gamma}$, then $[v_{\alpha}, v_{\beta}] = -\sum_{\gamma=1}^{r} c_{\alpha\beta}^{\gamma} v_{\gamma}$ for $\alpha, \beta = 1, \ldots, r$ (cf. [9]).

To determine the exact form of the Lie group action $\varphi: G \times N \to N$ as in (2.3), we impose

$$\varphi(\exp(\lambda_{\alpha}v_{\alpha}), x) = \varphi_{\alpha}(\lambda_{\alpha}, x), \qquad \alpha = 1, \dots, r, \qquad \forall x \in N,$$
(2.12)

where $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$. While we stay in a neighborhood U of the origin of G, where every element $g \in U$ can be written in the form

$$g = \exp(\lambda_1 v_1) \cdot \ldots \cdot \exp(\lambda_r v_r),$$

then the relations (2.12) and the properties of φ allow us to determine φ on a subset of $G \times N$. If we fix $x \in N$ and α in (2.12), the right-hand side of the equality turns into the integral curve of the vector field X_{α} with the initial condition x, this is why (2.12) holds.

Proposition 2.2 (see [9, 13] for details) Let g(t) be a solution to the system

$$\frac{dg}{dt} = \sum_{\alpha=1}^{r} b_{\alpha}(t) X_{\alpha}^{R}(g), \qquad t \in \mathbb{R}, \quad g \in G,$$
(2.13)

where $b_1(t), \ldots, b_r(t)$ are arbitrary t-dependent functions. Then, $x(t) = \varphi(g(t), x_0)$ is a solution of $X = \sum_{\alpha=1}^{r} b_{\alpha}(t) X_{\alpha}$ for every $x_0 \in N$. In particular, if one takes the solution g(t) that satisfies the initial condition g(0) = e, then x(t) is the solution of X such that $x(0) = x_0$.

Let us study a particularly relevant form of automorphic Lie systems that will be used hereafter. If \mathfrak{g} is a finite-dimensional Lie algebra, then Ado's theorem [2] guarantees that \mathfrak{g} is isomorphic to a matrix Lie algebra \mathfrak{g}_M . Let $\mathcal{V} = \{M_1, \ldots, M_r\}$ be a basis of $\mathfrak{g}_M \subset \mathcal{M}_n(\mathbb{R})$. As reviewed in Section 2.1, each M_α gives rise to a right-invariant vector field $X^R_\alpha(g) = M_\alpha g$, with $g \in G$, on G. These vector fields have the opposite commutation relations than the (matrix) elements of the basis.

In the case of matrix Lie groups, the system (2.10) takes a simpler form. Let Y(t) be the matrix associated with the element $g(t) \in G$. Using the right invariance property of each X_{α}^{R} , we have that

$$\frac{dY}{dt} = \sum_{\alpha=1}^{r} b_{\alpha}(t) X_{\alpha}^{R}(Y(t)) = \sum_{\alpha=1}^{r} b_{\alpha}(t) R_{Y(t)*,e} \left(X_{\alpha}^{R}(e) \right) = \sum_{\alpha=1}^{r} b_{\alpha}(t) R_{Y(t)*,e}(M_{\alpha}).$$

We can write the last term as

$$\sum_{\alpha=1}^{r} b_{\alpha}(t) R_{Y(t)*,e}(M_{\alpha}) = \sum_{\alpha=1}^{r} b_{\alpha}(t) M_{\alpha} Y(t),$$

in such a way that for matrix Lie groups, the system on the Lie group is

$$\frac{dY}{dt} = A(t)Y(t), \qquad Y(0) = I, \qquad \text{with} \quad A(t) = \sum_{\alpha=1}^{r} b_{\alpha}(t)M_{\alpha}, \qquad (2.14)$$

where I is the identity matrix (which corresponds with the neutral element of the matrix Lie group) and the matrices M_{α} form a finite-dimensional Lie algebra, which is anti-isomorphic to the VG Lie algebra of the system (by anti-isomorphic we mean that the systems have the same constants of structure but that they differ in one sign).

There exist various methods to solve system (2.10) analytically [69, §2.2], such as the Levi decomposition [43] or the theory of reduction of Lie systems [11, Theorem 2]. In some cases, it is relatively easy to solve it, as is the case where b_1, \ldots, b_r are constants. Nonetheless, we are interested in a numerical approach, since we will try to solve the automorphic Lie system with adapted geometric integrators. The solutions on the Lie group can be straightforwardly translated into solutions on the manifold for the Lie system defined on N via the Lie group action (2.1). This is the main idea behind the numerical integrator that we begin to depic in the following 7 step method, which finally will lead us to numerically integrate Lie systems on the manifold N, preserving its geometric properties.

Definition 2.2 (The 7 step method: Reduction procedure to automorphic Lie systems) The method can be itemized in the following seven steps:

- 1. Given a Lie system on a manifold N, we identify a VG Lie algebra of vector fields admitting a basis X_1, \ldots, X_r and associated with the Lie system on N.
- 2. We look for a matrix Lie algebra \mathfrak{g} that is isomorphic to the VG Lie algebra and determine a basis $\{M_1, \ldots, M_r\} \subset \mathcal{M}_n(\mathbb{R})$ with the opposite structure constants than the basis $\{X_1, \ldots, X_r\}$.
- 3. We integrate the vector fields X_1, \ldots, X_r to obtain their respective flows $\Phi_{\alpha} : \mathbb{R} \times N \to N$ with $\alpha = 1, \ldots, r$.

- 4. Using canonical coordinates of the second kind and the previous flows we construct the Lie group action $\varphi: G \times N \to N$ using expressions (2.12).
- 5. We define an automorphic Lie system \widehat{X}_{R}^{G} on the Lie group G associated with \mathfrak{g} as in (2.10).
- 6. We compute the solution g(t) of the system \widehat{X}_{R}^{G} that fulfills g(0) = e.
- 7. Finally, we recover the general solution for X on the manifold N by $x(t) = \varphi(g(t), x_0)$ for x_0 being an arbitrary element of N.

The 7-step method provides a solution of a Lie system on a manifold by means of one particular solution of an automorphic Lie systems on a Lie group and the action (2.1). It is important to emphasize that x(t) obtained in the last step of the 7 step method "lives" on the manifold N, and therefore carries all its geometric properties. In the next section we introduce how the implementation of the 7-step method is carried out numerically. This is accomplished via the Magnus expansion first and the Runge-Kutta-Munthe-Kaas (RKMK) method afterwards.

Let us provide a simple example to illustrate one of the nice features of Lie systems to analyze their properties.

• Step 1: Consider the nonautonomous system of differential equation on \mathbb{R}^2 of the form

$$\frac{dx}{dt} = b_1(t)x, \qquad \frac{dy}{dt} = b_2(t)y^k,$$

where k is odd and $b_1(t), b_2(t)$ are arbitrary t-dependent functions. The above system is a Lie system since it is associated with the t-dependent vector field $X = b_1(t)X_1 + b_2(t)X_2$, where

$$X_1 = x \frac{\partial}{\partial x}, \qquad X_2 = y^k \frac{\partial}{\partial x}$$

satisfy that $[X_1, X_2] = 0$. Then, X_1, X_2 span an abelian two-dimensional VG Lie algebra.

• Step 2: A matrix Lie algebra with a basis closing opposite constants of structure than X_1, X_2 is given by diagonal 2×2 matrices with the basis

$$M_1 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right], \qquad M_2 = \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right].$$

Then, the associated matrix Lie group is spanned by the product of exponentials of $\langle M_1, M_2 \rangle$, which gives invertible matrices

$$G_2 = \left\{ \left[\begin{array}{cc} a & 0 \\ 0 & b \end{array} \right], a, b > 0 \right\}.$$

• Step 3: The integration of the vector fields X_1, X_2 gives rise to a Lie group action $\varphi : G_2 \times \mathbb{R}^2 \to \mathbb{R}^2$. The integration of the vector fields X_1, X_2 for $k \neq 1$ gives the flows

$$\Phi_1 : (t, x, y) \in \mathbb{R} \times \mathbb{R}^2 \mapsto (xe^t, y) \in \mathbb{R}^2,$$

$$\Phi_2 : (t, x, y) \in \mathbb{R} \times \mathbb{R}^2 \mapsto \left(xe^t, [y^{1-k} + (1-k)t]^{1/(k-1)}\right) \in \mathbb{R}^2,$$

while for k = 1 the flows read

$$\Phi_1: (t, x, y) \in \mathbb{R} \times \mathbb{R}^2 \mapsto (xe^t, y) \in \mathbb{R}^2, \qquad \Phi_2: (t, x, y) \in \mathbb{R} \times \mathbb{R}^2 \mapsto (x, ye^t) \in \mathbb{R}^2.$$

It is worth noting that every finite-dimensional Lie algebra of vector fields gives rise to an integrable distribution. It can be added that X_1, X_2 span a generalized distribution \mathcal{D} on \mathbb{R}^2 given by

$$\mathcal{D}_{(x,0)} = \left\langle \frac{\partial}{\partial x} \right\rangle, \quad x \neq 0, \qquad \mathcal{D}_{(0,y)} = \left\langle \frac{\partial}{\partial y} \right\rangle, \quad y \neq 0,$$
$$\mathcal{D}_{(x,y)} = T_{(x,y)} \mathbb{R}^2, \quad xy \neq 0, \quad x, y \in \mathbb{R}.$$

This generalized distribution is integrable and its strata are given by: a) the point (0,0); the curves b) x = 0, y > 0, c) x = 0, y < 0, d) y = 0, x > 0, e) y = 0, x < 0; and the four connected components of the region of points (x, y) with $xy \neq 0$. The above strata are the orbits of φ .

• Step 4: Hence, the Lie group action for $k \neq 1$ reads $\varphi: G_2 \times \mathbb{R}^2 \to \mathbb{R}^2$ with

$$\varphi\left(\left[\begin{array}{cc} a & 0\\ 0 & b \end{array}\right], (x, y)\right) = \left(xa, [y^{1-k} + (1-k)\ln(b)]^{1/(k-1)}\right), \qquad \forall a, b > 0, \qquad \forall (x, y) \in \mathbb{R}^2, \ (2.15)$$

while the canonical coordinates of the second kind read $\lambda_1 = \ln a$ and $\lambda_2 = \ln b$. For the case k = 1, we obtain

$$\varphi\left(\left[\begin{array}{cc}a&0\\0&b\end{array}\right],(x,y)\right) = (xa,yb), \qquad \forall a,b>0, \qquad \forall (x,y) \in \mathbb{R}^2, \tag{2.16}$$

It is immediate that the fundamental vector fields of the Lie group action φ are given by the Lie algebra V.

• Step 5: For every possible value of k, the automorphic Lie system reads

$$\frac{d}{dt} \begin{bmatrix} a & 0\\ 0 & b \end{bmatrix} = \begin{bmatrix} b_1(t) & 0\\ 0 & b_2(t) \end{bmatrix} \begin{bmatrix} a & 0\\ 0 & b \end{bmatrix}, \quad a, b > 0.$$

The induced Lie system on the Lie algebra reads

$$\frac{d}{dt} \begin{bmatrix} \lambda_1 & 0\\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} b_1(t) & 0\\ 0 & b_2(t) \end{bmatrix} \begin{bmatrix} \lambda_1 & 0\\ 0 & \lambda_2 \end{bmatrix}, \qquad \lambda_1, \lambda_2 \in \mathbb{R}.$$

• Step 6: One can immediately integrate the above system numerically by means of some method to obtain the particular solution starting from the neutral element.

• Step 7: A numerical solution of the automorphic Lie system will give, via the associated Lie group action, an approximate solution of X that will respect the stratification.

The above example illustrates an interesting fact. The vector fields of the VG Lie algebra may not give rise to a globally defined Lie group action (see the denominator in (2.15)). In any case, the diffeomorphisms on the manifold \mathbb{R}^2 induced by the Lie group action, namely the maps $\varphi_g : (x, y) \in \mathbb{R}^2 \to \varphi(g, (x, y)) \in \mathbb{R}^2$ for $g \in G_2$, will respect the integral manifolds of \mathcal{D} . Moreover, the induced automorphic Lie system on G_2 can be solved locally around the neutral element to avoid that the action on the elements of a numerical solution will not be well-defined.

3 Numerical methods on matrix Lie groups

This section adapts known numerical methods on Lie groups to study automorphic Lie systems, which are defined by ordinary differential equations defined on Lie groups of the form (2.13). For this purpose, we start by reviewing briefly some fundamentals on numerical methods for ordinary differential equations and Lie groups [24, 36, 65], and later focus on two specific numerical methods on Lie groups, the Magnus expansion and RKMK methods [34, 37, 55, 56, 80]. We will rely on one-step methods with fixed time step. By that we mean that solutions x(t) of a dynamical system

$$\frac{dx}{dt} = f(t, x), \quad x(a) = x_0, \quad f \in \mathfrak{X}_t(N), \quad a \in \mathbb{R},$$
(3.1)

are approximated by a sequence of points $x_k = x(t_k) \in N$ with b > a, $h = (b-a)/\mathcal{N}$, $t_k = a + kh$ for $k = 0, \ldots, \mathcal{N}$, and

$$\frac{x_{k+1} - x_k}{h} = f_h(t_k, x_k, x_{k+1}), \tag{3.2}$$

where \mathcal{N} represents the number of steps into which our time interval is divided. We emphasize here that the left hand side of (3.2) symbolically represents a proper discretization of a tangent vector on a manifold. Note that we cannot "substract" elements of the manifold and we consider the expression on a local coordinate system. If N is a vector space, the minus sign recovers its usual meaning). We call h the time step, which is fixed, while $f_h : \mathbb{R} \times N \times N \to TN$ is a discrete vector field, which is a given approximation of f in (3.1). As usual, we shall denote the local truncation error by E_h , where

$$E_h = ||x_{k+1} - x(t_{k+1})|| \tag{3.3}$$

is the error at the step k. Note that for simplicity, $\|\cdot\|$ is a norm on a local coordinated neighborhood ¹ in N, and we say that the method is of order r if $E_h = \mathcal{O}(h^{r+1})$ for $h \to 0$, i.e. $\lim_{h\to 0} |E_h/h^{r+1}| < \infty$. Regarding the global error

$$E_{\mathcal{N}} = ||x_{\mathcal{N}} - x(b)||,$$

we shall say that the method is *convergent* of order r if $E_{\mathcal{N}} = \mathcal{O}(h^r)$, when $h \to 0$. As for the simulations, we pick the following norm in order to define the global error, that is

$$E_{\mathcal{N}} = \max_{k=1,\dots,\mathcal{N}} ||x(t_k) - x_k||.$$
(3.4)

Our purpose is to numerically solve the initial condition problem for system (2.14) defined on a matrix Lie group G of the form

$$\frac{dY}{dt} = A(t)Y \quad \text{with} \quad Y(0) = I, \tag{3.5}$$

where $Y \in G$, while A(t) is a given t-dependent matrix taking values in $\mathfrak{g} \cong T_e G$, and I is the identity matrix in G. That is, we are searching for a discrete sequence $\{Y_k\}_{k=0,\ldots,\mathcal{N}}$ of matrices in G. In an open neighborhood of the zero in $T_e G$, the exponential map defines a diffeomorphism onto an open subset U of the neutral element of G and the problem is equivalent on U to searching for a curve $\Omega(t)$ in \mathfrak{g} such that

$$Y(t) = \exp(\Omega(t)). \tag{3.6}$$

This ansatz helps us to transform (3.5), which is defined in a nonlinear space, into a new problem in a linear space, namely the Lie algebra $\mathfrak{g} \simeq T_e G$. This is expressed in the classical result by Magnus [48].

Theorem 3.1 (Magnus, 1954) The solution of the matrix Lie group (3.5) in G can be written for values of t close enough to zero, as $Y(t) = \exp(\Omega(t))$, where $\Omega(t)$ is the solution of the initial value problem

$$\frac{d\Omega}{dt} = \operatorname{dexp}_{\Omega(t)}^{-1}(A(t)), \qquad \Omega(0) = \mathbf{0}, \qquad (3.7)$$

where **0** is the zero element in T_eG .

When we are dealing with matrix Lie groups and Lie algebras, the map $dexp^{-1}$ is given by

$$\operatorname{dexp}_{\Omega}^{-1}(H) = \sum_{j=0}^{\infty} \frac{B_j}{j!} \operatorname{ad}_{\Omega}^j(H), \qquad (3.8)$$

where $\{B_j\}_{j=0,...,\infty}$ are the Bernoulli numbers and $\operatorname{ad}_{\Omega}(H) = [\Omega, H] = \Omega H - H \Omega$ for every $H \in \mathfrak{g}$. The convergence of the series (3.8) is ensured as long as a certain convergence condition is satisfied [48].

¹In matrix Lie groups and Lie algebras, the main cases we are dealing with, problems can be embedded or are defined in a vector space and we can use the restriction to our manifold of a norm associated with the vector space.

If we try to integrate (3.7) applying a numerical method directly (note that, now, we could employ one-step methods (3.2) safely), $\Omega(t)$ might sometimes drift too much away from the origin and the exponential map would not work. This would be a problem, since we are assuming that $\Omega(t)$ stays in a neighborhood of the origin of \mathfrak{g} where the exponential map defines a local diffeomorphism with the Lie group. Since we still do not know how to characterize this neighborhood, it is necessary to adopt a strategy that allows us to resolve (3.7) sufficiently close to the origin. The thing to do is to change the coordinate system in each iteration of the numerical method (Or keepking the time step *h* small enough, as we shall show when treating the Magnus methods). In the next lines we explain how this is achieved.

Consider now the restriction of the exponential map given by

$$\exp: U_{\mathfrak{g}} \subset \mathfrak{g} \to \exp(U_{\mathfrak{g}}) \subset G,$$
$$A \mapsto \exp(A)$$

so that this map establishes a diffeomorphism between an open neighborhood $U_{\mathfrak{g}}$ around the origin in \mathfrak{g} and its image. Since the elements of the matrix Lie group are invertible matrices, the map $U_{\mathfrak{g}} \to \exp(U_{\mathfrak{g}})Y_0 \subset G : A \mapsto \exp(A)Y_0$ from $U_{\mathfrak{g}} \subset \mathfrak{g}$ to the set

$$\exp(A)Y_0 = \{Y \in G : \exists X \in U_{\mathfrak{g}}, Y = XY_0\}$$

is also a diffeomorphism. This map gives rise to the so-called first-order canonical coordinates centered at Y_0 .

As well-known, the solutions of (3.7) are curves in \mathfrak{g} whose images by the exponential map are solutions to (3.5). In particular, the solution $\Omega^{(0)}(t)$ of system (3.5) such that $\Omega^{(0)}(0)$ is the zero matrix in $T_I G$, namely $\mathbf{0}$, corresponds with the solution $Y^{(e)}(t)$ of the system on G such that $Y^{(e)}(0) = I$. Now, for a certain $t = t_k$, the solution $\Omega^{(t_k)}(t)$ in \mathfrak{g} such that $\Omega^{(t_k)}(t_k) = \mathbf{0}$, corresponds with $Y^{(e)}(t)$ via first-order canonical coordinates centered at $Y^{(e)}(t_k) \in G$, since

$$\exp(\Omega^{(t_k)}(t_k))Y^{(e)}(t_k) = \exp(\mathbf{0})Y^{(e)}(t_k) = Y^{(e)}(t_k),$$

and the existence and uniqueness theorem guarantees $\exp(\Omega^{(0)}(t)) = \exp(\Omega^{(t_k)}(t))Y^{(e)}(t_k)$ around t_k . In this way, we can use the curve $\Omega^{(t_k)}(t)$ and the canonical coordinates centered on $Y^{(e)}(t_k)$ to obtain values for the solution of (3.5) in the proximity of $t = t_k$, instead of using $\Omega^{(0)}(t)$. Whilst the curve $\Omega^{(0)}(t)$ could be far from the origin of coordinates for t_k , we know that $\Omega^{(t_k)}(t)$ will be close, by definition. Applying this idea in each iteration of the numerical method, we are changing the curve in \mathfrak{g} to obtain the approximate solution of (3.5) while we stay near the origin (as long as the time step is small enough).

Thus, what is left is defining proper numerical methods for (3.7) whose solution, i.e. $\{\Omega_k\}_{k=0,\ldots,\mathcal{N}}$, via the exponential map, provides us with a numerical solution of (3.5) remaining in G. In other words, the general Lie group method defined this way [34, 37] can be set by the recursion

$$Y_{k+1} = e^{\Omega_k} Y_k, \qquad k = 0, 1, 2, \dots$$
 (3.9)

Next, we introduce two relevant families of numerical methods providing $\{\Omega_k\}_{k=0,\ldots,\mathcal{N}}$.

The Magnus method

Based on the work by Magnus, the Magnus method was introduced in [34, 35]. The starting point of this method is to resolve equation (3.7) by means of the Picard procedure. This method assures

that a given sequence of functions converges to the solution of (3.7) in a small enough neighborhood of $0 \in \mathbb{R}$. Operating, one obtains the *Magnus expansion*

$$\Omega(t) = \sum_{k=0}^{\infty} H_k(t), \qquad (3.10)$$

where each $H_k(t)$ is a linear combination of iterated commutators. The first three terms are given by

$$\begin{aligned} H_0(t) &= \int_0^t A(\xi_1) d\xi_1 \,, \\ H_1(t) &= -\frac{1}{2} \int_0^t \left[\int_0^{\xi_1} A(\xi_2) d\xi_2 \,, A(\xi_1) \right] d\xi_1 \,, \\ H_2(t) &= \frac{1}{12} \int_0^t \left[\int_0^{\xi_1} A(\xi_2) d\xi_2 \,, \left[\int_0^{\xi_1} A(\xi_2) d\xi_2 \,, A(\xi_1) \right] \right] d\xi_1 \\ &\quad + \frac{1}{4} \int_0^t \left[\int_0^{\xi_1} \left[\int_0^{\xi_1} A(\xi_2) d\xi_2 \,, A(\xi_1) \right] d\xi_2 \,, A(\xi_1) \right] d\xi_1 \end{aligned}$$

Note that the Magnus expansion (3.10) converges absolutely in a given norm for every $t \ge 0$ such that [37, p. 48]

$$\int_0^t \|A(\xi)\| d\xi \le \int_0^{2\pi} \frac{d\xi}{4 + \xi [1 - \cot(\xi/2)]} \approx 1,086868702.$$

In practice, if we work with the Magnus expansion we need a way to handle the infinite series and calculate the iterated integrals. Iserles and Nørsett proposed a method based on binary trees [34, 35]. In [37, §4.3] we can find a method to truncate the series in such a way that one obtains the desired order of convergence. Similarly, [37, §5] discusses in detail how the iterated integrals can be integrated numerically. In our case, for practical reasons we will implement the Magnus method following the guidelines of Blanes, Casas & Ros [6], which is based on a Taylor series of A(t) in (3.5) around the point t = h/2 (recall that, in the Lie group and Lie algebra equations we are setting the initial time $t_0 = a = 0$). With this technique one is able to achieve different orders of convergence. In particular, we will use the second and fourth order convergence methods [6, §3.2], although one can build up to eighth order methods.

The second-order approximation is

$$\exp(\Omega(h)) = \exp(ha_0) + \mathcal{O}(h^3)$$

and the forth-order one reads

$$\exp(\Omega(h)) = \exp\left(ha_0 + \frac{1}{12}h^3a_2 - \frac{1}{12}h^3[a_0, a_1]\right) + \mathcal{O}(h^5),$$

where $\Omega(0) = \mathbf{0}$ and

$$a_i = \frac{1}{i!} \left. \frac{d^i}{dt^i} A(t) \right|_{t=h/2} \qquad i = 0, 1, 2.$$

As we see from the definition, the first method computes the first and second derivative of matrix A(t). Applying the coordinate change in each iteration (3.9), we can implement it through the

following equations:

$$Y_{k+1} = \exp\left[hA\left(t_k + \frac{h}{2}\right)\right]Y_k. \quad [\text{Order 2}] \quad (3.11)$$

$$Y_{k+1} = \exp\left(ha_0 + h^3(a_2 - [a_0, a_1])\right)Y_k,$$

$$t_{1/2} = t_k + \frac{h}{2}, \quad a_0 = A(t_{1/2}), \quad a_1 = \frac{\dot{A}(t_{1/2})}{12}, \quad a_2 = \frac{\ddot{A}(t_{1/2})}{24}, \quad \left\{ \begin{array}{c} \text{[Order 4]} \\ \end{array} \right\}$$
(3.12)

where $\dot{A}(t_0)$, $\ddot{A}(t_0)$ stand for the first and second derivatives of A(t) in terms of t at t_0 . Note that the convergence order is defined for the Lie group dynamics (3.5). That is, when we say that the above methods are convergent of order 2, for instance, that means $E_{\mathcal{N}} = ||Y_{\mathcal{N}} - Y(b)|| = \mathcal{O}(h^2)$, with $h \to 0$, for a proper Lie matrix norm. It is worth noting that the approximations of A(t), given by derivatives of A(t) and their Lie brackets, belong to the Lie algebra. Moreover, it is quite apparent in this method that keeping h small enough ensures that $hA \in U_{\mathfrak{g}}$, i.e., the exponential of the Lie algebra element indeed belongs to the Lie group G.

The Runge-Kutta-Munthe-Kaas method

Changing the coordinate system in each step, as explained in previous sections, the classical RK methods applied to Lie groups give rise to the so-called Runge-Kutta-Munthe-Kaas (RKMK) methods [55, 56]. The equations that implement the method are

$$\Theta_{j} = h \sum_{l=1}^{s} a_{jl} F_{l},$$

$$F_{j} = \operatorname{dexp}_{\Theta_{j}}^{-1} (A(t_{k} + c_{j}h)),$$

$$j = 1, \dots, s,$$

$$\Theta = h \sum_{l=1}^{s} b_{l} F_{l},$$

$$Y_{k+1} = \operatorname{exp}(\Theta) Y_{k}.$$

where the constants $\{a_{jl}\}_{j,l=1}^{s}, \{b_l\}_{l=1}^{s}, \{c_j\}_{j=1}^{s}$ can be obtained from a Butcher's table [65, §11.8] (note that s is the number of stages of the usual RK methods). Apart from this, we have the consistency condition $\sum_{l=1}^{s} b_l = 1$. As the equation that we want to solve comes in the shape of an infinite series, it is necessary to study how we evaluate the function $dexp_{\Omega(t)}^{-1}$. For this, we need to use truncated series up to a certain order in such a way that the order of convergence of the underlying classical RK is preserved. Moreover, note that the truncated series do always belong to the Lie algebra, as they are given by linear combinations of Lie brackets of elements of the Lie algebra. If the classical RK is of order p and the truncated series of (3.7) is up to order j, such that $j \ge p - 2$, then the RKMK method is of order p (see [55, 56] and [23, Theorem 8.5, p. 124]). Again, this convergence order refers to the equation in the Lie group (3.5).

Let us now determine the RKMK method associated with the explicit Runge–Kutta whose Butcher's table is

that is a Runge-Kutta of order 4 (RK4). This implies that we need to truncate the series dexp⁻¹_{$\Omega(t)$} at j = 2:

$$\operatorname{dexp}_{\Omega}^{-1}(A) \approx A - \frac{1}{2}[\Omega, A] + \frac{1}{12}[\Omega, [\Omega, A]].$$
(3.13)

Then, the RKMK implementation for the given Butcher's table is

$$F_{1} = \operatorname{dexp}_{O_{n}}^{-1}(A(t_{k})),$$

$$F_{2} = \operatorname{dexp}_{\frac{1}{2}hF_{1}}^{-1}\left(A\left(t_{k} + \frac{1}{2}h\right)\right),$$

$$F_{3} = \operatorname{dexp}_{\frac{1}{2}hF_{2}}^{-1}\left(A\left(t_{k} + \frac{1}{2}h\right)\right),$$

$$F_{4} = \operatorname{dexp}_{hF_{3}}^{-1}(A(t_{k} + h)),$$

$$F_{4} = \operatorname{dexp}_{hF_{3}}^{-1}(A(t_{k} + h)),$$

$$G = \frac{h}{6}(F_{1} + 2F_{2} + 2F_{3} + F_{4}),$$

$$Y_{k+1} = \operatorname{exp}(\Theta)Y_{k},$$

$$(3.14)$$

where $dexp^{-1}$ is (3.13).

It is interesting to note that the method obtained in the previous section using the Magnus expansion (3.11) can be retrieved by a RKMK method associated with the following Butcher's table

$$\begin{array}{c|cccc}
0 & & \\
1/2 & 1/2 & \\
& 0 & 1 \\
\end{array}$$

Since it is an order 2 method, for the computation of dexp⁻¹ one can use dexp_{Ω}⁻¹(A) $\approx A$.

4 Numerical methods and automorphic Lie systems

So far, we have established in Procedure 2.2 how to construct an analytical solution of a Lie system N through a particular solution g(t) of an automorphic Lie system on a Lie group G that is based on the integration of the VG Lie algebra associated with the Lie system. More exactly, employing the Lie group action φ given in (2.1), we can use the particular solution g(t) to obtain the general solution of the Lie system on the manifold N. On the other hand, in Section 3 we have reviewed some methods in the literature providing a numerical approximation of the solution of (3.5) remaining in the Lie group G (which accounts for their most remarkable geometrical property).

Now, let us explain how we combine these two elements to construct our new numerical methods and solve (2.11). Let φ be the Lie group action constructed using (2.12) and consider the solution of the system (3.5) such that Y(0) = I. Numerically, we have shown that the solutions of (3.5) can be provided through the approximations of (3.8), say $\{\Omega_k\}_{k=0,\ldots,\mathcal{N}}$, and (3.9), as long as we stay close enough to the origin. As particular examples, we have picked the Magnus and RKMK methods in order to get $\{\Omega_k\}_{k=0,\ldots,\mathcal{N}}$ and, furthermore, the sequence $\{Y_k\}_{k=0,\ldots,\mathcal{N}}$. Next, we establish the scheme providing the numerical solution to automorphic Lie systems.

Definition 4.1 Let us consider a Lie system

$$\frac{dx}{dt} = \sum_{\alpha=1}^{r} b_{\alpha}(t) X_{\alpha}(x), \qquad t \in \mathbb{R}, \quad x \in N,$$
(4.1)

 $and \ let$

$$\frac{dY}{dt} = A(t)Y, \qquad A(t) = \sum_{\alpha=1}^{r} b_{\alpha}(t)M_{\alpha},$$

be its associated automorphic Lie system in matrix form close to the neutral element of G. We define the numerical solution to the Lie system, i.e., $\{x_k\}_{k=0,\ldots,N}$, via the algorithm given next.

Lie systems on Lie groups method

- 1: Initial data: $\mathcal{N}, h, A(t), Y_0 = I, \Omega_0 = \mathbf{0}.$ 2: Numerically solve $\frac{d\Omega}{dt} = \operatorname{dexp}_{\Omega}^{-1}A(t)$
- 3: Output $\{\Omega_k\}_{k=1,\ldots,\mathcal{N}}$
- 4: for k = 1, ..., N 1 do

$$Y_{k+1} = e^{\Omega_k} Y_k,$$

$$x_{k+1} = \varphi(Y_{k+1}, x_k),$$

5: end for 6: **Output:** $(x_1, x_2, ..., x_N)$.

At this point, we would like to highlight an interesting geometric feature of this method. On the one hand, the discretization is based on the numerical solution of the automorphic Lie group underlying the Lie system, which, itself, is founded upon the geometric structure of the latter. This numerical solution remains on G, i.e., $Y_k \in G$ for all k, due to the particular design of the Lie group methods (as long as h is small). Given this, our construction respects as well the geometrical structure of the Lie system, since, in principle, it evolves on a manifold N. We observe that the iteration

$$x_{k+1} = \varphi(Y_{k+1}, x_k)$$

leads to this preservation, since $x_{k+1} \in N$ as long as $Y_{k+1} \in G$ and $x_k \in N$ (we recall that $\varphi: G \times N \to N$). Note as well that the direct application of a one-step method (3.2) on a general Lie system (2.11) would destroy this structure, even if applied to an ambient Euclidean space.

For future reference, in regards of the Lie group methods (3.9), we shall refer to (3.11) as Magnus 2, to (3.12) as Magnus 4 and to (3.14) as, simply, RKMK (we recall that the last two methods are order 4 convergent).

5 Numerical integration on curved spaces

This section illustrate the effectiveness of our numerical scheme by applying it to a (κ_1, κ_2) parametric family of Lie systems on curved spaces (see [28] for details). Naturally, these curved spaces shall play the role of the manifold N where the Lie system evolves. Given that their intrinsic geometry is not trivial, they represent an optimal example of how our methods are better suited than others for the geometric preservation by our discrete solutions. After introducing a relevant class of Lie systems in curves spaces, Section 5.1 applies the 7 step method and the algorithm in Definition 4.1 to construct the geometry preserving numerical method.

For this, we start by considering a two-parametric family of 3D real Lie algebras, denoted by $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$, which depends on two real parameters, κ_1 and κ_2 . In the literature these Lie algebras are also known as CK Lie algebras [19, 21, 22, 30, 32, 64, 78] or quasisimple orthogonal algebras [31]. The Lie algebra $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$ admits a basis $\{P_1, P_2, J_{12}\}$ with structure constants

$$[J_{12}, P_1] = P_2, \qquad [J_{12}, P_2] = -\kappa_2 P_1, \qquad [P_1, P_2] = \kappa_1 J_{12}. \tag{5.1}$$

It is possible to rescale the basis of $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$ and reducing each parameter κ_a (a = 1, 2) to either +1, 0 or -1. The Lie algebra $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$ admits a representation as a matrix Lie algebra of 3×3 real matrices M satisfying [19]

$$M^T \mathbf{I}_{\kappa} + \mathbf{I}_{\kappa} M = 0, \qquad \mathbf{I}_{\kappa} = \operatorname{diag}(1, \kappa_1, \kappa_1 \kappa_2), \qquad \kappa = (\kappa_1, \kappa_2).$$
 (5.2)

If \mathbf{I}_{κ} is not degenerate, M is called an indefinite orthogonal Lie algebra $\mathfrak{so}(p,q)$, where p and q are the number of positive and negative eigenvalues of \mathbf{I}_{κ} . In particular, $\{P_1, P_2, J_{12}\}$ can be identified, respectively, with the matrices

$$P_1 = -\kappa_1 e_{01} + e_{10}, \quad P_2 = -\kappa_1 \kappa_2 e_{02} + e_{20}, \quad J_{12} = -\kappa_2 e_{12} + e_{21}, \tag{5.3}$$

where e_{ij} is the 3 × 3 matrix with a single non-zero entry 1 at row *i* and column *j* with i, j = 0, 1, 2.

The matrix exponential of the elements $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$ generate, by successive matrix multiplications of its elements, the referred to as CK Lie group $\mathrm{SO}_{\kappa_1,\kappa_2}(3)$. In particular, one has the one-parametric subgroups of the CK Lie group $\mathrm{SO}_{\kappa_1,\kappa_2}(3)$ of the form

$$\exp(\lambda_1 P_1) = \begin{pmatrix} C_{\kappa_1}(\lambda_1) & -\kappa_1 S_{\kappa_1}(\lambda_1) & 0\\ S_{\kappa_1}(\lambda_1) & C_{\kappa_1}(\lambda_1) & 0\\ 0 & 0 & 1 \end{pmatrix}, \ \exp(\lambda_2 P_2) = \begin{pmatrix} C_{\kappa_1 \kappa_2}(\lambda_2) & 0 & -\kappa_1 \kappa_2 S_{\kappa_1 \kappa_2}(\lambda_2)\\ 0 & 1 & 0\\ S_{\kappa_1 \kappa_2}(\lambda_2) & 0 & C_{\kappa_1 \kappa_2}(\lambda_2) \end{pmatrix}, \\ \exp(\lambda_3 J_{12}) = \begin{pmatrix} 1 & 0 & 0\\ 0 & C_{\kappa_2}(\lambda_3) & -\kappa_2 S_{\kappa_2}(\lambda_3)\\ 0 & S_{\kappa_2}(\lambda_3) & C_{\kappa_2}(\lambda_3) \end{pmatrix},$$
(5.4)

where the so-called κ -dependent cosine and sine functions [19, 30, 32] take the form

$$C_{\kappa}(\lambda) = \sum_{l=0}^{\infty} (-\kappa)^{l} \frac{\lambda^{2l}}{(2l)!} = \begin{cases} \cos\sqrt{\kappa}\,\lambda, & \kappa > 0, \\ 1, & \kappa = 0, \\ \operatorname{ch}\sqrt{-\kappa}\,\lambda, & \kappa < 0, \end{cases}$$
$$S_{\kappa}(\lambda) = \sum_{l=0}^{\infty} (-\kappa)^{l} \frac{\lambda^{2l+1}}{(2l+1)!} = \begin{cases} \frac{1}{\sqrt{\kappa}}\sin\sqrt{\kappa}\,\lambda, & \kappa > 0, \\ \lambda, & \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}}\operatorname{sh}\sqrt{-\kappa}\,\lambda, & \kappa < 0. \end{cases}$$

Similarly to standard trigonometry, the κ -tangent and the κ -versed sine (or versine) read

$$T_{\kappa}(\lambda) = \frac{S_{\kappa}(\lambda)}{C_{\kappa}(\lambda)}, \qquad V_{\kappa}(\lambda) = \frac{1}{\kappa} \left(1 - C_{\kappa}(\lambda)\right).$$
(5.5)

These κ -functions cover both the usual circular ($\kappa > 0$) and hyperbolic ($\kappa < 0$) trigonometric functions. Moreover, κ -functions reduce when $\kappa = 0$ to the parabolic functions $C_0(\lambda) = 1$, $S_0(\lambda) = T_0(\lambda) = \lambda$ and $V_0(\lambda) = \lambda^2/2$.

Some relations for the above κ -functions read

$$C_{\kappa}^{2}(\lambda) + \kappa S_{\kappa}^{2}(\lambda) = 1, \qquad C_{\kappa}(2\lambda) = C_{\kappa}^{2}(\lambda) - \kappa S_{\kappa}^{2}(\lambda), \qquad S_{\kappa}(2\lambda) = 2 S_{\kappa}(\lambda) C_{\kappa}(\lambda)$$

and their derivatives [30] are given by

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} C_{\kappa}(\lambda) = -\kappa S_{\kappa}(\lambda), \quad \frac{\mathrm{d}}{\mathrm{d}\lambda} S_{\kappa}(\lambda) = C_{\kappa}(\lambda), \quad \frac{\mathrm{d}}{\mathrm{d}\lambda} T_{\kappa}(\lambda) = \frac{1}{C_{\kappa}^{2}(\lambda)}, \quad \frac{\mathrm{d}}{\mathrm{d}\lambda} V_{\kappa}(\lambda) = S_{\kappa}(\lambda). \quad (5.6)$$

Define $H_0 = SO_{\kappa_2}(2)$ to be the Lie subgroup of $SO_{\kappa_1,\kappa_2}(3)$ obtained by matrix exponentiation of the Lie algebra $\mathfrak{h}_0 = \langle J_{12} \rangle$. The CK family of 2D homogeneous spaces is given by

$$\mathbf{S}^{2}_{[\kappa_{1}],\kappa_{2}} = \mathrm{SO}_{\kappa_{1},\kappa_{2}}(3)/\mathrm{SO}_{\kappa_{2}}(2).$$
(5.7)

The (possibly degenerate) metric defined by \mathbf{I}_{κ} in (5.2) on $T_e SO_{\kappa_1,\kappa_2}(3) \simeq \mathfrak{so}_{\kappa_1,\kappa_2}(3)$ can be extended to a right-invariant metric on the whole $SO_{\kappa_1,\kappa_2}(3)$ by right translation, and then projected onto $\mathbf{S}^2_{[\kappa_1],\kappa_2}$. Then, the CK family becomes a symmetric space relative to the obtained metric. Moreover, κ_1 becomes the constant (Gaussian) *curvature* of the space, while κ_2 determines the *signature* of the metric through diag $(+, \kappa_2)$.

The matrix realization induced by (5.4) leads to the identification of $SO_{\kappa_1,\kappa_2}(3)$ with the isometries of the bilinear form I_{κ} . More in detail,

$$g \in \mathrm{SO}_{\kappa_1,\kappa_2}(3) \Rightarrow g^T \mathbf{I}_{\kappa} g = \mathbf{I}_{\kappa_2}$$

which allows one to define a Lie group action of $SO_{\kappa_1,\kappa_2}(3)$ on \mathbb{R}^3 by isometries of \mathbf{I}_{κ} .

The subgroup $SO_{\kappa_2}(2) = \{\exp(\lambda J_{12}) : \lambda \in \mathbb{R}\}\)$ is the isotropy subgroup of the point O = (1, 0, 0), which is taken as the *origin* in the space $\mathbf{S}^2_{[\kappa_1],\kappa_2}$. Hence, $SO_{\kappa_1,\kappa_2}(3)$ becomes an isometry group of $\mathbf{S}^2_{[\kappa_1],\kappa_2}$.

The orbit of O is contained in the submanifold of \mathbb{R}^3 given by

$$\Sigma_{\kappa} = \{ v := (x_0, x_1, x_2) \in \mathbb{R}^3 : \mathbf{I}_{\kappa}(v, v) = x_0^2 + \kappa_1 x_1^2 + \kappa_1 \kappa_2 x_2^2 = 1 \}.$$
 (5.8)

This orbit can be identified with the space $\mathbf{S}^2_{[\kappa_1],\kappa_2}$. The coordinates $\{x_0, x_1, x_2\}$ on \mathbb{R}^3 that satisfy the constraint (5.8) on Σ_{κ} , are called *ambient*. In these variables, the metric on $\mathbf{S}^2_{[\kappa_1],\kappa_2}$ can be brought into the form

$$ds_{\kappa}^{2} = \frac{1}{\kappa_{1}} \left(dx_{0}^{2} + \kappa_{1} dx_{1}^{2} + \kappa_{1} \kappa_{2} dx_{2}^{2} \right) \Big|_{\Sigma_{\kappa}} = \frac{\kappa_{1} \left(x_{1} dx_{1} + \kappa_{2} x_{2} dx_{2} \right)^{2}}{1 - \kappa_{1} x_{1}^{2} - \kappa_{1} \kappa_{2} x_{2}^{2}} + dx_{1}^{2} + \kappa_{2} dx_{2}^{2}.$$
(5.9)

When $\kappa_1 = 0$, the manifold Σ_{κ} has two connected components with $x_0 \in \{-1, 1\}$, which ensures that ds_{κ}^2 is well-defined.

The ambient coordinates can be parametrized on Σ_{κ} through two intrinsic variables in different ways [28, 32]. Indeed, one may define the so called *geodesic parallel* $\{x, y\}$ or *geodesic polar* $\{r, \phi\}$ coordinates of a point $Q = (x_0, x_1, x_2)$ in $\mathbf{S}^2_{[\kappa_1],\kappa_2}$ that are defined via the action of the one-parametric subgroups (5.4) on O [32] given by

$$(x_0, x_1, x_2)^T = \exp(xP_1)\exp(yP_2)O^T = \exp(\phi J_{12})\exp(rP_1)O^T,$$

yielding

$$x_{0} = C_{\kappa_{1}}(x) C_{\kappa_{1}\kappa_{2}}(y) = C_{\kappa_{1}}(r), \qquad x_{1} = S_{\kappa_{1}}(x) C_{\kappa_{1}\kappa_{2}}(y) = S_{\kappa_{1}}(r) C_{\kappa_{2}}(\phi),$$
$$x_{2} = S_{\kappa_{1}\kappa_{2}}(y) = S_{\kappa_{1}}(r) S_{\kappa_{2}}(\phi).$$

Using these relations in the metric (5.9) and applying (5.6), one obtains

$$\mathrm{d}s_{\boldsymbol{\kappa}}^2 = \mathrm{C}_{\kappa_1\kappa_2}^2(y)\mathrm{d}x^2 + \kappa_2\mathrm{d}y^2 = \mathrm{d}r^2 + \kappa_2\mathrm{S}_{\kappa_1}^2(r)\mathrm{d}\phi^2.$$

Different values of (κ_1, κ_2) lead to different spaces. The case $\kappa_2 > 0$ gives rise to Riemannian spaces. Any case with $\kappa_2 > 0$ and $\kappa_1 < 0$ leads to a two-sheeted hyperboloids. Its upper sheet is called \mathbf{H}^2 , namely the part with $x_0 \ge 1$, the Lobachevsky space. Meanwhile, $\kappa_1 = 0$ describes two Euclidean planes $x_0 = \pm 1$. We will call the one with $x_0 = +1$ Euclidean space \mathbf{E}^2 . The instances with $\kappa_2 < 0$ define pseudo-Riemannian spaces or Lorentzian spacetimes. In this case, a positive Gaussian curvature κ_1 induces a (1+1)D anti-de Sitter spacetime \mathbf{AdS}^{1+1} ; if $\kappa_1 < 0$, we find the (1+1)D de Sitter spacetime \mathbf{M}^{1+1} ; or the flat case with $\kappa_1 = 0$, aka the (1+1)D Minkowskian spacetime \mathbf{M}^{1+1} . If $\kappa_2 = 0$ ($c = \infty$), we encounter Semi-Riemannian spaces or Newtonian spacetimes, in which the metric (5.9) is degenerate and its kernel is an integrable foliation of $\mathbf{S}^2_{[\kappa_1],0}$ that is invariant under the action of the CK group $\mathrm{SO}_{\kappa_1,0}(3)$ on $\mathbf{S}^2_{[\kappa_1],0}$. There appears a well-defined subsidiary metric $\mathrm{d}s'^2 = \mathrm{d}s^2_{\kappa}/\kappa_2$ restricted to each leaf, which in the coordinates (x, y) read [32]

$$ds^2 = dx^2$$
, $ds'^2 = dy^2$ on $x = constant$.

For $\kappa_1 > 0$ we find the (1+1)D oscillating Newton-Hook (NH) spacetime \mathbf{NH}^{1+1}_+ , and for $\kappa_1 < 0$ we obtain the (1+1)D expanding NH spacetime \mathbf{NH}^{1+1}_- . The flat space with $\kappa_1 = 0$ is just the Galilean \mathbf{G}^{1+1} .

5.1 A class of Lie systems on curved spaces

Our procedure consists in defining a Lie system X_{κ} possessing a Vessiot-Guldberg Lie algebra V_{κ} consisting of infinitesimal symmetries of the metric of the CK space $\mathbf{S}^2_{[\kappa_1],\kappa_2}$. The fundamental vector fields of the Lie group action of $\mathrm{SO}_{\kappa}(3)$ on \mathbb{R}^3 by isometries of \mathbf{I}_{κ} are Lie symmetries of ds^2_{κ} . Since the action is linear, the fundamental vector fields can be obtained straightforwardly from the 3D matrix representation (5.3). In ambient coordinates (x_0, x_1, x_2) , they read [32],

$$P_1 = \kappa_1 x_1 \frac{\partial}{\partial x_0} - x_0 \frac{\partial}{\partial x_1}, \qquad P_2 = \kappa_1 \kappa_2 x_2 \frac{\partial}{\partial x_0} - x_0 \frac{\partial}{\partial x_2}, \qquad J_{12} = \kappa_2 x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}. \tag{5.10}$$

•. Step 1. In this case, we are going to use a VG Lie algebra to determine the system we want to analyze. Indeed, the most general Lie system related to V_{κ} reads

$$X = b_1(t)P_1 + b_2(t)P_2 + b_{12}(t)J_{12}, (5.11)$$

where the vector fields P_1 , P_2 , J_{12} correspond with those in (5.10), and the associated VG Lie algebra has structure constants (5.1). According to the theory of Lie systems [47], the integral curves of the time-dependent vector field (5.11) are described by the system of ordinary differential equations

$$\begin{cases} \frac{dx_0}{dt} = b_1(t)\kappa_1 x_1 + b_2(t)\kappa_1 \kappa_2 x_2, \\ \frac{dx_1}{dt} = -b_1(t)x_0 + b_{12}(t)\kappa_2 x_2, \\ \frac{dx_2}{dt} = -b_2(t)x_0 - b_{12}(t)x_1 \end{cases}$$
(5.12)

and

$$\frac{d(x_0^2 + \kappa_1 x_1^2 + \kappa_1 \kappa_2 x_2^2)}{dt} = 2\left(x_0 \frac{dx_0}{dt} + \kappa_1 x_1 \frac{dx_1}{dt} + \kappa_1 \kappa_2 x_2 \frac{dx_2}{dt}\right) = 0$$

which yields that $I(x_0, x_1, x_2) = x_0^2 + \kappa_1 x_1^2 + \kappa_1 \kappa_2 x_2^2$ is a constant of motion the Lie system (5.12). This invariant will be of utmost importance to show the efficiency of our method when preserving geometric invariants under numerical integration. Note that determining (5.11) for the system (5.12) can be considered to be the final aim of step 1 of our 7-step method.

• Step 2. It is the moment to consider the following set of matrices

$$M_{P_1} = -\begin{pmatrix} 0 & -\kappa_1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}, \qquad M_{P_2} = -\begin{pmatrix} 0 & 0 & -\kappa_1 \kappa_2\\ 0 & 0 & 0\\ 1 & 0 & 0 \end{pmatrix}, \qquad M_{J_{12}} = -\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -\kappa_2\\ 0 & 1 & 0 \end{pmatrix}, \quad (5.13)$$

that have the opposite commutation relations than the vector fields in (5.1), i.e.,

$$[M_{P_1}, M_{P_2}] = -\kappa_1 M_{J_{12}}, \qquad [M_{J_{12}}, M_{P_1}] = M_{P_2}, \qquad [M_{J_{12}}, M_{P_2}] = -\kappa_2 M_{P_1}.$$

These are the matrices that we will use as a basis of the Lie algebra of the Lie group to be consider.

• Step 3. Let us integrate the vector fields P_1, P_2 , and J_{12} . To find the flow associated with P_1 , let us solve the system of differential equations $\{dx_0/dt = \kappa_1 x_1, dx_1/dt = -x_0, dx_2/dt = 0\}$ with initial conditions $(x_0(0), x_1(0), x_2(0))$. The solution is

$$\begin{cases} x_0(t) = x_0(0) \operatorname{C}_{\kappa_1}(t) + \kappa_1 x_1(0) \operatorname{S}_{\kappa_1}(t), \\ x_1(t) = x_1(0) \operatorname{C}_{\kappa_1}(t) - x_0(0) \operatorname{S}_{\kappa_1}(t), \\ x_2(t) = x_2(0), \end{cases}$$

and the flow $\Phi_{P_1} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ associated with P_1 can be expressed in the following manner

$$\Phi_{P_1}(t, (x_0(0), x_1(0), x_2(0))) = (x_0, x_1, x_2), \text{ with } \begin{cases} x_0 = x_0(0) \operatorname{C}_{\kappa_1}(t) + \kappa_1 x_1(0) \operatorname{S}_{\kappa_1}(t), \\ x_1 = x_1(0) \operatorname{C}_{\kappa_1}(t) - x_0(0) \operatorname{S}_{\kappa_1}(t), \\ x_2 = x_2(0). \end{cases}$$
(5.14)

Similarly, we calculate the flows for P_2 and J_{12} to obtain

$$\Phi_{P_2}(t, (x_0(0), x_1(0), x_2(0))) = (x_0, x_1, x_2), \text{ with } \begin{cases} x_0 = x_0(0) \operatorname{C}_{\kappa_1 \kappa_2}(t) + \kappa_1 \kappa_2 x_2(0) \operatorname{S}_{\kappa_1 \kappa_2}(t), \\ x_1 = x_1(0), \\ x_2 = x_2(0) \operatorname{C}_{\kappa_1 \kappa_2}(t) - x_0(0) \operatorname{S}_{\kappa_1 \kappa_2}(t), \end{cases}$$
(5.15)

$$\Phi_{J_{12}}(t, (x_0(0), x_1(0), x_2(0))) = (x_0, x_1, x_2), \text{ with } \begin{cases} x_0 = x_0(0), \\ x_1 = x_1(0) \operatorname{C}_{\kappa_2}(t) + \kappa_2 x_2(0) \operatorname{S}_{\kappa_2}(t), \\ x_2 = x_2(0) \operatorname{C}_{\kappa_2}(t) - x_1(0) \operatorname{S}_{\kappa_2}(t). \end{cases}$$
(5.16)

• Step 4. We now obtain the Lie group action associated with V_{κ} . Let us briefly review how to obtain the Lie group action related to V_{κ} . Given an element of the Lie algebra, the canonical coordinates of the second kind permit us to obtain a point in the group (near the origin of coordinates and the neutral element of the algebra, respectively). That is, we have a correspondence between a point in the algebra $M \in \mathfrak{g}$ determined by the coordinates $(\lambda_1, \lambda_2, \lambda_3)$, i.e.

$$M = \lambda_1 M_{P_1} + \lambda_2 M_{P_2} + \lambda_3 M_{J_{12}} \in \mathfrak{g}$$

and the point $g \in G$ determined by the same coordinates

$$g = \exp(\lambda_1 M_{P_1}) \exp(\lambda_2 M_{P_2}) \exp(\lambda_3 M_{J_{12}}) \in G.$$

If we calculate the exponential of $M_{P_1}, M_{P_2}, M_{J_{12}}$, we obtain (5.4).

By multiplying the exponential of these matrices, we obtain the canonical coordinates of the second kind

$$\exp(\lambda_1 M_{P_1}) \exp(\lambda_2 M_{P_2}) \exp(\lambda_3 M_{J_{12}}) = \begin{pmatrix} C_{\kappa_1}(\lambda_1) C_{\kappa_1 \kappa_2}(\lambda_2) & * & * \\ S_{\kappa_1}(\lambda_1) C_{\kappa_1 \kappa_2}(\lambda_2) & * & * \\ S_{\kappa_1 \kappa_2}(\lambda_2) & C_{\kappa_1 \kappa_2}(\lambda_2) S_{\kappa_2}(\lambda_3) & C_{\kappa_1 \kappa_2}(\lambda_2) C_{\kappa_2}(\lambda_3) \end{pmatrix}, \quad (5.17)$$

where we have omitted some matrix entries that are not further needed. In this way, given a point on the group, we can work out the parameters $\{\lambda_1, \lambda_2, \lambda_3\}$. First, we take the entries g_{11} y g_{21} and define $g = g_{21}/g_{11}$. So, λ_1 can be expressed as

$$\lambda_{1} = \begin{cases} \frac{\arctan g\sqrt{\kappa_{1}}}{\sqrt{\kappa_{1}}}, & \text{if } \kappa_{1} > 0, \\ g, & \text{if } \kappa_{1} = 0, \\ \frac{1}{2\sqrt{-\kappa_{1}}} \log \left(\frac{1+g\sqrt{-\kappa_{1}}}{1-g\sqrt{-\kappa_{1}}}\right), & \text{if } \kappa_{1} < 0. \end{cases}$$
(5.18)

With the term g_{13} , we can obtain λ_2 as

$$\lambda_{2} = \begin{cases} \frac{\arccos g_{13}\sqrt{\kappa_{1}\kappa_{2}}}{\sqrt{\kappa_{1}\kappa_{2}}}, & \text{if } \kappa_{1}\kappa_{2} > 0, \\ g_{13}, & \text{if } \kappa_{1}\kappa_{2} = 0, \\ \frac{\log \left(g_{13}\sqrt{-\kappa_{1}\kappa_{2}} + \sqrt{-g_{13}^{2}\kappa_{1}\kappa_{2} + 1}\right)}{\sqrt{-\kappa_{1}\kappa_{2}}}, & \text{if } \kappa_{1}\kappa_{2} < 0. \end{cases}$$
(5.19)

And lastly, analogously, defining $g = g_{32}/g_{33}$ we can obtain λ_3 as

$$\lambda_{3} = \begin{cases} \frac{\arctan g\sqrt{\kappa_{2}}}{\sqrt{\kappa_{2}}}, & \text{if } \kappa_{2} > 0, \\ g, & \text{if } \kappa_{2} = 0, \\ \frac{1}{2\sqrt{-\kappa_{2}}} \log \left(\frac{1+g\sqrt{-\kappa_{2}}}{1-g\sqrt{-\kappa_{2}}}\right), & \text{if } \kappa_{2} < 0. \end{cases}$$
(5.20)

With all of this, by definition, the Lie group action $\varphi : G \times \mathbb{R}^3 \to \mathbb{R}^3$ in a point $g \in G$ and $\mathbf{x}(0) = (x_0(0), x_1(0), x_2(0)) \in \mathbb{R}^3$ is computed as

$$\varphi(g, \boldsymbol{x}(0)) = \varphi(\exp(\lambda_1 M_{P_1}) \exp(\lambda_2 M_{P_2}) \exp(\lambda_3 M_{J_{12}}), \boldsymbol{x}(0)) = \varphi(\exp(\lambda_1 M_{P_1}), \varphi(\exp(\lambda_2 M_{P_2}), \varphi(\exp(\lambda_3 M_{J_{12}}), \boldsymbol{x}(0)))) = \Phi_{P_1}(\lambda_1, \Phi_{P_2}(\lambda_2, \Phi_{J_{12}}(\lambda_3, \boldsymbol{x}(0)))).$$

At this point it is interesting to observe that the three vector fields P_1 , P_2 y J_{12} share the same invariant with the system (5.12). This is,

$$I(x_0, x_1, x_2) = I\left(\Phi_i(t, (x_0(0), x_1(0), x_2(0)))\right) \quad \forall t \in \mathbb{R}, \quad \forall (x_0(0), x_1(0), x_2(0)) \in \mathbb{R}^3, \quad (5.21)$$

for each of the flows Φ_i associated with $\{P_1, P_2, J_{12}\}$. As we have just depicted, the action from the Lie group to the manifold is constructed as the composition of three flows. Moreover, it is apparent that, in spite we are using the "Euclidean-like" notation \mathbb{R}^3 for our manifold N in this example, it is obvious that it carries a nontrivial geometric structure, as we have already shown. Therefore, our numerical scheme preserves the invariant.

• Steps 5, 6 and 7. Given all these elements, we can implement our numerical scheme. Instead of solving (5.12), we will solve the following differential equation on the Lie group

$$\frac{dY}{dt} = A(t)Y(t), \qquad Y(0) = I,$$
(5.22)

with $A(t) = b_1(t)M_{P_1} + b_2(t)M_{P_2} + b_{12}(t)M_{J_{12}}$.

Let us assume that we are in the k-th interaction. This means that we know \boldsymbol{x}_k and Y_k . To calculate the next point, we apply the numerical scheme (5.22) with the initial condition $Y(0) = Y_k$, obtaining Y_{k+1} , and being able to compute $\boldsymbol{x}_{k+1} = \varphi(Y_{k+1}, \boldsymbol{x}_k)$.

5.2 Numerical integration of a particular example

Let us apply our method to (5.12) with the following coefficients

$$b_1(t) = t^2$$
, $b_2(t) = \sin t$, $b_{12}(t) = \log(t+1)$,

and constants with values ($\kappa_1 = 0.8, \kappa_2 = -0.5$), and initial condition $x_0 = (1, 1, 1)$, for the interval [3, 4] and step size h = 0.1.

With these parameters our scheme provides the following solution, which is shown overlapped with another solution calculated with a very small step. We also show the solution obtained with a classical 4th-order Runge-Kutta applied directly to the system, i.e., (5.12).



As we observe in the bottom right plot, the geometric quantity I(x, y, z) is exactly preserved, which is not the case with a classical 4th-order RK scheme applied directly to the original Lie system. Once again, this geometric preservation is achieved by means of the specific design of the integrator, which is the main point of our work. Finally, we show a convergence plot (in logarithmic scale) of our scheme for the x component. We employ the definition of the global error E_N given in (3.4), where it is enough to consider an Euclidean norm, since we are in \mathbb{R}^3 (with nontrivial curvature). We observe convergence, as in the case of the other two components.



6 Initial conditions, long term behavior, and numerical methods

To observe more critically when and how our methods are more appropriate than other ones, let us consider the nonautonomous system of differential equation on \mathbb{R}^2 of the form

$$\frac{dx}{dt} = b_1(t)y + b_2(t)(x^2 + y^2 - 1)x, \qquad \frac{dy}{dt} = -b_1(t)x + b_2(t)(x^2 + y^2 - 1)y.$$
(6.1)

where $b_1(t), b_2(t)$ are arbitrary t-dependent functions. What we are going to show is that our methods, which are adapted to the geometric features of (6.1), more accurately reflect its long-term behavior and the dependence on initial conditions of particular solutions.

System (6.1) is a Lie system since it is associated with the *t*-dependent vector field $X = b_1(t)X_1 + b_2(t)X_2$, where

$$X_1 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}, \qquad X_2 = (x^2 + y^2 - 1) \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)$$

satisfy that $[X_1, X_2] = 0$. Then, X_1, X_2 span an abelian two-dimensional VG Lie algebra. Additionally, X_1, X_2 span a generalized distribution \mathcal{D} on \mathbb{R}^2 given by

$$\mathcal{D}_{(x,y):x^2+y^2=1} = \left\langle y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right\rangle, \qquad \mathcal{D}_{(x,y):x^2+y^2 \neq \{1,0\}} = T_{(x,y)} \mathbb{R}^2, \qquad \mathcal{D}_{(0,0)} = 0.$$

Recall that every finite-dimensional Lie algebra of vector fields gives rise to an involutive distribution. This generalized distribution is also integrable and its strata are given by the circle $x^2 + y^2 = 1$, the point (0,0), and the regions $0 < x^2 + y^2 < 1$ and $x^2 + y^2 > 1$.

The integration of the vector fields X_1, X_2 gives rise to a Lie group action that respects the leaves of \mathcal{D} . The matrix Lie algebra with a basis closing opposite constants of structure than X_1, X_2 is spanned by the basis

$$M_1 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right], \qquad M_2 = \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right].$$

This is the matrix Lie algebra of diagonal 2×2 matrices. Then, its matrix Lie group is spanned by the product of an arbitrary number of exponentials of elements in $\langle M_1, M_2 \rangle$, which gives the Lie group of invertible matrices

$$G_2 = \left\{ \left[\begin{array}{cc} a & 0 \\ 0 & b \end{array} \right] : a, b > 0 \right\}.$$

The integration of the vector fields X_1, X_2 leads to the flows

$$\Phi_1: (t, x, y) \in \mathbb{R} \times \mathbb{R}^2 \mapsto (x \cos t + y \sin t, -x \sin t + y \cos t) \in \mathbb{R}^2,$$

$$\Phi_2: (t, x, y) \in \mathbb{R} \times \mathbb{R}^2 \mapsto \frac{(x, y)}{\sqrt{(x^2 + y^2 - (x^2 + y^2 - 1)e^{2t}}} \in \mathbb{R}^2.$$

And the associated Lie group action reads $\varphi: G_2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is such that (0,0) is left invariant by G_2 , while

$$\varphi\left(\left[\begin{array}{cc} a & 0\\ 0 & b \end{array}\right], (x, y)\right) = \frac{(x \cos\ln a + y \sin\ln a, -x \sin\ln a + y \cos\ln a)}{\sqrt{x^2 + y^2 - (x^2 + y^2 - 1)b^2)}},\tag{6.2}$$

where the canonical coordinates of the second kind read $\lambda_1 = \ln a$ and $\lambda_2 = \ln b$. This action leaves invariant the circle $x^2 + y^2 = 1$ and the point x = y = 0. It is important to remark that the Lie group action is local and, given (x, y), the action is only defined for the values of a, b that do not make the denominator in (6.2) to be negative or zero. In fact, when b is such that the denominator tends to zero, the image of the Lie group action tends to infinite. Moreover, the form of X_1, X_2 and the associated distribution \mathcal{D} shows that the solutions of (6.1) must remain in one the strata of \mathcal{D} . This will be respected by our methods, but the Runge-Kutta and other numeric methods do need to do so: the circular solution does not need to be respected and solutions may change from one orbit of φ to another, which violates one of the fundamental features of (6.1). This clearly shows that generic numerical methods will not respect geometric features of differential equations, which can be extremely important, for instance, when the particular solutions depend strongly on the initial conditions.

The automorphic Lie system associated with (6.1) reads

$$\frac{d}{dt} \begin{bmatrix} a & 0\\ 0 & b \end{bmatrix} = \begin{bmatrix} b_1(t) & 0\\ 0 & b_2(t) \end{bmatrix} \begin{bmatrix} a & 0\\ 0 & b \end{bmatrix}, \quad a, b > 0.$$

and the induced Lie system on the abelian Lie algebra \mathbb{R}^2 becomes

$$\frac{d}{dt} \begin{bmatrix} \lambda_1 & 0\\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} b_1(t) & 0\\ 0 & b_2(t) \end{bmatrix} \begin{bmatrix} \lambda_1 & 0\\ 0 & \lambda_2 \end{bmatrix}, \qquad \lambda_1, \lambda_2 \in \mathbb{R}.$$

Then, the numerical solution of the automorphic Lie system will give, via the associated Lie group action φ , an approximate solution of X that will respect the stratification.

To verify the usefulness of our method, let us analyse the initial problem with the initial conditions (0, 1) and the *t*-dependent coefficients $b_1(t) = (1 + t^2)$ and $b_2(t) = e^t$. Note that a Runge-Kutta method will show that a solution with initial condition in $x^2 + y^2 = 1$ will finally escape from that circle and make it to move far away from the real solution.



7 Conclusions and outlook

Given the wide range of spaces, and geometries, where the mathematical and physical dynamical systems evolve, it is always worth to take care of its intrinsic properties when passing to the "discrete" side in order to obtain an approximate solution. As extensively showed in the literature, this results on some computational and dynamical benefits. This is the spirit of geometric integration, and the one we uphold in this article, where we take advantage of the geometric structure of Lie systems in order to propose a 7-step method to analytically solve them (mainly, the possibility to reduce such systems to equivalent ones on a Lie group), plus a geometric numeric integrator. We have proven its geometric properties with a wide class of Lie systems evolving on curved spaces.

We have also studied examples whose particular solutions may strongly depend on the initial conditions and our methods are more appropriate to study the short- and long-term behavior than other numerical non-geometrical methods.

As for future work, it shall be worth wondering about the numerical features of the integrator, such as consistency and convergence, besides finding new examples which may be of interest in mathematics, physics or other applied sciences. Moreover, it is interesting that linear systems of differential equations have been used to study differential equations close to equilibrium points. Such systems are Lie systems. It will be interesting to use approximations up to second order of differential equations close to equilibrium points, which may potentially lead to Lie systems that reflect more appropriately the properties of the systems under study and the use of the methods of this work (cf. [47] for Lie systems on the plane). There is no classification of Lie algebras of smooth vector fields on the plane whose elements vanish at a certain point (cf. [20]). This is a very interesting topic from the point of view of stability analysis and numerical methods for Lie systems to be developed in a further work.

Acknowledgements

J. de Lucas acknowledges a Simons–CRM professorship funded by the Simons Foundation and the Centre de Recherches Mathématiques (CRM) of the Université de Montréal. J. de Lucas would like to thank for the hospitality shown by the members and staff of the CRM during his stay. We also would like to thank the anonymous referees for their suggestions, which allowed us to significantly improve the quality, relevance, and clarity of this work.

Data availability

The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

References

- P.A. Absil, R. Mahony, and R. Sepulchre, Riemannian Geometry of Grassmann Manifolds with a View on Algorithmic Computation, Acta Appl. Math. 80:199-220, 2004.
- [2] I.D. Ado, The representation of Lie algebras by matrices, Uspekhi Matematicheskikh Nauk 2:159-173, 1947.
- [3] R.M. Angelo and W.F. Wresziński, Two-level quantum dynamics, integrability and unitary NOT gates, Phys. Rev. A 72:034105, 2005.
- [4] A. Ballesteros, J.F. Cariñena, F.J. Herranz, J. de Lucas, C. Sardón, From constants of motion to superposition rules for Lie-Hamilton systems, J. Phys. A 46:285203, 2013.
- [5] S. Blanes, F. Casas, J.A. Oteo and J. Ros, The Magnus expansion and some of its application, Phys. Rep. 470:5-6, 2006.
- S. Blanes, F. Casas and J. Ros, Improved high order integrators based on the Magnus expansion, BIT Nume. Math. 40:434-450, 2000.
- [7] S. Blanes and P.C. Moan, Practical Symplectic Partitioned Runge-Kutta and Runge-Kutta-Nyström Methods, J. Computational Appl. Math. 142:313-330, 2002.

- [8] A. Blasco, F.J. Herranz, J. de Lucas and C. Sardón, *Lie-Hamilton systems on the plane:* Applications and superposition rules, J. Phys. A **48**:345202, 2015.
- [9] J.F. Cariñena, J. Grabowski and G. Marmo, Lie-Scheffers Systems: a Geometric Approach, Napoli Series in Physics and Astrophysics, Bibliopolis, Naples, 2000.
- [10] J.F. Cariñena, J. Grabowski and G. Marmo, Superposition rules, Lie theorem and partial differential equations, Rep. Math. Phys. 60:237-258, 2007.
- [11] J.F. Cariñena, J. Grabowski and A. Ramos, Reduction of t-dependent systems admitting a superposition principle, Acta Appl. Math. 66:67-87, 2001.
- [12] J.F. Cariñena and J. de Lucas, Applications of Lie systems in dissipative Milne-Pinney equations, Int. J. Geom. Meth. Modern Phys. 6:683-699, 2009.
- [13] J.F. Cariñena and J. de Lucas, Lie Systems: Theory, Generalisations, and Applications, Diss. Mathematicae 479, Warsaw, 2011.
- [14] J.F. Cariñena, J. de Lucas and C. Sardón, A new Lie systems approach to second-order Riccati equations, Int. J. Geom. Meth. Modern Phys. 9:1260007, 2011.
- [15] J.F. Cariñena and A. Ramos, Integrability of the Riccati equation from a group theoretical viewpoint, Int. J. Mod. Phys. A 14:1935-1951, 1999.
- [16] J. Cortés and S. Martínez, Non-holonomic integrators, Nonlinearity 14:1365-1392, 2001.
- [17] M.L. Curtis, *Matrix groups*, Springer, New York, 1984.
- [18] S. Domínguez, P. Campoy, J.M. Sebastián and A. Jiménez, Control en el Espacio de Estado, Pearson, Educación, 2006.
- [19] A. Ballesteros, F.J. Herranz, M.A. del Olmo, and M. Santander, Quantum structure of the motion groups of the two-dimensional Cayley-Klein geometries, J. Phys. A 26:5801-5823, 1993.
- [20] A. González, N. Kamran, and P.J. Olver, Lie algebras of vector fields on the plane, Proc. London Math. Soc. 64:339-368, 1992.
- [21] N.A. Gromov, Contractions and analytical continuations of the classical groups. Unified approach, Komi Scienfic Center, Syktyvkar, 1992 (in Russian).
- [22] N.A. Gromov and V.I. Man'ko, The Jordan-Schwinger representations of Cayley-Klein groups. I. The orthogonal groups, J. Math. Phys. 31:1047-1053, 1990.
- [23] E. Hairer, C. Lubich and G. Wanner, Geometric Numerical Integration, Springer-Verlag, Berlin-Heidelberg, 2006.
- [24] E. Hairer, S.P. Nørsett and G. Wanner, Solving Ordinary Differential Equations I: Nonstiff Problems, Springer-Verlag, Berlin, 1993.
- [25] B.C. Hall, Lie Groups, Lie Algebras, and Representations: An Elementary Introduction, Springer, Cham, 2015.
- [26] J. Harnad, P. Winternitz and R.L. Anderson, Superposition principles for matrix Riccati equations, J. Math. Phys. 24:1062, 1983.
- [27] R. Hartshorne, Foundations of Projective Geometry, W.A. Benjamin, Inc., New York, 1967.
- [28] F.J. Herranz, J. de Lucas and M. Tobolski, Lie-Hamilton systems on curved spaces: A geometrical approach, J. Phys. A 50:495201, 2017.

- [29] F.J. Herranz, M. de Montigny, M.A. del Olmo and M. Santander, Cayley-Klein algebras as graded contractions of so(N + 1), J. Phys. A 27:2515-2526, 1994.
- [30] F.J. Herranz, R. Ortega and M. Santander, Trigonometry of spacetimes: a new self-dual approach to a curvature/signature (in)dependent trigonometry, J. Phys. A 33:4525-4551, 2000.
- [31] F.J. Herranz and M. Santander, Casimir invariants for the complete family of quasisimple orthogonal algebras, J. Phys. A 30:5411-5426, 1997.
- [32] F.J. Herranz and M. Santander, Conformal symmetries of spacetimes, J. Phys. A 35:6601-6618, 2002.
- [33] V. Hussin, J. Beckers, L. Gagnon and P. Winternitz, Superposition formulas for nonlinear superequations, J. Math. Phys. 31:2528-2534, 1990.
- [34] A. Iserles and S.P. Nørsett, On the solution of linear differential equations in Lie groups, Phil. Trans Royal Soc. A 357:983-1020, 1999.
- [35] A. Iserles, S.P. Nørsett and A.F. Rasmussen, t-symmetry and high-order Magnus methods, Technical Report 1998/NA06, DAMTP, University of Cambridge, 1998.
- [36] E. Isaacson and H.B. Keller, Analysis of Numerical Methods, John Wiley & Sons, New York-London-Sydney, 1966.
- [37] A. Iserles, H. Munthe-Kaas, S. Nørsett and A. Zanna, *Lie-group methods*, Acta Num. 9:215-365, 2000.
- [38] V. Kučera, A Review of the Matrix Riccati Equation, Kybernetika 9:42-61, 1973.
- [39] L. Königsberger, Uber die einer beliebigen differentialgleichung erster Ordnung angehörigen selb-ständigen Transcendenten, Acta Math.3:1-48, 1883.
- [40] J. Lange and J. de Lucas, Geometric models for Lie-Hamilton systems on \mathbb{R}^2 , Mathematics **2019**:7, 1053.
- [41] J.A. Lázaro-Camí and J.P. Ortega, Superposition rules and stochastic Lie-Scheffers systems, Ann. Inst. H. Poincaré Probab. Stat. 45:910-931, 2009.
- [42] J.M. Lee, Introduction to Smooth Manifolds, Graduate Texts in Mathematics 218, Springer-Verlag, New York, 2003.
- [43] E.E. Levi, *Sulla struttura dei gruppi finiti e continui*, Atti della Reale Accademia delle Scienze di Torino, 1905.
- [44] S. Lie, Sophus Lie's 1880 Transformation group paper, (Translated by M. Ackerman and comments by R. Hermann), Lie Groups: History, frontiers and applications I, Math. Sci. Press, Mass, 1975.
- [45] S. Lie and G. Scheffers, Vorlesungen über continuierliche Gruppen mit geometrischen und anderen Anwendungen, Teubner, Leipzig, 1893.
- [46] J. de Lucas and A.M. Grundland, A Lie systems approach to the Riccati hierarchy and partial differential equations, J. Differential Equations 263:299-337, 2017.
- [47] J. de Lucas and C. Sardón, A Guide to Lie Systems with Compatible Geometric Structures, World Scientific, Singapore, 2020.
- [48] W. Magnus, On the exponential solution of differential equations for a linear operator, Comm. Pure Appl. Math. 7:649-673, 1954.

- [49] R. Mahony, T. Hamel, and J.M. Pflimlin, Complementary Kalman filtering on matrix Lie groups, IEEE Trans. Aut. Control 53:1203-1218, 2008.
- [50] M. Mallick, G.S. Chirikjian, and R. Mahony, A new numerical integration algorithm for rigid body dynamics on Lie groups, IEEE Transactions on Robotics 30:799-812, 2014.
- [51] J.C. Marrero, D. Martín de Diego and E. Martínez, Discrete Lagrangian and Hamiltonian mechanics on Lie groupoids, Nonlinearity 19:1313-1348, 2006.
- [52] J.E. Marsden and M. West, Discrete mechanics and variational integrators, Acta Num. 10:357-514, 2001.
- [53] R. McLachlan and G.R.W. Quispel, *Splitting methods*, Acta Num. 11:341-434, 2002.
- [54] R.I. McLachlan, Higher Order Symplectic Integrators, Nonlinearity 8:513-525, 1995.
- [55] H. Munthe-Kaas, Runge-Kutta methods on Lie groups, BIT Numerical Mathematics 38:92-111, 1998.
- [56] H. Munthe-Kaas, High order Runge-Kutta methods on manifolds, J. Appl. Num. Maths. 29:115-127, 1999.
- [57] H.Z. Munthe-Kaas and B. Owren, On the construction of numerical methods preserving adiabatic invariants and related quantities, Math. Comp. 65:1353-1373, 1996.
- [58] R.M. Murray and S. Arimoto, Numerical Solution of the Nonlinear Optimal Control Problem for Rigid Body Motion, in Proceedings of the 29th IEEE Conference on Decision and Control 1990, IEEE, Honolulu, 2000, pp. 2038-2043.
- [59] A. Odzijewicz and A.M. Grundland, The Superposition Principle for the Lie Type first-order PDEs, Rep. Math. Phys. 45:293-306, 2000.
- [60] R.S. Palais, A Global Formulation of the Lie Theory of Transformation Groups, Mem. Amer. Math. Soc. 22, Amer. Math. Soc., Providence, 1957.
- [61] A. Pandey, A. Ghose-Choudhury and P. Guha, *Chiellini integrability and quadratically damped oscillators*, Int. J. Non-Linear Mech. **92**:153-159, 2017.
- [62] A.V. Penskoi and P. Winternitz, Discrete matrix Riccati equations with super- position formulas, J. Math. Anal. Appl. 294:533–547, 2004.
- [63] G. Pietrzkowski, Explicit solutions of the a₁-type Lie-Scheffers system and a general Riccati equation, J. Dyn. Control Systems 18:551-571, 2012.
- [64] B.A. Rozenfel'd, A History of Non-Euclidean Geometry, Springer, New York, 1988.
- [65] A. Quarteroni, R. Sacco and F. Saleri, Numerical Mathematics, Springer-Verlag, New York, 2007.
- [66] D.W. Rand and P. Winternitz, Nonlinear superposition principles: a new numerical method for solving matrix Riccati equations, Comput. Phys. Comm. 33:305-328, 1984.
- [67] W.T. Reid, Riccati Differential Equations, Academic, New York, 1972.
- [68] J.M. Sanz-Serna, Symplectic integrators for Hamiltonian problems: an overview, Acta Num. 1:243-286, 1992.
- [69] C. Sardón, Lie systems, Lie symmetries and reciprocal transformations, PhD Thesis, Universidad de Salamanca, Salamanca, 2015.

- [70] S. Sastry and R.M. Murray, Nonlinear feedback control for the stabilization of the attitude of a rigid body, IEEE Trans. Aut. Control 32:412-417, 1987.
- [71] D.H. Sattinger and O.L. Weaver, Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics Appl. Math. Sci. 61, Springer-Verlag, Berlin, 1986.
- [72] J. Sola, J. Kuffner, and S.K. Agrawal, A Quaternion-based Interpolation Technique for Smooth and Visually Appealing Motion Planning on Lie Groups in: Proceedings of the 2018 IEEE International Conference on Robotics and Automation (ICRA), Brisbane, 2018, pp. 2181-2187.
- [73] E.D. Sontag, Mathematical Control Theory: Deterministic Finite Dimensional Systems, Springer-Verlag, New York, 1998.
- [74] V.S. Varadarajan, Lie Groups, Lie Algebras, and Their Representations, Graduate Texts in Mathematics 102, Springer, Los Angeles, 1985.
- [75] L. Verlet, Computer "Experiments" on Classical Fluids. I. Thermodynamical Properties of Lennard-Jones Molecules, Phys. Rev. 159:98-103 (1967).
- [76] P. Winternitz, Lie groups and solutions of nonlinear differential equations in Nonlinear phenomena(Oaxtepec, 1982), Lect. Notes Phys. 189, Springer, Berlin, 1983, pp. 263-331.
- [77] P. Winternitz, Nonlinear action of Lie groups and superposition rules for nonlinear differential equations, Phys. A 114:105–113 (1982).
- [78] I.M. Yaglom, A simple non-Euclidean geometry and its physical basis, Springer, New York, 1979.
- [79] H. Yoshida, Construction of Higher Order Symplectic Integrators, Phys. Lett. A 150:262-268, 1990.
- [80] A. Zanna, Collocation and relaxed collocation for the Fer and Magnus expansions, J. Numer. Anal. 36:1145-1182, 1999.