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Martingale Solutions in Stochastic Fluid–Structure Interaction

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Abstract
We consider a viscous incompressible fluid interacting with a linearly elastic shell of Koiter type which is located at some part of the boundary. Recently models with stochastic perturbation in the shell equation have been proposed in the literature but only analysed in simplified cases. We investigate the full model with transport noise, where (a part of) the boundary of the fluid domain is randomly moving in time. We prove the existence of a weak martingale solution to the underlying system.

Keywords Incompressible Navier–Stokes equation · Transport noise · Fluid–structure interaction

Mathematics Subject Classification 76D05 · 76D09 · 74F10 · 60H15

1 Introduction

The mathematical analysis of systems of partial differential equations arising from fluid–structure interaction has seen a vast progress in the last two decades. This is motivated by a variety of applications, for instance, in biomechanics Bodnár et al. (2014), hydro-dynamics Chakrabarti (2002), aero-elasticity Dowell (2015) and hemo-mechanics Formaggia et al. (2001).

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1.1 Deterministic Models

We are interested in the case where a viscous incompressible fluid interacts with an elastic structure located at a part of the boundary of the fluid's domain $\mathcal{O} \subset \mathbb{R}^3$ denoted by Γ. The structure reacts to the forces imposed by the fluid at the boundary. Assuming that this deformation only acts in the normal direction and denoting by $\mathbf{n}$ the outer unit normal at the reference domain, $\mathcal{O}$ is deformed to the domain $\mathcal{O}_{\eta(t)}$ defined through its boundary

$$\partial\mathcal{O}_{\eta(t)} := \{\mathbf{y} + \eta(t,\mathbf{y})\mathbf{n}(\mathbf{y}) : \mathbf{y} \in \Gamma\}. \tag{1.1}$$

Here $\eta : (t,\mathbf{y}) : I \times \Gamma \mapsto \eta(t,\mathbf{y}) \in \mathbb{R}$ describes the deformation of the structure and $I := (0,T)$, for some $T > 0$ denotes a time interval. For technical simplification we will suppose that Γ is the whole boundary and identify it with the two-dimensional torus (the precise geometric set-up is presented in Sect. 2.2).

As a prototype, let us consider the following problem where the equation for the shell can be seen as a linearised version of Koiter's model (neglecting lower order terms for simplicity and setting all positive physical constants to 1). In the unknowns

$$\begin{aligned}\mathbf{u} &: (t,\mathbf{x}) : I \times \mathcal{O}_\eta \mapsto \mathbf{u}(t,\mathbf{x}) \in \mathbb{R}^3,\\ \pi &: (t,\mathbf{x}) : I \times \mathcal{O}_\eta \mapsto \pi(t,\mathbf{x}) \in \mathbb{R},\end{aligned}$$

accounting for the fluid's velocity field and pressure, respectively (defined on a moving space-time cylinder), it reads as (for simplicity we neglect volume forces in the fluid equations)

$$\operatorname{div}_{\mathbf{x}}\mathbf{u} = 0, \tag{1.2}$$

$$\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbf{u} = \Delta_{\mathbf{x}}\mathbf{u} - \nabla_{\mathbf{x}}\pi, \tag{1.3}$$

$$\partial_t^2\eta + \Delta_{\mathbf{y}}^2\eta = -\mathbf{n}^\top(\mathbb{T}(\mathbf{u},\pi)\mathbf{n}_\eta)\circ\boldsymbol{\varphi}_\eta|\det(\nabla_{\mathbf{y}}\boldsymbol{\varphi}_\eta)|. \tag{1.4}$$

The system is complemented by the kinematic boundary condition

$$\mathbf{u}\circ\boldsymbol{\varphi}_\eta = \mathbf{n}\partial_t\eta \qquad \text{on } I \times \Gamma \tag{1.5}$$

at the fluid–structure interface as well as initial conditions for (1.3)–(1.4) and periodic boundary conditions for (1.4). Here $\mathbb{T}(\mathbf{u},\pi) = (\nabla_{\mathbf{x}}\mathbf{u} + \nabla_{\mathbf{x}}\mathbf{u}^\top) - \pi\mathbb{I}_{3\times 3}$ is the stress tensor of the fluid. The vectors $\mathbf{n}$ and $\mathbf{n}_\eta$ denote the normal vectors on $\mathcal{O}$ and $\mathcal{O}_\eta$, respectively. The function $\boldsymbol{\varphi}_\eta$ gives the coordinate transform from $\Gamma \to \partial\mathcal{O}_\eta$. The existence of a weak solution to (1.2)–(1.5) has been shown in Lengeler and Ružička (2014) (see also Muha and Canić (2013) for the case of a cylindrical shell model). It satisfies the energy balance

$$\begin{aligned}&\tfrac{1}{2}\int_{\mathcal{O}_\eta}|\mathbf{u}(t)|^2\,\mathrm{d}\mathbf{x}+\int_0^t\int_{\mathcal{O}_\eta}|\nabla_{\mathbf{x}}\mathbf{u}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}s+\tfrac{1}{2}\int_\Gamma|\partial_t\eta(t)|^2\,\mathrm{d}\mathbf{y}+\tfrac{1}{2}\int_\Gamma|\Delta_{\mathbf{y}}\eta(t)|^2\,\mathrm{d}\mathbf{y}\\&\qquad\le\tfrac{1}{2}\int_{\mathcal{O}_{\eta(0)}}|\mathbf{u}(0)|^2\,\mathrm{d}\mathbf{x}+\tfrac{1}{2}\int_\Gamma|\partial_t\eta(0)|^2\,\mathrm{d}\mathbf{y}+\tfrac{1}{2}\int_\Gamma|\Delta_{\mathbf{y}}\eta(0)|^2\,\mathrm{d}\mathbf{y}\end{aligned}\tag{1.6}$$

for a.a. $t\in I$, from which one can easily deduce the function spaces in which the weak solution lives. The easier case of an elastic plate (where the reference geometry is flat) has been studied before in Grandmont (2008). The main advancement in Lengeler and Ružička (2014) is a new compactness method which eventually allows to establish compactness of the velocity field. On account of the deformed space-time cylinder on which the problem is posed, it is impossible to apply the standard Aubin-Lions compactness lemma in order to pass to the limit in the convective term of approximate solutions. Interestingly, this issue is ultimately linked to the divergence-free constraint (1.2). Without it, the compactness can simply be localised thus completely removing the difficulties posed by the moving boundary, see Breit and Schwarzacher (2018) where the compressible Navier–Stokes equations are studied. In Muha and Schwarzacher (2022) (where even the fully nonlinear Koiter model is considered), the compactness argument from Lengeler and Ružička (2014) has been replaced by an abstract compactness criterion which is more in the spirit of the classical Aubin-Lions result and thus allows for wider applications. Let us finally remark that all the results just mentioned hold under the assumption that there is no self-intersection of the structure (which can always be avoided if $\|\eta\|_{L^\infty_{\mathbf{y}}}$ is not too large).

1.2 Stochastic Models

It was recently suggested in Kuan and Čanić (2022) to consider a stochastic perturbation in (1.4) to account for random effects in real-life problems and uncertainty in the data. A first step towards a well-posedness theory for such stochastic fluid–structure interaction models is done in Kuan and Čanić (2023), where the 2D time-dependent Stokes equations are linearly coupled to a structure described by a stochastic 1D wave equation. Although this is only a simplified model (and the boundary is not moving in time) the analysis is already quite advanced. As already indicated above, the geometry breaks down if η causes a self-intersection of the domain. In the simplified case, where the reference domain is a box and the deformation only occurs in the vertical direction, this happens exactly when the value of $-\eta$ coincides with the height of the box. If η has a Gaussian distribution as in Kuan and Čanić (2023), this can always happen (though maybe only with a low probability) no matter how short the time horizon is. This issue may be circumvented by studying the local-in-time well-posedness of the problem which is done in Tawri and Čanić (2023). The authors of Tawri and Čanić (2023) study the interaction of an elastic plate (the reference geometry is flat) with the 2D Navier–Stokes equation. The existence time is a random variable about which the only available information is $\mathbb{P}$-a.s. positivity.

In this paper, we aim for the natural next step by considering the full model (1.2)–(1.5) globally in time, where (1.4) is subject to some Gaussian noise.[1] We take a different perspective to Kuan and Čanić (2022, 2023); Tawri and Čanić (2023) and do not consider stochasticity entering as an external force but as an intrinsic property of the system. Thus we consider transport noise in the shell Eq. (see (1.9) below). It has the very appealing feature of being energy conservative. If the initial data are deterministic (or simply bounded in probability), we have a pathwise control over the energy and the restrictions on the time interval are the same as in previous deterministic papers such as Grandmont (2008), Lengeler and Ružička (2014), Muha and Schwarzacher (2022). Transport noise has a clear physical meaning in fluid mechanical transport processes, see Cotter et al. (2017) and Holm (2015; 2020) as well as Chen et al. (2023), Crisan et al. (2019), Flandoli and Pappalettera (2021). Depending on the particular structure, it can be conservative with respect to several important quantities such as energy, enstrophy and circulation. Note that this is excluded in the case of an Itô noise. Also, it has been observed that transport noise has regularising effects on certain ill-posed PDEs, see Flandoli et al. (2010) and Flandoli and Luo (2021). Nevertheless, the role of transport noise for elastic materials must be further explored. Understanding the role of noise in the shell Eq. (1.9) is motivated by Kuan and Čanić (2022, 2023); Tawri and Čanić (2023) and we are only at the beginning stages.

Our goal is to construct on random space-time cylinders $\Omega \times I \times \mathcal{O}_\eta$ and $\Omega \times I \times \Gamma$, a global weak solution triple (note that the pressure does not enter the weak formulation)

$$
\begin{aligned}
\mathbf{u} &: (\omega, t, \mathbf{x}) : \Omega \times I \times \mathcal{O}_\eta \mapsto \mathbf{u}(\omega, t, \mathbf{x}) \in \mathbb{R}^3, \\
\pi &: (\omega, t, \mathbf{x}) : \Omega \times I \times \mathcal{O}_\eta \mapsto \pi(\omega, t, \mathbf{x}) \in \mathbb{R}, \\
\eta &: (\omega, t, \mathbf{y}) : \Omega \times I \times \Gamma \mapsto \eta(\omega, t, \mathbf{y}) \in \mathbb{R},
\end{aligned}
$$

representing the fluid's velocity, the fluid's pressure and the structure displacement of the coupled fluid–structure system given by

$$
\mathrm{div}_{\mathbf{x}} \mathbf{u} = 0, \tag{1.7}
$$

$$
\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} = \Delta_{\mathbf{x}} \mathbf{u} - \nabla_{\mathbf{x}} \pi, \tag{1.8}
$$

$$
\mathrm{d}\partial_t \eta + (\Delta_{\mathbf{y}}^2 \eta + g_\eta)\,\mathrm{d}t + ((\varkappa \cdot \nabla_{\mathbf{y}})\partial_t \eta) \circ \mathrm{d}B_t = 0, \tag{1.9}
$$

with $g_\eta = -\mathbf{n}^\top (\mathbb{T}(\mathbf{u}, \pi)\mathbf{n}_\eta) \circ \boldsymbol{\varphi}_\eta |\det(\nabla_{\mathbf{y}} \boldsymbol{\varphi}_\eta)|$. Here, Ω is a sample space of a filtered probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t \geq 0}, \mathbb{P})$ with associated expectation $\mathbb{E}(\cdot)$. Equation (1.9) contains a Stratonovich differential of a real-valued Brownian motion (B_t) and $\varkappa$ is a given solenoidal (incompressible) vector field (i.e. $\mathrm{div}_{\mathbf{y}} \varkappa = 0$)[2] in $\mathbb{R}^2$. The initial conditions for (1.7)–(1.9) are

$$
\eta(\cdot, 0, \cdot) = \eta_0(\cdot, \cdot), \quad \partial_t \eta(\cdot, 0, \cdot) = \eta_1(\cdot, \cdot) \qquad \text{in } \Omega \times \Gamma, \tag{1.10}
$$

$$
\mathbf{u}(\cdot, 0, \cdot) = \mathbf{u}_0(\cdot, \cdot) \qquad \text{in } \Omega \times \mathcal{O}_{\eta_0}. \tag{1.11}
$$

[1] One can very well add a suitable stochastic term in (1.3). This is covered by our analysis as long as it is energy conservative.

[2] One can allow a (possibly infinite) sum of stochastic transport terms in (1.9) without affecting the analysis.

With regards to boundary conditions, we supplement the shell Eq. (1.9) with periodic boundary conditions and impose

$$\mathbf{u} \circ \boldsymbol{\varphi}_\eta = \mathbf{n}\partial_t \eta \qquad \text{on } \Omega \times I \times \Gamma \tag{1.12}$$

at the fluid–structure interface. Note that (1.7)–(1.12) is a free-boundary problem where the boundary of the fluid domain is moving randomly in time.

In the 3D case, regularity and uniqueness of solutions to (1.7)–(1.12) is certainly out of reach (at least globally in time) so that one can only hope for the existence of weak martingale solutions. Here weak refers to the analytical concept of distributional derivatives, whereas martingale solutions refers to solutions which are weak in the probabilistic sense (they do not exist on a given stochastic basis; the latter becomes an integral part of the solution). Such a concept is very common in stochastic evolutionary problems (even on the level of ordinary stochastic differential equations), whenever uniqueness of the underlying system is unavailable.

1.3 The Weak Formulation

A first rather philosophical question is to come up with an analytically weak formulation for the problem. In fluid–structure interaction problems, the space of test functions typically depends on the structure displacement η (the test function for the fluid sub-problem and the structure sub-problem must match at the interface as in (1.5) and (1.12)). On the other hand, in stochastic PDEs it is common to work with spatial test functions. This is also our preferred point of view as an η-dependence of the test functions in our case means that they depend on time and are also random. The idea now is to start with a pair of test functions $(\phi, \boldsymbol{\phi})$ on the reference domain (that is $\phi : \Gamma \to \mathbb{R}$ and $\boldsymbol{\phi} : \mathcal{O} \to \mathbb{R}^3$) with the correct boundary condition and transform $\boldsymbol{\phi}$ to the moving domain. An obvious choice, therefore, is the Hanzawa transform $\boldsymbol{\Psi}_\eta : \mathcal{O} \to \mathcal{O}_\eta$ which we formally introduce in Sect. 2.2. Unfortunately, it has the disadvantage of destroying the divergence-free constraint on the test functions. At the level of weak solutions this cannot be remedied through the recovery of the pressure function as the latter only exists as a distribution on the solenoidal test functions. Thus, we use instead the Piola transform

$$\mathscr{J}_\eta \mathbf{v} = \big(\nabla_{\mathbf{x}} \boldsymbol{\Psi}_\eta (\det \nabla_{\mathbf{x}} \boldsymbol{\Psi}_\eta)^{-1} \mathbf{v}\big) \circ \boldsymbol{\Psi}_\eta^{-1}, \tag{1.13}$$

which preserves the solenoidability of a function $\mathcal{O} \to \mathbb{R}^3$. Using now $(\iota_\eta \phi, \mathscr{J}_{\eta(t)} \boldsymbol{\phi})$ with $\iota_\eta = (\det \nabla_{\mathbf{x}} \boldsymbol{\Psi}_\eta)^{-1}$ we obtain the following weak formulation

$$\begin{aligned} \mathrm{d}\bigg(\int_{\mathcal{O}_\eta} \mathbf{u} \cdot \mathscr{J}_{\eta(t)} \boldsymbol{\phi} \, \mathrm{d}\mathbf{x} + \int_\Gamma \partial_t \eta \, \iota_\eta \phi \, \mathrm{d}\mathbf{y} \bigg) &= \int_\Gamma \big(\partial_t \eta \, \partial_t (\iota_\eta \phi) - \Delta_{\mathbf{y}} (\iota_\eta \phi) \, \Delta_{\mathbf{y}} \eta \big) \, \mathrm{d}\mathbf{y} \, \mathrm{d}t \\ &+ \int_{\mathcal{O}_\eta} \Big(\mathbf{u} \cdot \partial_t (\mathscr{J}_{\eta(t)} \boldsymbol{\phi}) + ((\mathbf{u} \cdot \nabla_{\mathbf{x}}) \mathscr{J}_{\eta(t)} \boldsymbol{\phi}) \cdot \mathbf{u} - \nabla_{\mathbf{x}} \mathbf{u} : \nabla_{\mathbf{x}} \mathscr{J}_{\eta(t)} \boldsymbol{\phi} \Big) \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\ &- \int_\Gamma \partial_t \eta ((\varkappa \cdot \nabla_{\mathbf{y}}) \iota_\eta \phi) \, \mathrm{d}\mathbf{y} \circ \mathrm{d}B_t \end{aligned} \tag{1.14}$$

for all test functions $(\phi, \boldsymbol{\phi})$ (which clearly depend only on space), see Sect. 2.4 for the derivation. One easily shows that a dense set of pairs of test functions $(\phi, \boldsymbol{\phi})$ with the correct boundary condition leads to a dense set of pairs of test functions $(\iota_\eta \phi, \mathcal{J}_{\eta(t)}\boldsymbol{\phi})$ on the moving domain with the right boundary condition, see (Lengeler and Ružička 2014, page 237). Thus our weak formulation is consistent with the strong formulation. However, the Piola transform behaves like $\nabla_{\mathbf{x}}\boldsymbol{\Psi}_\eta$ and inherits the regularity of $\nabla_{\mathbf{y}}\eta$ so that we require more regularity on η. Since the embedding $W^{2,2}(\Gamma) \hookrightarrow W^{1,\infty}(\Gamma)$ fails in two dimensions, $\nabla_{\mathbf{x}}\boldsymbol{\Psi}_\eta$ is not bounded. Thus the information from (1.6) is not sufficient to give a meaning to all the terms in (1.14). Hence we need additional regularity. A crucial point in our approach is, therefore, to establish an estimate for the $L^2(I; W^{2+s,2}(\Gamma))$-norm of η for all $s \in (0, 1/2)$. One can easily check by using Hölder's inequality and Sobolev's embedding that this information, together with (1.6), is sufficient to define all integrals appearing in (1.14). Different to (1.6) this estimate is not independent of the transport noise and hence only holds in expectation. As it turns out, the regularity of the terms arising from the transport noise have just enough regularity to close the estimate. Details can be found in Sect. 5.1. Concluding this discussion, the additional fractional differentiability of the shell displacement must be included in the definition of a solution, see Definition 2.

1.4 Plan

This work straddles different fields of mathematics including fluid mechanics, partial differential equations, differential geometry and stochastic analysis. In order to make this work as self-contained as possible, we collect in Sect. 2 useful results in the different fields of mathematics that are essential in establishing our result. We begin by giving a rigorous interpretation of the Stratonovich integral in (1.9) after which we introduce the geometric setup for the fluid–structure system (1.7)–(1.9). We also present the functional analytic framework (function spaces on moving domains) and present some key results necessary for our analysis (extension operators). Finally, we make precise, the notion of a solution that we are interested in (Definition 2) and state our main result (Theorem 5).

The proof of our main result can be summarised into three main steps. In Sect. 3, we consider an extension of the fluid–structure system that incorporates 'artificial' regularising terms in the shell Eq. (1.9). We then construct a solution to the linearised version of this extended system using a Galerkin approximation and stochastic compactness tools. We then move to Sect. 4 where we use a fixed-point argument to remove the linearisation performed in the previous section and obtain a solution to the fully nonlinear system (with the extra regularising terms in the shell equation). To complete the proof of the main result, we pass to the limit in the regularisation parameter in Sect. 5 to finally obtain a solution for (1.7)–(1.12). In each stage we apply a refined stochastic compactness method which is based on Jakubowski's extension of the Skorokhod representation theorem Jakubowski (1998). In our case, it is crucial to re-interpret the compactness lemma from Muha and Schwarzacher (2022) in the context of tightness of probability measures (see Sects. 3.2 and 4.1).

Remark 1 (The 2D case) One might wonder whether it is possible to show the global-in-time existence of strong pathwise solutions to (1.7)–(1.12) (solutions existing on a given stochastic basis). In the 2D case, the existence of global-in-time strong solutions to (1.2)–(1.5) has been shown in Breit (2022) with additional dissipation (see also Grandmont and Hillairet (2016) for previous results for elastic plates) and in Schwarzacher and Su (2023) for the case of plates (but without dissipation). In all cases, the approach is heavily based on taking temporal derivatives of $\mathbf{u}$ and $\partial_t\eta$, which is not possible for (1.7)–(1.12). Hence, it is unclear whether one should even expect such a result here.

2 Mathematical Framework and the Main Result

2.1 Stratonovich Integrals

Let $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis with a complete, right-continuous filtration and let (B_t) represent a real-valued Brownian motion relative to $(\mathfrak{F}_t)$. We consider a smooth solenoidal vector field $\varkappa : \Gamma \to \mathbb{R}^2$. If $\xi \in L^2(\Omega; C(\overline{I}; W^{1,2}(\Gamma)))$ is $(\mathfrak{F}_t)$-adapted, the stochastic integral

$$\int_0^t \varkappa \cdot \nabla_{\mathbf{y}}\xi \; \mathrm{d}B_s$$

is well defined in the sense of Itô with values in $L^2(\Gamma)$. If we only have $\xi \in L^2(\Omega; C(\overline{I}; L^2(\Gamma)))$ one can use the identity $(\varkappa \cdot \nabla_{\mathbf{y}})\xi = \mathrm{div}_{\mathbf{y}}(\varkappa\xi)$ and define the stochastic integral

$$\int_0^t \mathrm{div}_{\mathbf{y}}(\xi\varkappa) \; \mathrm{d}B_s$$

with values in $W^{-1,2}(\Gamma)$. We define the Stratonovich integrals in (1.9) by means of the Itô-Stratonovich correction, that is

$$\begin{aligned}\int_0^t \int_\Gamma \xi\varkappa \cdot \nabla_{\mathbf{y}}\phi \,\mathrm{d}\mathbf{y} \circ \mathrm{d}B_s = &\int_0^t \int_\Gamma \xi\varkappa \cdot \nabla_{\mathbf{y}}\phi \,\mathrm{d}\mathbf{y}\; \mathrm{d}B_s \\ &+ \frac{1}{2}\Big\langle\Big\langle \int_\Gamma \xi\varkappa \cdot \nabla_{\mathbf{y}}\phi \,\mathrm{d}\mathbf{y}, B_t\Big\rangle\Big\rangle_t\end{aligned} \tag{2.1}$$

for $\phi \in W^{1,2}(\Gamma)$. Here $\langle\langle\cdot,\cdot\rangle\rangle_t$ denotes the cross variation. We compute now the cross variations by means of (1.9). If $\xi = \partial_t\eta$, where η solves (1.9), we have for all $\phi \in W^{2,2}(\Gamma)$ and $t \in I$,

$$\int_\Gamma \partial_t\eta\varkappa \cdot \nabla_{\mathbf{y}}\phi \,\mathrm{d}\mathbf{y} = \int_\Gamma \partial_t\eta(0)\varkappa \cdot \nabla_{\mathbf{y}}\phi \,\mathrm{d}\mathbf{y} - \int_0^t \int_\Gamma \Delta_{\mathbf{y}}\eta\Delta_{\mathbf{y}}((\varkappa \cdot \nabla_{\mathbf{y}})\phi) \,\mathrm{d}\mathbf{y}\,\mathrm{d}s$$

$$
\begin{aligned}
&- \int_0^t \int_\Gamma g_\eta((\varkappa \cdot \nabla_{\mathbf{y}})\phi) \, \mathrm{d}\mathbf{y} \, \mathrm{d}s \\
&+ \frac{1}{2}\left\langle\!\!\left\langle \int_0^t \int_\Gamma \partial_t \eta \, ((\varkappa \cdot \nabla_{\mathbf{y}})(\varkappa \cdot \nabla_{\mathbf{y}})\phi) \, \mathrm{d}\mathbf{y}, B_t \right\rangle\!\!\right\rangle_t \\
&+ \int_0^t \int_\Gamma \partial_t \eta \, ((\varkappa \cdot \nabla_{\mathbf{y}})(\varkappa \cdot \nabla_{\mathbf{y}})\phi) \, \mathrm{d}\mathbf{y} \, \mathrm{d}B_s.
\end{aligned}
$$

Only the last term on the right contributes to the cross-variation when inserted in (2.1). Plugging the previous considerations together, we obtain

$$
\int_0^t (\varkappa \cdot \nabla_{\mathbf{y}})\partial_t \eta \, \circ \, \mathrm{d}B_s = \int_0^t (\varkappa \cdot \nabla_{\mathbf{y}})\partial_t \eta \, \mathrm{d}B_s - \frac{1}{2}\int_0^t (\varkappa \cdot \nabla_{\mathbf{y}})(\varkappa \cdot \nabla_{\mathbf{y}})\partial_t \eta \, \mathrm{d}s.
$$

to be understood in $W^{-1,2}(\Gamma)$ or $W^{-2,2}(\Gamma)$, respectively, depending on the regularity of $\partial_t \eta$.

2.2 Geometric Setup

The spatial domain $\mathcal{O}$ is assumed to be an open bounded subset of $\mathbb{R}^3$ with smooth boundary $\partial\mathcal{O}$ and an outer unit normal $\mathbf{n}$. We assume that $\partial\mathcal{O}$ can be parametrised by an injective mapping $\boldsymbol{\varphi} \in C^k(\Gamma; \mathbb{R}^3)$ for some sufficiently large $k \in \mathbb{N}$, where Γ is the two-dimensional torus. We suppose for all points $\mathbf{y} = (y_1, y_2) \in \Gamma$ that the pair of vectors $\partial_i \boldsymbol{\varphi}(\mathbf{y})$, $i = 1, 2$, is linearly independent. For a point $\mathbf{x}$ in the neighbourhood of $\partial\mathcal{O}$, we define the functions $\mathbf{y}$ and s by

$$
\mathbf{y}(\mathbf{x}) = \arg\min_{\mathbf{y} \in \Gamma} |\mathbf{x} - \boldsymbol{\varphi}(\mathbf{y})|, \quad s(\mathbf{x}) = (\mathbf{x} - \mathbf{p}(\mathbf{x})) \cdot \mathbf{n}(\mathbf{y}(\mathbf{x})),
$$

where we used the projection $\mathbf{p}(\mathbf{x}) = \boldsymbol{\varphi}(\mathbf{y}(\mathbf{x}))$. We define $L > 0$ to be the largest number such that s, $\mathbf{y}$ and $\mathbf{p}$ are well defined on S_L, where

$$
S_L = \{\mathbf{x} \in \mathbb{R}^n : \operatorname{dist}(\mathbf{x}, \partial\mathcal{O}) < L\}. \tag{2.2}
$$

Due to the smoothness of $\partial\mathcal{O}$ for L small enough we have $|s(\mathbf{x})| = \min_{\mathbf{y} \in \Gamma} |\mathbf{x} - \boldsymbol{\varphi}(\mathbf{y})|$ for all $\mathbf{x} \in S_L$. This implies that $S_L = \{s\mathbf{n}(\mathbf{y}) + \mathbf{y} : (s, \mathbf{y}) \in (-L, L) \times \Gamma\}$. For a given function $\eta : I \times \Gamma \to \mathbb{R}$, we parametrise the deformed boundary by

$$
\boldsymbol{\varphi}_\eta(t, \mathbf{y}) = \boldsymbol{\varphi}(\mathbf{y}) + \eta(t, \mathbf{y})\mathbf{n}(\mathbf{y}), \quad \mathbf{y} \in \Gamma, \; t \in I. \tag{2.3}
$$

With an abuse of notation, we define the deformed space-time cylinder as $I \times \mathcal{O}_\eta = \bigcup_{t \in I}\{t\} \times \mathcal{O}_{\eta(t)} \subset \mathbb{R}^4$. The corresponding function spaces for variable domains are defined as follows.

Definition 1 (Function spaces) For $I = (0, T)$, $T > 0$, and $\eta \in C(\overline{I} \times \omega)$ with $\|\eta\|_{L^\infty(I\times\Gamma)} < L$, we define for $1 \le p, r \le \infty$

$$L^p(I; L^r(\mathcal{O}_\eta)) := \big\{v \in L^1(I \times \mathcal{O}_\eta) : \; v(t, \cdot) \in L^r(\mathcal{O}_{\eta(t)}) \text{ for a.e. } t, \\ \|v(t,\cdot)\|_{L^r(\mathcal{O}_{\eta(t)})} \in L^p(I)\big\},$$

$$L^p(I; W^{1,r}(\mathcal{O}_\eta)) := \big\{v \in L^p(I; L^r(\mathcal{O}_\eta)) : \; \nabla_{\mathbf{x}} v \in L^p(I; L^r(\mathcal{O}_\eta))\big\}.$$

To establish a relationship between the fixed domain and the time-dependent domain, we introduce the Hanzawa transform $\boldsymbol{\Psi}_\eta : \mathcal{O} \to \mathcal{O}_\eta$ defined by

$$\boldsymbol{\Psi}_\eta(\mathbf{x}) = \begin{cases} \mathbf{p}(\mathbf{x}) + \big(s(\mathbf{x}) + \eta(\mathbf{y}(\mathbf{x}))\phi(s(\mathbf{x}))\big)\mathbf{n}(\mathbf{y}(\mathbf{x})) & \text{if } \operatorname{dist}(\mathbf{x}, \partial\mathcal{O}) < L, \\ \mathbf{x} & \text{elsewhere.} \end{cases} \tag{2.4}$$

for any $\eta : \omega \to (-L, L)$. Here $\phi \in C^\infty(\mathbb{R})$ is such that $\phi \equiv 0$ in a neighbourhood of $-L$ and $\phi \equiv 1$ in a neighbourhood of 0. The other variables $\mathbf{p}$, s and $\mathbf{n}$ are as defined earlier in this section. A straightforward verification shows that the inverse of $\boldsymbol{\Psi}_{\eta(t)}$ is $\boldsymbol{\Psi}_{-\eta(t)}$.

In order to obtain a weak formulation for the fluid–structure system, we also introduce the Piola transform

$$\mathcal{J}_\zeta \mathbf{v} = \big(\nabla_{\mathbf{x}}\boldsymbol{\Psi}_\zeta(\det\nabla_{\mathbf{x}}\boldsymbol{\Psi}_\zeta)^{-1}\mathbf{v}\big) \circ \boldsymbol{\Psi}_\zeta^{-1} \tag{2.5}$$

of a vector field $\mathbf{v} : \mathcal{O} \to \mathbb{R}^3$ with respect to a mapping $\zeta : \Gamma \to \mathbb{R}$. The Piola transform is invertible with inverse

$$\mathcal{J}_\zeta^{-1}\mathbf{v} = \big((\nabla_{\mathbf{x}}\boldsymbol{\Psi}_\zeta)^{-1}(\det\nabla_{\mathbf{x}}\boldsymbol{\Psi}_\zeta)\mathbf{v}\big) \circ \boldsymbol{\Psi}_\zeta. \tag{2.6}$$

It preserves vanishing boundary values as well as the divergence-free property of a function. In order to compensate for the additional factor $(\det\nabla_{\mathbf{x}}\boldsymbol{\Psi}_\zeta)^{-1}$ in the trace of $\mathcal{J}_\zeta\mathbf{v}$, we define the mapping

$$\iota_\zeta\phi := (\det\nabla_{\mathbf{x}}\boldsymbol{\Psi}_\zeta \circ \boldsymbol{\varphi}_\zeta)^{-1}\phi$$

for a function $\phi : \Gamma \to \mathbb{R}$. If $\boldsymbol{\phi} \circ \boldsymbol{\varphi} = \phi$ on Γ it follows that $(\mathcal{J}_\eta\boldsymbol{\phi}) \circ \boldsymbol{\varphi}_\eta = \iota_\eta\phi$ on Γ. Thus a pair of test functions $(\phi, \boldsymbol{\phi})$ with the correct boundary condition leads to a pair of test functions $(\iota_\eta\phi, \mathcal{J}_{\eta(t)}\boldsymbol{\phi})$ on the moving domain with the right boundary condition. Also, a dense set of test functions on the reference domain leads to a dense set of test functions on the moving domain; see (Lengeler and Ružička 2014, page 237).

We finish this section by recalling the following Aubin-Lions type lemma which is shown in (Muha and Schwarzacher 2022, Theorem 5.1. & Remark 5.2.) and slightly reformulated for our purposes.

Theorem 1 *Let X, Z be two Banach spaces, such that $X' \subset Z'$. Assume that $f_n : I \to X$ and $g_n : I \to X'$, such that $g_n \in L^\infty(I; Z')$ uniformly. Moreover assume the following:*

(a) *The* boundedness*: for some $s \in [1, \infty]$ we have that (f_n) is bounded in $L^s(X)$ and (g_n) in $L^{s'}(X')$.*
(b) *The* approximability-condition *is satisfied: For every $\kappa \in (0, 1]$ there exists a $f_{n,\kappa} \in L^s(I; X) \cap L^1(I; Z)$, such that for every $\epsilon \in (0, 1)$ there exists some $\kappa_\epsilon \in (0, 1)$ (depending only on ϵ) such that*

$$\|f_n - f_{n,\kappa}\|_{L^s(I;X)} \le \epsilon \textit{ for all } \kappa \in (0, \kappa_\epsilon]$$

and for every $\kappa \in (0, 1]$ there is some $C(\kappa)$ such that

$$\|f_{n,\kappa}\|_{L^1(I;Z)}\, dt \le C(\kappa).$$

(c) *The* equi-continuity *of g_n: We require that there exists some $\alpha \in (0, 1]$, functions A_n with $A_n \in L^1(I)$ uniformly, such that for every $\kappa > 0$ that there exist some $C(\kappa) > 0$ and some $n_\kappa \in \mathbb{N}$ such that for $\tau > 0$, $n \ge n_\kappa$ and a.e. $t \in [0, T - \tau]$*

$$\left|\tau^{-1}\int_0^\tau \langle g_n(t) - g_n(t+s), f_{n,\kappa}(t)\rangle_{X',X}\, ds\right| \le C(\kappa)\tau^\alpha(A_n(t) + 1).$$

(d) *The* compactness assumption *is satisfied: $X' \hookrightarrow\hookrightarrow Z'$. More precisely, every uniformly bounded sequence in X' has a strongly converging subsequence in Z'.*

Then there is a subsequence, such that

$$\int_0^T \langle f_n, g_n\rangle_{X,X'}\, dt \to \int_0^T \langle f, g\rangle_{X,X'}\, dt.$$

2.3 Solenoidal Extension

In this section, we present a linear solenoidal extension operator that maps boundary elements of a spatial domain into the interior. For this end, we first consider the *corrector map*

$$\mathscr{K}_\eta : L^1(\Gamma) \to \mathbb{R}, \qquad \mathscr{K}_\eta(\xi) = \frac{\int_{\mathcal{A}_\kappa} \xi(\mathbf{y}(\mathbf{x}))\lambda_\eta(t, \mathbf{x})\, \mathrm{d}\mathbf{x}}{\int_{\mathcal{A}_\kappa} \lambda_\eta(t, \mathbf{x})\, \mathrm{d}\mathbf{x}},$$

where $\lambda_\eta \ge 0$ for $(t, \mathbf{x}) \in I \times \mathcal{A}_\kappa$ is an appropriately chosen weight function, cf. (Muha and Schwarzacher (2022), Eq. (3.3)), and $\mathcal{A}_\kappa := S_{\kappa/2} \setminus S_\kappa$. It satisfies

$$\begin{aligned}
&\|\mathscr{K}_\eta(\xi)\|_{L^q(I)} \lesssim \|\xi\|_{L^q(I;L^1(\Gamma))}\\
&\|\partial_t\mathscr{K}_\eta(\xi)\|_{L^q(I)} \lesssim \|\partial_t\xi\|_{L^q(I;L^1(\Gamma))} + \|\xi\partial_t\eta\|_{L^q(I;L^1(\Gamma))}
\end{aligned}$$

for all $q \in [1, \infty]$. The corrector $\mathscr{K}_\eta$ above preconditions the boundary data to be compatible with the interior solenoidality. The following is proved in (Muha and Schwarzacher (2022), Prop. 3.3) and it provides a solenoidal extension. For that, we introduce the solenoidal space $W^{1,1}_{\mathrm{div}_\mathbf{x}}(\mathcal{O} \cup S_\alpha) := \{\mathbf{w} \in W^{1,1}(\mathcal{O} \cup S_\alpha) : \mathrm{div}_\mathbf{x}\mathbf{w} = 0\}$.

Proposition 2 *For a given $\eta \in L^\infty(I; W^{1,2}(\Gamma))$ with $\|\eta\|_{L^\infty(I\times\Gamma)} < \alpha < L$, there is a linear operator*

$$\mathscr{F}^\eta : \{\xi \in L^1(I; W^{1,1}(\Gamma)) : \mathscr{K}_\eta(\xi) = 0\} \to L^1(I; W^{1,1}_{div_{\boldsymbol{x}}}(\mathcal{O} \cup S_\alpha)),$$

such that the tuple $(\mathscr{F}^\eta(\xi - \mathscr{K}_\eta(\xi)), \xi - \mathscr{K}_\eta(\xi))$ satisfies

$$\begin{aligned} \mathscr{F}^\eta(\xi - \mathscr{K}_\eta(\xi)) &\in L^\infty(I; L^2(\mathcal{O}_\eta)) \cap L^2(I; W^{1,2}_{div_{\boldsymbol{x}}}(\mathcal{O}_\eta)), \\ \xi - \mathscr{K}_\eta(\xi) &\in L^\infty(I; W^{2,2}(\Gamma)) \cap W^{1,\infty}(I; L^2(\Gamma)), \\ (\mathscr{F}^\eta(\xi - \mathscr{K}_\eta(\xi)) \circ \boldsymbol{\varphi}_\eta &= \boldsymbol{n}(\xi - \mathscr{K}_\eta(\xi)), \\ \partial_t(\mathscr{F}^\eta(\xi - \mathscr{K}_\eta(\xi)) &\in L^2(I; L^2(\mathcal{O}_\eta)), \\ \mathscr{F}^\eta(\xi - \mathscr{K}_\eta(\xi))(t,x) &= 0 \text{ for } (t,x) \in I \times (\mathcal{O} \setminus S_\alpha) \end{aligned}$$

provided we have $\xi \in L^\infty(I; W^{2,2}(\Gamma)) \cap W^{1,\infty}(I; L^2(\Gamma))$. In particular, we have the estimates

$$\begin{aligned} \|\mathscr{F}^\eta(\xi - \mathscr{K}_\eta(\xi))\|_{L^q(I;W^{1,p}(\mathcal{O}\cup S_\alpha))} &\lesssim \|\xi\|_{L^q(I;W^{1,p}(\Gamma))} \\ &\quad + \|\xi \nabla_{\boldsymbol{y}} \eta\|_{L^q(I;L^p(\Gamma))}, \qquad (2.7) \\ \|\partial_t \mathscr{F}^\eta(\xi - \mathscr{K}_\eta(\xi))\|_{L^q(I;L^p(\mathcal{O}\cup S_\alpha))} &\lesssim \|\partial_t \xi\|_{L^q(I;L^p(\Gamma))} \\ &\quad + \|\xi \partial_t \eta\|_{L^q(I;L^p(\Gamma))}, \qquad (2.8) \end{aligned}$$

for any $p \in (1, \infty)$, $q \in [1, \infty]$.

The following result is a consequence of Proposition 2.

Corollary 3 *Let the assumptions of Proposition 2 be satisfied and in addition, let $a, r \in [2, \infty]$, $p, q \in (1, \infty)$ and $s \in [0, 1]$, and assume that $\eta \in L^r(I; W^{2,a}(\Gamma)) \cap W^{1,r}(I; L^a(\Gamma))$. Let $\xi \in W^{s,p}(\Gamma)$ and let ξ_δ be a smooth approximation of ξ in Γ. Then $\mathcal{E}^\eta_\delta(\xi) := \mathscr{F}^\eta(\xi_\delta - \mathscr{K}_\eta(\xi_\delta))$ satisfies all the conclusions in Proposition 2. In particular,*

$$\|\partial_t \mathcal{E}^\eta_\delta(\xi)\|_{L^r(I;L^a(\mathcal{O}\cup S_\alpha))} \lesssim \|(\xi_\delta)\partial_t \eta\|_{L^r(I;L^a(\Gamma))}$$

and

$$\|\mathcal{E}^\eta_\delta(\xi) - \mathscr{F}^\eta(\xi - \mathscr{K}_\eta(\xi))\|_{L^p(\mathcal{O}\cup S_\alpha)} \lesssim \|\xi_\delta - \xi\|_{L^p(\Gamma)}$$

holds uniformly in $t \in I$.

For the final statement of this subsection, borrowed from (Muha and Schwarzacher (2022), Lemma 3.5), we first introduce the following fractional difference quotient in space in the direction $\mathbf{e}_i$ given by $\Delta^s_h f(\mathbf{y}) = h^{-s}(f(\mathbf{y} + \mathbf{e}_i h) - f(\mathbf{y}))$ for some $h > 0$. Now, we define

$$D^{s,\mathscr{K}}_{-h,h}\eta := \Delta^s_{-h}\Delta^s_h \eta - \mathscr{K}_\eta(\Delta^s_{-h}\Delta^s_h \eta),$$

where $s \in (0, \frac{1}{2})$ and the result is as follows:

Lemma 4 *Let the assumptions of Proposition 2 be satisfied and in addition, let $p, \tilde{a} \in (1, \infty)$ be such that $p' < \tilde{a} \le \frac{3p'}{3-p'}$ if $p' < 3$, and $p' < \tilde{a} < \infty$ otherwise. Furthermore, assume that $\eta \in C^{0,\theta}(\Gamma) \cap W^{1,\frac{\tilde{a}p}{\tilde{a}p-\tilde{a}-p}}(\Gamma)$ and $\boldsymbol{u} \in W^{1,p'}(\mathcal{O}_\eta)$. Then*

$$\left| \int_{\mathcal{O}_\eta} \boldsymbol{u} \cdot \mathscr{F}^\eta (D^{s,\mathscr{K}}_{-h,h}\xi)\, dx \right| \le \Big(h^{\theta - s} + \|\Delta_h^s \eta\|_{W^{1,\frac{\tilde{a}p}{\tilde{a}p-\tilde{a}-p}}(\Gamma)} \Big) \|\boldsymbol{u}\|_{W^{1,p'}(\mathcal{O}_\eta)} \|\xi\|_{L^p(\Gamma)}$$

and when $\partial_t \xi \in L^p(\Gamma)$,

$$\begin{aligned} \left| \int_{\mathcal{O}_\eta} \boldsymbol{u} \cdot \partial_t \mathscr{F}^\eta (D^{s,\mathscr{K}}_{-h,h}\xi)\, dx \right| &\lesssim \Big(h^{\theta - s} + \|\Delta_h^s \eta\|_{W^{1,\frac{\tilde{a}p}{\tilde{a}p-\tilde{a}-p}}(\Gamma)} \Big) \|\boldsymbol{u}\|_{W^{1,p'}(\mathcal{O}_\eta)} \|\partial_t \xi\|_{L^p(\Gamma)} \\ &\quad + \Big(\big\| \, |\Delta_h^s \xi(t)| \, |\partial_t \eta| \, \big\|_{L^{\tilde{a}}(\Gamma)} + \big\| \Delta_h^s \xi(t) \big\|_{L^1(\Gamma)} \big\| \partial_t \eta \big\|_{L^1(\Gamma)} \Big) \|\boldsymbol{u}\|_{W^{1,p'}(\mathcal{O}_\eta)}. \end{aligned}$$

Here, the constants only depends on α, L and $\|\eta\|_{C^{0,\theta}(\Gamma)}$.

2.4 Weak Martingale Solutions

We are interested in a solution to (1.7)–(1.9) that is weak in the probabilistic sense and also weak in the deterministic sense. From the probabilistic point of view, this means that the stochastic basis is also an unknown of the system and from the deterministic angle, we want a distributional solution of the system integrated against a deterministic test function pair $(\phi, \boldsymbol{\phi}) \in W^{2,2}(\Gamma) \times W^{1,2}_{\mathrm{div}_\mathbf{x}}(\mathcal{O})$ that satisfies $\boldsymbol{\phi} \circ \boldsymbol{\varphi} = \phi \mathbf{n}$ at the fluid–structure interface Γ.

We are now deriving the weak formulation of the coupled system assuming we have a sufficiently regular solution at hand. Since the momentum Eq. (1.8) is merely a random PDE rather than a SPDE, and advected by the large-scale incompressible vector field, we can directly apply Reynolds transport theorem Harouna and Mémin (2017) to obtain for $(\iota_\eta \phi, \mathscr{J}_{\eta(t)}\boldsymbol{\phi})$ (recalling the definitions from Sect. 2.2)

$$\begin{aligned} \mathrm{d} \int_{\mathcal{O}_\eta} \mathbf{u} \cdot \mathscr{J}_{\eta(t)}\boldsymbol{\phi}\, \mathrm{d}\mathbf{x} &= \int_{\mathcal{O}_\eta} \partial_t \mathbf{u} \cdot \mathscr{J}_{\eta(t)}\boldsymbol{\phi}\, \mathrm{d}\mathbf{x}\, \mathrm{d}t + \int_{\mathcal{O}_\eta} \mathbf{u} \cdot \partial_t (\mathscr{J}_{\eta(t)}\boldsymbol{\phi})\, \mathrm{d}\mathbf{x}\, \mathrm{d}t \\ &\quad + \int_{\mathcal{O}_\eta} (\mathbf{u} \cdot \nabla_\mathbf{x})(\mathbf{u} \cdot \mathscr{J}_{\eta(t)}\boldsymbol{\phi})\, \mathrm{d}\mathbf{x}\, \mathrm{d}t. \end{aligned}$$

We can now use the momentum Eq. (1.8) and the divergence-free condition on $\boldsymbol{\phi}$ (which transfers to $\mathscr{J}_{\eta(t)}\boldsymbol{\phi}$) to obtain

$$\begin{aligned} \int_{\mathcal{O}_\eta} \partial_t \mathbf{u} \cdot \mathscr{J}_{\eta(t)}\boldsymbol{\phi}\, \mathrm{d}\mathbf{x}\, \mathrm{d}t &= - \int_{\mathcal{O}_\eta} ((\mathbf{u} \cdot \nabla_\mathbf{x})\mathbf{u}) \cdot \mathscr{J}_{\eta(t)}\boldsymbol{\phi}\, \mathrm{d}\mathbf{x}\, \mathrm{d}t - \int_{\mathcal{O}_\eta} \nabla_\mathbf{x}\mathbf{u} : \nabla_\mathbf{x}\mathscr{J}_{\eta(t)}\boldsymbol{\phi}\, \mathrm{d}\mathbf{x}\, \mathrm{d}t \\ &\quad + \int_{\mathcal{O}_\eta} \mathrm{div}_\mathbf{x}(\mathbb{T}(\mathbf{u}, \pi)\mathscr{J}_{\eta(t)}\boldsymbol{\phi})\, \mathrm{d}\mathbf{x}\, \mathrm{d}t \end{aligned}$$

with the latter satisfying

$$\int_{\mathcal{O}_\eta} \mathrm{div}_{\mathbf{x}}(\mathbb{T}(\mathbf{u},\pi)\mathcal{J}_{\eta(t)}\boldsymbol{\phi})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_{\partial\mathcal{O}_\eta} \mathbf{n}_\eta \cdot (\mathbb{T}(\mathbf{u},\pi)\mathcal{J}_{\eta(t)}\boldsymbol{\phi})\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t$$
$$= \int_\Gamma g_\eta \phi\,\mathrm{d}\mathbf{y}\,\mathrm{d}t.$$

To obtain a distributional formulation for the shell Eq. (1.9), we first transform it into the Itô equation

$$\mathrm{d}\partial_t\eta + \left[\Delta_{\mathbf{y}}^2\eta + g_\eta - \tfrac{1}{2}((\varkappa\cdot\nabla_{\mathbf{y}})(\varkappa\cdot\nabla_{\mathbf{y}})\partial_t\eta)\right]\mathrm{d}t + ((\varkappa\cdot\nabla_{\mathbf{y}})\partial_t\eta)\,\mathrm{d}B_t = 0,$$

cf. the discussion in Sect. 2.1. If we now use Itô's formula, we obtain

$$\mathrm{d}\int_\Gamma \partial_t\eta\,\iota_\eta\phi\,\mathrm{d}\mathbf{y} = -\int_\Gamma \iota_\eta\phi\left[\Delta_{\mathbf{y}}^2\eta + g_\eta - \tfrac{1}{2}((\varkappa\cdot\nabla_{\mathbf{y}})(\varkappa\cdot\nabla_{\mathbf{y}})\partial_t\eta)\right]\mathrm{d}\mathbf{y}\,\mathrm{d}t$$
$$+ \int_\Gamma \partial_t\eta\,\partial_t(\iota_\eta\phi)\,\mathrm{d}\mathbf{y} + \int_\Gamma ((\varkappa\cdot\nabla_{\mathbf{y}})\partial_t\eta)\iota_\eta\phi\,\mathrm{d}\mathbf{y}\,\mathrm{d}B_t,$$

where due to the periodicity of the boundary of Γ,

$$\int_\Gamma \iota_\eta\phi\Delta_{\mathbf{y}}^2\eta\,\mathrm{d}\mathbf{y}\,\mathrm{d}t = \int_\Gamma \Delta_{\mathbf{y}}\iota_\eta\phi\Delta_{\mathbf{y}}\eta\,\mathrm{d}\mathbf{y}\,\mathrm{d}t.$$

If we now use the identity $(\mathbf{v}_1\cdot\nabla_{\mathbf{x}})(\mathbf{v}_2\cdot\mathbf{v}_3) - ((\mathbf{v}_1\cdot\nabla_{\mathbf{x}})\mathbf{v}_2)\cdot\mathbf{v}_3 = ((\mathbf{v}_1\cdot\nabla_{\mathbf{x}})\mathbf{v}_3)\cdot\mathbf{v}_2$, it follows that

$$\begin{aligned}
\mathrm{d}\Big(\int_{\mathcal{O}_\eta} \mathbf{u}\cdot\mathcal{J}_{\eta(t)}\boldsymbol{\phi}\,\mathrm{d}\mathbf{x} + \int_\Gamma \partial_t\eta\,\iota_\eta\phi\,\mathrm{d}\mathbf{y}\Big) &= \int_\Gamma \big(\partial_t\eta\,\partial_t(\iota_\eta)\phi - \Delta_{\mathbf{y}}\iota_\eta\phi\Delta_{\mathbf{y}}\eta\big)\,\mathrm{d}\mathbf{y}\,\mathrm{d}t\\
&+ \int_{\mathcal{O}_\eta}\Big(\mathbf{u}\cdot\partial_t(\mathcal{J}_{\eta(t)}\boldsymbol{\phi}) + ((\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathcal{J}_{\eta(t)}\boldsymbol{\phi})\cdot\mathbf{u} - \nabla_{\mathbf{x}}\mathbf{u} : \nabla_{\mathbf{x}}\mathcal{J}_{\eta(t)}\boldsymbol{\phi}\Big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&+ \frac{1}{2}\int_\Gamma \partial_t\eta\,((\varkappa\cdot\nabla_{\mathbf{y}})(\varkappa\cdot\nabla_{\mathbf{y}})(\iota_\eta\phi))\,\mathrm{d}\mathbf{y}\,\mathrm{d}t\\
&- \int_\Gamma \partial_t\eta\,((\varkappa\cdot\nabla_{\mathbf{y}})\iota_\eta\phi)\,\mathrm{d}\mathbf{y}\,\mathrm{d}B_t.
\end{aligned} \tag{2.9}$$

Note that $\mathrm{div}_{\mathbf{x}}(\mathcal{J}_{\eta(t)}\boldsymbol{\phi}) = 0$ and thus, no pressure term appears in the weak formulation. The term containing $\partial_t\mathcal{J}_\eta$ is still not well defined and needs to be rewritten. First of all, we have

$$\begin{aligned}
\mathcal{J}_\eta\boldsymbol{\phi} &= \nabla_{\mathbf{x}}\boldsymbol{\Psi}_\eta\circ\boldsymbol{\Psi}_\eta^{-1}(\det\nabla_{\mathbf{x}}\boldsymbol{\Psi}_\eta\circ\boldsymbol{\Psi}_\eta^{-1})^{-1}\boldsymbol{\phi}\circ\boldsymbol{\Psi}_\eta^{-1}\\
&= \nabla_{\mathbf{x}}\boldsymbol{\Psi}_\eta^{-1}(\det\nabla_{\mathbf{x}}\boldsymbol{\Psi}_\eta^{-1})^{-1}\boldsymbol{\phi}\circ\boldsymbol{\Psi}_\eta^{-1}\\
&= \nabla_{\mathbf{x}}\boldsymbol{\Psi}_{-\eta}(\det\nabla_{\mathbf{x}}\boldsymbol{\Psi}_{-\eta})^{-1}\boldsymbol{\phi}\circ\boldsymbol{\Psi}_{-\eta}
\end{aligned}$$

so that

$$
\begin{aligned}
\partial_t(\mathcal{J}_\eta \boldsymbol{\phi}) = {} & \partial_t \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta}(\det \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta})^{-1} \boldsymbol{\phi} \circ \boldsymbol{\Psi}_{-\eta} \\
& - \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta}(\det \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta})^{-2} \operatorname{tr}((\operatorname{cof} \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta})^\top \partial_t \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta}) \boldsymbol{\phi} \circ \boldsymbol{\Psi}_{-\eta} \\
& + \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta}(\det \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta})^{-1} \nabla_{\mathbf{x}} \boldsymbol{\phi} \circ \boldsymbol{\Psi}_{-\eta} \partial_t \boldsymbol{\Psi}_{-\eta}.
\end{aligned}
$$

By using Gauß theorem, we obtain

$$
\begin{aligned}
& \int_{\mathcal{O}_\eta} \mathbf{u} \cdot \partial_t \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta}(\det \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta})^{-1} \boldsymbol{\phi} \circ \boldsymbol{\Psi}_{-\eta} \, \mathrm{d}\mathbf{x} \\
& \qquad = \int_{\partial \mathcal{O}_\eta} ((\mathbf{u} \cdot \partial_t \boldsymbol{\Psi}_{-\eta})(\det \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta})^{-1} \boldsymbol{\phi} \circ \boldsymbol{\Psi}_{-\eta}) \cdot \mathbf{n}_\eta \, \mathrm{d}\mathcal{H}^2 \\
& \qquad \quad - \int_{\mathcal{O}_\eta} \partial_t \boldsymbol{\Psi}_{-\eta} \cdot \operatorname{div}_{\mathbf{x}} \big(\mathbf{u} \otimes (\det \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta})^{-1} \boldsymbol{\phi} \circ \boldsymbol{\Psi}_{-\eta} \big) \, \mathrm{d}\mathbf{x}
\end{aligned}
$$

and similarly

$$
\begin{aligned}
& \int_{\mathcal{O}_\eta} \mathbf{u} \cdot \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta}(\det \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta})^{-2} \operatorname{tr}((\operatorname{cof} \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta})^\top \partial_t \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta}) \boldsymbol{\phi} \circ \boldsymbol{\Psi}_{-\eta} \, \mathrm{d}\mathbf{x} \\
& \quad = \int_{\mathcal{O}_\eta} \sum_{j=1}^3 \sum_{i=1}^3 ((\operatorname{cof} \partial_j \Psi_{-\eta}^i) \partial_t \partial_j \Psi_{-\eta}^i) \mathbf{u} \cdot \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta}(\det \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta})^{-2} \boldsymbol{\phi} \circ \boldsymbol{\Psi}_{-\eta} \, \mathrm{d}\mathbf{x} \\
& \quad = \int_{\mathcal{O}_\eta} \sum_{j=1}^3 \partial_j \Big(\sum_{i=1}^3 ((\operatorname{cof} \partial_j \Psi_{-\eta}^i) \partial_t \Psi_{-\eta}^i) \mathbf{u} \cdot \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta}(\det \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta})^{-2} \boldsymbol{\phi} \circ \boldsymbol{\Psi}_{-\eta} \Big) \, \mathrm{d}\mathbf{x} \\
& \qquad - \int_{\mathcal{O}_\eta} \sum_{j=1}^3 \sum_{i=1}^3 \partial_t \Psi_{-\eta}^i \partial_j \Big((\operatorname{cof} \partial_j \Psi_{-\eta}^i) \mathbf{u} \cdot (\nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta}(\det \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta})^{-2} \boldsymbol{\phi} \circ \boldsymbol{\Psi}_{-\eta}) \Big) \, \mathrm{d}\mathbf{x} \\
& \quad = \int_{\partial \mathcal{O}_\eta} \sum_{j=1}^3 \Big(\sum_{i=1}^3 ((\operatorname{cof} \partial_j \Psi_{-\eta}^i) \partial_t \Psi_{-\eta}^i) \mathbf{u} \cdot \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta}(\det \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta})^{-2} \boldsymbol{\phi} \circ \boldsymbol{\Psi}_{-\eta} \Big) n_\eta^j \, \mathrm{d}\mathcal{H}^2 \\
& \qquad - \int_{\mathcal{O}_\eta} \sum_{j=1}^3 \sum_{i=1}^3 \partial_t \Psi_{-\eta}^i \partial_j \Big((\operatorname{cof} \partial_j \Psi_{-\eta}^i) \mathbf{u} \cdot (\nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta}(\det \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{-\eta})^{-2} \boldsymbol{\phi} \circ \boldsymbol{\Psi}_{-\eta}) \Big) \, \mathrm{d}\mathbf{x}
\end{aligned}
$$

where $\Psi_{-\eta}^i$ is the i-th component of $\boldsymbol{\Psi}_{-\eta}$ and n_η^j that of $\mathbf{n}_\eta$. The last term of $\partial_t(\mathcal{J}_\eta \boldsymbol{\phi})$ does not require such an integration by parts. Combining Hölder's inequality with Sobolev's embedding and using that $\boldsymbol{\Psi}_\eta$ has the same regularity as η, one easily checks that for a weak solution with regularity as below, all terms are well defined.

With this preparation, we now give the precise notion of a solution.

Definition 2 (Weak martingale solution) Let $(\eta_0, \eta_1, \mathbf{u}_0, \varkappa)$ be a dataset such that

$$\begin{aligned} &\eta_0 \in W^{2,2}(\Gamma) \text{ with } \|\eta_0\|_{L^\infty(\Gamma)} < L, \quad \eta_1 \in L^2(\Gamma), \\ &\mathbf{u}_0 \in L^2_{\mathrm{div}_\mathbf{x}}(\mathcal{O}_{\eta_0}) \text{ is such that } \mathbf{u}_0 \circ \boldsymbol{\varphi}_{\eta_0} = \eta_1 \mathbf{n} \text{ on } \Omega \times \Gamma, \\ &\|\varkappa\|_{W^{1,\infty}(\Gamma)} \lesssim 1. \end{aligned} \tag{2.10}$$

We call $((\Omega, \mathfrak{F}, (\mathfrak{F})_{t\geq 0}, \mathbb{P}), \eta, \mathbf{u})$ a *weak martingale solution* of (1.7)–(1.9) with data $(\eta_0, \eta_1, \mathbf{u}_0, \varkappa)$ provided that the following holds:

(a) $(\Omega, \mathfrak{F}, (\mathfrak{F})_{t\geq 0}, \mathbb{P})$ is a stochastic basis with a complete right-continuous filtration;
(b) B_t is an $(\mathfrak{F}_t)$-Brownian motion;
(c) the shell function η is $(\mathfrak{F}_t)$-adapted with $\|\eta\|_{L^\infty(I\times\Gamma)} < L$ a.s. and for all $s \in (0, 1/2)$

$$\eta \in L^\infty\big(I; W^{2,2}(\Gamma) \cap L^2(I; W^{2+s,2}(\Gamma))\big), \qquad \partial_t \eta \in C_w\big(\overline{I}; L^2(\Gamma)\big) \qquad a.s.;$$

(d) the velocity $\mathbf{u}$ is $(\mathfrak{F}_t)$-adapted with $\mathbf{u} \circ \boldsymbol{\varphi}_\eta = \mathbf{n}\partial_t \eta$ on $I \times \Gamma$ a.s.

$$\mathbf{u} \in C_w\big(\overline{I}; L^2_{\mathrm{div}_\mathbf{x}}(\mathcal{O}_\eta)\big) \cap L^2\big(I; W^{1,2}(\mathcal{O}_\eta)\big) \qquad a.s.;$$

(e) Equation (2.9) holds a.s. for all $(\phi, \boldsymbol{\phi}) \in W^{2,2}(\Gamma) \times W^{1,2}_{\mathrm{div}_\mathbf{x}}(\mathcal{O})$ with $\boldsymbol{\phi} \circ \boldsymbol{\varphi} = \phi \mathbf{n}$ on Γ.
(e) The energy inequality holds in the sense that

$$\begin{aligned} &\frac{1}{2} \int_{\mathcal{O}_{\eta(t)}} |\mathbf{u}(t)|^2 \,\mathrm{d}\mathbf{x} + \int_0^t \int_{\mathcal{O}_{\eta(s)}} |\nabla_\mathbf{x} \mathbf{u}|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}s + \frac{1}{2} \int_\Gamma |\partial_t \eta(t)|^2 \,\mathrm{d}\mathbf{y} + \frac{1}{2} \int_\Gamma |\Delta_\mathbf{y} \eta(t)|^2 \,\mathrm{d}\mathbf{y} \\ &\quad \leq \frac{1}{2} \int_{\mathcal{O}_{\eta_0}} |\mathbf{u}_0|^2 \,\mathrm{d}\mathbf{x} + \frac{1}{2} \int_\Gamma |\eta_1|^2 \,\mathrm{d}\mathbf{y} + \frac{1}{2} \int_\Gamma |\Delta_\mathbf{y} \eta_0|^2 \,\mathrm{d}\mathbf{y} \end{aligned} \tag{2.11}$$

a.s. for a.a. $t \in I$.

The following is our main result.

Theorem 5 *Let $(\eta_0, \eta_1, \boldsymbol{u}_0, \varkappa)$ be a dataset such that (2.10) holds. Then there is a weak martingale solution of (1.7)–(1.9) with data $(\eta_0, \eta_1, \boldsymbol{u}_0, \varkappa)$ in the sense of Definition 2. The interval of existence is of the form $\overline{I} = (0, t)$, where $t < T$ only if $\lim_{s\to t} \|\eta(s)\|_{L^\infty(\Gamma)} = L$ a.s. in Ω_0 for some $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) > 0$.*

3 The Linearised Problem

In the first instant, we wish to construct a weak solution to a system with a regularised geometry and a regularised convection term. Here, by a *regularised geometry*, we mean a regularization of a solution to a *given* shell equation and not the solution to our anticipated shell Eq. (1.9). Thus, we aim at solving the system

$$\mathrm{div}_{\mathbf{x}}\mathbf{u} = 0, \tag{3.1}$$

$$\partial_t \mathbf{u} + (\mathbf{u}_\epsilon \cdot \nabla_{\mathbf{x}})\mathbf{u} = \Delta_{\mathbf{x}}\mathbf{u} - \nabla_{\mathbf{x}}\pi, \tag{3.2}$$

$$\mathrm{d}\partial_t \eta + (\epsilon \mathcal{L}'(\eta) + \epsilon \partial_t \Delta_{\mathbf{y}}^2 \eta + \Delta_{\mathbf{y}}^2 \eta + g_{\eta_\epsilon})\,\mathrm{d}t + ((\varkappa \cdot \nabla_{\mathbf{y}})\partial_t \eta) \circ \mathrm{d}B_t = 0, \tag{3.3}$$

in $I \times \mathcal{O}_{\eta_\epsilon}$ where

$$g_{\eta_\epsilon} = \mathbf{n}^\top (\mathbb{T}(\mathbf{u},\pi)\mathbf{n}_{\eta_\epsilon}) \circ \boldsymbol{\varphi}_{\eta_\epsilon} |\det(\nabla_{\mathbf{y}}\boldsymbol{\varphi}_{\eta_\epsilon})|, \quad \mathbb{T}(\mathbf{u},\pi) = (\nabla_{\mathbf{x}}\mathbf{u} + \nabla_{\mathbf{x}}\mathbf{u}^\top) - \pi \mathbb{I}_{3\times 3},$$

$\mathcal{L}'$ is the operator given $\int_\Gamma \mathcal{L}'(\eta)\phi \,\mathrm{d}\mathbf{y} = \int_\Gamma \nabla_{\mathbf{y}}^3 \eta : \nabla_{\mathbf{y}}^3 \phi \,\mathrm{d}\mathbf{y}$ for all $\phi \in W^{3,2}(\Gamma)$, and $\epsilon > 0$ is a fixed regularisation parameter. With some slight abuse of notation we denote by f_ϵ the regularisation of a function on the fluid domain (which is previously extended by zero to the whole space) as well as the regularization of a function defined on $I \times \Gamma$. The regularisation is taken with respect to space and time, where the temporal regularization is taken backwards in extending functions to $(-\infty, T)$ by their values at time 0. A martingale solution to (3.1)–(3.3) can be defined analogously to Definition 2. We aim to show the following result (the proof of Theorem 6 can be found in the next section).

Theorem 6 *Let* $(\eta_0, \eta_1, \boldsymbol{u}_0, \varkappa)$ *be a dataset such that* (2.10) *holds and we have additionally* $\eta_0 \in W^{3,2}(\Gamma)$. *Then there is a weak martingale solution of* (3.1)–(3.3) *with data* $(\eta_0, \eta_1, \boldsymbol{u}_0, \varkappa)$. *The interval of existence is of the form* $\overline{I} = (0, t)$, *where* $t < T$ *only if* $\lim_{s\to t} \|\eta(s)\|_{L^\infty(\Gamma)} = L$ *a.s. in* Ω_0 *for some* $\Omega_0 \subset \Omega$ *with* $\mathbb{P}(\Omega_0) > 0$.

In order to solve (3.1)–(3.3), we linearise the problem by replacing the regularised velocity in the convective term with a regularization of a *given* velocity field $\mathbf{v} \in \mathbb{R}^3$. We also replace the regularised geometry with a regularised geometry with respect to a given structure displacement ζ with an initial state $\zeta(0,\cdot) = \eta_0$. The corresponding regularization of the pair $(\zeta, \mathbf{v})$ is denoted by $(\zeta_\epsilon, \mathbf{v}_\epsilon)$. The solution we seek will be constructed as the limit $N \to \infty$ of the solution $(\eta^N, \mathbf{u}^N)$ to a finite dimensional Galerkin approximation system incorporating these regularizing terms. Since this is a linear system we aim to construct a probabilistically strong solution defined on a stochastic basis $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$ and driven by a given Brownian motion (B_t) relative to $(\mathfrak{F}_t)$. Suppose that $(\zeta, \mathbf{v})$ (and thus its regularization $(\zeta_\epsilon, \mathbf{v}_\epsilon)$) are a given pair of $(\mathfrak{F}_t)$-progressively measurable[3] random variables with values in $C(\overline{I} \times \Gamma) \times L^2(I; L^2(\mathcal{O} \cup S_\alpha))$ belonging to $L^p(\Omega)$ for some sufficiently large p where we suppose that ϵ is small enough such that $\|\zeta_\epsilon\|_{L^\infty(I\times\Gamma)} < \alpha < L$ a.s. We now look for an $(\mathfrak{F}_t)$-progressively measurable process $(\eta, \mathbf{u})$ with values in the space

$$\begin{aligned} &W^{1,2}(I; W^{2,2}(\Gamma)) \times L^\infty(I; W^{3,2}(\Gamma)) \cap W^{1,\infty}(I; L^2(\Gamma)) \\ &\qquad \times L^\infty(I; L^2(\Omega_{\zeta_\epsilon})) \cap L^2(I; W^{1,2}_{\mathrm{div}_{\mathbf{x}}}(\Omega_{\zeta_\epsilon})) \end{aligned} \tag{3.4}$$

[3] To be understood in the sense of random distributions, cf. (Breit et al. (2018), Chapter 2.8).

such that

$$
\begin{aligned}
&\mathrm{d}\Big(\int_{\mathcal{O}_{\zeta_\epsilon}}\mathbf{u}\cdot\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}+\int_\Gamma\partial_t\eta\,\iota_{\zeta_\epsilon}\phi\,\mathrm{d}\mathbf{y}\Big)\\
&\quad=\int_\Gamma\big(\partial_t\eta\,\partial_t(\iota_{\zeta_\epsilon}\phi)-\Delta_{\mathbf{y}}\iota_{\zeta_\epsilon}\phi\,\Delta_{\mathbf{y}}\eta-\epsilon\Delta_{\mathbf{y}}\iota_{\zeta_\epsilon}\phi\,\partial_t\Delta_{\mathbf{y}}\eta\big)\,\mathrm{d}\mathbf{y}\,\mathrm{d}t\\
&\qquad-\epsilon\int_\Gamma\nabla_{\mathbf{y}}^3\iota_{\zeta_\epsilon}\phi:\nabla_{\mathbf{y}}^3\eta\,\mathrm{d}\mathbf{y}\,\mathrm{d}t+\int_\Gamma\Big(\frac{1}{2}\mathbf{n}_{\zeta_\epsilon}\cdot\mathbf{n}\iota_{\zeta_\epsilon}\phi\,\partial_t\zeta_\epsilon\,\partial_t\eta\,|\det(\nabla_{\mathbf{y}}\boldsymbol{\varphi}_{\zeta_\epsilon})|\Big)\,\mathrm{d}\mathbf{y}\,\mathrm{d}t\\
&\qquad+\int_{\mathcal{O}_{\zeta_\epsilon}}\Big(\mathbf{u}\cdot\partial_t(\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{\phi})-\frac{1}{2}((\mathbf{v}_\epsilon\cdot\nabla_{\mathbf{x}})\mathbf{u})\cdot(\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{\phi})\Big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\qquad+\int_{\mathcal{O}_{\zeta_\epsilon}}\Big(\frac{1}{2}((\mathbf{v}_\epsilon\cdot\nabla_{\mathbf{x}})\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{\phi})\cdot\mathbf{u}-\nabla_{\mathbf{x}}\mathbf{u}:\nabla_{\mathbf{x}}(\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{\phi})\Big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\qquad+\frac{1}{2}\int_\Gamma((\varkappa\cdot\nabla_{\mathbf{y}})(\varkappa\cdot\nabla_{\mathbf{y}})\partial_t\eta)\iota_{\zeta_\epsilon}\phi\,\mathrm{d}\mathbf{y}\,\mathrm{d}t+\int_\Gamma((\varkappa\cdot\nabla_{\mathbf{y}})\partial_t\eta)\iota_{\zeta_\epsilon}\phi\,\mathrm{d}\mathbf{y}\,\mathrm{d}B_t
\end{aligned}\tag{3.5}
$$

for all $(\phi,\boldsymbol{\phi})\in W^{3,2}(\Gamma)\times W^{1,2}_{\mathrm{div}_{\mathbf{x}}}(\mathcal{O})$ with $\boldsymbol{\phi}\circ\boldsymbol{\varphi}=\phi\mathbf{n}$ on Γ. Moreover, we require $\mathbf{u}\circ\boldsymbol{\varphi}_{\zeta_\epsilon}=\mathbf{n}\partial_t\eta$ on $I\times\Gamma$.

Theorem 7 *Let $(\eta_0,\eta_1,\boldsymbol{u}_0,\varkappa)$ be a dataset such that (2.10) holds and we have additionally $\eta_0\in W^{3,2}(\Gamma)$. Let $(\Omega,\mathfrak{F},(\mathfrak{F}_t)_{t\geq0},\mathbb{P})$ be a stochastic basis with a complete, right-continuous filtration and let (B_t) be an $(\mathfrak{F}_t)$-Brownian motion. Then there is a unique probabilistically strong solution of (3.5) with data $(\eta_0,\eta_1,\boldsymbol{u}_0,\varkappa)$. The interval of existence is of the form $\overline{I}=(0,t)$, where $t<T$ only if $\lim_{s\to t}\|\eta(s)\|_{L^\infty(\Gamma)}=L$ a.s. in Ω_0 for some $\Omega_0\subset\Omega$ with $\mathbb{P}(\Omega_0)>0$.*

It will turn out that the solution from Theorem 7 satisfies the energy equality

$$
\begin{aligned}
&\tfrac{1}{2}\int_{\mathcal{O}_{\zeta_\epsilon}}|\mathbf{u}(t)|^2\,\mathrm{d}\mathbf{x}+\int_0^t\int_{\mathcal{O}_{\zeta_\epsilon}}|\nabla_{\mathbf{x}}\mathbf{u}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}s+\epsilon\int_0^t\int_\Gamma|\partial_s\Delta_{\mathbf{y}}\eta|^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}s\\
&\quad+\int_\Gamma\Big(\tfrac{1}{2}|\partial_t\eta(t)|^2+\tfrac{1}{2}|\Delta_{\mathbf{y}}\eta(t)|^2+\epsilon|\nabla_{\mathbf{y}}^3\eta(t)|^2\Big)\,\mathrm{d}\mathbf{y}\\
&\quad=\tfrac{1}{2}\int_{\mathcal{O}_{\zeta_\epsilon(0)}}|\mathbf{u}_0|^2\,\mathrm{d}\mathbf{x}+\int_\Gamma\Big(\tfrac{1}{2}|\eta_1|^2+\tfrac{1}{2}|\Delta_{\mathbf{y}}\eta_0|^2+\epsilon|\nabla_{\mathbf{y}}^3\eta_0|\Big)\,\mathrm{d}\mathbf{y}
\end{aligned}\tag{3.6}
$$

a.s. for a.a. $t\in I$.

The aim of the following subsection is to construct a Galerkin approximation of (3.1)–(3.3) on a given stochastic basis $(\Omega,\mathfrak{F},(\mathfrak{F})_{t\geq0},\mathbb{P})$, while its limit passage (and thus the proof of Theorem 6) can be found in the next section.

3.1 The Linearised Galerkin Problem

Let us now explain in which function spaces we seek the finite dimensional objects $(\eta^N,\mathbf{n}^N)$. Let $(Y_i)_{i\in\mathbb{N}}$ be a basis of $W^{2,2}(\Gamma)$ and let $(\mathbf{X}_i)_{i\in\mathbb{N}}$ be a basis of $W^{1,2}_{0,\mathrm{div}_{\mathbf{x}}}(\mathcal{O})$.

Clearly, there exists divergence-free vector fields $\mathbf{Y}_i$ that are solving Stokes systems in the reference domain $\mathcal{O}$ with boundary data $(Y_i\mathbf{n}) \circ \boldsymbol{\varphi}^{-1}$. We then set

$$\boldsymbol{w}_i = \begin{cases} \mathbf{X}_i : i \text{ even}, \\ \mathbf{Y}_i : i \text{ odd} \end{cases} \tag{3.7}$$

and set $w_i = \boldsymbol{w}_i \circ \boldsymbol{\varphi}|_\Gamma \cdot \mathbf{n}$. Now we take the pair $(\zeta, \mathbf{v})$ from

$$L^\infty(\Omega; C(\overline{I} \times \Gamma)) \times L^\infty(\Omega; L^2(I; L^2(\mathbb{R}^3)))$$

being $(\mathfrak{F}_t)$-progressively measurable. We search for $(\alpha_i^N)_{k,N\in\mathbb{N}} : \Omega \times \overline{I} \to \mathbb{R}^n$ such that

$$\mathbf{u}^N = \sum_{i=1}^N \alpha_i^N (\mathcal{J}_{\zeta_\epsilon(t)} \boldsymbol{w}_i) \quad \text{and} \quad \eta^N(t,\cdot) = \sum_{i=1}^N \int_0^t \alpha_i^N \iota_{\zeta_\epsilon} w_i \, \mathrm{d}s + \eta_0$$

solve the equations[4]

$$\begin{aligned}
&\mathrm{d}\Big(\int_{\mathcal{O}_{\zeta_\epsilon}} \mathbf{u}^N \cdot \mathcal{J}_{\zeta_\epsilon(t)} \boldsymbol{w}_j \, \mathrm{d}\mathbf{x} + \int_\Gamma \partial_t \eta^N \, \iota_{\zeta_\epsilon} w_j \, \mathrm{d}\mathbf{y} \Big) \\
&\quad = \int_\Gamma \big(\partial_t \eta^N \, \partial_t(\iota_{\zeta_\epsilon} w_j) - \Delta_\mathbf{y} \iota_{\zeta_\epsilon} w_j \Delta_\mathbf{y} \eta^N - \epsilon \Delta_\mathbf{y} \iota_{\zeta_\epsilon} w_j \, \partial_t \Delta_\mathbf{y} \eta^N \big) \, \mathrm{d}\mathbf{y} \, \mathrm{d}t \\
&\qquad - \epsilon \int_\Gamma \nabla_\mathbf{y}^3 \iota_{\zeta_\epsilon} w_j : \nabla_\mathbf{y}^3 \eta^N \, \mathrm{d}\mathbf{y} \, \mathrm{d}t + \int_\Gamma \Big(\frac{1}{2} \mathbf{n}_{\zeta_\epsilon} \cdot \mathbf{n}^\top \iota_{\zeta_\epsilon} w_j \, \partial_t \zeta_\epsilon \, \partial_t \eta^N \, |\det(\nabla_\mathbf{y} \boldsymbol{\varphi}_{\zeta_\epsilon})| \Big) \, \mathrm{d}\mathbf{y} \, \mathrm{d}t \\
&\qquad + \int_{\mathcal{O}_{\zeta_\epsilon}} \Big(\mathbf{u}^N \cdot \partial_t (\mathcal{J}_{\zeta_\epsilon(t)} \boldsymbol{w}_j) - \frac{1}{2} ((\mathbf{v}_\epsilon \cdot \nabla_\mathbf{x}) \mathbf{u}^N) \cdot (\mathcal{J}_{\zeta_\epsilon(t)} \boldsymbol{w}_j) \Big) \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\
&\qquad + \int_{\mathcal{O}_{\zeta_\epsilon}} \Big(\frac{1}{2} ((\mathbf{v}_\epsilon \cdot \nabla_\mathbf{x}) \mathcal{J}_{\zeta_\epsilon(t)} \boldsymbol{w}_j) \cdot \mathbf{u}^N - \nabla_\mathbf{x} \mathbf{u}^N : \nabla_\mathbf{x} (\mathcal{J}_{\zeta_\epsilon(t)} \boldsymbol{w}_j) \Big) \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\
&\qquad + \frac{1}{2} \int_\Gamma ((\varkappa \cdot \nabla_\mathbf{y})(\varkappa \cdot \nabla_\mathbf{y}) \partial_t \eta^N) \iota_{\zeta_\epsilon} w_j \, \mathrm{d}\mathbf{y} \, \mathrm{d}t + \int_\Gamma ((\varkappa \cdot \nabla_\mathbf{y}) \partial_t \eta^N) \iota_{\zeta_\epsilon} w_j \, \mathrm{d}\mathbf{y} \, \mathrm{d}B_t
\end{aligned} \tag{3.8}$$

for $1 \leq j \leq N$ with an initial condition $\alpha_i^N(0)$ which is such that

$$\partial_t \eta^N(0,\cdot) \to \eta_1 \quad \text{in} \quad L^2(\Gamma), \tag{3.9}$$

$$\mathbf{u}^N(0,\cdot) \to \mathbf{u}_0 \quad \text{in} \quad L^2(\mathcal{O}_{\zeta_\epsilon(0)}). \tag{3.10}$$

Note that the derivation of the weak formulation (3.8) is slightly different to the derivation of (2.9) due to the differences in their respective advective terms. The

[4] We neglect the dependency of the unknown $(\eta^N, \mathbf{u}^N)$ on ϵ at this point for simplicity.

treatment of the former advection term goes as follows. By using the trivial identity

$$\int_{\mathcal{O}_{\zeta_\epsilon}} ((\mathbf{v}_\epsilon \cdot \nabla_{\mathbf{x}})\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_j) \cdot \mathbf{u}^N \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \frac{1}{2}\int_{\mathcal{O}_{\zeta_\epsilon}} ((\mathbf{v}_\epsilon \cdot \nabla_{\mathbf{x}})\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_j) \cdot \mathbf{u}^N \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \frac{1}{2}\int_{\mathcal{O}_{\zeta_\epsilon}} ((\mathbf{v}_\epsilon \cdot \nabla_{\mathbf{x}})\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_j) \cdot \mathbf{u}^N \,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$

we rewrite the last term as follows

$$\begin{aligned}
\int_{\mathcal{O}_{\zeta_\epsilon}} &((\mathbf{v}_\epsilon \cdot \nabla_{\mathbf{x}})\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_j) \cdot \mathbf{u}^N \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_{\mathcal{O}_{\zeta_\epsilon}} \mathrm{div}_{\mathbf{x}}(\mathbf{v}_\epsilon \otimes \mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_j) \cdot \mathbf{u}^N \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&= \int_{\partial\mathcal{O}_{\zeta_\epsilon}} \mathbf{n}_{\zeta_\epsilon} \cdot ([\mathbf{v}_\epsilon \otimes \mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_j]\mathbf{u}^N)\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t - \int_{\mathcal{O}_{\zeta_\epsilon}} (\mathbf{v}_\epsilon \otimes \mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_j) : \nabla_{\mathbf{x}}\mathbf{u}^N \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&= \int_{\partial\mathcal{O}_{\zeta_\epsilon}} \mathbf{n}_{\zeta_\epsilon} \cdot ([\mathbf{v}_\epsilon \circ \boldsymbol{\varphi}_{\zeta_\epsilon} \circ \boldsymbol{\varphi}_{\zeta_\epsilon}^{-1}((\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_j) \circ \boldsymbol{\varphi}_{\zeta_\epsilon} \circ \boldsymbol{\varphi}_{\zeta_\epsilon}^{-1})^\top]\mathbf{u}^N \circ \boldsymbol{\varphi}_{\zeta_\epsilon} \circ \boldsymbol{\varphi}_{\zeta_\epsilon}^{-1})\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t \\
&\quad - \int_{\mathcal{O}_{\zeta_\epsilon}} ((\mathbf{v}_\epsilon \cdot \nabla_{\mathbf{x}})\mathbf{u}^N) \cdot \mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_j \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&= \int_{\Gamma} \mathbf{n}_{\zeta_\epsilon} \cdot \mathbf{n}^\top \iota_{\zeta_\epsilon} w_j \,\partial_t\zeta_\epsilon\, \partial_t\eta^N \,|\det(\nabla_{\mathbf{y}}\boldsymbol{\varphi}_{\zeta_\epsilon})|\,\mathrm{d}\mathbf{y}\,\mathrm{d}t \\
&\quad + \int_{\mathcal{O}_{\zeta_\epsilon}} ((\mathbf{v}_\epsilon \cdot \nabla_{\mathbf{x}})\mathbf{u}^N) \cdot \mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_j \,\mathrm{d}\mathbf{x}\,\mathrm{d}t
\end{aligned}$$

where we have used $\mathbf{u}^N \circ \boldsymbol{\varphi}_{\zeta_\epsilon} = \mathbf{n}\partial_t\eta^N$ and $\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_j \circ \boldsymbol{\varphi}_{\zeta_\epsilon} = \iota_{\zeta_\epsilon} w_j\mathbf{n}$ in the last step. This explains the presence of the Jacobian determinant in (3.8). Moving on, we note that Eq. (3.8) is equivalent to

$$\begin{aligned}
\mathrm{d}&\Big[\sum_{i=1}^N \alpha_i^N \Big(\int_{\mathcal{O}_{\zeta_\epsilon}} \mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_i \cdot \mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_j \,\mathrm{d}\mathbf{x} + \int_\Gamma \iota_{\zeta_\epsilon} w_i\, \iota_{\zeta_\epsilon} w_j \,\mathrm{d}\mathbf{y}\Big)\Big] \\
&= \sum_{i=1}^N \int_{\mathcal{O}_{\zeta_\epsilon}} \alpha_i^N \Big(\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_i \cdot \partial_t(\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_j) - \frac{1}{2}((\mathbf{v}_\epsilon \cdot \nabla_{\mathbf{x}})\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_i) \cdot (\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_j)\Big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad + \sum_{i=1}^N \int_{\mathcal{O}_{\zeta_\epsilon}} \alpha_i^N \Big(\frac{1}{2}((\mathbf{v}_\epsilon \cdot \nabla_{\mathbf{x}})\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_j) \cdot (\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_i) - \nabla_{\mathbf{x}}(\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_i) : \nabla_{\mathbf{x}}(\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_j)\Big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad + \sum_{i=1}^N \frac{\alpha_i^N}{2} \int_\Gamma \mathbf{n}_{\zeta_\epsilon} \cdot \mathbf{n}^\top \iota_{\zeta_\epsilon} w_j \,\partial_t\zeta_\epsilon\, \iota_{\zeta_\epsilon} w_i \,|\det(\nabla_{\mathbf{y}}\boldsymbol{\varphi}_{\zeta_\epsilon})|\,\mathrm{d}\mathbf{y}\,\mathrm{d}t - \int_\Gamma \Delta_{\mathbf{y}}\eta_0\, \Delta_{\mathbf{y}}\iota_{\zeta_\epsilon} w_j \,\mathrm{d}\mathbf{y}\,\mathrm{d}t \\
&\quad - \sum_{i=1}^N \int_\Gamma \int_0^t \alpha_i^N(s) \Delta_{\mathbf{y}}\iota_{\zeta_\epsilon} w_i(s)\, \Delta_{\mathbf{y}}\iota_{\zeta_\epsilon} w_j(t)\,\mathrm{d}s\,\mathrm{d}\mathbf{y}\,\mathrm{d}t - \epsilon\int_\Gamma \nabla_{\mathbf{y}}^3\eta_0 : \nabla_{\mathbf{y}}^3\iota_{\zeta_\epsilon} w_j \,\mathrm{d}\mathbf{y}\,\mathrm{d}t \\
&\quad - \epsilon\sum_{i=1}^N \int_\Gamma \int_0^t \alpha_i^N(s) \nabla_{\mathbf{y}}^3\iota_{\zeta_\epsilon} w_i(s) : \nabla_{\mathbf{y}}^3\iota_{\zeta_\epsilon} w_j(t)\,\mathrm{d}s\,\mathrm{d}\mathbf{y}\,\mathrm{d}t - \epsilon\sum_{i=1}^N \int_\Gamma \alpha_i^N \Delta_{\mathbf{y}}\iota_{\zeta_\epsilon} w_i \Delta_{\mathbf{y}}\iota_{\zeta_\epsilon} w_j \,\mathrm{d}\mathbf{y}\,\mathrm{d}t \\
&\quad + \sum_{i=1}^N \frac{\alpha_i^N}{2} \int_\Gamma ((\varkappa \cdot \nabla_{\mathbf{y}})(\varkappa \cdot \nabla_{\mathbf{y}})\iota_{\zeta_\epsilon} w_i)\iota_{\zeta_\epsilon} w_j \,\mathrm{d}\mathbf{y}\,\mathrm{d}t + \sum_{i=1}^N \alpha_i^N \int_\Gamma ((\varkappa \cdot \nabla_{\mathbf{y}})\iota_{\zeta_\epsilon} w_i)\iota_{\zeta_\epsilon} w_j \,\mathrm{d}\mathbf{y}\,\mathrm{d}B_t.
\end{aligned}$$

To simplify notations, we drop the summation signs by employing Einstein's summation convention. Then for

$$
\begin{aligned}
(a_{ij}(t)) &:= \int_{\mathcal{O}_{\zeta_\epsilon}} (\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_i)\cdot(\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_j)\,\mathrm{d}\mathbf{x} + \int_\Gamma \iota_{\zeta_\epsilon} w_i\, \iota_{\zeta_\epsilon} w_j\,\mathrm{d}\mathbf{y},\\
(b_{ij}(t)) &:= \int_{\mathcal{O}_{\zeta_\epsilon}} \Big((\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_i)\cdot\partial_t(\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_j) - \frac{1}{2}((\mathbf{v}_\epsilon\cdot\nabla_{\mathbf{x}}(\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_i)\cdot(\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_j)\Big)\,\mathrm{d}\mathbf{x},\\
&\quad + \int_{\mathcal{O}_{\zeta_\epsilon}} \Big(\frac{1}{2}((\mathbf{v}_\epsilon\cdot\nabla_{\mathbf{x}}(\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_j)\cdot(\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_i) - \nabla_{\mathbf{x}}(\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_i):\nabla_{\mathbf{x}}(\mathcal{J}_{\zeta_\epsilon(t)}\boldsymbol{w}_j)\Big)\,\mathrm{d}\mathbf{x}\\
&\quad + \frac{1}{2}\int_\Gamma \mathbf{n}_{\zeta_\epsilon}\cdot\mathbf{n}^\top \iota_{\zeta_\epsilon} w_j\,\partial_t\zeta_\epsilon\,\iota_{\zeta_\epsilon} w_i\,|\det(\nabla_{\mathbf{y}}\boldsymbol{\varphi}_{\zeta_\epsilon})|\,\mathrm{d}\mathbf{y} - \epsilon\int_\Gamma \Delta_{\mathbf{y}}\iota_{\zeta_\epsilon} w_i\,\Delta_{\mathbf{y}}\iota_{\zeta_\epsilon} w_j\,\mathrm{d}\mathbf{y}\\
&\quad + \frac{1}{2}\int_\Gamma ((\varkappa\cdot\nabla_{\mathbf{y}})(\varkappa\cdot\nabla_{\mathbf{y}})\iota_{\zeta_\epsilon} w_i)\iota_{\zeta_\epsilon} w_j\,\mathrm{d}\mathbf{y},\\
(c_{ij}(t,s)) &:= -\int_\Gamma \Delta_{\mathbf{y}}\iota_{\zeta_\epsilon} w_i(s)\,\Delta_{\mathbf{y}}\iota_{\zeta_\epsilon} w_j(t)\,\mathrm{d}\mathbf{y} - \epsilon\int_\Gamma \nabla_{\mathbf{y}}^3\iota_{\zeta_\epsilon} w_i(s):\nabla_{\mathbf{y}}^3\iota_{\zeta_\epsilon} w_j(t)\,\mathrm{d}\mathbf{y},\\
(d_j(t)) &:= -\int_\Gamma \Delta_{\mathbf{y}}\eta_0\,\Delta_{\mathbf{y}}\iota_{\zeta_\epsilon} w_j\,\mathrm{d}\mathbf{y} - \epsilon\int_\Gamma \nabla_{\mathbf{y}}^3\eta_0:\nabla_{\mathbf{y}}^3\iota_{\zeta_\epsilon} w_j\,\mathrm{d}\mathbf{y},\\
(e_{ij}(t)) &:= \int_\Gamma ((\varkappa\cdot\nabla_{\mathbf{y}})\iota_{\zeta_\epsilon} w_i)\iota_{\zeta_\epsilon} w_j\,\mathrm{d}\mathbf{y},
\end{aligned}
$$

we can rewrite the above as the following system of SDEs

$$
\begin{aligned}
\int_I \mathrm{d}\big[\alpha_i^N(t)(a_{ij}(t))\big] &= \int_I \alpha_i^N(t)(b_{ij}(t))\,\mathrm{d}t - \int_I\int_0^t \alpha_i^N(s)(c_{ij}(t,s))\,\mathrm{d}s\,\mathrm{d}t\\
&\quad - \int_I (d_j(t))\,\mathrm{d}t + \int_I \alpha_i^N(t)(e_{ij}(t))\,\mathrm{d}B_t.
\end{aligned} \tag{3.11}
$$

Since the coefficient matrix $(a_{ij}(t))$ is symmetric and positive definite, it is invertible. As the problem is linear, we can infer the existence of a unique global solution, cf. (Prévôt and Röckner (2007), Theorem 3.1.1.). Moreover, we have the following energy estimate which is obtained by applying Itô's formula[5] to the process $t \mapsto \frac{1}{2}\int_{\mathcal{O}_{\zeta_\epsilon}} |\mathbf{u}^N(t)|^2\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Gamma |\partial_t\eta^N(t)|^2\,\mathrm{d}\mathbf{y}$: it holds

$$
\begin{aligned}
&\tfrac{1}{2}\int_{\mathcal{O}_{\zeta_\epsilon}} |\mathbf{u}^N(t)|^2\,\mathrm{d}\mathbf{x} + \int_0^t\int_{\mathcal{O}_{\zeta_\epsilon}} |\nabla_{\mathbf{x}}\mathbf{u}^N|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}s + \epsilon\int_0^t\int_\Gamma |\partial_s\Delta_{\mathbf{y}}\eta^N|^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}s\\
&\quad + \int_\Gamma \Big(\tfrac{1}{2}|\partial_t\eta^N(t)|^2 + \tfrac{1}{2}|\Delta_{\mathbf{y}}\eta^N(t)|^2 + \epsilon|\nabla_{\mathbf{y}}^3\eta^N(t)|^2\Big)\,\mathrm{d}\mathbf{y}\\
&\quad = \tfrac{1}{2}\int_{\mathcal{O}_{\eta_0}} |\mathbf{u}_0|^2\,\mathrm{d}\mathbf{x} + \int_\Gamma \Big(\tfrac{1}{2}|\eta_1|^2 + \tfrac{1}{2}|\Delta_{\mathbf{y}}\eta_0|^2 + \epsilon|\nabla_{\mathbf{y}}^3\eta_0|\Big)\,\mathrm{d}\mathbf{y}
\end{aligned} \tag{3.12}
$$

5 This can be rewritten in terms of the α_i^N, cf. (3.11), such that a finite dimensional version is sufficient. Note that the coefficients in (3.11) are random but differentiable in time.

a.s. for a.a. $t \in I$. From the above, we then obtain

$$\sup_I \left(\|\eta^N\|^2_{W^{2,2}(\Gamma)} + \epsilon \|\eta^N\|^2_{W^{3,2}(\Gamma)} \right) \lesssim 1, \tag{3.13}$$

$$\sup_I \|\partial_t \eta^N\|^2_{L^2(\Gamma)} \lesssim 1, \tag{3.14}$$

$$\epsilon \int_I \|\partial_t \eta^N\|^2_{W^{2,2}(\Gamma)} \, \mathrm{d}t \lesssim 1, \tag{3.15}$$

$$\sup_I \|\mathbf{u}^N\|^2_{L^2(\mathcal{O}_{\zeta_\epsilon})} \lesssim 1, \tag{3.16}$$

$$\int_I \|\nabla_{\mathbf{x}} \mathbf{u}^N\|^2_{L^2(\mathcal{O}_{\zeta_\epsilon})} \, \mathrm{d}t \lesssim 1. \tag{3.17}$$

In addition, for any $s \in (0, \frac{1}{2})$, it follows from $\mathbf{u}^N \circ \boldsymbol{\varphi}_{\zeta_\epsilon} = \mathbf{n}\partial_t \eta^N$, (3.17) and the trace theorem that

$$\int_I \|\partial_t \eta^N\|^2_{W^{s,2}(\Gamma)} \, \mathrm{d}t \lesssim 1 \tag{3.18}$$

holds.

3.2 Tightness of $\partial_t \eta^N$

The effort of this subsection is to prove tightness of the law of $\partial_t \eta^N$ on L^2 in order to pass to the limit in the stochastic integral. We define the projection $\mathcal{P}^N$ and the extension $\mathscr{F}_N^{\zeta_\epsilon}$ (for a given $\zeta : \omega \to (-L, L)$)

$$\mathcal{P}^N b = \sum_{k=1}^N \alpha_k(b) \iota_{\zeta_\epsilon} w_k, \quad \mathscr{F}_N^{\zeta_\epsilon} b = \sum_{k=1}^N \alpha_k(b) \mathcal{J}_{\zeta_\epsilon} \boldsymbol{w}_k,$$

where $\alpha_k(b) = \langle b, \iota_{\zeta_\epsilon} w_k \rangle_{W^{3,2}(\Gamma)}$ if $w_k = Y_\ell$ for some $\ell \in \mathbb{N}$ and $\alpha_k(b) = 0$ otherwise. Obviously, we have $\mathscr{F}_N^{\zeta_\epsilon} b \circ \boldsymbol{\varphi}_{\zeta_\epsilon} = \mathbf{n}\mathcal{P}^N b$ for any $b \in W^{3,2}(\Gamma)$. We have by definition,

$$\|\mathcal{P}^N b\|^2_{W^{3,2}(\Gamma)} \le \|b\|^2_{W^{3,2}(\Gamma)} \quad \forall b \in W^{3,2}(\Gamma). \tag{3.19}$$

The eigenvalue equation for the basis vectors implies additionally that

$$\|\mathcal{P}^N b\|^2_{L^2(\Gamma)} \lesssim \|b\|^2_{L^2(\Gamma)} \quad \forall b \in L^2(\Gamma). \tag{3.20}$$

By interpolation, we obtain an estimate on $W^{s,2}(\Gamma)$ for any $s \in [0, 3]$, that is

$$\|\mathcal{P}^N b\|_{W^{s,2}(\Gamma)} \lesssim \|b\|_{W^{s,2}(\Gamma)}. \tag{3.21}$$

Finally, for $\zeta_\epsilon \in W^{3,2}(\Gamma)$ with $\|\zeta_\epsilon\|_{L^\infty(\Gamma)} < \alpha$ for $\alpha \in (0, L)$ we have

$$
\begin{aligned}
\|\mathscr{F}_N^{\zeta_\epsilon} b\|^2_{W^{s+1/2,2}(\mathcal{O}\cup S_\alpha)} &= \Big\| \sum_{k=1}^N \alpha_k(b)\mathcal{J}_{\zeta_\epsilon}\boldsymbol{w}_k \Big\|^2_{W^{s+1/2,2}(\mathcal{O}\cup S_\alpha)} \\
&\lesssim \Big\| \sum_{k=1}^N \alpha_k(b)\iota_{\zeta_\epsilon} w_k \Big\|^2_{W^{s,2}(\Gamma)} \\
&\leq \|b\|^2_{W^{s,2}(\Gamma)}
\end{aligned}
\tag{3.22}
$$

for all $b \in W^{s,2}(\Gamma)$ and for any $s \in [0, 3]$. Here we used that $\sum_{k=1}^N \alpha_k(b)\boldsymbol{w}_k$ solves the homogeneous Stokes problem with boundary datum $\sum_{k=1}^N \alpha_k(b)w_k\mathbf{n}$ in $\mathcal{O}$ and well-known elliptic estimates (see (Galdi (2011), Chapter IV)). Similarly, we obtain for $\zeta_\epsilon^1, \zeta_\epsilon^2 \in W^{1,2}(\Gamma)$

$$
\begin{aligned}
&\|\mathscr{F}_N^{\zeta_\epsilon^1} b - \mathscr{F}_N^{\zeta_\epsilon^2} b\|^2_{L^2(\mathcal{O}\cup S_\alpha)} \\
&\quad = \Big\| (\mathcal{J}_{\zeta_\epsilon^1} - \mathcal{J}_{\zeta_\epsilon^2}) \sum_{k=1}^N \alpha_k(b)\boldsymbol{w}_k \Big\|^2_{L^2(\mathcal{O}\cup S_\alpha)} \\
&\quad \lesssim \|(\zeta_\epsilon^1, \zeta_\epsilon^2)\|^4_{W^{1,\infty}(\Gamma)} \|\zeta_\epsilon^1 - \zeta_\epsilon^2\|^2_{L^2(\Gamma)} \Big\| \sum_{k=1}^N \alpha_k(b)\boldsymbol{w}_k \Big\|^2_{W^{1,\infty}(\mathcal{O})} \\
&\qquad + \|(\zeta_\epsilon^1, \zeta_\epsilon^2)\|^2_{W^{1,\infty}(\Gamma)} \|\nabla_{\mathbf{y}}(\zeta_\epsilon^1 - \zeta_\epsilon^2)\|^2_{L^2(\Gamma)} \Big\| \sum_{k=1}^N \alpha_k(b)\boldsymbol{w}_k \Big\|^2_{L^\infty(\mathcal{O})} \\
&\quad \lesssim (1 + \|(\zeta_\epsilon^1, \zeta_\epsilon^2)\|^4_{W^{1,\infty}(\Gamma)}) \|\zeta_\epsilon^1 - \zeta_\epsilon^2\|^2_{W^{1,2}(\Gamma)} \Big\| \sum_{k=1}^N \alpha_k(b)\boldsymbol{w}_k \Big\|^2_{W^{3,2}(\mathcal{O})} \\
&\quad \lesssim (1 + \|(\zeta_\epsilon^1, \zeta_\epsilon^2)\|^4_{W^{1,\infty}(\Gamma)}) \|\zeta_\epsilon^1 - \zeta_\epsilon^2\|^2_{W^{1,2}(\Gamma)} \Big\| \sum_{k=1}^N \alpha_k(b) w_k \Big\|^2_{W^{3,2}(\Gamma)} \\
&\quad \leq (1 + \|(\zeta_\epsilon^1, \zeta_\epsilon^2)\|^4_{W^{1,\infty}(\Gamma)}) \|\zeta_\epsilon^1 - \zeta_\epsilon^2\|^2_{W^{1,2}(\Gamma)} \|b\|^2_{W^{3,2}(\Gamma)}
\end{aligned}
\tag{3.23}
$$

for all $b \in W^{3,2}(\Gamma)$. Following Lengeler and Ružička (2014), we write

$$
\begin{aligned}
&\int_I \int_{\mathcal{O}_{\zeta_\epsilon}} |\mathbf{u}^N|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_I \int_\Gamma |\partial_t \eta^N|^2 \,\mathrm{d}\mathbf{y}\,\mathrm{d}t \\
&\quad = \int_I \int_{\mathcal{O}\cup S_\alpha} \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}} \mathbf{u}^N \cdot \mathscr{F}_N^{\zeta_\epsilon} \partial_t \eta^N \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_I \int_\Gamma |\partial_t \eta^N|^2 \,\mathrm{d}\mathbf{y}\,\mathrm{d}t \\
&\qquad + \int_I \int_{\mathcal{O}\cup S_\alpha} \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}} \mathbf{u}^N \cdot (\mathbf{u}^N - \mathscr{F}_N^{\zeta_\epsilon} \partial_t \eta^N) \,\mathrm{d}\mathbf{x}\,\mathrm{d}t.
\end{aligned}
\tag{3.24}
$$

We consider the space $X := L^2(\Gamma) \times W^{-s,2}(\mathcal{O} \cup S_\alpha)$ with $s \in (0, 1/4)$. In order to apply Theorem 1 yielding tightness of the corresponding laws we need to equip $L^2(I; X' \times X)$ with an unconventional topology which we denote by $\tau_\sharp$ and define the convergence $\rightarrow^\sharp$ as follows. We say that

$$((\zeta^N, \mathbf{v}^N), (\xi^N, \mathbf{w}^N)) \rightarrow^\sharp ((\zeta, \mathbf{v}), (\xi, \mathbf{w})) \text{ in } L^2(I; X' \times X)$$

provided that

$$((\zeta^N, \mathbf{v}^N), (\xi^N, \mathbf{w}^N)) \rightharpoonup ((\zeta, \mathbf{v}), (\xi, \mathbf{w})) \text{ in } L^2(I; X' \times X)$$

and it holds

$$\begin{aligned} &\int_I \langle \mathbf{v}^N, \mathbf{w}^N \rangle_{W^{s,2}, W^{-s,2}} \,\mathrm{d}t + \int_I \int_\Gamma \zeta^N \xi^N \,\mathrm{d}\mathbf{y}\,\mathrm{d}t \\ &\quad \longrightarrow \int_I \langle \mathbf{v}, \mathbf{w} \rangle_{W^{s,2}, W^{-s,2}} \,\mathrm{d}t + \int_I \int_\Gamma \zeta\, \xi \,\mathrm{d}\mathbf{y}\,\mathrm{d}t. \end{aligned} \tag{3.25}$$

Since this topology is finer than the weak topology on $L^2(I; X' \times X)$, it is clear that $(L^2(I; X' \times X), \tau_\sharp)$ is a quasi-Polish space such that Jakubowski's version of the Skorokhod representation theorem applies. We obtain the following result concerning tightness.

Lemma 8 *The laws of*

$$((\partial_t \eta^N, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}} \boldsymbol{u}^N), (\partial_t \eta^N, \mathscr{F}_N^{\zeta_\epsilon} \partial_t \eta^N)) \quad \textit{and} \quad ((\partial_t \eta_n, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}} \boldsymbol{u}^N), (0, \boldsymbol{u}^N - \mathscr{F}_N^{\zeta_\epsilon} \partial_t \eta^N))$$

on $(L^2(I; X' \times X), \tau_\sharp)$ *are tight.*

Proof According to Theorem 1, we first need boundedness in $L^2(I; X' \times X)$. By (3.13)–(3.18) and the properties of the projection and extension operators above this follows immediately (even uniformly in probability). Note that the extension by zero is a bounded operator on $W^{s,2}$ for $s < \frac{1}{4}$. For (b) we observe that we may assume that a regulariser $b \mapsto (b)_\kappa$ exists such that for any $s, a \in \mathbb{R}$

$$\|b - (b)_\kappa\|_{W^{a,2}(\Gamma)} \lesssim \kappa^{s-a} \|b\|_{W^{s,2}(\Gamma)}, \quad b \in W^{s,2}(\Gamma). \tag{3.26}$$

The estimate is well known for $a, s \in \mathbb{N}_0$, while the general case follows by interpolation and duality. Next we introduce the mollification operator on $\partial_t \eta^N$ by considering for $\kappa > 0$ and $N \in \mathbb{N}$, $f_N(t) := (\mathcal{P}^N(\partial_t \eta^N(t)), \mathscr{F}_N^{\zeta_\epsilon(t)}(\mathcal{P}^N(\partial_t \eta^N(t))))$ and set

$$f_{N,\kappa}(t) := (\mathcal{P}^N((\partial_t \eta^N(t))_\kappa), \mathscr{F}_N^{\zeta_\epsilon(t)}(\mathcal{P}^N((\partial_t \eta^N(t))_\kappa))).$$

We find by the continuity of the mollification operator from (3.26), the continuity of the projection operator from (3.21) and the estimate for the extension operator (3.22) that for a.e. $t \in I$ and $s < s_0 < 1/2$

$$\|f_{N,\kappa} - f_N\|_{L^2(\Gamma)\times W^{s,2}(\mathcal{O}\cup S_{L/2})} \lesssim \kappa^{s_0-s}\|\partial_t\eta^N\|_{W^{s_0,2}(\Gamma)}, \tag{3.27}$$

which can be made arbitrarily small in L^2 by choosing κ appropriately, cf. (3.18). Similarly, we have

$$\|f_{N,\kappa}\|_{W^{1,2}(\Gamma)\times W^{1,2}(\mathcal{O}\cup S_{L/2})} \lesssim \kappa^{-1}\|\partial_t\eta^N\|_{L^2(\Gamma)}.$$

Clearly, if $f_N \rightharpoonup f$ in $L^2(I; X)$ then we can deduce a converging subsequence such that $f_{N,\kappa} \rightharpoonup f_\kappa$ (for some f_κ) in $L^2(I; X)$ for any $\kappa > 0$, which implies (b).

For (c) we have to control $\langle g_N(t) - g_N(s), f_{N,\kappa}(t)\rangle$, where $g_N(t) := (\partial_t\eta^N(t), \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon(t)}}\mathbf{u}^N(t))$ and hence we decompose

$$\begin{aligned}
&\langle g_N(t) - g_N(s), f_{N,\kappa}(t)\rangle \\
&\quad = \big(\langle g_N(t), (\mathcal{P}^N((\partial_t\eta^N(t))_\kappa), \mathscr{F}_N^{\zeta_\epsilon(t)}(\mathcal{P}^N((\partial_t\eta^N(t))_\kappa)))\rangle \\
&\qquad - \langle g_N(s), (\mathcal{P}^N((\partial_t\eta^N(t))_\kappa), \mathscr{F}_N^{\zeta_\epsilon(s)}(\mathcal{P}^N((\partial_t\eta^N(t))_\kappa)))\rangle\big) \\
&\qquad + \langle g_N(s), (0, \mathscr{F}_N^{\zeta_\epsilon(s)}(\mathcal{P}^N((\partial_t\eta^N(t))_\kappa)) - \mathscr{F}_N^{\zeta_\epsilon(t)}(\mathcal{P}^N((\partial_t\eta^N(t))_\kappa)))\rangle \\
&\quad =: (I) + (II).
\end{aligned}$$

We begin estimating (II) to find that

$$\begin{aligned}
(II) &= \int_{\mathcal{O}_{\zeta_\epsilon(s)}} \mathbf{u}^N(s)\cdot\Big(\mathscr{F}_N^{\zeta_\epsilon(s)}(\mathcal{P}^N((\partial_t\eta^N(t))_\kappa)) - \mathscr{F}_N^{\zeta_\epsilon(t)}(\mathcal{P}^N((\partial_t\eta^N(t))_\kappa))\Big)\,\mathrm{d}\mathbf{x} \\
&\lesssim \|\mathbf{u}^N(s)\|_{L^2(\mathcal{O}_{\zeta_\epsilon(s)})}(1 + \|\zeta_\epsilon\|^2_{L^\infty(I;W^{1,\infty}(\Gamma))})\|\zeta_\epsilon(t) - \zeta_\epsilon(s)\|_{W^{1,2}(\Gamma)} \\
&\quad \times \|\mathcal{P}^N((\partial_t\eta^N(t))_\kappa)\|_{W^{3,2}(\Gamma)}
\end{aligned}$$

using (3.23). By (3.26) and (3.21) the last term can be estimated by

$$\|\mathcal{P}^N((\partial_t\eta^N(t))_\kappa)\|_{W^{3,2}(\Gamma)} \leq c\|(\partial_t\eta^N(t))_\kappa\|_{W^{3,2}(\Gamma)} \leq c\kappa^{-3}\|\partial_t\eta^N(t)\|_{L^2(\Gamma)},$$

which is bounded by (3.14). Due to this as well as (3.14) and (3.16) we further have that

$$\begin{aligned}
|(II)| &\leq c(\kappa)|t-s|(1 + \|\zeta_\epsilon\|^3_{W^{1,\infty}(I;W^{1,\infty}(\Gamma))}) \\
&\leq c(\kappa,\epsilon)|t-s|(1 + \|\zeta\|^3_{C(\overline{I}\times\Gamma)}).
\end{aligned}$$

In the last step, we used standard properties of the mollification recalling that ϵ is a fixed parameter on this level. It follows by Markov's inequality for $K > 0$

$$\mathbb{P}\big(|t-s|^{-1}|(II)| \geq K\big) \leq \frac{c(\kappa,\epsilon)}{K}\mathbb{E}\Big[1 + \|\zeta\|^3_{C(\overline{I}\times\Gamma)}\Big] \leq \frac{c(\kappa,\epsilon)}{K},$$

which can be made arbitrarily small for large K provided $\zeta \in L^3(\Omega; C(\overline{I} \times \Gamma))$. The term (I) is estimated using the test function $f_{N,\kappa}$ obtaining

$$
\begin{aligned}
&\int_t^s (I)\,\mathrm{d}r \\
&= -\int_t^s \int_\Gamma \Delta_{\mathbf{y}}(\mathcal{P}^N((\partial_t \eta^N(t))_\kappa)\Delta_{\mathbf{y}}\eta^N \,\mathrm{d}\mathbf{y}\,\mathrm{d}r \\
&\quad -\epsilon \int_t^s \int_\Gamma \Big(\Delta_{\mathbf{y}}(\mathcal{P}^N((\partial_t \eta^N(t))_\kappa)\partial_t \Delta_{\mathbf{y}}\eta^N + \nabla_{\mathbf{y}}^3(\mathcal{P}^N((\partial_t \eta^N(t))_\kappa) : \nabla_{\mathbf{y}}^3 \eta^N\Big)\,\mathrm{d}\mathbf{y}\,\mathrm{d}r \\
&\quad +\int_t^s \int_\Gamma \Big(\frac{1}{2}\mathbf{n}_{\zeta_\epsilon}\cdot \mathbf{n}^\top(\mathcal{P}^N((\partial_t \eta^N(t))_\kappa)\partial_t \zeta_\epsilon\, \partial_t \eta^N \,|\det(\nabla_{\mathbf{y}}\boldsymbol{\varphi}_{\zeta_\epsilon})|\Big)\,\mathrm{d}\mathbf{y}\,\mathrm{d}r \\
&\quad +\int_t^s \int_{\mathcal{O}_{\zeta_\epsilon}} \mathbf{u}^N \cdot \partial_r \mathscr{F}_N^{\zeta_\epsilon(r)}(\mathcal{P}^N((\partial_t \eta^N(t))_\kappa))\,\mathrm{d}\mathbf{x}\,\mathrm{d}r \\
&\quad -\int_t^s \int_{\mathcal{O}_{\zeta_\epsilon}} \frac{1}{2}((\mathbf{v}_\epsilon\cdot\nabla_{\mathbf{x}})\mathbf{u}^N)\cdot \mathscr{F}_N^{\zeta_\epsilon}(\mathcal{P}^N((\partial_r \eta^N(t))_\kappa))\,\mathrm{d}\mathbf{x}\,\mathrm{d}r \\
&\quad +\int_t^s \int_{\mathcal{O}_{\zeta_\epsilon}} \frac{1}{2}((\mathbf{v}_\epsilon\cdot\nabla_{\mathbf{x}})\mathscr{F}_N^{\zeta_\epsilon}(\mathcal{P}^N((\partial_t \eta^N(t))_\kappa)))\cdot \mathbf{u}^N\,\mathrm{d}\mathbf{x}\,\mathrm{d}r \\
&\quad -\int_t^s \int_{\mathcal{O}_{\zeta_\epsilon}} \nabla_{\mathbf{x}}\mathbf{u}^N : \nabla_{\mathbf{x}}\mathscr{F}_N^{\zeta_\epsilon}(\mathcal{P}^N((\partial_t \eta^N(t))_\kappa))\,\mathrm{d}\mathbf{x}\,\mathrm{d}r \\
&\quad +\frac{1}{2}\int_t^s \int_\Gamma ((\varkappa\cdot\nabla_{\mathbf{y}})(\varkappa\cdot\nabla_{\mathbf{y}})\partial_t \eta^N)(\mathcal{P}^N((\partial_t \eta^N(t))_\kappa)\,\mathrm{d}\mathbf{y}\,\mathrm{d}r \\
&\quad +\int_t^s \int_\Gamma ((\varkappa\cdot\nabla_{\mathbf{y}})\partial_t \eta^N)(\mathcal{P}^N((\partial_t \eta^N(t))_\kappa)\,\mathrm{d}\mathbf{y}\,\mathrm{d}B_r.
\end{aligned}
\tag{3.28}
$$

Due to (3.26), (3.21) and (3.22) and the uniform estimates (3.13)–(3.18), it is clear that all terms can be estimated by $c(\kappa,\epsilon)|t-s|^{1/2}(1 + \|\zeta\|_{C(\overline{I}\times\Gamma)} + \|\nabla_{\mathbf{x}}\mathbf{u}^N\|_{L^2(I\times\mathcal{O}_{\zeta_\epsilon})})$ except for the last one. Here we have by Markov's inequality and Itô-isometry (using also $\mathrm{div}_{\mathbf{y}}\varkappa = 0$)

$$
\begin{aligned}
&\mathbb{P}\bigg(|t-s|^{-1/2}\bigg|\int_t^s \int_\Gamma ((\varkappa\cdot\nabla_{\mathbf{y}})\partial_t \eta^N)\mathcal{P}^N(\partial_t \eta^N(t))_\kappa\,\mathrm{d}\mathbf{y}\,\mathrm{d}B_r\bigg| \geq K\bigg) \\
&\qquad \leq \frac{1}{K^2}\mathbb{E}\bigg||t-s|^{-1/2}\int_t^s \int_\Gamma \partial_t \eta^N((\varkappa\cdot\nabla_{\mathbf{y}})\mathcal{P}^N(\partial_t \eta^N(t))_\kappa)\,\mathrm{d}\mathbf{y}\,\mathrm{d}B_r\bigg|^2 \\
&\qquad = \frac{1}{K^2}\mathbb{E}\bigg[|t-s|^{-1}\int_t^s \bigg|\int_\Gamma \partial_t \eta^N((\varkappa\cdot\nabla_{\mathbf{y}})\mathcal{P}^N(\partial_t \eta^N(t))_\kappa)\,\mathrm{d}\mathbf{y}\bigg|^2\,\mathrm{d}r\bigg] \\
&\qquad \lesssim \frac{1}{K^2}\mathbb{E}\bigg[|t-s|^{-1}\int_t^s \|\partial_t \eta^N\|_{L^2(\Gamma)}\|\nabla_{\mathbf{y}}\mathcal{P}^N(\partial_t \eta^N(t))_\kappa\|_{L^2(\Gamma)}\,\mathrm{d}r\bigg] \\
&\qquad \lesssim \frac{\kappa^{-1}}{K^2}\mathbb{E}\bigg[|t-s|^{-1}\int_t^s \|\partial_t \eta^N(r)\|_{L^2(\Gamma)}\|\partial_t \eta^N(t)\|_{L^2(\Gamma)}\,\mathrm{d}r\bigg] \lesssim \frac{\kappa^{-1}}{K^2}
\end{aligned}
$$

using (3.14). This can be made arbitrarily small for K large. Now we set $Z := W^{s_0,2}(\Gamma) \times W^{s_0,2}(\mathcal{O} \cup S_\alpha)$, where $s_0 \in (s, \frac{1}{4})$. Noticing that property (d) from Theorem 1 follows by the usual compactness in (negative) Sobolev spaces, we conclude tightness of the law of $((\partial_t \eta^N, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}} \mathbf{u}^N), (\partial_t \eta^N, \mathscr{F}_N^{\zeta_\epsilon} \partial_t \eta^N))$ on $(L^2(I; X' \times X), \tau_\sharp)$. The tightness for $((0, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}} \mathbf{u}^N), (0, \mathbf{u}^N - \mathscr{F}_N^{\zeta_\epsilon} \partial_t \eta^N))$ follows along the same line, the only difference being the regularisation of

$$f_N := \big(0, \mathbf{u}^N - \mathscr{F}_N^{\zeta_\epsilon} \partial_t \eta^N\big).$$

While $\mathscr{F}_N^{\zeta_\epsilon} \partial_t \eta^N$ can be replaced by $\mathscr{F}_N^{\zeta_\epsilon(s)}(\mathcal{P}^N((\partial_t \eta^N(t))_\kappa))$ as above, we need to regularise $\mathbf{u}^N$ accordingly to preserve the homogeneous boundary conditions of f_N. Recalling the definition of $\boldsymbol{w}_i$ from (3.7), we define $\mathbf{X}_i^\kappa \in W^{1,2}_{0,\mathrm{div}_\mathbf{x}}(\mathcal{O})$ as a spatial regularisation of $\mathbf{X}_i$ and $\mathbf{Y}_i^\kappa$ as the solution to the Stokes problem with boundary datum $Y_i^\kappa \mathbf{n}$ (which inherits the regularity of Y_i^κ). Then we set

$$\boldsymbol{w}_i^\kappa = \begin{cases} \mathbf{X}_i^\kappa : i \text{ even}, \\ \mathbf{Y}_i^\kappa : i \text{ odd}. \end{cases} \tag{3.29}$$

If $\mathbf{u}^N = \sum_{i=1}^N \alpha_i^N \mathcal{J}_{\zeta_\epsilon} \boldsymbol{w}_i$ we set

$$f_{N,\kappa} = \left(0, \sum_{i=1}^N \alpha_i^N \mathcal{J}_{\zeta_\epsilon} \boldsymbol{w}_i^\kappa - \mathscr{F}_N^{\zeta_\epsilon}(\mathcal{P}^N((\partial_t \eta^N(t))_\kappa))\right).$$

Hence $f_{N,\kappa}$ has zero boundary conditions and thus only the fluid part is seen in the expression $\langle g_N(t) - g_N(s), f_{N,\kappa}(t)\rangle$ (where $g_N := (\partial_t \eta^N, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}} \mathbf{u}^N)$ as above). In particular, the noise is not seen and we conclude

$$\langle g_N(t) - g_N(s), f_{N,\kappa}(t)\rangle \leq c(\kappa, \epsilon)|t-s|^{1/6}(1 + \|\zeta\|_{C(\overline{I}\times\Gamma)} + \|\nabla_\mathbf{x} \mathbf{u}^N\|_{L^2(I\times\mathcal{O}_{\zeta_\epsilon})})$$

obtaining again the claimed tightness. □

3.3 Stochastic Compactness

With the bounds from (3.13)–(3.18) in hand, we wish to obtain compactness. For this, we define the path space

$$\chi = \chi_B \times \chi_\mathbf{u}^2 \times \chi_{\nabla\mathbf{u}} \times \chi_\eta \times \chi_\zeta \times \chi_{f,g}^2$$

where

$$\begin{aligned}
&\chi_B = C(\overline{I}), \quad \chi_\mathbf{u} = \big(L^2(I; L^2(\mathcal{O} \cup S_\alpha)), w\big), \quad \chi_{\nabla\mathbf{u}} = \big(L^2(I; L^2(\mathcal{O} \cup S_\alpha)), w\big),\\
&\chi_\eta = \big(W^{1,\infty}(I; L^2(\Gamma)), w^*\big) \cap \big(L^\infty(I; W^{3,2}(\Gamma)), w^*\big) \cap \big(W^{1,2}(I; W^{2,2}(\Gamma)), w\big),\\
&\chi_\zeta = C(\overline{I} \times \Gamma), \quad \chi_{f,g} = \big(L^2(I; X' \times X), \tau_\sharp\big).
\end{aligned}$$

From (3.13)–(3.18) (together with Alaoglu–Bourbaki Theorem) we obtain the following.

Lemma 9 *For fixed $\epsilon > 0$, the joint law*

$$\left\{\mathcal{L}\left[B_t, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}}\boldsymbol{u}^N, \boldsymbol{v}, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}}\nabla_{\boldsymbol{x}}\boldsymbol{u}^N, \eta^N, \zeta, \begin{pmatrix} ((\partial_t\eta^N, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}}\boldsymbol{u}^N), (\partial_t\eta^N, \mathscr{F}_N^{\zeta_\epsilon}\partial_t\eta^N)) \\ ((\partial_t\eta^N, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}}\boldsymbol{u}^N), (0, \boldsymbol{u}^N - \mathscr{F}_N^{\zeta_\epsilon}\partial_t\eta^N)) \end{pmatrix}\right]; \; N \in \mathbb{N}\right\}$$

is tight on χ.

Now we use Jakubowski's version of the Skorokhod representation theorem, see Jakubowski (1998), to infer the following result (we refer to (Ondreját (2010), Theorem A.1) for a statement which combines Prokhorov's and Skorokhod's theorem for quasi-Polish spaces) which one obtains after taking a non-relabelled subsequence.

Proposition 10 *There exists a complete probability space $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbb{P}})$ with χ-valued random variables*

$$\tilde{\boldsymbol{\Theta}}^N := \left[\tilde{B}_t^N, \mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon^N}}\tilde{\boldsymbol{u}}^N, \tilde{\boldsymbol{v}}^N, \mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon^N}}\nabla_{\boldsymbol{x}}\tilde{\boldsymbol{u}}^N, \tilde{\eta}^N, \tilde{\zeta}^N, \begin{pmatrix} ((\partial_t\tilde{\eta}^N, \mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon^N}}\tilde{\boldsymbol{u}}^N), (\partial_t\tilde{\eta}^N, \mathscr{F}_N^{\tilde{\zeta}_\epsilon^N}\partial_t\tilde{\eta}^N)) \\ ((\partial_t\tilde{\eta}^N, \mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon^N}}\tilde{\boldsymbol{u}}^N), (0, \tilde{\boldsymbol{u}}^N - \mathscr{F}_N^{\tilde{\zeta}_\epsilon^N}\partial_t\tilde{\eta}^N)) \end{pmatrix}\right]$$

for $N \in \mathbb{N}$ and

$$\tilde{\boldsymbol{\Theta}} := \left[\tilde{B}_t, \mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon}}\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}, \mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon}}\nabla_{\boldsymbol{x}}\tilde{\boldsymbol{u}}, \tilde{\eta}, \tilde{\zeta}, \begin{pmatrix} ((\partial_t\tilde{\eta}, \mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon}}\tilde{\boldsymbol{u}}), (\partial_t\tilde{\eta}, \mathscr{F}^{\tilde{\zeta}_\epsilon}\partial_t\tilde{\eta})) \\ ((\partial_t\tilde{\eta}, \mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon}}\tilde{\boldsymbol{u}}), (0, \tilde{\boldsymbol{u}} - \mathscr{F}^{\tilde{\zeta}_\epsilon}\partial_t\tilde{\eta})) \end{pmatrix}\right]$$

such that

(a) For all $n \in \mathbb{N}$ the law of $\tilde{\boldsymbol{\Theta}}^N$ on χ is given by

$$\mathcal{L}\left[B_t, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}}\boldsymbol{u}^N, \boldsymbol{v}, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}}\nabla_{\boldsymbol{x}}\boldsymbol{u}^N, \eta^N, \zeta, \begin{pmatrix} ((\partial_t\eta^N, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}}\boldsymbol{u}^N), (\partial_t\eta^N, \mathscr{F}_N^{\zeta_\epsilon}\partial_t\eta^N)) \\ ((\partial_t\eta^N, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}}\boldsymbol{u}^N), (0, \boldsymbol{u}^N - \mathscr{F}_N^{\zeta_\epsilon}\partial_t\eta^N)) \end{pmatrix}\right]$$

(b) $\tilde{\boldsymbol{\Theta}}^N$ converges $\tilde{\mathbb{P}}$-almost surely to $\tilde{\boldsymbol{\Theta}}$ in the topology of χ, i.e.

$$\begin{aligned} \tilde{B}_t^N &\to \tilde{B}_t \quad \text{in} \quad C(\overline{I})\ \tilde{\mathbb{P}}\text{-a.s.}, \\ \mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon^N}}\tilde{\boldsymbol{u}}^N,\ \tilde{\boldsymbol{v}}^N &\rightharpoonup \mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon}}\tilde{\boldsymbol{u}},\ \tilde{\boldsymbol{v}} \quad \text{in} \quad L^2(I; L^2(\mathcal{O} \cup S_\alpha))\ \tilde{\mathbb{P}}\text{-a.s.}, \\ \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon^N}}\nabla_{\boldsymbol{x}}\tilde{\boldsymbol{u}}^N &\rightharpoonup \mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon}}\nabla_{\boldsymbol{x}}\tilde{\boldsymbol{u}} \quad \text{in} \quad L^2(I; L^2(\mathcal{O} \cup S_\alpha))\ \tilde{\mathbb{P}}\text{-a.s.}, \\ \tilde{\eta}^N &\rightharpoonup^* \tilde{\eta} \quad \text{in} \quad L^\infty(I; W^{3,2}(\Gamma))\ \tilde{\mathbb{P}}\text{-a.s.}, \\ \tilde{\eta}^N &\rightharpoonup^* \tilde{\eta} \quad \text{in} \quad W^{1,\infty}(I; L^2(\Gamma))\ \tilde{\mathbb{P}}\text{-a.s.}, \\ \tilde{\eta}^N &\rightharpoonup \tilde{\eta} \quad \text{in} \quad W^{1,2}(I; W^{2,2}(\Gamma))\ \tilde{\mathbb{P}}\text{-a.s.}, \\ \tilde{\zeta}^N &\to \tilde{\zeta} \quad \text{in} \quad C(\overline{I} \times \Gamma)\ \tilde{\mathbb{P}}\text{-a.s.}, \end{aligned} \tag{3.30}$$

as well as (recalling the definition of $\tau_\sharp$ from (3.25))

$$\begin{aligned}&\int_I\int_{\mathcal{O}\cup S_\alpha}\mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon^N}}\boldsymbol{u}^N\cdot\mathscr{F}_N^{\tilde{\zeta}_\epsilon^N}\partial_t\tilde{\eta}^N\,d\boldsymbol{x}\,dt+\int_I\int_\Gamma|\partial_t\tilde{\eta}^N|^2\,d\boldsymbol{y}\,dt\\&\quad\longrightarrow\int_I\int_{\mathcal{O}\cup S_\alpha}\mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon}}\tilde{\boldsymbol{u}}\cdot\mathscr{F}^{\tilde{\zeta}_\epsilon}\partial_t\tilde{\eta}\,d\boldsymbol{x}\,dt+\int_I\int_\Gamma|\partial_t\tilde{\eta}|^2\,d\boldsymbol{y}\,dt\end{aligned}\tag{3.31}$$

and

$$\begin{aligned}&\int_I\int_{\mathcal{O}\cup S_\alpha}\mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon^N}}\tilde{\boldsymbol{u}}^N\cdot(\tilde{\boldsymbol{u}}^N-\mathscr{F}_N^{\tilde{\zeta}_\epsilon^N}\partial_t\tilde{\eta}^N)\,d\boldsymbol{x}\,dt\\&\quad\longrightarrow\int_I\int_{\mathcal{O}\cup S_\alpha}\mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon}}\tilde{\boldsymbol{u}}\cdot(\tilde{\boldsymbol{u}}-\mathscr{F}^{\tilde{\zeta}_\epsilon}\partial_t\tilde{\eta})\,d\boldsymbol{x}\,dt\end{aligned}\tag{3.32}$$

$\tilde{\mathbb{P}}$-*a.s.*

Now we introduce the filtration on the new probability space, which ensures the correct measurabilities of the new random variables. Let $(\tilde{\mathfrak{F}}_t)_{t\geq 0}$ and $(\tilde{\mathfrak{F}}_t^N)_{t\geq 0}$ be the $\tilde{\mathbb{P}}$-augmented canonical filtration on the variables $\tilde{\boldsymbol{\Theta}}$ and $\tilde{\boldsymbol{\Theta}}^N$, respectively, that is[6]

$$\tilde{\mathfrak{F}}_t=\sigma\big[\sigma_t(\tilde{B}_t)\cup\sigma_t(\mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon}}\tilde{\mathbf{u}})\cup\sigma_t(\tilde{\mathbf{v}})\cup\sigma_t(\mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon}}\nabla_{\mathbf{x}}\tilde{\mathbf{u}})\cup\sigma_t(\tilde{\eta})\cup\sigma_t(\tilde{\zeta})\big]$$

for $t\in I$ and similarly for $\tilde{\mathfrak{F}}_t^N$. By (Breit et al. (2018), Theorem 2.9.1) the weak equation continuous to hold on the new probability space. Combining (5.13) and (3.32) we have

$$\begin{aligned}&\int_I\int_{\mathcal{O}_{\tilde{\zeta}_\epsilon^N}}|\tilde{\mathbf{u}}^N|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\int_I\int_\Gamma|\partial_t\tilde{\eta}^N|^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}t\\&\quad\longrightarrow\int_I\int_{\mathcal{O}_{\tilde{\zeta}}}|\tilde{\mathbf{u}}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\int_I\int_\Gamma|\tilde{\partial}_t\eta|^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}t\end{aligned}$$

$\tilde{\mathbb{P}}$-a.s. By uniform convexity of the L^2-norm this implies

$$\partial_t\tilde{\eta}^N\to\partial_t\tilde{\eta}\quad\text{in}\quad L^2(I;L^2(\Gamma))\ \tilde{\mathbb{P}}\text{-a.s.}$$

This is sufficient to pass to the limit in the weak formulation of the equations (note that all terms except for the stochastic integral can be treated by using (3.30)).

Since we have a linear problem at hand, we may now take appropriate subsequence and pass to the limit in (3.8). Thus we obtain a martingale solution to (3.5). In order to obtain a probabilistically strong solution we prove in the following subsections pathwise uniqueness as well as convergence in probability of the original sequence.

[6] Some of the variables are not continuous in time, for those one can deifne σ_t as the history of a random distribution, cf. (Breit et al. (2018), Chapter 2.8)

3.4 Pathwise Uniqueness

We are now going to prove that any martingale solution satisfies the energy equality (3.6). Pathwise uniqueness is then a direct consequence of the linearity of the problem.

Proposition 11 *Suppose that $(\eta, \boldsymbol{u})$ is a weak martingale solution to (3.5). Then it holds $\mathbb{P}$-a.s.*

$$
\begin{aligned}
&\tfrac{1}{2}\int_{\mathcal{O}_{\zeta_\epsilon}} |\boldsymbol{u}(t)|^2\, d\boldsymbol{x} + \int_0^t \int_{\mathcal{O}_{\zeta_\epsilon}} |\nabla_{\boldsymbol{x}} \boldsymbol{u}|^2\, d\boldsymbol{x}\, ds + \epsilon \int_0^t \int_\Gamma |\partial_s \Delta_{\boldsymbol{y}} \eta|^2\, d\boldsymbol{y}\, ds \\
&\quad + \int_\Gamma \Big(\tfrac{1}{2}|\partial_t \eta(t)|^2 + \tfrac{1}{2}|\Delta_{\boldsymbol{y}} \eta(t)|^2 + \epsilon |\nabla_{\boldsymbol{y}}^3 \eta(t)|^2 \Big)\, d\boldsymbol{y} \\
&\quad = \tfrac{1}{2}\int_{\mathcal{O}_{\zeta_\epsilon(0)}} |\boldsymbol{u}_0|^2\, d\boldsymbol{x} + \int_\Gamma \Big(\tfrac{1}{2}|\eta_1|^2 + \tfrac{1}{2}|\Delta_{\boldsymbol{x}} \eta_0|^2 + \epsilon |\nabla_{\boldsymbol{x}}^3 \eta_0| \Big)\, d\boldsymbol{y}
\end{aligned}
$$

for a.a. $t \in I$.

Proof We rewrite (3.5) as an equation for $(\partial_t \bar{\eta}, \bar{\mathbf{v}})$, where $\bar{\eta} := \int_0^t \iota_{\zeta_\epsilon} \partial_t \eta \,\mathrm{d}s$ and

$$
\bar{\mathbf{v}} := \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{\zeta_\epsilon}^\top (|\det(\nabla_{\mathbf{x}} \boldsymbol{\Psi}_{\zeta_\epsilon})|^{-1} \nabla_{\mathbf{x}} \boldsymbol{\Psi}_{\zeta_\epsilon}) \bar{\mathbf{u}}
$$

with $\bar{\mathbf{u}} := \mathcal{J}_{\zeta_\epsilon}^{-1} \mathbf{u}$ which reads as

$$
\begin{aligned}
&\mathrm{d}\left\langle \begin{pmatrix} \partial_t \bar{\eta} \\ \bar{\mathbf{v}} \end{pmatrix} \cdot \begin{pmatrix} \phi \\ \boldsymbol{\phi} \end{pmatrix} \right\rangle_{\mathbb{H}} \\
&\quad = \mathscr{L}\begin{pmatrix} \phi \\ \boldsymbol{\phi} \end{pmatrix} + \int_\Gamma ((\varkappa \cdot \nabla_{\mathbf{y}}) \partial_t \eta) \iota_{\zeta_\varepsilon} \phi \,\mathrm{d}\mathbf{y}\,\mathrm{d}B_t \\
&\quad := \int_\Gamma \big(\partial_t \eta\, \partial_t(\iota_{\zeta_\epsilon}) \phi - \Delta_{\mathbf{y}} \iota_{\zeta_\epsilon} \phi \Delta_{\mathbf{y}} \eta - \epsilon \Delta_{\mathbf{y}} \iota_{\zeta_\epsilon} \phi\, \partial_t \Delta_{\mathbf{y}} \eta \big)\,\mathrm{d}\mathbf{y}\,\mathrm{d}t \\
&\qquad - \epsilon \int_\Gamma \nabla_{\mathbf{y}}^3 \iota_{\zeta_\epsilon} \phi : \nabla_{\mathbf{y}}^3 \eta \,\mathrm{d}\mathbf{y}\,\mathrm{d}t + \int_\Gamma \Big(\frac{1}{2} \mathbf{n}_{\zeta_\epsilon} \cdot \mathbf{n}^\top \iota_{\zeta_\epsilon} \phi\, \partial_t \zeta_\epsilon\, \partial_t \eta\, |\det(\nabla_{\mathbf{y}} \boldsymbol{\varphi}_{\zeta_\epsilon})| \Big)\,\mathrm{d}\mathbf{y}\,\mathrm{d}t \\
&\qquad + \int_{\mathcal{O}_{\zeta_\epsilon(t)}} \Big(\mathbf{u} \cdot \partial_t(\mathcal{J}_{\zeta_\epsilon(t)}) \boldsymbol{\phi} - \frac{1}{2}((\mathbf{v}_\epsilon \cdot \nabla_{\mathbf{x}}) \mathbf{u}) \cdot (\mathcal{J}_{\zeta_\epsilon(t)} \boldsymbol{\phi}) \Big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\qquad + \int_{\mathcal{O}_{\zeta_\epsilon(t)}} \Big(\frac{1}{2}((\mathbf{v}_\epsilon \cdot \nabla_{\mathbf{x}}) \mathcal{J}_{\zeta_\epsilon(t)} \boldsymbol{\phi}) \cdot \mathbf{u} - \nabla_{\mathbf{x}} \mathbf{u} : \nabla_{\mathbf{x}} (\mathcal{J}_{\zeta_\varepsilon(t)} \boldsymbol{\phi}) \Big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\qquad + \frac{1}{2} \int_\Gamma ((\varkappa \cdot \nabla_{\mathbf{y}})(\varkappa \cdot \nabla_{\mathbf{y}}) \partial_t \eta) \iota_{\zeta_\varepsilon} \phi \,\mathrm{d}\mathbf{y}\,\mathrm{d}t - \int_\Gamma ((\varkappa \cdot \nabla_{\mathbf{y}}) \partial_t \eta) \iota_{\zeta_\varepsilon} \phi \,\mathrm{d}\mathbf{y}\,\mathrm{d}B_t
\end{aligned} \tag{3.33}
$$

where

$$
\mathbb{H} := \big\{ (\xi, \mathbf{w}) \in W^{2,2}(\Gamma) \times W^{1,2}_{\mathrm{div}_{\mathbf{x}}}(\mathcal{O}) : \mathbf{w} \circ \boldsymbol{\varphi} = \xi \mathbf{n} \text{ on } \Gamma \big\}.
$$

Note that $\mathbb{H}$ is a Hilbert space (as closed subset of the Hilbert space $W^{2,2}(\Gamma) \times W^{1,2}_{\mathrm{div}_{\mathbf{x}}}(\mathcal{O})$) and $\mathscr{L} \in \mathbb{H}'$. Hence we get from Itô's formula in Hilbert spaces (see

(Da Prato and Zabczyk (2014), Theorem 4.17)) applied to the mapping

$$t \mapsto \frac{1}{2}\int_{\mathcal{O}} \bar{\mathbf{v}} \cdot |\det(\nabla_{\mathbf{x}}\boldsymbol{\Psi}_{\zeta_\epsilon})|(\nabla_{\mathbf{x}}\boldsymbol{\Psi}_{\zeta_\epsilon}^\top \nabla_{\mathbf{x}}\boldsymbol{\Psi}_{\zeta_\epsilon}^\top)^{-1}\bar{\mathbf{v}}\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Gamma \left|\frac{\partial_t \bar{\eta}}{\iota_{\zeta_\epsilon}}\right|^2 \mathrm{d}\mathbf{y},$$

noticing that $|\det(\nabla_{\mathbf{x}}\boldsymbol{\Psi}_{\zeta_\epsilon})|(\nabla_{\mathbf{x}}\boldsymbol{\Psi}_{\zeta_\epsilon}^\top \nabla_{\mathbf{x}}\boldsymbol{\Psi}_{\zeta_\epsilon}^\top)^{-1}\bar{\mathbf{v}} = \bar{\mathbf{u}}$,

$$\begin{aligned}
&\frac{1}{2}\int_{\mathcal{O}_{\zeta_\epsilon(t)}} |\mathbf{u}|^2\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Gamma |\partial_t\eta|^2\,\mathrm{d}\mathbf{y} \\
&\quad = \frac{1}{2}\int_{\mathcal{O}} \bar{\mathbf{v}} \cdot |\det(\nabla_{\mathbf{x}}\boldsymbol{\Psi}_{\zeta_\epsilon})|(\nabla_{\mathbf{x}}\boldsymbol{\Psi}_{\zeta_\epsilon}^\top \nabla_{\mathbf{x}}\boldsymbol{\Psi}_{\zeta_\epsilon}^\top)^{-1}\bar{\mathbf{v}}\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Gamma \left|\frac{\partial_t \bar{\eta}}{\iota_{\zeta_\epsilon}}\right|^2 \mathrm{d}\mathbf{y} \\
&\quad = \frac{1}{2}\int_{\mathcal{O}} \bar{\mathbf{v}}(0)\cdot\bar{\mathbf{u}}(0)\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Gamma |\eta_1|^2\,\mathrm{d}\mathbf{y} + \int_0^t \mathscr{L}\begin{pmatrix}\iota_{\zeta_\epsilon}^{-2}\partial_t\bar{\eta} \\ \bar{\mathbf{u}}\end{pmatrix}\mathrm{d}s \\
&\qquad + \int_0^t\int_\Gamma ((\varkappa\cdot\nabla_{\mathbf{y}})\partial_t\eta)\partial_t\eta\,\mathrm{d}\mathbf{y}\,\mathrm{d}B_s + \frac{1}{2}\int_0^t\int_\Gamma |(\varkappa\cdot\nabla_{\mathbf{y}})\partial_t\eta|^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}s \\
&\qquad + \frac{1}{2}\int_{\mathcal{O}} \bar{\mathbf{v}}\cdot\partial_t\big(|\det(\nabla_{\mathbf{x}}\boldsymbol{\Psi}_{\zeta_\epsilon})|(\nabla_{\mathbf{x}}\boldsymbol{\Psi}_{\zeta_\epsilon}^\top \nabla_{\mathbf{x}}\boldsymbol{\Psi}_{\zeta_\epsilon}^\top)^{-1}\big)\bar{\mathbf{v}}\,\mathrm{d}\mathbf{x} - \int_\Gamma \frac{|\partial_t\bar{\eta}|^2}{\iota_{\zeta_\epsilon}^3}\partial_t\iota_{\zeta_\epsilon}\,\mathrm{d}\mathbf{y}.
\end{aligned}$$

Noticing various cancellations such as

$$\begin{aligned}
\int_{\mathcal{O}} \bar{\mathbf{v}}\cdot\partial_t\big(|\det(\nabla_{\mathbf{x}}\boldsymbol{\Psi}_{\zeta_\epsilon})|(\nabla_{\mathbf{x}}\boldsymbol{\Psi}_{\zeta_\epsilon}^\top \nabla_{\mathbf{x}}\boldsymbol{\Psi}_{\zeta_\epsilon}^\top)^{-1}\big)\bar{\mathbf{v}}\,\mathrm{d}\mathbf{x} &= \int_{\mathcal{O}_{\zeta_\epsilon(t)}} \mathbf{u}\cdot(\partial_t\mathcal{J}_{\zeta_\epsilon(t)}\bar{\mathbf{u}})\,\mathrm{d}\mathbf{x} \\
\int_\Gamma \frac{|\partial_t\bar{\eta}|^2}{\iota_{\zeta_\epsilon}^3}\partial_t\iota_\zeta\,\mathrm{d}\mathbf{y} &= \int_\Gamma \partial_t\eta\,(\partial_t\iota_{\zeta_\varepsilon})\iota_{\zeta_\epsilon}^{-2}\partial_t\bar{\eta}\,\mathrm{d}\mathbf{y}
\end{aligned}$$

we obtain

$$\begin{aligned}
\frac{1}{2}\int_{\mathcal{O}_{\zeta_\epsilon(t)}} |\mathbf{u}|^2\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Gamma |\partial_t\eta|^2\,\mathrm{d}\mathbf{y} =& \frac{1}{2}\int_{\mathcal{O}_{\zeta_\epsilon(0)}} |\mathbf{u}_0|^2\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Gamma |\eta_1|^2\,\mathrm{d}\mathbf{y} \\
&- \int_\Gamma \Delta_{\mathbf{y}}\iota_{\zeta_\varepsilon}\partial_t\bar{\eta}\,\Delta_{\mathbf{y}}\eta\,\mathrm{d}\mathbf{y} \\
&- \int_0^t\int_{\mathcal{O}_\zeta} \nabla_{\mathbf{x}}\mathbf{u} : \nabla_{\mathbf{x}}(\mathcal{J}_{\zeta_\varepsilon(t)}\bar{\mathbf{u}})\,\mathrm{d}\mathbf{x}\,\mathrm{d}s.
\end{aligned}$$

The claim follows on noticing that $\mathbf{u} = \mathcal{J}_{\zeta_\varepsilon(t)}\bar{\mathbf{u}}$ and $\Delta_{\mathbf{y}}(\iota_{\zeta_\varepsilon}\partial_t\eta) = \partial_t\Delta_{\mathbf{y}}\bar{\eta}$. □

3.5 Convergence in Probability

In order to complete the proof of Theorem 6, we make use of (Breit et al. (2018), Chapter 2, Theorem 2.10.3) which is a generalization of the Gyöngy–Krylov characterization of convergence in probability introduced in Gyöngy and Krylov (1996) to quasi-Polish spaces. It applies to situations where pathwise uniqueness and existence of a martingale solution are valid and allows to establish existence of a probabilistically strong solution. We consider two sequences (N_n), $(N_m) \subset \mathbb{N}$ diverging to infinity. Let

$$\boldsymbol{\vartheta}^N := \left[\mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}}\mathbf{u}^N, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}}\nabla_{\mathbf{x}}\mathbf{u}^N, \eta^N, \begin{pmatrix} ((\partial_t\eta^N, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}}\mathbf{u}^N), (\partial_t\eta^N, \mathscr{F}_N^{\zeta_\epsilon}\partial_t\eta^N)) \\ ((\partial_t\eta^N, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}}\mathbf{u}^N), (\partial_t\eta^N, \mathbf{u}^N - \mathscr{F}_N^{\zeta_\epsilon}\partial_t\eta^N)) \end{pmatrix}\right],$$

for $N \in \mathbb{N}$ and set

$$\boldsymbol{\vartheta}^n := \boldsymbol{\vartheta}^{N_n}, \quad \boldsymbol{\vartheta}^m := \boldsymbol{\vartheta}^{N_m}.$$

We also set $\mathbf{u}^n := \mathbf{u}^{N_n}$, $\eta^n := \eta^{N_n}$ and $\mathbf{u}^m := \mathbf{u}^{N_m}$, $\eta^m := \eta^{N_m}$. We consider the collection of joint laws of $(\boldsymbol{\vartheta}^n, \boldsymbol{\vartheta}^m, \mathbf{v}, \zeta, B_t)$ on the extended path space

$$\chi_{\mathtt{ext}} := \left(\chi_{\mathbf{u}} \times \chi_{\nabla\mathbf{u}} \times \chi_\eta \times \chi_{f,g}^2\right)^2 \times \chi_{\mathbf{u}} \times \chi_\eta \times \chi_B.$$

Similarly to Lemma 9 we obtain tightness of

$$\{\mathcal{L}[\boldsymbol{\vartheta}^n, \boldsymbol{\vartheta}^m, \mathbf{v}, \zeta, B_t);\ n, m \in \mathbb{N}\}$$

on $\chi_{\mathtt{ext}}$. Let us take any subsequence $(\boldsymbol{\vartheta}^{n_k}, \boldsymbol{\vartheta}^{m_k}, \mathbf{v}, \zeta, B_t)$. By the Jakubowski-Skorokhod representation theorem we infer for a non-relabelled subsequence the existence of a probability space $(\bar{\Omega}, \bar{\mathfrak{F}}, \bar{\mathbb{P}})$ with a sequence of random variables $(\hat{\boldsymbol{\vartheta}}^{n_k}, \check{\boldsymbol{\vartheta}}^{m_k}, \overline{\mathbf{v}}^k, \overline{\zeta}^k, \overline{B}_t^k)$ converging almost surely in $\chi_{\mathtt{ext}}$ to a random variable $(\hat{\boldsymbol{\vartheta}}, \check{\boldsymbol{\vartheta}}, \overline{\mathbf{v}}, \overline{\zeta}, \overline{B}_t)$. Moreover,

$$\mathcal{L}[\hat{\boldsymbol{\vartheta}}^{n_k}, \check{\boldsymbol{\vartheta}}^{m_k}, \overline{\mathbf{v}}^k, \overline{\zeta}^k, \overline{B}_t^k] = \mathcal{L}[\boldsymbol{\vartheta}^{n_k}, \boldsymbol{\vartheta}^{m_k}, \mathbf{v}^k, \zeta^k, B_t^k]$$

on $\chi_{\mathtt{ext}}$ for all $k \in \mathbb{N}$. Observe that, in particular, $\mathcal{L}[\hat{\boldsymbol{\vartheta}}^{n_k}, \check{\boldsymbol{\vartheta}}^{m_k}, \overline{\mathbf{v}}^k, \overline{\zeta}^k, \overline{B}_t^k]$ converges weakly to the measure $\mathcal{L}[\hat{\boldsymbol{\vartheta}}, \check{\boldsymbol{\vartheta}}, \overline{\mathbf{v}}, \overline{\zeta}, \overline{B}_t]$. As in Sect. 3.3 we can show that the limit objects are martingale solutions to (3.5) defined on the same stochastic basis $(\bar{\Omega}, \bar{\mathfrak{F}}, (\bar{\mathfrak{F}}_t), \bar{\mathbb{P}})$, where $(\bar{\mathfrak{F}}_t)_{t\geq 0}$ is the $\bar{\mathbb{P}}$-augmented canonical filtration of $(\hat{\boldsymbol{\vartheta}}, \check{\boldsymbol{\vartheta}}, \overline{\mathbf{v}}, \overline{\zeta}, \overline{B}_t)$. We employ the pathwise uniqueness result from Proposition 11. Therefore, the solutions $(\hat{\eta}, \hat{\mathbf{u}})$ and $(\check{\eta}, \check{\mathbf{u}})$ coincide $\bar{\mathbb{P}}$-a.s. and we have

$$\mathcal{L}[\hat{\boldsymbol{\vartheta}}, \check{\boldsymbol{\vartheta}}, \overline{\mathbf{v}}, \overline{\zeta}, \overline{B}_t]\Big((\boldsymbol{\vartheta}_1, \boldsymbol{\vartheta}_2, \mathbf{v}, \zeta, B_t) \in \chi_{\mathtt{ext}} :\ (\eta_1, \mathbf{u}_1) = (\eta_2, \mathbf{u}_2)\Big) = \bar{\mathbb{P}}((\hat{\eta}, \hat{\mathbf{u}}) = (\check{\eta}, \check{\mathbf{u}})) = 1.$$

Now, we have all in hand to apply the Gyöngy–Krylov theorem from (Breit et al. (2018), Chapter 2, Theorem 2.10.3). It implies that the original sequence $\boldsymbol{\vartheta}^N$ defined on the initial probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ converges in probability in the topology of $\chi_{\mathbf{u}} \times \chi_{\nabla\mathbf{u}} \times \chi_\eta \times \chi_{f,g}^2$ to the random variable

$$\boldsymbol{\vartheta} := \left[\mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}}\mathbf{u}, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}}\nabla_{\mathbf{x}}\mathbf{u}, \eta, \begin{pmatrix} ((\partial_t\eta, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}}\mathbf{u}), (\partial_t\eta, \mathscr{F}^{\zeta_\epsilon}\partial_t\eta)) \\ ((\partial_t\eta, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}}\mathbf{u}), (0, \mathbf{u} - \mathscr{F}^{\zeta_\epsilon}\partial_t\eta)) \end{pmatrix}\right].$$

Therefore, we finally deduce that $(\eta, \mathbf{u})$ is a probabilistically strong solution to (3.5). This completes the proof of Theorem 7.

4 The Nonlinear Regularised Problem

The aim of this section is to obtain a solution of the regularised problem thus completing the proof of Theorem 6. This is done via a fixed point argument for which the main point is proving compactness of the mapping $(\zeta, \mathbf{v}) \mapsto (\eta, \mathbf{u})$. Given a bounded sequence $(\zeta_n, \mathbf{v}_n)$ this is achieved in three steps:

- We first need to establish tightness of the probability laws, where the main difficulty arises for the velocity field.
- We apply the stochastic compactness method based on Jakubowski's extension of the Skorokhod representation theorem. This yields a.s. convergence on a new probability space.
- We apply a Gyöngy–Krylov type argument to obtain convergence in probability of the original sequence on the original probability space. This requires the pathwise uniqueness from Proposition 11.

Suppose there is a sequence of processes $(\zeta^n, \mathbf{v}^n)$ which are $(\mathfrak{F}_t)$-progressively measurable and bounded in

$$L^p(\Omega; C(\overline{I} \times \Gamma)) \times L^p(\Omega; L^2(I; L^2(\mathcal{O} \cup S_\alpha)))$$

for some sufficiently large p. Now apply Theorem 7 yielding a sequence $(\eta^n, \mathbf{u}^n)$ of solutions to (3.5). By the energy equality from Proposition 11 we obtain

$$\sup_I \|\eta^n\|^2_{W^{2,2}(\Gamma)} + \epsilon \sup_I \|\eta^n\|^2_{W^{3,2}(\Gamma)} \lesssim 1, \tag{4.1}$$

$$\sup_I \|\partial_t \eta^n\|^2_{L^2(\Gamma)} \lesssim 1, \tag{4.2}$$

$$\epsilon \int_I \|\partial_t \eta^n\|^2_{W^{2,2}(\Gamma)} \,\mathrm{d}t \lesssim 1, \tag{4.3}$$

$$\sup_I \|\mathbf{u}^n\|^2_{L^2(\mathcal{O}_{\zeta^n_\epsilon})} \lesssim 1, \tag{4.4}$$

$$\int_I \|\nabla_{\mathbf{x}} \mathbf{u}^n\|^2_{L^2(\mathcal{O}_{\zeta^n_\epsilon})} \,\mathrm{d}t \lesssim 1. \tag{4.5}$$

In addition, for any $s \in (0, \frac{1}{2})$, it follows from $\mathbf{u}^n \circ \boldsymbol{\varphi}_{\zeta^n_\epsilon} = \mathbf{n}\partial_t \eta^n$, (4.5) and the trace theorem that

$$\int_I \|\partial_t \eta^n\|^2_{W^{s,2}(\Gamma)} \,\mathrm{d}t \lesssim 1 \tag{4.6}$$

holds.

4.1 Tightness of the Velocity Sequence

The effort of this subsection is to prove tightness of the law of $\mathbf{u}^n$. Similarly to Lemma 8 we obtain the following result.

Lemma 12 *The laws of*

$$((\partial_t \eta^n, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon^n}} \boldsymbol{u}^n), (\partial_t \eta^n, \mathscr{F}^{\zeta_\epsilon^n} \partial_t \eta^n)) \quad and \quad ((\partial_t \eta^n, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon^n}} \boldsymbol{u}^n), (0, \boldsymbol{u}^n - \mathscr{F}^{\zeta_\epsilon^n} \partial_t \eta^n))$$

on $(L^2(I; X' \times X), \tau_\sharp)$ *are tight.*

Proof As in Lemma 8 we must verify the assumptions from Theorem 1. First of all, boundedness in $L^2(I; X' \times X)$ follows from By (4.1)–(4.6) and the properties of the extension $\mathscr{F}^{\zeta_n}$ from Proposition 2. For (b) we consider again the regularisation of $\partial_t \eta^n$ with parameter $\kappa > 0$ and set for $n \in \mathbb{N}$

$$f_{n,\kappa}(t) := ((\partial_t \eta^n(t))_\kappa, \mathcal{E}_\kappa^{\zeta_\epsilon^n}(\partial_t \eta^n(t))),$$

where $\mathcal{E}_\kappa^{\zeta_\epsilon^n}$ is given in Corollary 3. We find by the continuity of the mollification operator from (3.26) and the continuity of $\mathcal{E}_\kappa^{\zeta_\epsilon^n}$ from Corollary 3 that for a.e. $t \in (0, T)$ and $s < 1/2$

$$\| f_{n,\kappa} - f_n \|_{L^2(\Gamma) \times L^2(\mathcal{O} \cup S_\alpha)} \le c\kappa^s \|\partial_t \eta^n\|_{W^{s,2}(\Gamma)}, \tag{4.7}$$

which can be made arbitrarily small in L^2 by choosing κ appropriately, cf. (4.6). Similarly, we have

$$\| f_{n,\kappa} \|_{W^{1,2}(\Gamma) \times W^{1,2}(\Omega \cup S_\alpha)} \le c\kappa^{-1} \|\partial_t \eta^n\|_{L^2(\Gamma)}.$$

Clearly, if $f_N \rightharpoonup f$ in $L^2(I; X)$ then we can deduce a converging subsequence such that $f_{N,\kappa} \rightharpoonup f_\kappa$ (for some f_κ) in $L^2(I; X)$ for any $\kappa > 0$, which implies (b).

For (c) we have to control $\langle g_n(t) - g_n(s), f_{n,\kappa}(t) \rangle$, where $g_n := (\partial_t \eta^n, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon^n}} \mathbf{u}^n)$ and hence decompose

$$\begin{aligned}
&\langle g_n(t) - g_n(s), f_{n,\kappa}(t) \rangle \\
&\quad = \big(\langle g_n(t), (\partial_t \eta^n(t))_\kappa, \mathcal{E}_\kappa^{\zeta_\epsilon^n(t)}(\partial_t \eta^n(t))) \rangle \\
&\qquad - \langle g_n(s), (\partial_t \eta_n(t))_\kappa, \mathcal{E}_\kappa^{\zeta_\epsilon^n(s)}(\partial_t \eta^n(t))) \rangle \big) \\
&\qquad + \langle g_N(s), (0, \mathcal{E}_\kappa^{\zeta_\epsilon^n(t)}(\partial_t \eta^n(t))) - \mathcal{E}_\kappa^{\zeta_\epsilon^n(s)}(\partial_t \eta^n(t)) \rangle =: (I) + (II).
\end{aligned}$$

We begin estimating (II) to find that

$$\begin{aligned}
(II) &= \int_{\mathcal{O}_{\zeta_\epsilon^n(s)}} \mathbf{u}^n(s) \cdot \int_t^s \partial_r \mathcal{E}_\kappa^{\zeta_\epsilon^n(r)}(\partial_t \eta^n(t)) \, \mathrm{d}r \, \mathrm{d}\mathbf{x} \\
&\le c \|\mathbf{u}^n(s)\|_{L^2(\mathcal{O}_{\zeta_\epsilon^n(s)})} \|\partial_t (\zeta_\epsilon^n)_\kappa\|_{L^\infty(I \times \Gamma)} \|\partial_t \eta^n(t)\|_{L^\infty(I; L^2(\Gamma))} |t - s| \\
&\le c(\kappa) \|\mathbf{u}^n(s)\|_{L^2(\mathcal{O}_{\zeta_\epsilon^n(s)})} \|\partial_t \zeta_\epsilon^n\|_{L^\infty(I; L^2(\Gamma)} \|\partial_t \eta^n(t)\|_{L^\infty(I; L^2(\Gamma))} |t - s|.
\end{aligned}$$

using Corollary 3. By (4.2) and (4.4) we thus get

$$\begin{aligned}|(II)| &\leq c(\kappa)\|\partial_t\zeta_\epsilon^n\|_{L^\infty(I;L^2(\Gamma))}|t-s|\\ &\leq c(\kappa,\epsilon)\|\zeta^n\|_{L^\infty(I;L^2(\Gamma))}|t-s|.\end{aligned}$$

As in the proof of Lemma 8 we conclude for $K>0$ that

$$\mathbb{P}\Big(|t-s|^{-1}|(II)|\geq K\Big)\leq \frac{c(\kappa,\epsilon)}{K},$$

provided $\zeta\in L^1(\Omega;L^\infty(I;L^2(\Gamma)))$. The term (I) is estimated using (3.5) obtaining (this can be justified by Itô's formula similarly to the proof of Proposition 11)

$$\begin{aligned}(I)=&-\int_t^s\int_\Gamma\big(\epsilon\Delta_{\mathbf{y}}(\partial_t\eta^n(t))_\kappa\,\partial_t\Delta_{\mathbf{y}}\eta^n+\Delta_{\mathbf{y}}(\partial_t\eta^n(t))_\kappa\,\Delta_{\mathbf{y}}\eta^n\big)\,\mathrm{d}\mathbf{y}\,\mathrm{d}r\\
&-\epsilon\int_t^s\int_\Gamma\nabla_{\mathbf{y}}^3(\partial_t\eta^n(t))_\kappa:\nabla_{\mathbf{y}}^3\eta^n\,\mathrm{d}\mathbf{y}\,\mathrm{d}r\\
&+\int_t^s\int_\Gamma\frac{1}{2}\mathbf{n}_{\zeta_\epsilon^n}\cdot\mathbf{n}(\partial_t\eta^n(t))_\kappa\,\partial_t\zeta_\epsilon\,\partial_t\eta^n\,|\det(\nabla_{\mathbf{y}}\boldsymbol{\varphi}_{\zeta_\epsilon})|\,\mathrm{d}\mathbf{y}\,\mathrm{d}r\\
&+\int_t^s\int_{\mathcal{O}_{\zeta_\epsilon}}\Big(\mathbf{u}\cdot\partial_r\mathcal{E}_\kappa^{\zeta_\epsilon^n(r)}(\partial_t\eta^n(t)))-\frac{1}{2}((\mathbf{v}_\epsilon\cdot\nabla_{\mathbf{x}})\mathbf{u})\cdot\mathcal{E}_\kappa^{\zeta_\epsilon^n(r)}(\partial_t\eta^n(t))\Big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}r\\
&+\int_t^s\int_{\mathcal{O}_\zeta}\Big(\frac{1}{2}((\mathbf{v}_\epsilon\cdot\nabla_{\mathbf{x}})\mathcal{E}_\kappa^{\zeta_\epsilon^n(r)}(\partial_t\eta^n(t)))\cdot\mathbf{u}-\nabla_{\mathbf{x}}\mathbf{u}:\nabla_{\mathbf{x}}\mathcal{E}_\kappa^{\zeta_\epsilon^n(r)}(\partial_t\eta^n(t)))\Big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}r\\
&+\frac{1}{2}\int_t^s\int_\Gamma((\varkappa\cdot\nabla_{\mathbf{y}})(\varkappa\cdot\nabla_{\mathbf{y}})\partial_t\eta)(\partial_t\eta^n(t))_\kappa\,\mathrm{d}\mathbf{y}\,\mathrm{d}r\\
&-\int_t^s\int_\Gamma((\varkappa\cdot\nabla_{\mathbf{y}})\partial_t\eta^n)(\partial_t\eta^n(t))_\kappa\,\mathrm{d}\mathbf{y}\,\mathrm{d}B_r.\end{aligned}\tag{4.8}$$

Due to (3.26), Corollary 3 and the uniform estimates (4.1)–(4.6), it is clear that all terms can be estimated by $c(\kappa)|t-s|^{1/2}(1+\|\zeta^n\|_{C(I\times\Gamma)}+\|\nabla_{\mathbf{x}}\mathbf{u}^n\|_{L^2(\mathcal{O}_{\zeta_\epsilon^n})})$ except for the last one. As in the proof of Lemma 8 it can be controlled via

$$\begin{aligned}\mathbb{P}\bigg(|t-s|^{-1/2}\bigg|\int_t^s\int_\Gamma((\varkappa\cdot\nabla_{\mathbf{y}})\partial_t\eta^n)(\partial_t\eta^n)_\kappa\,\mathrm{d}\mathbf{y}\,\mathrm{d}B_r\bigg|>K\bigg)&\\
\lesssim\frac{\kappa^{-1}}{K^2}\mathbb{E}\bigg[|t-s|^{-1/2}\int_t^s\|\partial_t\eta^n(r)\|_{L^2(\Gamma)}\|\nabla_{\mathbf{y}}(\partial_t\eta^n(t))_\kappa\|_{L^2(\Gamma)}\,\mathrm{d}r\bigg]&\\
\lesssim\frac{1}{K^2}\mathbb{E}\bigg[|t-s|^{-1/2}\int_t^s\|\partial_t\eta^n(r)\|_{L^2(\Gamma)}\|\partial_t\eta^n(t)\|_{L^2(\Gamma)}\,\mathrm{d}r\bigg]\lesssim\frac{\kappa^{-1}}{K^2}&\end{aligned}$$

using (4.2). This can be made arbitrarily small for K large. Noticing that property (d) from Theorem 1 follows by the usual compactness in (negative) Sobolev spaces, we conclude tightness of the law of $((\partial_t\eta^n,\mathbb{I}_{\mathcal{O}_{\zeta_\epsilon^n}}\mathbf{u}^n),(\partial_t\eta^n,\mathscr{F}^{\zeta_\epsilon^n}\eta^N))$ on $(L^2(I;X'\times X),\tau_\sharp)$.

The tightness for $((\partial_t\eta^n,\mathbb{I}_{\mathcal{O}_{\zeta_\epsilon^n}}\mathbf{u}^n),(0,\mathbf{u}^n-\mathscr{F}^{\zeta_\epsilon^n}\partial_t\eta^n))$ follows along the same lines, the only difference being the regularisation of

$$f_n:=(0,\mathbf{u}^n-\mathscr{F}^{\zeta_\epsilon^n}\partial_t\eta^n).$$

While $\mathscr{F}^{\zeta_\epsilon^n}\partial_t\eta^n$ can be replaced by $\mathcal{E}_\kappa^{\zeta_\epsilon^n(s)}(\partial_t\eta^n(t))$ as above, we need to regularise $\mathbf{u}^n$ accordingly to preserve the homogeneous boundary conditions on f_n, which can by done by using (Lengeler and Ružička (2014), Proposition 2.28 and Lemma A.13). This leads to a function $f_{n,\kappa}$ which has zero boundary conditions and thus only the fluid part is seen in the expression $\langle g_n(t) - g_n(s), f_{n,\kappa}(t)\rangle$ (where $g_n := (\partial_t\eta^n, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon^n}}\mathbf{u}^n)$ as above). In particular, the noise is not seen and we conclude that

$$\langle g_n(t) - g_n(s), f_{n,\kappa}(t)\rangle \leq c(\kappa)|t-s|^{1/2}(1 + \|\zeta^n\|_{C(\overline{I}\times\Gamma)} + \|\nabla_{\mathbf{x}}\mathbf{u}^n\|_{L^2(\mathcal{O}_{\zeta_\epsilon^n})})$$

and we obtain again the claimed tightness. $\square$

4.2 Passage to the Limit

With the bounds from (4.1)–(4.6) at hand, we wish to obtain compactness. We consider the same path space χ as in Sect. 3.3. From (3.13)–(3.18) (together the Alaoglu-Burbaki Theorem) and Lemma 8 we obtain similarly to Proposition 10:

Proposition 13 *There exists a complete probability space $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbb{P}})$ with χ-valued random variables*[7]

$$\tilde{\mathbf{\Theta}}^n := \left[\tilde{B}_t^n, \mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon^n}}\tilde{\boldsymbol{u}}^n, \tilde{\boldsymbol{v}}^n, \mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon^n}}\nabla_{\boldsymbol{x}}\tilde{\boldsymbol{u}}^n, \tilde{\eta}^n, \tilde{\zeta}^n, \begin{pmatrix} ((\partial_t\tilde{\eta}^n, \mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon^n}}\tilde{\boldsymbol{u}}^n), (\partial_t\tilde{\eta}^n, \mathscr{F}^{\tilde{\zeta}_\epsilon^n}\partial_t\tilde{\eta}^n)) \\ ((0, \mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon^n}}\tilde{\boldsymbol{u}}^n), (\partial_t\tilde{\eta}^n, \tilde{\boldsymbol{u}}^n - \mathscr{F}^{\tilde{\zeta}_\epsilon^n}\partial_t\tilde{\eta}^n)) \end{pmatrix}\right],$$

for $n \in \mathbb{N}$ and

$$\tilde{\mathbf{\Theta}} := \left[\tilde{B}_t, \mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon}}\tilde{\boldsymbol{u}}^n, \tilde{\boldsymbol{v}}, \mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon}}\nabla_{\boldsymbol{x}}\tilde{\boldsymbol{u}}, \tilde{\eta}, \tilde{\zeta}, \begin{pmatrix} ((\partial_t\eta, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}}\boldsymbol{u}^n), (\partial_t\tilde{\eta}, \mathscr{F}^{\tilde{\zeta}_\epsilon}\partial_t\tilde{\eta})) \\ ((0, \mathbb{I}_{\mathcal{O}_{\tilde{\eta}}}\tilde{\boldsymbol{u}}), (\partial_t\tilde{\eta}, \tilde{\boldsymbol{u}} - \mathscr{F}^{\tilde{\zeta}_\epsilon}\partial_t\tilde{\eta})) \end{pmatrix}\right]$$

such that

(a) for all $n \in \mathbb{N}$ the law of $\tilde{\mathbf{\Theta}}^n$ on χ is given by

$$\mathcal{L}\left[B_t, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon^n}}\boldsymbol{u}^n, \boldsymbol{v}^n, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon^n}}\nabla_{\boldsymbol{x}}\boldsymbol{u}^n, \eta^n, \zeta^n, \begin{pmatrix} ((\partial_t\eta^n, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon^n}}\boldsymbol{u}^n), (\partial_t\eta^n, \mathscr{F}^{\zeta^n}\partial_t\eta^n)) \\ ((0, \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon^n}}\boldsymbol{u}^n), (\partial_t\eta^n, \boldsymbol{u}^n - \mathscr{F}^{\zeta_\epsilon^n}\partial_t\eta^n)) \end{pmatrix}\right]$$

[7] The fact that the variable $\mathscr{F}^{\tilde{\zeta}_\epsilon^n}\partial_t\tilde{\eta}^n$ can be represented in that form follows from the measureability of $\mathscr{F}$ on the pathspace, cf. Proposition 2.

(b) $\tilde{\boldsymbol{\Theta}}^n$ *converges* $\tilde{\mathbb{P}}$*-almost surely to* $\tilde{\boldsymbol{\Theta}}$ *in the topology of* χ*, i.e.*

$$\begin{aligned}
\tilde{B}^n_t &\to \tilde{B}_t \quad \text{in} \quad C(\overline{I})\ \tilde{\mathbb{P}}\text{-a.s.},\\
\mathbb{I}_{\mathcal{O}_{\tilde{\zeta}^n_\epsilon}}\tilde{\boldsymbol{u}}^n,\ \tilde{\boldsymbol{v}}^n &\rightharpoonup \mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon}}\tilde{\boldsymbol{u}},\ \tilde{\boldsymbol{v}} \quad \text{in} \quad L^2(I; L^2(\mathcal{O}\cup S_\alpha))\ \tilde{\mathbb{P}}\text{-a.s.},\\
\mathbb{I}_{\mathcal{O}_{\tilde{\zeta}^n_\epsilon}}\nabla_{\boldsymbol{x}}\tilde{\boldsymbol{u}}^n &\rightharpoonup \mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon}}\nabla_{\boldsymbol{x}}\tilde{\boldsymbol{u}} \quad \text{in} \quad L^2(I; L^2(\mathcal{O}\cup S_\alpha))\ \tilde{\mathbb{P}}\text{-a.s.},\\
\tilde{\eta}^n &\rightharpoonup^* \tilde{\eta} \quad \text{in} \quad L^\infty(I; W^{3,2}(\Gamma))\ \tilde{\mathbb{P}}\text{-a.s.},\\
\tilde{\eta}^n &\rightharpoonup^* \tilde{\eta} \quad \text{in} \quad W^{1,\infty}(I; L^2(\Gamma))\ \tilde{\mathbb{P}}\text{-a.s.},\\
\tilde{\eta}^n &\rightharpoonup \tilde{\eta} \quad \text{in} \quad W^{1,2}(I; W^{2,2}(\Gamma))\ \tilde{\mathbb{P}}\text{-a.s.},\\
\tilde{\zeta}^n &\to \tilde{\zeta} \quad \text{in} \quad W^{1,2}(I\times\Gamma)\ \tilde{\mathbb{P}}\text{-a.s.},
\end{aligned} \tag{4.9}$$

as well as (recalling the definition of $\tau_\sharp$ *from* (3.25)*)*

$$\begin{aligned}
&\int_I\int_{\mathcal{O}\cup S_\alpha}\mathbb{I}_{\mathcal{O}_{\tilde{\zeta}^n_\epsilon}}\boldsymbol{u}^n\cdot\mathscr{F}^{\tilde{\zeta}^n_\epsilon}\partial_t\tilde{\eta}^n\,d\boldsymbol{x}\,dt+\int_I\int_\Gamma|\partial_t\tilde{\eta}^n|^2\ d\boldsymbol{y}\,dt\\
&\quad\longrightarrow\int_I\int_{\mathcal{O}\cup S_\alpha}\mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon}}\tilde{\boldsymbol{u}}\cdot\mathscr{F}^{\tilde{\zeta}_\epsilon}\partial_t\eta\,d\boldsymbol{x}\,dt+\int_I\int_\Gamma|\partial_t\tilde{\eta}|^2\ d\boldsymbol{y}\,dt
\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}
&\int_I\int_{\mathcal{O}\cup S_\alpha}\mathbb{I}_{\mathcal{O}_{\tilde{\zeta}^n_\epsilon}}\tilde{\boldsymbol{u}}^n\cdot(\tilde{\boldsymbol{u}}^n-\mathscr{F}^{\tilde{\zeta}^n_\epsilon}\partial_t\tilde{\eta}^n)\,d\boldsymbol{x}\,dt\\
&\quad\longrightarrow\int_I\int_{\mathcal{O}\cup S_\alpha}\mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon}}\tilde{\boldsymbol{u}}\cdot(\tilde{\boldsymbol{u}}-\mathscr{F}^{\tilde{\zeta}_\epsilon}\partial_t\tilde{\eta})\,d\boldsymbol{x}\,dt
\end{aligned} \tag{4.11}$$

$\tilde{\mathbb{P}}$*-a.s.*

Now we introduce the filtration on the new probability space, which ensures the correct measurabilities of the new random variables. Let $(\tilde{\mathfrak{F}}_t)_{t\geq 0}$ and $(\tilde{\mathfrak{F}}^n_t)_{t\geq 0}$ be the $\tilde{\mathbb{P}}$-augmented canonical filtration on the variables $\tilde{\boldsymbol{\Theta}}$ and $\tilde{\boldsymbol{\Theta}}^n$, respectively, that is[8]

$$\tilde{\mathfrak{F}}_t=\sigma\big[\sigma_t(\tilde{B}_t)\cup\sigma_t(\mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon}}\tilde{\mathbf{u}})\cup\sigma_t(\tilde{\mathbf{v}})\cup\sigma_t(\mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon}}\nabla_{\mathbf{x}}\tilde{\mathbf{u}})\cup\sigma_t(\tilde{\eta})\cup\sigma_t(\tilde{\zeta})\big]$$

for $t\in I$ and similarly for $\tilde{\mathfrak{F}}^n_t$. By (Breit et al. (2018), Theorem 2.9.1) the weak equation continuous to hold on the new probability space. Combining (4.10) and (4.11) we have

$$\begin{aligned}
&\int_I\int_{\mathcal{O}_{\tilde{\zeta}^n_\epsilon}}|\tilde{\mathbf{u}}^n|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\int_I\int_\Gamma|\partial_t\tilde{\eta}^n|^2\ \mathrm{d}\mathbf{y}\,\mathrm{d}t\\
&\quad\longrightarrow\int_I\int_{\mathcal{O}_{\tilde{\zeta}_\epsilon}}|\tilde{\mathbf{u}}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\int_I\int_\Gamma|\partial_t\tilde{\eta}|^2\ \mathrm{d}\mathbf{y}\,\mathrm{d}t
\end{aligned}$$

[8] Some of the variables are not continuous in time. For those, one can define σ_t as the history of a random distribution, cf. (Breit et al. (2018), Chapter 2.8)

$\tilde{\mathbb{P}}$-a.s. By uniform convexity of the L^2-norm this implies

$$\begin{aligned}
\tilde{\eta}^n &\to \tilde{\eta} \quad \text{in} \quad W^{1,2}(I; L^2(\Gamma)),\\
\mathbb{I}_{\mathcal{O}_{\tilde{\zeta}^n_\epsilon}} \tilde{\mathbf{u}}^n &\to \mathbb{I}_{\mathcal{O}_{\tilde{\zeta}_\epsilon}} \tilde{\mathbf{u}} \quad \text{in} \quad L^2(I; L^2(\mathcal{O} \cup S_\alpha)) \ \tilde{\mathbb{P}}\text{-a.s..}
\end{aligned}$$

We can now apply Proposition 11 and argue as in Sect. 3.5 to obtain convergence of the original sequence, in particular,

$$\begin{aligned}
\eta^n &\to \eta \quad \text{in} \quad C(\overline{I} \times \Gamma) \quad \text{in probability},\\
\mathbb{I}_{\mathcal{O}_{\zeta^n_\epsilon}} \mathbf{u}^n &\to \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}} \mathbf{u} \quad \text{in} \quad L^2(I; L^2(\mathcal{O} \cup S_\alpha)) \quad \text{in probability}.
\end{aligned}$$

This yields due the uniform-in-probability estimates (for arbitrary $p < \infty$)

$$\begin{aligned}
\eta^n &\to \eta \quad \text{in} \quad L^p(\Omega; C(\overline{I} \times \Gamma)),\\
\mathbb{I}_{\mathcal{O}_{\zeta^n_\epsilon}} \mathbf{u}^n &\to \mathbb{I}_{\mathcal{O}_{\zeta_\epsilon}} \mathbf{u} \quad \text{in} \quad L^p(\Omega; L^2(I; L^2(\mathcal{O} \cup S_\alpha))),
\end{aligned}$$

which gives the desired compactness of the fixed-point map.

5 The Limit $\varepsilon \to 0$

For fixed $\epsilon > 0$, Theorem 6 yields the existence of a probabilistically strong solution $(\eta^\epsilon, \mathbf{u}^\epsilon)$ to the regularised fluid–structure system defined on a given stochastic basis. Using the energy balance we obtain

$$\left(\sup_I \|\eta^\epsilon\|^2_{W^{2,2}(\Gamma)} + \epsilon \sup_I \|\eta^\epsilon\|^2_{W^{3,2}(\Gamma)}\right) \lesssim 1, \tag{5.1}$$

$$\sup_I \|\partial_t \eta^\epsilon\|^2_{L^2(\Gamma)} \lesssim 1, \tag{5.2}$$

$$\epsilon \int_I \|\partial_t \eta^\epsilon\|^2_{W^{2,2}(\Gamma)} \,\mathrm{d}t \lesssim 1, \tag{5.3}$$

$$\sup_I \|\mathbf{u}^\epsilon\|^2_{L^2(\mathcal{O}_{(\eta^\epsilon)_\epsilon})} \lesssim 1, \tag{5.4}$$

$$\int_I \|\nabla_{\mathbf{x}} \mathbf{u}^\epsilon\|^2_{L^2(\mathcal{O}_{(\eta^\epsilon)_\epsilon})} \,\mathrm{d}t \lesssim 1. \tag{5.5}$$

In addition, for any $s \in (0, \frac{1}{2})$, it follows from $\mathbf{u}^\epsilon \circ \boldsymbol{\varphi}_{(\eta^\epsilon)_\epsilon} = \mathbf{n}\partial_t \eta^\epsilon$, (4.5) and the trace theorem that

$$\int_I \|\partial_t \eta^\epsilon\|^2_{W^{s,2}(\Gamma)} \,\mathrm{d}t \lesssim 1. \tag{5.6}$$

Moreover, we can argue as in Proposition 12 to obtain:

Lemma 14 *The laws of*

$$((\partial_t\eta^\epsilon, \mathbb{I}_{\mathcal{O}_{(\eta^\epsilon)_\epsilon}}\boldsymbol{u}^\epsilon), (\partial_t\eta^\epsilon, \mathscr{F}^{\eta^\epsilon}\partial_t\eta^\epsilon)) \quad and \quad ((\partial_t\eta^\epsilon, \mathbb{I}_{\mathcal{O}_{(\eta^\epsilon)_\epsilon}}\boldsymbol{u}^\epsilon), (\partial_t\eta^\epsilon, \boldsymbol{u}^\epsilon - \mathscr{F}^{\eta^\epsilon}\partial_t\eta^\epsilon))$$

on $(L^2(I; X' \times X), \tau_\sharp)$ *are tight.*

Unfortunately, (5.1)–(5.6) is no longer sufficient which is why we first improve the regularity of the shell by adapting a method from Muha and Schwarzacher (2022) (which we also applied in Breit and Mensah (2021)).

5.1 Higher Regularity

As already explained in the introduction, the regularity arising from (5.1)–(5.6) is not sufficient to control the terms involving the Piola transform in the weak equation. Thus we aim at improving the spatial regularity of η implementing ideas from Muha and Schwarzacher (2022). Finally, for some $h > 0$, we let $\Delta_h^s f(\mathbf{y}) = h^{-s}(f(\mathbf{y}+\mathbf{e}_i h) - f(\mathbf{y}))$ for $i = 1, 2$ represent the fractional difference quotient in space in the direction $\mathbf{e}_i$. Now, for

$$D^{s,\mathscr{K}}_{-h,h} f := \Delta^s_{-h}\Delta^s_h f - \mathscr{K}_\eta(\Delta^s_{-h}\Delta^s_h f),$$

where $s \in (0, \frac{1}{2})$, we consider the following as test function

$$(\boldsymbol{\phi}, \phi) = \big(\mathcal{J}^{-1}_{\eta^\epsilon}\big(\mathscr{F}^{(\eta^\epsilon)_\epsilon}(D^{s,\mathscr{K}}_{-h,h}\eta^\epsilon)\big),\ \iota^{-1}_{\eta^\epsilon} D^{s,\mathscr{K}}_{-h,h}\eta^\epsilon\big).$$

in the weak formulation of the regularised fluid–structure system (this can be justified by Itô's formula similarly to the proof of Proposition 11). By making the fourth order term the subject, we obtain

$$\begin{aligned}
&\int_\Gamma \Delta_{\mathbf{y}}\mathscr{K}_\eta(\Delta^s_{-h}\Delta^s_h\eta^\epsilon)\Delta_{\mathbf{y}}\eta^\epsilon\,\mathrm{d}\mathbf{y}\,\mathrm{d}t - \int_\Gamma \Delta_{\mathbf{y}}(\Delta^s_{-h}\Delta^s_h\eta^\epsilon)\Delta_{\mathbf{y}}\eta^\epsilon\,\mathrm{d}\mathbf{y}\,\mathrm{d}t \\
&\quad + \epsilon\int_\Gamma \nabla^3_{\mathbf{y}}\mathscr{K}_\eta(\Delta^s_{-h}\Delta^s_h\eta^\epsilon) : \nabla^3_{\mathbf{y}}\eta^\epsilon\,\mathrm{d}\mathbf{y}\,\mathrm{d}t - \epsilon\int_\Gamma \nabla^3_{\mathbf{y}}(\Delta^s_{-h}\Delta^s_h\eta^\epsilon) : \nabla^3_{\mathbf{y}}\eta^\epsilon\,\mathrm{d}\mathbf{y}\,\mathrm{d}t \\
&\quad + \epsilon\int_\Gamma \Delta_{\mathbf{y}}\mathscr{K}_\eta(\Delta^s_{-h}\Delta^s_h\eta^\epsilon)\partial_t\Delta_{\mathbf{y}}\eta^\epsilon\,\mathrm{d}\mathbf{y}\,\mathrm{d}t - \epsilon\int_\Gamma \Delta_{\mathbf{y}}(\Delta^s_{-h}\Delta^s_h\eta^\epsilon)\partial_t\Delta_{\mathbf{y}}\eta^\epsilon\,\mathrm{d}\mathbf{y}\,\mathrm{d}t \\
&= \mathrm{d}\int_{\mathcal{O}_{(\eta^\epsilon)_\epsilon}} \mathbf{u}^\epsilon\cdot\mathscr{F}^{(\eta^\epsilon)_\epsilon}(D^{s,\mathscr{K}}_{-h,h}\eta^\epsilon(t))\,\mathrm{d}\mathbf{x} + \mathrm{d}\int_\Gamma \partial_t\eta^\epsilon(D^{s,\mathscr{K}}_{-h,h}\eta^\epsilon(t))\,\mathrm{d}\mathbf{y} \\
&\quad - \int_\Gamma \partial_t\eta^\epsilon\partial_t(D^{s,\mathscr{K}}_{-h,h}\eta^\epsilon)\,\mathrm{d}\mathbf{y}\,\mathrm{d}t \\
&\quad - \frac{1}{2}\int_\Gamma \mathbf{n}_{\eta^\epsilon}\cdot\mathbf{n}^\top(D^{s,\mathscr{K}}_{-h,h}\eta^\epsilon)\,\partial_t(\eta^\epsilon)_\epsilon\,\partial_t\eta^\epsilon\,|\det(\nabla_{\mathbf{y}}\boldsymbol{\varphi}_{(\eta^\epsilon)_\epsilon})|\,\mathrm{d}\mathbf{y}\,\mathrm{d}t \\
&\quad - \int_{\mathcal{O}_{(\eta^\epsilon)_\epsilon}} \mathbf{u}^\epsilon\cdot\partial_t(\mathscr{F}^{(\eta^\epsilon)_\epsilon}(D^{s,\mathscr{K}}_{-h,h}\eta^\epsilon))\,\mathrm{d}\mathbf{x}\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{\mathcal{O}_{(\eta^\epsilon)_\epsilon}}(((\mathbf{u}^\epsilon)_\epsilon\cdot\nabla_{\mathbf{x}})\mathbf{u}^\epsilon)\cdot\mathscr{F}^{(\eta^\epsilon)_\epsilon}(D^{s,\mathscr{K}}_{-h,h}\eta^\epsilon)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&-\frac{1}{2}\int_{\mathcal{O}_{(\eta^\epsilon)_\epsilon}}(((\mathbf{u}^\epsilon)_\epsilon\cdot\nabla_{\mathbf{x}})\mathscr{F}^{(\eta^\epsilon)_\epsilon}(D^{s,\mathscr{K}}_{-h,h}\eta^\epsilon))\cdot\mathbf{u}^\epsilon\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&+\int_{\mathcal{O}_{(\eta^\epsilon)_\epsilon}}\nabla_{\mathbf{x}}\mathbf{u}^\epsilon:\nabla_{\mathbf{x}}\mathscr{F}^{(\eta^\epsilon)_\epsilon}(D^{s,\mathscr{K}}_{-h,h}\eta^\epsilon)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&-\frac{1}{2}\int_\Gamma((\varkappa\cdot\nabla_{\mathbf{x}})(\varkappa\cdot\nabla_{\mathbf{x}})\partial_t\eta^\epsilon)D^{s,\mathscr{K}}_{-h,h}\eta^\epsilon\,\mathrm{d}\mathbf{y}\,\mathrm{d}t+\int_\Gamma((\varkappa\cdot\nabla_{\mathbf{x}})\partial_t\eta^\epsilon)D^{s,\mathscr{K}}_{-h,h}\eta^\epsilon\,\mathrm{d}\mathbf{y}\,\mathrm{d}B_t\\
&=: I_1\,\mathrm{d}t+\ldots+I_{10}\,\mathrm{d}t.
\end{aligned}
\tag{5.7}
$$

a.s. First of all, note that

$$
\begin{aligned}
&\int_\Gamma\big(\Delta_{\mathbf{y}}(\Delta^s_{-h}\Delta^s_h\eta^\epsilon)\Delta_{\mathbf{y}}\eta^\epsilon+\epsilon\nabla^3_{\mathbf{y}}(\Delta^s_{-h}\Delta^s_h\eta^\epsilon):\nabla^3_{\mathbf{y}}\eta^\epsilon+\epsilon\Delta_{\mathbf{y}}(\Delta^s_{-h}\Delta^s_h\eta^\epsilon)\partial_t\Delta_{\mathbf{y}}\eta^\epsilon\big)\,\mathrm{d}\mathbf{y}\,\mathrm{d}t\\
&\quad=-\int_\Gamma|\Delta_{\mathbf{y}}\Delta^s_h\eta^\epsilon|^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}t-\epsilon\int_\Gamma|\nabla^3_{\mathbf{y}}\Delta^s_h\eta^\epsilon|^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}t-\frac{\epsilon}{2}\,\mathrm{d}\int_\Gamma|\Delta_{\mathbf{y}}\Delta^s_h\eta^\epsilon|^2\,\mathrm{d}\mathbf{y}
\end{aligned}
$$

and since the corrector $\mathscr{K}_{\eta^\epsilon}$ is spatially independent,

$$
\int_\Gamma\Delta_{\mathbf{y}}\mathscr{K}_{\eta^\epsilon}(\Delta^s_{-h}\Delta^s_h\eta^\epsilon)\Delta_{\mathbf{y}}\eta^\epsilon\,\mathrm{d}\mathbf{y}\,\mathrm{d}t=0.
$$

Similarly, the two remaining ϵ-terms are zero. We now wish to take the p-th moment of the time integral of (5.7) where $p\geq 1$. To begin with, we have

$$
\begin{aligned}
\mathbb{E}\Big(\int_I I_1\,\mathrm{d}t\Big)^p&\lesssim\mathbb{E}\Big(\sup_I\|\mathbf{u}^\epsilon\|_{L^2(\mathcal{O}_{(\eta^\epsilon)_\epsilon})}\sup_I\|\Delta^s_{-h}\Delta^s_h\eta^\epsilon\|_{L^2(\Gamma)}\Big)^p\\
&\lesssim\mathbb{E}\sup_I\|\mathbf{u}^\epsilon\|^{2p}_{L^2(\mathcal{O}_{(\eta^\epsilon)_\epsilon})}+\mathbb{E}\sup_I\|\eta^\epsilon\|^{2p}_{W^{2,2}(\Gamma)}
\end{aligned}
$$

by Proposition 2, where the right-hand side is uniformly bounded by (5.1) and (5.4). Moreover,

$$
\begin{aligned}
\mathbb{E}\Big(\int_I I_2\,\mathrm{d}t\Big)^p&\lesssim\mathbb{E}\Big(\sup_I\|\partial_t\eta^\epsilon\|_{L^2(\Gamma)}\Big[\sup_I\|\Delta^s_{-h}\Delta^s_h\eta^\epsilon\|_{L^2(\Gamma)}+\sup_I|\mathscr{K}_{\eta^\epsilon}(\Delta^s_{-h}\Delta^s_h\eta^\epsilon)|\Big]\Big)^p\\
&\lesssim\mathbb{E}\Big(\sup_I\|\partial_t\eta^\epsilon\|_{L^2(\Gamma)}\Big[\sup_I\|\eta^\epsilon\|_{W^{2,2}(\Gamma)}+\sup_I\|\Delta^s_{-h}\Delta^s_h\eta^\epsilon\|_{L^1(\Gamma)}\Big]\Big)^p\\
&\lesssim\mathbb{E}\sup_I\|\partial_t\eta^\epsilon\|^{2p}_{L^2(\Gamma)}+\mathbb{E}\sup_I\|\eta^\epsilon\|^{2p}_{W^{2,2}(\Gamma)}
\end{aligned}
$$

which is again uniformly bounded. Since we assume that η (and consequently also $\partial_t\eta$) has mean value zero and $\mathscr{K}_{\eta^\epsilon}$ maps to spatially independent functions we have

$\int_\Gamma \partial_t\eta^\epsilon\, \partial_t\mathscr{K}_{\eta^\epsilon}(\Delta_{-h}^s\Delta_h^s\eta^\epsilon)\,\mathrm{d}\mathbf{y} = 0$. Thus it holds

$$-I_3 = \int_\Gamma |\partial_t\Delta_h^s\eta^\epsilon|^2\,\mathrm{d}\mathbf{y} \lesssim \|\partial_t\eta^\epsilon\|^2_{W^{s,2}(\Gamma)},$$

which is uniformly bounded in $L^p(\Omega; L^1(I))$ by (5.6). Recalling the definition of $\boldsymbol{\varphi}_\eta$ from (2.3) we further have by (5.1), (5.2) and 2D Sobolev embeddings

$$\begin{aligned}
\mathbb{E}\Big(\int_I I_4\,\mathrm{d}t\Big) &\lesssim \mathbb{E}\Big(\int_I \|\Delta_{-h}^s\Delta_h^s\eta^\epsilon\|_{L^\infty(\Gamma)}\big(1+\|\nabla_\mathbf{y}\eta^\epsilon\|_{L^3(\Gamma)}\big)\|\partial_t\eta^\epsilon\|^2_{L^3(\Gamma)}\,\mathrm{d}t\Big)^p\\
&\lesssim \mathbb{E}\Big(\int_I \|\eta^\epsilon\|_{W^{2,2}(\Gamma)}\|\partial_t\eta^\epsilon\|^2_{W^{s,2}(\Gamma)}\,\mathrm{d}t\Big)^p\\
&\lesssim \mathbb{E}\Big(\int_I \|\partial_t\eta^\epsilon\|^2_{W^{s,2}(\Gamma)}\,\mathrm{d}t\Big)^p
\end{aligned}$$

which is again bounded by (5.6). By using Lemma 4 (with $p = p' = 2$, $\theta = s$ and $\tilde{a} = 6$), and (5.1)–(5.5) (together with the embedding $W^{2,2}(\Gamma) \hookrightarrow C^{0,\theta}(\Gamma)$ for all $\theta < 1$) we have that (for $\delta > 0$ arbitrary)

$$\begin{aligned}
&\mathbb{E}\Big(\int_I I_5\,\mathrm{d}t\Big)^p\\
&\quad\lesssim \mathbb{E}\bigg(\int_I \Big(1+\|\Delta_{-h}^s\Delta_h^s\eta^\epsilon\|_{W^{1,3}(\Gamma)}\Big)\|\partial_t\eta^\epsilon\|_{L^2(\Gamma)}\|\mathbf{u}^\epsilon\|_{W^{1,2}(\mathcal{O}_{(\eta^\epsilon)_\epsilon})}\,\mathrm{d}t\\
&\qquad+\int_I \|(\Delta_{-h}^s\Delta_h^s\eta^\epsilon)\partial_t\eta^\epsilon\|_{L^{6/5}(\Gamma)}\|\mathbf{u}^\epsilon\|_{W^{1,2}(\mathcal{O}_{(\eta^\epsilon)_\epsilon})}\,\mathrm{d}t\bigg)^p\\
&\quad\lesssim \mathbb{E}\bigg(\sup_I \|\partial_t\eta^\epsilon\|_{L^2(\Gamma)}\int_I \big(1+\|\Delta_h^s\eta^\epsilon\|_{W^{s+1,3}(\Gamma)}\big)\|\mathbf{u}^\epsilon\|_{W^{1,2}(\mathcal{O}_{(\eta^\epsilon)_\epsilon})}\,\mathrm{d}t\\
&\qquad+\sup_I \|\partial_t\eta^\epsilon\|_{L^2(\Gamma)}\int_I \|\Delta_{-h}^s\Delta_h^s\eta^\epsilon\|_{L^3(\Gamma)}\|\mathbf{u}^\epsilon\|_{W^{1,2}(\mathcal{O}_{(\eta^\epsilon)_\epsilon})}\,\mathrm{d}t\bigg)^p\\
&\quad\lesssim \mathbb{E}\bigg(\int_I \big(1+\|\Delta_h^s\eta^\epsilon\|_{W^{s+1,3}(\Gamma)}+\|\eta^\epsilon\|_{W^{2s,3}(\Gamma)}\big)\|\mathbf{u}^\epsilon\|_{W^{1,2}(\mathcal{O}_{(\eta^\epsilon)_\epsilon})}\,\mathrm{d}t\bigg)^p\\
&\quad\le \delta\mathbb{E}\bigg(\int_I \|\Delta_h^s\eta^\epsilon\|^2_{W^{2,2}(\Gamma)}\,\mathrm{d}t\bigg)^p + c(\delta)\bigg(\int_I \big(\|\eta^\epsilon\|^2_{W^{2,2}(\Gamma)}+\|\mathbf{u}^\epsilon\|^2_{W^{1,2}(\mathcal{O}_{(\eta^\epsilon)_\epsilon})}\big)\,\mathrm{d}t\bigg)^p
\end{aligned}$$

where we have used the continuous embedding $W^{2,2}(\Gamma) \hookrightarrow W^{s+1,3}(\Gamma)$, $s \in (0, \frac{1}{2})$. The first term can be absorbed for δ small enough, while the second done is uniformly bounded, cf. (5.1) and (5.5). Next, it follows from Proposition 2 that

$$\begin{aligned}
\sup_I \|\mathscr{F}^{(\eta^\epsilon)_\epsilon}(D^{s,\mathscr{K}}_{-h,h}\eta^\epsilon)\|_{W^{1,2}(\mathcal{O}_{(\eta^\epsilon)_\epsilon})} &\lesssim \sup_I \Big(\|\Delta_{-h}^s\Delta_h^s\eta^\epsilon\|_{W^{1,2}(\Gamma)}+\|(\Delta_{-h}^s\Delta_h^s\eta^\epsilon)\nabla_\mathbf{y}\eta^\epsilon\|_{L^2(\Gamma)}\Big)\\
&\lesssim \sup_I \Big(\|\eta^\epsilon\|_{W^{2,2}(\Gamma)}+\|\Delta_{-h}^s\Delta_h^s\eta^\epsilon\|_{L^\infty(\Gamma)}\|\eta^\epsilon\|_{W^{1,2}(\Gamma)}\Big)\\
&\lesssim \sup_I \Big(\|\eta^\epsilon\|_{W^{2,2}(\Gamma)}+\|\eta^\epsilon\|^2_{W^{2,2}(\Gamma)}\Big)
\end{aligned}$$

therefore,

$$
\begin{aligned}
&\mathbb{E}\Big(\int_I (I_6+I_7+I_8)\,\mathrm{d}t\Big)^p\\
&\quad\lesssim \mathbb{E}\bigg(\sup_I \|\mathscr{F}^{(\eta^\epsilon)_\epsilon}(D^{s,\mathscr{K}}_{-h,h}\eta^\epsilon)\|_{W^{1,2}(\mathcal{O}_{\eta^\epsilon})}\int_I \big(\|\mathbf{u}^\epsilon\|^2_{W^{1,2}(\mathcal{O}_{(\eta^\epsilon)_\epsilon})}+\|\mathbf{u}^\epsilon\|_{W^{1,2}(\mathcal{O}_{(\eta^\epsilon)_\epsilon})}\big)\,\mathrm{d}t\bigg)^p\\
&\quad\lesssim \mathbb{E}\sup_I\|\eta^\epsilon\|^{4p}_{W^{2,2}(\Gamma)}+\mathbb{E}\bigg(\int_I\|\mathbf{u}^\epsilon\|^2_{W^{1,2}(\mathcal{O}_{(\eta^\epsilon)_\epsilon})}\,\mathrm{d}t\bigg)^p.
\end{aligned}
$$

This is uniformly bounded by (5.1) and (5.5). Furthermore, since $\mathscr{K}_{\eta^\epsilon}$ is spatially independent, and Γ is endowed with periodic boundary condition,

$$
\begin{aligned}
\mathbb{E}\Big(\int_I I_9\,\mathrm{d}t\Big)^p &= \mathbb{E}\bigg(\int_I\int_\Gamma \partial_t\eta^\epsilon\,(\varkappa\cdot\nabla_{\mathbf{y}})(\varkappa\cdot\nabla_{\mathbf{y}})\Delta^s_{-h}\Delta^s_h\eta^\epsilon\,\mathrm{d}\mathbf{y}\,\mathrm{d}t\bigg)^p\\
&\lesssim \mathbb{E}\bigg(\int_I \|\partial_t\eta^\epsilon\|_{W^{s,2}(\Gamma)}\|(\varkappa\cdot\nabla_{\mathbf{y}})(\varkappa\cdot\nabla_{\mathbf{y}})\Delta^s_{-h}\Delta^s_h\eta^\epsilon\|_{W^{-s,2}(\Gamma)}\,\mathrm{d}t\bigg)^p\\
&\leq \delta\mathbb{E}\bigg(\int_I\|\Delta^s_h\eta^\epsilon\|^2_{W^{2,2}(\Gamma)}\,\mathrm{d}t\bigg)^p\\
&\qquad + c(\delta)\,\mathbb{E}\bigg(\int_I\|\partial_t\eta^\epsilon\|^2_{W^{s,2}(\Gamma)}\,\mathrm{d}t\bigg)^p,
\end{aligned}
\tag{5.8}
$$

where $\delta>0$ is arbitrary. Lastly, since $\mathscr{K}_{\eta^\epsilon}$ is spatially independent,

$$
\int_I I_{10}\,\mathrm{d}t = -\int_I\int_\Gamma \partial_t\eta^\epsilon((\varkappa\cdot\nabla_{\mathbf{y}})\Delta^s_{-h}\Delta^s_h\eta^\epsilon)\,\mathrm{d}\mathbf{y}\,\mathrm{d}B_t.
$$

Therefore, by the Burkholder–Davis–Gundy inequality

$$
\begin{aligned}
\mathbb{E}\sup_I\Big|\int_I I_{10}\,\mathrm{d}t\Big|^p &\lesssim \mathbb{E}\bigg(\int_I\Big(\int_\Gamma \partial_t\eta^\epsilon((\varkappa\cdot\nabla_{\mathbf{y}})\Delta^s_{-h}\Delta^s_h\eta^\epsilon)\,\mathrm{d}\mathbf{y}\Big)^2\,\mathrm{d}t\bigg)^{\frac{p}{2}}\\
&\lesssim \mathbb{E}\bigg(\int_I\|\partial_t\eta^\epsilon\|^2_{L^2(\Gamma)}\|\nabla_{\mathbf{y}}\Delta^s_{-h}\Delta^s_h\eta^\epsilon\|^2_{L^2(\Gamma)}\,\mathrm{d}t\bigg)^{\frac{p}{2}}\\
&\lesssim \mathbb{E}\bigg(\Big(\sup_I\|\partial_t\eta^\epsilon\|^2_{L^2(\Gamma)}\Big)^{\frac{p}{2}}\Big(\sup_I\|\nabla_{\mathbf{y}}\Delta^s_{-h}\Delta^s_h\eta^\epsilon\|^2_{L^2(\Gamma)}\Big)^{\frac{p}{2}}\bigg)\\
&\lesssim \mathbb{E}\sup_I\|\partial_t\eta^\epsilon\|^{2p}_{L^2(\Gamma)}+\mathbb{E}\sup_I\|\eta^\epsilon\|^{2p}_{W^{2,2}(\Gamma)},
\end{aligned}
\tag{5.9}
$$

bounded by (5.1) and (5.2). If we combine everything, we obtain for $s\in(0,\frac{1}{2})$,

$$
\begin{aligned}
\mathbb{E}\Big(\int_I\|\Delta^s_h\eta^\epsilon\|^2_{W^{s+2,2}(\Gamma)}\,\mathrm{d}t\Big)^p &+ \epsilon\mathbb{E}\Big(\int_I\|\Delta^s_h\eta^\epsilon\|^2_{W^{3,2}(\Gamma)}\,\mathrm{d}t\Big)^p\\
&+\epsilon\mathbb{E}\Big(\sup_I\|\Delta^s_h\eta^\epsilon\|^2_{W^{2,2}(\Gamma)}\Big)^p\lesssim 1
\end{aligned}
\tag{5.10}
$$

uniformly in h and thus

$$\mathbb{E}\Big(\int_I \|\eta^\epsilon\|^2_{W^{s+2,2}(\Gamma)}\,\mathrm{d}t\Big)^p + \epsilon\mathbb{E}\Big(\int_I \|\eta^\epsilon\|^2_{W^{s+3,2}(\Gamma)}\,\mathrm{d}t\Big)^p + \epsilon\mathbb{E}\Big(\sup_I \|\eta^\epsilon\|^2_{W^{s+2,2}(\Gamma)}\Big)^p \lesssim 1 \tag{5.11}$$

for all $p \geq 1$.

5.2 Passage to the Limit

With the bounds from (5.1)–(5.6) at hand, we wish to obtain compactness. For this, we define the path space

$$\chi_{\mathtt{high}} = \chi_B \times \chi_{\mathbf{u}} \times \chi_{\nabla\mathbf{u}} \times \chi_\eta^{\mathtt{high}} \times \chi_{f,g}^2$$

where

$$\chi_\eta^{\mathtt{high}} = \big(W^{1,\infty}(I;L^2(\Gamma)), w^*\big) \cap \big(L^\infty(I;W^{2,2}(\Gamma)), w^*\big) \cap \big(L^2(I;W^{s+2,2}(\Gamma)), w\big) \cap \big(W^{1,2}(I;W^{s,2}(\Gamma)), w\big)$$

with $s \in (0, \frac{1}{2})$. From (5.1)–(5.6), Lemma 14 and (5.11) we obtain similarly to Proposition 13:

Proposition 15 *There exists a complete probability space $(\tilde\Omega, \tilde{\mathfrak{F}}, \tilde{\mathbb{P}})$ with χ-valued random variables*

$$\tilde{\mathbf{\Theta}}^{\epsilon_n} := \Bigg[\tilde B_t^n, \mathbb{I}_{\mathcal{O}_{(\tilde\eta^{\epsilon_n})_\epsilon}}\tilde{\boldsymbol{u}}^{\epsilon_n}, \mathbb{I}_{\mathcal{O}_{(\tilde\eta^{\epsilon_n})_\epsilon}}\nabla_{\boldsymbol{x}}\tilde{\boldsymbol{u}}^{\epsilon_n}, \tilde\eta^{\epsilon_n}, \begin{pmatrix} ((\partial_t\tilde\eta^{\epsilon_n}, \mathbb{I}_{\mathcal{O}_{(\tilde\eta^{\epsilon_n})_\epsilon}}\tilde{\boldsymbol{u}}^{\epsilon_n}), (\partial_t\tilde\eta^{\epsilon_n}, \mathscr{F}^{(\tilde\eta^{\epsilon_n})_\epsilon}\partial_t\tilde\eta^{\epsilon_n})) \\ ((\partial_t\tilde\eta^{\epsilon_n}, \mathbb{I}_{\mathcal{O}_{(\tilde\eta^{\epsilon_n})_\epsilon}}\tilde{\boldsymbol{u}}^{\epsilon_n}), (0, \tilde{\boldsymbol{u}}^{\epsilon_n} - \mathscr{F}^{(\tilde\eta^{\epsilon_n})_\epsilon}\partial_t\tilde\eta^{\epsilon_n})) \end{pmatrix}\Bigg],$$

for $n \in \mathbb{N}$ and

$$\tilde{\mathbf{\Theta}} := \Bigg[\tilde B_t, \mathbb{I}_{\mathcal{O}_{\tilde\eta}}\tilde{\boldsymbol{u}}, \mathbb{I}_{\mathcal{O}_{\tilde\eta}}\nabla_{\boldsymbol{x}}\tilde{\boldsymbol{u}}, \tilde\eta, \begin{pmatrix} ((\partial_t\tilde\eta, \mathbb{I}_{\mathcal{O}_{\tilde\eta}}\tilde{\boldsymbol{u}}), (\partial_t\tilde\eta, \mathscr{F}^{\tilde\eta}\partial_t\tilde\eta)) \\ ((0, \mathbb{I}_{\mathcal{O}_{\tilde\eta}}\tilde{\boldsymbol{u}}), (\partial_t\tilde\eta, \tilde{\boldsymbol{u}} - \mathscr{F}^{\tilde\eta}\partial_t\tilde\eta)) \end{pmatrix}\Bigg]$$

such that

(a) *For all $n \in \mathbb{N}$ the law of $\tilde{\mathbf{\Theta}}^{\epsilon_n}$ on χ is given by*

$$\mathcal{L}\Bigg[B_t^n, \mathbb{I}_{\mathcal{O}_{(\eta^{\epsilon_n})_{\epsilon_n}}}\boldsymbol{u}^{\epsilon_n}, \mathbb{I}_{\mathcal{O}_{(\eta^{\epsilon_n})_{\epsilon_n}}}\nabla_{\boldsymbol{x}}\boldsymbol{u}^{\epsilon_n}, \eta^{\epsilon_n}, \begin{pmatrix} ((\partial_t\eta^{\epsilon_n}, \mathbb{I}_{\mathcal{O}_{(\eta^{\epsilon_n})_{\epsilon_n}}}\boldsymbol{u}^{\epsilon_n}), (\partial_t\eta^{\epsilon_n}, \mathscr{F}^{(\eta^{\epsilon_n})_{\epsilon_n}}\partial_t\eta^{\epsilon_n})) \\ ((0, \mathbb{I}_{\mathcal{O}_{(\eta^{\epsilon_n})_{\epsilon_n}}}\boldsymbol{u}^{\epsilon_n}), (\partial_t\eta^{\epsilon_n}, \boldsymbol{u}^{\epsilon_n} - \mathscr{F}^{(\eta^{\epsilon_n})_{\epsilon_n}}\partial_t\eta^{\epsilon_n})) \end{pmatrix}\Bigg]$$

(b) $\tilde{\boldsymbol{\Theta}}^{\epsilon_n}$ *converges* $\tilde{\mathbb{P}}$*-almost surely to* $\tilde{\boldsymbol{\Theta}}$ *in the topology of* $\chi_{\mathtt{high}}$, *i.e.*

$$\begin{aligned}
\tilde{B}_t^n &\to \tilde{B}_t \quad in \quad C(\overline{I})\ \tilde{\mathbb{P}}\text{-}a.s.,\\
\mathbb{I}_{\mathcal{O}_{(\tilde{\eta}^{\epsilon_n})_\epsilon}}\tilde{\boldsymbol{u}}^{\epsilon_n} &\rightharpoonup \mathbb{I}_{\mathcal{O}_{\tilde{\eta}}}\tilde{\boldsymbol{u}} \quad in \quad L^2(I;L^2(\mathcal{O}\cup S_\alpha))\ \tilde{\mathbb{P}}\text{-}a.s.,\\
\mathbb{I}_{\mathcal{O}_{(\eta^{\epsilon_n})_\epsilon}}\nabla_{\boldsymbol{x}}\tilde{\boldsymbol{u}}^{\epsilon_n} &\rightharpoonup \mathbb{I}_{\mathcal{O}_{\tilde{\eta}}}\nabla_{\boldsymbol{x}}\tilde{\boldsymbol{u}} \quad in \quad L^2(I;L^2(\mathcal{O}\cup S_\alpha))\ \tilde{\mathbb{P}}\text{-}a.s.,\\
\tilde{\eta}^{\epsilon_n} &\rightharpoonup^* \tilde{\eta} \quad in \quad L^\infty(I;W^{2,2}(\Gamma))\ \tilde{\mathbb{P}}\text{-}a.s.,\\
\tilde{\eta}^{\epsilon_n} &\rightharpoonup^* \tilde{\eta} \quad in \quad W^{1,\infty}(I;L^2(\Gamma))\ \tilde{\mathbb{P}}\text{-}a.s.,\\
\tilde{\eta}^{\epsilon_n} &\rightharpoonup \tilde{\eta} \quad in \quad W^{1,2}(I;W^{s,2}(\Gamma))\ \tilde{\mathbb{P}}\text{-}a.s.,\\
\tilde{\eta}^{\epsilon_n} &\rightharpoonup \tilde{\eta} \quad in \quad L^2(I;W^{s+2,2}(\Gamma))\ \tilde{\mathbb{P}}\text{-}a.s.,
\end{aligned} \tag{5.12}$$

as well as (recalling the definition of $\tau_\sharp$ *from (3.25))*

$$\begin{aligned}
&\int_I\int_{\mathcal{O}\cup S_\alpha}\mathbb{I}_{\mathcal{O}_{(\tilde{\eta}^{\epsilon_n})_\epsilon}}\boldsymbol{u}^{\epsilon_n}\cdot\mathscr{F}^{(\tilde{\eta}^{\epsilon_n})_\epsilon}\partial_t\tilde{\eta}^{\epsilon_n}\,d\boldsymbol{x}\,dt+\int_I\int_\Gamma|\partial_t\tilde{\eta}^{\epsilon_n}|^2\ d\boldsymbol{y}\,dt\\
&\quad\longrightarrow\int_I\int_{\mathcal{O}\cup S_\alpha}\mathbb{I}_{\mathcal{O}_{\tilde{\eta}}}\tilde{\boldsymbol{u}}\cdot\mathscr{F}^{\tilde{\eta}}\partial_t\eta\,d\boldsymbol{x}\,dt+\int_I\int_\Gamma|\partial_t\tilde{\eta}|^2\ d\boldsymbol{y}\,dt
\end{aligned} \tag{5.13}$$

and

$$\begin{aligned}
&\int_I\int_{\mathcal{O}\cup S_\alpha}\mathbb{I}_{\mathcal{O}_{(\tilde{\eta}^{\epsilon_n})_\epsilon}}\tilde{\boldsymbol{u}}^{\epsilon_n}\cdot(\tilde{\boldsymbol{u}}^{\epsilon_n}-\mathscr{F}^{(\tilde{\eta}^{\epsilon_n})_\epsilon}\partial_t\tilde{\eta}^{\epsilon_n})\,d\boldsymbol{x}\,dt\\
&\quad\longrightarrow\int_I\int_{\mathcal{O}\cup S_\alpha}\mathbb{I}_{\mathcal{O}_{\tilde{\eta}}}\tilde{\boldsymbol{u}}\cdot(\tilde{\boldsymbol{u}}-\mathscr{F}^{\tilde{\eta}}\partial_t\tilde{\eta})\,d\boldsymbol{x}\,dt
\end{aligned} \tag{5.14}$$

$\tilde{\mathbb{P}}$*-a.s.*

Similar to the analysis after Propositions 10 and 13, we can define the $\tilde{\mathbb{P}}$-augmented canonical filtrations $(\tilde{\mathfrak{F}}_t)_{t\geq 0}$ and $(\tilde{\mathfrak{F}}_t^\epsilon)_{t\geq 0}$ on the variables $\tilde{\boldsymbol{\Theta}}$ and $\tilde{\boldsymbol{\Theta}}^\epsilon$, respectively, which ensures the correct measurabilities of the new random variables.

By (Breit et al. (2018), Theorem 2.9.1) the weak equation continuous to hold on the new probability space. Combining (5.13) and (3.32) we have

$$\begin{aligned}
&\int_I\int_{\mathcal{O}_{(\tilde{\eta}^{\epsilon_n})_\epsilon}}|\tilde{\mathbf{u}}^{\epsilon_n}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\int_I\int_\Gamma|\partial_t\tilde{\eta}^{\epsilon_n}|^2\ \mathrm{d}\mathbf{y}\,\mathrm{d}t\\
&\quad\longrightarrow\int_I\int_{\mathcal{O}_{\tilde{\eta}}}|\tilde{\mathbf{u}}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\int_I\int_\Gamma|\tilde{\partial}_t\eta|^2\ \mathrm{d}\mathbf{y}\,\mathrm{d}t
\end{aligned}$$

$\tilde{\mathbb{P}}$-a.s. By uniform convexity of the L^2-norm this implies

$$\begin{aligned}
\tilde{\eta}^{\epsilon_n} &\to \tilde{\eta} \quad \text{in} \quad W^{1,2}(I;L^2(\Gamma)),\\
\mathbb{I}_{\mathcal{O}_{(\tilde{\eta}^{\epsilon_n})_\epsilon}}\tilde{\mathbf{u}}^{\epsilon_n} &\to \mathbb{I}_{\mathcal{O}_{\tilde{\eta}}}\tilde{\mathbf{u}} \quad \text{in} \quad L^2(I;L^2(\mathcal{O}\cup S_\alpha))\ \tilde{\mathbb{P}}\text{-a.s.}.
\end{aligned}$$

This is sufficient to pass to the limit and the weak formulation of the equations (note that all terms except for the convective term can be treated by (4.9)). As far as the energy balance is concerned, this is even easier since there is no noise. By (4.9) and the lower semi-continuity of the involved quantities we can pass to the limit (obtaining only an inequality).

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Declarations

Conflict of interest The authors declare no competing interests.

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