Strategy-proofness versus efficiency in exchange economies: general domain properties and applications

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Abstract

We identify general domain properties that induce the non-existence of *efficient, strategy-proof,* and *non-dictatorial* rules in the 2-agent exchange economy. Applying these properties, we establish impossibility results in several restricted domains; for example, the intertemporal exchange problem (without saving technology) with preferences represented by the discounted sum of a temporal utility function, the "risk sharing problem" with risk averse expected utility preferences, the CES-preference domain, etc. None of the earlier studies applies to these examples.

Keywords: *Strategy-proofness*; *efficiency*; dictatorship; exchange economy; domain.

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1 Introduction

In the "exchange economy", an allocation rule, or simply, a *rule*, associates with each profile of agents' preferences a *single* desirable allocation, a list of individual consumption bundles. We refer to the set of admissible preference profiles as the *domain*. We are interested in the following two basic requirements of rules. The first is *efficiency*, the requirement that no one can be made better off without anyone else being made worse off. The second is *strategy-proofness* (Gibbard, 1973, Satterthwaite, 1975), the requirement that truthful representation of one's preference always weakly dominates any admissible misrepresentation.

A number of earlier studies have shown impossibilities of satisfying the two requirements together with other standard equity criteria. In particular, in the 2-agent case, Dasgupta, Hammond, and Maskin (1979), Zhou (1991a), and Schummer (1997) show that there is no *efficient* and *strategy-proof* rules satisfying the minimal equity criterion, "non-dictatorship"; a rule is *dictatorial* if there is an agent, the dictator, who always receives his best bundle.[1] However, their results are not fully satisfactory because they provide no implication for various interesting allocation problems in which agents' preferences are restricted for some intuitive or technical reasons.

For example, in the "intertemporal exchange problem" (without saving technology), we often consider preferences that are represented in the additively separable form by temporal utility functions and discount factors. In the "risk sharing problem", we often consider preferences that are represented in the expected utility form by strictly convex ("risk aversion") utility indices and subjective probability distributions over states. Also, in many applications, we focus on preferences that satisfy technical conditions such as "smoothness", "continuous differentiability of utility functions", "quasilinearity", etc.

Our main objective is to strengthen the impossibility result for the 2-agent exchange economy by identifying general domain properties that are sufficient for the impossibility. These properties are satisfied by the domains considered in the earlier studies by Dasgupta, Hammond, and Maskin (1979), Zhou (1991a), and Schummer (1997); our result simplifies their proofs. More importantly, our domain properties are applicable to several restricted domains such as the intertemporal exchange problem, the risk sharing problem, the domain of "CES preferences", and the domain of quasilinear, strictly convex, and smooth preferences, etc., while none of the earlier studies applies to them.

The seminal study by Hurwicz (1972) shows that in the 2-agent and 2-good

[1] See also Hurwicz (1972), Satterthwaite and Sonnenschein (1981), Hurwicz and Walker (1990), and Barberà and Jackson (1995).

exchange economy, there exists no *efficient* and *strategy-proof* rule satisfying *individual rationality*, the requirement that everyone should be at least as well off as in his endowment. Dasgupta, Hammond, and Maskin (1979) strengthen his result by replacing *individual rationality* with *non-dictatorship.* However, their conclusion crucially relies on the admissibility of "discontinuous" preferences, while Hurwicz's result pertains to preferences satisfying the classical assumptions, "continuity", "monotonicity", and "convexity".

Zhou (1991) reinforces the impossibility result by Dasgupta, Hammond, and Maskin (1979), considering the classical domain consisting of continuous, strictly monotonic, and strictly convex preferences. When preferences are strictly monotonic, this conclusion extends to any larger domain, as he remarks. A natural question addressed by Schummer (1997) is whether the impossibility applies to smaller, yet interesting, domains. He shows that the impossibility continues to hold both in the domain of "homothetic" preferences and in the domain of "linear" preferences (preferences with linear utility functions).

The arguments used by Zhou (1991) and Schummer (1997) crucially rely on the admissibility of "kinked" preferences.[2] So their results do not apply, for example, to domains consisting of only *smooth* and strictly convex preferences. On the other hand, Schummer (1997) crucially relies on the homotheticity restriction. So, his result does not apply to other restricted domains, for example, the domain consisting of only *quasilinear* and strictly convex preferences. Our domain properties do not necessarily require that kinked or homothetic preferences be admissible. They are applicable not only to all the above domains but various other restricted domains as we show in the application of our main result.

Several recent authors bring out some important domain properties in different perspectives of their studies on *strategy-proofness.* In a voting model, Barberà, Sonnenschein, and Zhou (1991) identify the unique maximal domain in which a class of rules, called "voting by committees", are *strategy-proof.* The maximal domain issue is studied also by Berga and Serizawa (2000) in the 1-dimensional public choice model. In a linear production model, Maniquet and Sprumont (1999) identify domain properties under which their characterization results apply.

Most of the earlier studies focus on "product domains", Cartesian products of families of individual preferences.[3] Product domains do not capture the interdependency, or correlation, of preferences across agents, which is common in reality. Such an interdependency arises especially when agents share identical cultural or historic background relevant to their preferences. Thus, it is standard

[2]The only exception is the linear preference domain in Schummer (1997).

[3]Or "independent domains" (Moore, 1993, p 214).

in implementation theory to capture such an interdependency by considering non-product domains: see Moore (1993) for a broad survey of literature. Therefore, we do not restrict our attention only to product domains; our domain properties are stated for possibly, non-product domains. *Strategy-proofness* is a necessary condition for the implementability in dominant strategy equilibrium both in the product domain case and in the non-product domain case. It is also sufficient in the product domain case, while it is not in the non-product domain case.

This paper is composed of five sections. In Section 2, we introduce the model and basic concepts. In Section 3, we define general domain properties and establish our main result. In Section 4, we provide some applications. We conclude in Section 5.

2 The model and basic concepts

We consider l-good exchange economies, $l \geq 2$, with social endowment $\Omega \in \mathbb{R}^l_{++}$ and two agents. Let $N \equiv \{1, 2\}$ be the set of agents. Let $Z \equiv \{z \in \mathbb{R}^{l \cdot 2}_{+} : \sum_N z_i = \Omega\}$ be the set of feasible allocations. Let $Z_0 \equiv \{z_i \in \mathbb{R}^l_+ : 0 \leqq z_i \leqq \Omega\}$ be the set of possible consumption bundles for each agent.[4] We use z, z', z'', etc. to denote allocations: z_i denotes i's bundle at z. Notation $-i$ refers to the agent other than i. Each agent has a ***preference***, a complete and transitive binary relation over $\mathbb{R}^l_+$. Preferences are *continuous, strictly monotonic over* $\mathbb{R}^l_{++}$, and *convex*.[5] Let $\mathcal{R}$ be the class of all such preferences. A preference $R_i \in \mathcal{R}$ is *strictly monotonic* if for all $z_i, z_i' \in \mathbb{R}^l_+$, $z_i \geq z_i'$ implies $z_i \ P_i \ z_i'$. Let $I_i(z_i)$ be the set of all bundles indifferent to z_i under R_i.

A ***domain*** $\mathcal{D}$ is a subset of $\mathcal{R}^N$. Let $\mathcal{D}(R_{-i}) \equiv \{R' \in \mathcal{D} : R'_{-i} = R_{-i}\}$, $\mathcal{D}_i(R_{-i}) \equiv \{R_i : (R_i, R_{-i}) \in \mathcal{D}\}$, and $\mathcal{D}_i \equiv \{R_i :$ for some R_{-i}, $(R_i, R_{-i}) \in \mathcal{D}\}$. Since we keep the social endowment fixed, an ***economy*** can be characterized by a preference profile in $\mathcal{D}$. A social choice rule, or simply a ***rule***, over $\mathcal{D}$ is a function $\varphi \colon \mathcal{D} \to Z$ associating with each economy a feasible allocation.

A domain $\mathcal{D}$ is a ***product domain*** if for each $i \in N$, there exists $\mathcal{D}_i \subseteq \mathcal{R}$ such that $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$. We do not restrict ourselves to product domains. However, the following feature of product domain is important in our result.

[4] We denote elements of Z_0 by z_i, z_0, x, y etc. Vector inequalities, $\leqq$, $\leq$, $<$, are defined as follows. Let $x, y \in \mathbb{R}^l$. Then $x \leqq y$ if for all $k \in \{1, \cdots, l\}$, $x_k \leq y_k$. We write $x \leq y$ if $x \leqq y$ and $x \neq y$. We write $x < y$ if for all $k \in \{1, \cdots, l\}$, $x_k < y_k$.

[5] We use R_i to denote agent i's preference and P_i and I_i to denote its strict and indifference relations respectively. A preference R_i is *strictly monotonic over* $\mathbb{R}^l_{++}$ if for all $z_i, z_i' \in \mathbb{R}^l_{++}$, $z_i \geq z_i'$ implies $z_i \ P_i \ z_i'$, where the vector inequality $z_i \geq z_i'$ means that each component of z_i is weakly larger than each component of z_i' and $z_i \neq z_i'$. It is *convex* if for all $z_i, z_i' \in \mathbb{R}^l_{++}$ with $z_i \ R_i \ z_i'$ and all $\lambda \in [0, 1]$, $\lambda z_i + (1 - \lambda) z_i' \ R_i \ z_i'$.

Let $R, R' \in \mathcal{D}$. Profile R' is a ***unilateral variation of R*** if $R'_1 = R_1$ or $R'_2 = R_2$. A unilateral variation of R, R', is ***0-indifference-monotonic for i*** if $I'_i(0) \supseteq I_i(0)$.[6] Profile R' is ***reachable from R through iterative unilateral variations*** if there exists a finite sequence of profiles $(R^1, \cdots, R^n)$ in $\mathcal{D}$ such that $R^1 = R$, $R^n = R'$, and for all $k \in \{2, \cdots, n\}$, R^k is a unilateral variation of R^{k-1}. A domain $\mathcal{D}$ is ***everywhere reachable***** if for all $i \in N$ and all $R, R' \in \mathcal{D}$ with $I_i(0) \subseteq I'_i(0)$, R' is reachable from R through iterative unilateral variations that are 0-indifference-monotonic for i. Note that every product domain $\mathcal{D}$ is *everywhere reachable**, since for all $R, R' \in \mathcal{D}$, R' is reachable from R through any one of two iterative unilateral variations $(R_1, R_2) \to (R'_1, R_2) \to (R'_1, R'_2)$ and $(R_1, R_2) \to (R_1, R'_2) \to (R'_1, R'_2)$.

We next define our two main requirements of rules. Given $R \in \mathcal{D}$, an allocation $z \in Z$ is ***efficient for R*** if there exists no $z' \in Z$ such that for all $i \in N$, z'_i R_i z_i and for some $j \in N$, z'_j P_j z_j. Let $P(R)$ be the set, called ***Pareto set***, of all efficient allocations for R. For all $i \in N$, let $P_i(R) \equiv \{z_i \in \mathbb{R}^l_+ : (z_i, z_{-i}) \in Z$ for some $z_{-i} \in \mathbb{R}^l_+\}$. A rule $\varphi\colon \mathcal{D} \to Z$ satisfies ***efficiency*** if for all $R \in \mathcal{D}$, $\varphi(R) \in P(R)$.

In order to define the next requirement, consider agent $i \in N$ with preference R_i. Let R_{-i} and R'_i be such that (R_i, R_{-i}) and $(R'_i, R_{-i}) \in \mathcal{D}$. Consider a rule $\varphi\colon \mathcal{D} \to Z$. Let $z \equiv \varphi(R_i, R_{-i})$ and $z' \equiv \varphi(R'_i, R_{-i})$. Agent i will have an incentive to represent his true preference as opposed to the misrepresentation with R'_i if z_i R_i z'_i. We refer to this condition as ***i's incentive compatibility condition associated with*** $\boldsymbol{(R_i, R'_i, z_i)}$, where R_i is i's true preference, R'_i is a misrepresentation, and z_i is the "truthful outcome". We require that incentive compatibility condition should never be violated. Formally, a rule $\varphi\colon \mathcal{D} \to Z$ satisfies ***strategy-proofness*** if for all $i \in N$ and all $R, R' \in \mathcal{D}$ with $R_{-i} = R'_{-i}$, $\varphi_i(R)$ R_i $\varphi_i(R')$.

We show that every *efficient* and *strategy-proof* rule has the following displeasing feature. A rule $\varphi\colon \mathcal{D} \to Z$ is ***dictatorial over*** $\boldsymbol{\mathcal{D}^*} \subseteq \mathcal{D}$ if there exists $i \in N$ such that for all $R \in \mathcal{D}^*$ and all $z \in Z$, $\varphi_i(R)$ R_i z_i. The rule is ***dictatorial*** if it is dictatorial over the entire domain. Since preferences are strictly monotonic over $\mathbb{R}^l_{++}$, a rule φ is dictatorial over $\mathcal{D}^*$ if and only if there exists $i \in N$ such that for all $R \in \mathcal{D}^*$, $\varphi_i(R) = \Omega$.

We use the following notation. For all $R_i \in \mathcal{R}$ and all $z_i \in \mathbb{R}^l_+$, let $\boldsymbol{UC(R_i, z_i)} \equiv \{x \in Z_0 : x \; R_i \; z_i\}$ and $\boldsymbol{UC^0(R_i, z_i)} \equiv \{x \in Z_0 : x \; P_i \; z_i\}$ be the "(constrained) upper contour set of R_i at z_i" and the "strict upper contour set", respectively. Let $\boldsymbol{LC(R_i, z_i)} \equiv \{x \in Z_0 : z_i \; R_i \; x\}$ and $\boldsymbol{LC^0(R_i, z_i)} \equiv \{x \in Z_0 : z_i \; P_i \; x\}$ be the "lower contour set of R_i at z_i" and the "strictly lower contour set".

[6] Two sets $I_i(0)$ and $I'_i(0)$ are indifference sets through 0 for R_i and R'_i respectively.

3 The main result

We consider domain $\mathcal{D} \subseteq \mathcal{R}^N$ that has a subdomain $\bar{\mathcal{D}} \subseteq \mathcal{D}$ and a *reference set* $M \subseteq Z$ satisfying the following three properties. For all $i \in N$, let $M_i \equiv \{z_i \in \mathbb{R}^l_+ : (z_i, z_{-i}) \in M \text{ for some } z_{-i} \in \mathbb{R}^l_+\}$.

The first property is that the reference set M is the Pareto set for at least one economy in $\bar{\mathcal{D}}$ with strictly monotonic preferences.

A1, Potential efficiency: There exists $R \in \bar{\mathcal{D}}$ such that $P(R) = M$ and both R_1 and R_2 are strictly monotonic.

The second property is that each agent can always make M be the Pareto set by announcing a preference admissible in $\bar{\mathcal{D}}$.

A2, Attainability: For all $i \in N$ and all $R_{-i} \in \bar{\mathcal{D}}_{-i}$, there exists $R_i \in \bar{\mathcal{D}}_i(R_{-i})$ such that $P(R_i, R_{-i}) = M$.

The third property is stated in terms of the following notions. Two incentive compatibility conditions associated with (R_i, R'_i, z_i) and (R'_i, R_i, z'_i) imply that $z'_i \in LC(R_i, z_i) \cap UC(R'_i, z_i)$. Therefore, given R_{-i} and the truthful outcome z_i for R_i, the set of incentive compatible outcomes for R'_i coincides with $LC(R_i, z_i) \cap UC(R'_i, z_i)$. Thus we call $LC(R_i, z_i) \cap UC(R'_i, z_i)$ the ***incentive compatibility set associated with*** $\boldsymbol{(R_i, R'_i, z_i)}$. For all $R \in \mathcal{D}$, all $i \in N$, and all $z \in P(R)$, $R'_i \in \mathcal{D}(R_{-i})$ is a ***local transformation of*** $\boldsymbol{R_i}$ ***at*** $\boldsymbol{z_i}$ ***relative to*** $\boldsymbol{R_{-i}}$ if z_i is the unique *efficient* bundle for (R'_i, R_{-i}), in i's incentive compatibility set associated with (R_i, R'_i, z_i), that is, $P_i(R'_i, R_{-i}) \cap LC(R_i, z_i) \cap UC(R'_i, z_i) = \{z_i\}$ (see Figure 1). A preference R_i of agent i exhibits ***crossly local dominance of*** $\boldsymbol{z'_i}$ ***relative to*** $\boldsymbol{(R_{-i}, R'_{-i}, z_i)}$ if agent i with R_i prefers z'_i to every allocation that is *efficient* for (R_i, R'_{-i}) and is in $-i$'s incentive compatibility set associated with $(R_{-i}, R'_{-i}, z_{-i})$, that is, $P_i(R_i, R'_{-i}) \cap \{x \in Z_0 : \Omega - x \in LC(R_{-i}, z_{-i}) \cap UC(R'_{-i}, z_{-i})\} \subset LC^0(R_i, z'_i)$ (see Figure 1).

Our next property states that for any two profiles R and R' with the Pareto set M and for any two *efficient* allocations z and z', there exist an agent $i \in N$ and his preference $\bar{R}_i$ that is a local transformation of R_i at z_i relative to R_{-i} and exhibits crossly local dominance of z'_i relative to R_{-i}, R'_{-i}, and z_i (see Figure 1).

A3, Transformability with crossly local dominance: For all $R, R' \in \bar{\mathcal{D}}$ and all $z, z' \in M$, if $P(R) = P(R') = M$ and $z \neq z'$, then there exist $i \in N$ and $\bar{R}_i \in \bar{\mathcal{D}}_i(R_{-i}) \cap \bar{\mathcal{D}}_i(R'_{-i})$ such that
(i) $P_i(\bar{R}_i, R_{-i}) \cap LC(R_i, z_i) \cap UC(\bar{R}_i, z_i) = \{z_i\}$;
(ii) $P_i(\bar{R}_i, R'_{-i}) \cap \{x \in Z_0 : \Omega - x \in LC(R_{-i}, z_{-i}) \cap UC(R'_{-i}, z_{-i})\} \subset LC^0(\bar{R}_i, z'_i)$.

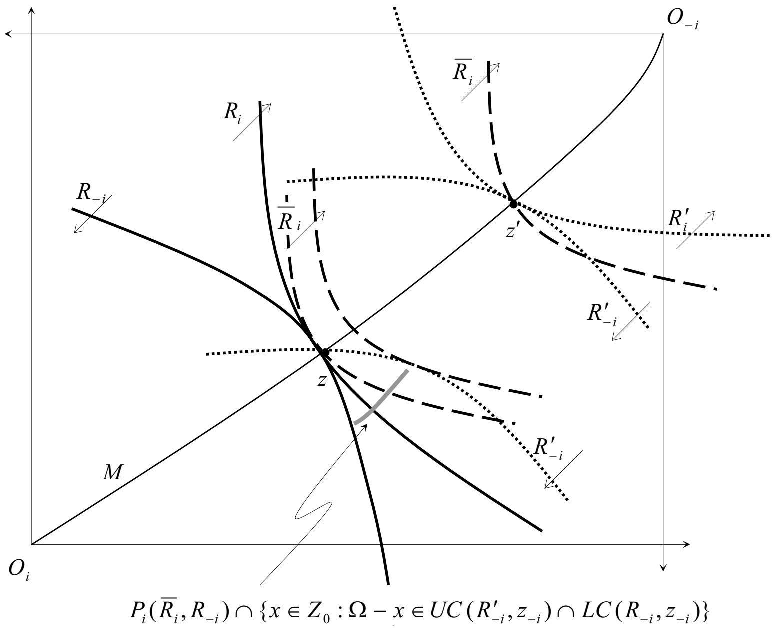

Figure 1: A3, Transformability with crossly local dominance.

There are domains that satisfy the above three properties and over which we do have *efficient, strategy-proof*, and *non-dictatorial* rules.

Example 1 *Risk sharing with an objective probability distribution and aggregate certainty*: Let l be the number of states. Each state $k = 1, \cdots, l$ is realized with probability π_k. Each bundle $x \in \mathbb{R}^l_+$ is a state-contingent commodity. Let $\mathcal{R}_*$ be the family of all preferences $R_0 \in \mathcal{R}$ that has the following "expected utility" representation: there exists a *concave* function $u_0 \colon \mathbb{R}_+ \to \mathbb{R}$ such that for all $x, x' \in \mathbb{R}^l_+$, $x \; R_0 \; x' \Leftrightarrow \sum_{k=1}^{l} \pi_k u_0(x_k) \geq \sum_{k=1}^{l} \pi_k u_0(x'_k)$. Suppose *aggregate certainty,* that is, $\Omega_1 = \cdots = \Omega_l$. Let $\bar{m}$ be the constant aggregate wealth across states.

Note that the equal division $((\frac{\bar{m}}{2}, \cdots, \frac{\bar{m}}{2}), (\frac{\bar{m}}{2}, \cdots, \frac{\bar{m}}{2}))$ is *efficient* for all profiles in $\mathcal{R}^N_*$. Let $\varphi^{\text{ed}} \colon \mathcal{R}^N_* \to Z$ be the *equal division rule*, that is, for all $R \in \mathcal{R}^N_*$, $\varphi^{\text{ed}}(R) \equiv ((\frac{\bar{m}}{2}, \cdots, \frac{\bar{m}}{2}), (\frac{\bar{m}}{2}, \cdots, \frac{\bar{m}}{2}))$. Then φ^{ed} is *efficient* and *strategy-proof* over $\mathcal{R}^N_*$.

Let $M \equiv \{z \in Z : \text{for all } i \in N, \, z_{i1} = \cdots = z_{il}\}$. We now show that $\mathcal{R}^N_*$ and M satisfy the above three properties. *A1* and *A2* are trivial. Let $R, R' \in \mathcal{R}^N_*$ be such that $P(R) = P(R') = M$. Let $z, z' \in M$ and $z \neq z'$. Then since M_i is a monotonic path for each $i \in N$, without loss of generality we may assume $z_1 < z'_1$. Since z_1 is on the 45°-line, there exists $\bar{R}_1 \in \mathcal{R}_*$ such that $\bar{R}_1$ is *strictly convex* and $LC(R_1, z_1) \cap UC(\bar{R}_1, z_1) = \{z_1\}$. Then (i) of *A3* holds. Since $\bar{R}_1$

is strictly convex, $P_1\left(\bar{R}_1, R_2'\right) = M_1$. Therefore, $P_1(\bar{R}_1, R_2') \cap \{x \in Z_0 : \Omega - x \in LC(R_2, z_2) \cap UC(R_2', z_2)\} = \{z_1\}$. Hence (ii) of *A3* also holds. $\square$

We now introduce additional domain properties that will induce the non-existence of rules satisfying *efficiency, strategy-proofness*, and *non-dictatorship.* The following notation is useful. Let $R_0 \in \mathcal{R}$ and all $z_0 \in Z_0$. For all $p \in \mathbb{R}^l_{++}$, let $H(p, z_0) \equiv \{x \in Z_0 : p \cdot x \geq p \cdot z_0\}$. We say that $\boldsymbol{p}$ (or the hyperplane normal to p) ***supports*** $\boldsymbol{R_0}$ ***at*** $\boldsymbol{z_0}$ if $UC(R_0, z_0) \subseteq H(p, z_0)$. Note that since $UC(\cdot)$ and $H(\cdot)$ are defined as subsets of Z_0, although p supports R_0 at z_0, there can be a bundle x outside Z_0, which is preferred to z_0, or $x \ R_0 \ z_0$, but satisfies $p \cdot x < p \cdot z_0$. Let $\boldsymbol{\nabla R_0(z_0)} \equiv \{p \in \mathbb{R}^l_{++} : UC(R_0, z_0) \subseteq H(p, z_0)\}$ be the set of all vectors supporting R_0 at z_0.[7] For all $R \in \mathcal{D}$ and all $z \in Z$, let $\boldsymbol{\nabla R(z)} \equiv \{p \in \mathbb{R}^l_{++} : UC(R_1, z_1) \subseteq H(p, z_1) \text{ and } UC(R_2, z_2) \subseteq H(p, z_2)\}$ be the set of all vectors supporting both R_1 at z_1 and R_2 at z_2. Note that when z is *efficient for* R, $\nabla R(z) \neq \varnothing$.

A domain is ***flexible*** if there exist a subdomain $\bar{\mathcal{D}}$ and a reference set M satisfying *A1, A2, A3*, and the following two properties, F1 and F2. Condition F1 states that for any preference and any bundle, there is an admissible local transformation with sufficiently flat indifference curve at the bundle (see Figure 2 (a)). This property obviously implies the admissibility of local transformation.

F1: For all $R \in \bar{\mathcal{D}}$, all $i \in N$, all $z \in P(R)$, and all $x \in \mathbb{R}^l_+$, if for some $p \in \nabla R(z)$, $p \cdot z_i < p \cdot x$, then there exists $R_i' \in \bar{\mathcal{D}}_i(R_{-i})$ such that $P_i(R_i', R_{-i}) \cap LC(R_i, z_i) \cap UC(R_i', z_i) = \{z_i\}$ and $x \ P_i' \ z_i$.

Next condition F2 states that given an agent i and an allocation $d \in M$, there exists a profile $R \in \bar{\mathcal{D}}$ whose Pareto set intersects with M only at $(0, \Omega)$ and $(\Omega, 0)$, and such that whenever an efficient allocation $z \neq d$ for R happens to have d on the hyperplane supporting R at z (see Figure 2 (b)), changing i's preference is admissible so that for the new profile (R_i', R_{-i}), such a coincidence never happens at any efficient allocation in i's incentive compatibility set associated with (R_i, R_i', z_i), $LC(R_i, z_i) \cap UC(R_i', z_i)$ (see Figure 2 (b)).

F2: For all $i \in N$ and all $d \in M$, there exists $R \in \bar{\mathcal{D}}$ such that (i) $P_i(R) \cap M_i = \{0, \Omega\}$ and (ii) if $z \in P(R) \backslash \{d\}$ and $p \cdot z_i = p \cdot d_i$ for all $p \in \nabla R(z)$, then there exists $R_i' \in \bar{\mathcal{D}}_i(R_{-i})$ such that for all $z_i' \in P_i(R_i', R_{-i}) \cap LC(R_i, z_i) \cap UC(R_i', z_i)$, $p' \cdot z_i' \neq p' \cdot d_i$, for some $p' \in \nabla(R_i', R_{-i})(z')$.

We next provide an example of *flexible* domain and an example of "inflexibility".

[7]When R_0 has a differentiable representation u_0 and z_0 is an interior bundle of Z_0, $\nabla R_0(z_0)$ is simply the gradient of u_0 at z_0, that is, $\nabla u_0(z_0) \equiv (\partial u_0(z_0)/\partial x_1, \cdots, \partial u_0(z_0)/\partial x_l)$.

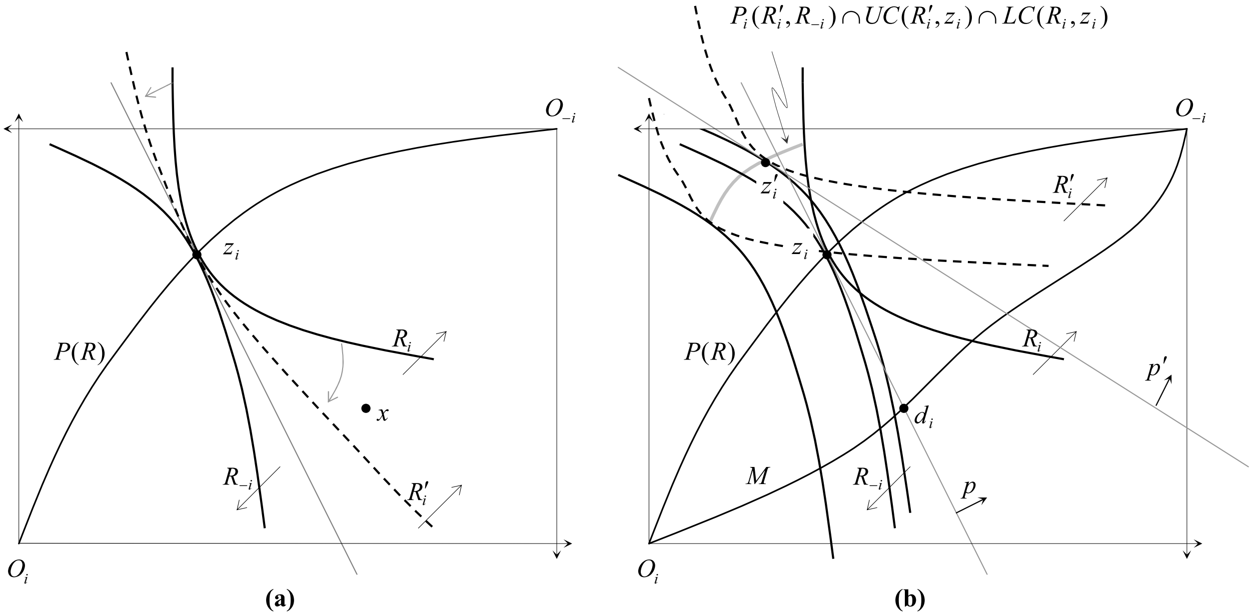

Figure 2: Flexibility. (a) Figure for F1: $P_i(R'_i, R_{-i}) \cap LC(R_i, z_i) \cap UC(R'_i, z_i) = \{z_i\}$ and $x \; P'_i \; z_i$. (b) Figure for F2: $P_i(R) \cap M_i = \{0, \Omega\}$, $z \in P(R) \backslash \{d\}$ and $p \cdot z_i = p \cdot d_i$ for all $p \in \nabla R(z)$. Note that for all $z'_i \in P_i(R'_i, R_{-i}) \cap LC(R_i, z_i) \cap UC(R'_i, z_i)$, $p' \cdot z'_i \neq p' \cdot d_i$, for some $p' \in \nabla(R'_i, R_{-i})(z')$.

Example 2 *Homothetic preferences*: Let $\mathcal{R}_H$ be the family of all *homothetic* preferences that are *smooth*,[8] *strictly convex*, and *strictly monotonic over* $\mathbb{R}^l_{++}$. Let $\bar{\mathcal{D}} \equiv \mathcal{R}^N_H$ and $M \equiv \{z \in Z : \text{for some } \lambda \in [0,1],\; z_1 = \lambda\Omega + (1-\lambda)\,0 \text{ and } z_2 = \Omega - z_1\}$. We will show that $\bar{\mathcal{D}}$ and M satisfy *A1, A2, A3,* F1, and F2; so $\mathcal{R}^N_H$ is *flexible.*

For all $R_i \in \mathcal{R}_H$, if $R_{-i} = R_i$, $P\,(R_i, R_{-i}) = M$. Hence *A1* and *A2* hold.

In order to show *A3*, let $R, R' \in \bar{\mathcal{D}}$ and $z, z' \in M$ be such that $P(R) = P(R') = M$ and $z \neq z'$. Without loss of generality, let $z_1 < z'_1$. When $z_1 = 0$, if we let $\bar{R}_1 = R_1$, then (i) and (ii) of *A3* hold. Now suppose $z_1 \neq 0$. Then let R_1^{Leon} be the Leontieff-type preference with the locus of kinks equal to M_1. Then (i) and (ii) holds with $\bar{R}_1 = R_1^{\text{Leon}}$. Note that $R_1^{\text{Leon}} \notin \mathcal{R}_H$ but that $\mathcal{R}_H$ contains a sequence of preferences, which is composed of local transformations of R_1 at z_1 relative to R_{-1} and, at the same time, converges to R_1^{Leon}. Therefore, there exist a local transformation of R_1, $\bar{R}_1$, which is sufficiently close to R_1^{Leon} so that (i) and (ii) of *A3* can be satisfied.

[8]A preference $R_0 \in \mathcal{R}$ is ***smooth*** if for all $x \in \mathbb{R}^l_{++}$, there is a unique $p \in \Delta^{l-1}$ such that for all $y \in \mathbb{R}^l_{+}$, if $y \; R_0 \; x$, then $p \cdot y \geq p \cdot x$..

For all $x \in \mathbb{R}^l_{++}$ and all $p \in \mathbb{R}^l_{++}$, there exists a sequence of preferences in $\mathcal{R}_H$, which are supported by p at x and converge to the linear preference associated with normal vector p. Therefore F1 holds. Now we only have to verify F2. We show F2 for the 2-good case. However, our argument can be easily extended to the l-good case.

Let $i \equiv 1$ and $d \in M$. Let R be the preference such that $P_1(R) = \{z_1 \in Z_0 : z_{11} = 0 \text{ or } z_{12} = \Omega_2\}$ and the slope of indifference curves of R_1 over $P_1(R)$ is bounded *above* by $-\delta < 0$. Clearly R satisfies (i) of F2. Let $z \in P(R) \setminus \{d\}$ be such that for all $p \in \nabla R(z)$, $p \cdot z_1 = p \cdot d_1$. Then there exists $R'_1 \in \mathcal{R}_H$ such that the slopes of indifference curves of R'_1 is bounded *below* by $-\delta$. Then clearly, $P(R'_1, R_2) = P(R)$ and since $P_1(R)$ is a boundary and monotonic path of the Edgeworth box, $P_1(R'_1, R_2) \cap LC(R_1, z_1) \cap UC(R'_1, z_1) = \{z_1\}$. Since for all $p \in \nabla R(z)$, $p \cdot z_1 = p \cdot d_1$, and the indifference curve of R'_1 through z_1 is flatter at z_1 than the indifference curve of R_1, there exists $p' \in \nabla(R'_i, R_{-i})(z')$ such that $p' \cdot z_1 \neq p' \cdot d_1$. Therefore, (ii) of F2 also holds. $\square$

Example 3 *Linear preferences*: Let $\mathcal{R}_L$ be the family of preferences that are represented by linear utility functions. Let $\bar{\mathcal{D}} \equiv \mathcal{R}_L^N$. Then we show that for any reference set M, $\bar{\mathcal{D}}$ and M do not satisfy F1. Let $R_1 = R_2 \in \mathcal{R}_L$. Then $P(R) = Z$. Let $z \in P(R)$ be such that $z_1 \in \mathbb{R}^L_{++}$. Then for all $R'_1 \in \mathcal{R}_L$, either $z_1 \notin P_1(R'_1, R_2)$ or $R'_1 = R_1$; hence there is no local transformation of R_1 at z_1 relative to $R_2 (= R_1)$. $\square$

We show in Lemma 1 that F1 and F2 imply the following more general property.

Double transformability: For all $i \in N$ and all $d \in M$, there exists $R \in \bar{\mathcal{D}}$ such that (i) $P_i(R) \cap M_i = \{0, \Omega\}$ and (ii) for all $z \in P(R)$ with $z_i \neq d_i$, there exists $R'_i \in \bar{\mathcal{D}}_i(R_{-i})$ such that if $z'_i \in P_i(R'_i, R_{-i}) \cap UC(R'_i, z_i) \cap LC(R_i, z_i)$, then either

(ii-1) for some $R''_i \in \bar{\mathcal{D}}_i(R_{-i})$,
$$P_i(R''_i, R_{-i}) \cap LC(R'_i, z'_i) \cap UC(R''_i, z'_i) = \{z'_i\} \text{ and } d_i \; P''_i \; z'_i,$$
or

(ii-2) for some $R'_{-i} \in \bar{\mathcal{D}}_{-i}(R'_i)$,
$$P_{-i}(R'_i, R'_{-i}) \cap LC(R_{-i}, z'_{-i}) \cap UC(R'_{-i}, z'_{-i}) = \{z'_{-i}\} \text{ and } d_{-i} \; P'_{-i} \; z'_{-i}.$$

Double transformability has wider applicability. For example, as we saw in Example 3, there is no reference path M such that $\mathcal{R}_L^N$ and M satisfy *flexibility*. However, our conclusion in Section 4.1 shows that there exists a reference path M such that $\mathcal{R}_L^N$ and M satisfy *double transformability*.

A domain $\mathcal{D}$ is ***rich*** if there exist a subdomain $\bar{\mathcal{D}}$ and a reference set M satisfying *A1, A2, A3*, and *double transformability.*

Lemma 1: *Every flexible domain is rich.*

Proof: Let $\mathcal{D}$ be *flexible* with respect to a subdomain $\bar{\mathcal{D}} \subseteq \mathcal{D}$ and a reference set $M \subseteq Z$. Let $i \in N$ and $d \in M$. By F2, there exists $R \in \bar{\mathcal{D}}$ such that $P_i(R) \cap M_i = \{0, \Omega\}$. Let $z \in P(R)$ with $z_i \neq d_i$. We divide into three cases.

Case 1: *There exists* $p \in \nabla R(z)$ *such that* $p \cdot z_i < p \cdot d_i$. Then by F1, there exists $R_i' \in \bar{\mathcal{D}}_i(R_{-i})$ such that $P_i(R_i', R_{-i}) \cap LC(R_i, z_i) \cap UC(R_i', z_i) = \{z_i\}$ and $d_i \ P_i' \ z_i$. Therefore, if we let $R_i'' \equiv R_i'$, then (ii-1) of *double transformability* holds. □

Case 2: *There exists* $p \in \nabla R(z)$ *such that* $p \cdot z_i > p \cdot d_i$. By F1, there exists $R_i' \in \bar{\mathcal{D}}_i(R_{-i})$ such that $P_i(R_i', R_{-i}) \cap LC(R_i, z_i) \cap UC(R_i', z_i) = \{z_i\}$. Since $p \cdot z_{-i} < p \cdot d_{-i}$, then applying F1 for (R_i', R_{-i}) and agent $-i$, there exists $R_{-i}' \in \bar{\mathcal{D}}_{-i}(R_i')$ such that $P_i(R_i', R_{-i}') \cap LC(R_{-i}, z_{-i}) \cap UC(R_{-i}', z_i) = \{z_{-i}\}$ and $d_{-i} \ P_{-i}' \ z_{-i}$. Therefore, if we let $R_i' \equiv R_i$, then (ii-2) of *double transformability* holds. □

Case 3: *For all* $p \in \nabla R(z)$, $p \cdot z_i = p \cdot d_i$. By F2, there exists $R_i' \in \bar{\mathcal{D}}_i(R_{-i})$ such that for all $z_i' \in P_i(R_i', R_{-i}) \cap LC(R_i, z_i) \cap UC(R_i', z_i)$, $p' \cdot z_i' \neq p' \cdot d_i$, for some $p' \in \nabla(R_i', R_{-i})(z')$. If $z_i' \in P_i(R_i', R_{-i}) \cap LC(R_i, z_i) \cap UC(R_i', z_i)$ and $p' \cdot z_i' < p' \cdot d_i$, for some $p' \in \nabla(R_i', R_{-i})(z')$, then by F1, there exists $R_i'' \in \bar{\mathcal{D}}_i(R_{-i})$ such that $P_i(R_i'', R_{-i}) \cap LC(R_i', z_i) \cap UC(R_i'', z_i) = \{z_i'\}$ and $d_i \ P_i'' \ z_i'$. Hence (ii-1) of *double transformability* holds. On the other hand, if $z_i' \in P_i(R_i', R_{-i}) \cap LC(R_i, z_i) \cap UC(R_i', z_i)$ and $p' \cdot z_i' > p' \cdot d_i$, for some $p' \in \nabla(R_i', R_{-i})(z')$, then $p' \cdot z_{-i}' < p' \cdot d_{-i}$. Now applying F1 for (R_i', R_{-i}) and agent $-i$, there exists $R_{-i}' \in \bar{\mathcal{D}}_{-i}(R_i')$ such that $P_i(R_i', R_{-i}') \cap LC(R_{-i}, z_{-i}') \cap UC(R_{-i}', z_i') = \{z_{-i}'\}$ and $d_{-i} \ P_{-i}' \ z_{-i}'$. Hence (ii-2) of *double transformability* holds. □ *Q.E.D.*

Zhou (1991) and Schummer (1997) establish an invariance property of *efficient* and *strategy-proof* rule with respect to "Maskin monotonic" transformations of preferences.[9] Lemma 2 states an even stronger invariance property related with local transformation. A rule φ is ***invariant with respect to local transformation*** if for all $R \in \mathcal{D}$ and all $i \in N$, if R_i' is a local transformation of R_i at z_i relative to R_{-i}, then $\varphi(R_i', R_{-i}) = \varphi(R)$.

Lemma 2: *Every efficient and strategy-proof rule is invariant with respect to local transformation.*

[9]Let $R \in \mathcal{D}$. Let $z \in P(R)$. A preference R_i' is a (strong) *Maskin monotonic transformation of* R_i *at* z if $LC(R_i', z_i) \supseteq LC(R_i, z_i)$ and $LC(R_i, z_i) \cap UC(R_i', z_i) = \{z_i\}$. Clearly, such R_i' is a local transformation of R_i at z_i relative to R_{-i}. However, there are various local transformations that are not Maskin monotonic transformations.

Proof: Let φ be an *efficient* and *strategy-proof* rule. Let $z \equiv \varphi(R)$, $i \in N$, and R_i' be a local transformation of R_i at z_i relative to R_{-i}, that is, $P_i(R_i', R_{-i}) \cap LC(R_i, z_i) \cap UC(R_i', z_i) = \{z_i\}$. Let $z' \equiv \varphi(R_i', R_{-i})$. By the two incentive compatibility conditions associated with (R_i, R_i', z_i) and (R_i', R_i, z_i'), $z_i' \in LC(R_i, z_i) \cap UC(R_i', z_i)$. Therefore by *efficiency,* $z_i' \in P_i(R_i', R_{-i}) \cap LC(R_i, z_i) \cap UC(R_i', z_i) = \{z_i\}$ and so $z_i' = z_i$. Hence $\varphi(R_i', R_{-i}) = \varphi(R)$. *Q.E.D.*

Next, we show that if the domain is *rich* with respect to a subdomain $\bar{\mathcal{D}}$ and a set M, then for a rule to be *efficient* and *strategy-proof,* it should always pick a fixed allocation for each economy in $\bar{\mathcal{D}}$ with Pareto set M.

Lemma 3: *Let $\mathcal{D}$ be rich with respect to $\bar{\mathcal{D}} \subseteq \mathcal{D}$ and $M \subset Z$. Let $\varphi\colon \mathcal{D} \to Z$ be efficient and strategy-proof. Then for all $R, R' \in \bar{\mathcal{D}}$, if $P(R) = P(R') = M$, then $\varphi(R) = \varphi(R')$.*[10]

Proof: Let $R, R' \in \bar{\mathcal{D}}$ be such that $P(R) = P(R') = M$. Let $z \equiv \varphi(R)$ and $z' \equiv \varphi(R')$. Suppose to the contrary $z_1 \neq z_1'$. By *A3,* there exists $\bar{R}_1 \in \bar{\mathcal{D}}_1(R_2) \cap \bar{\mathcal{D}}_1(R_2')$ such that (i) $P_1(\bar{R}_1, R_2) \cap LC(R_1, z_1) \cap UC(\bar{R}_1, z_1) = \{z_1\}$ and (ii) $P_1(\bar{R}_1, R_2') \cap \{x \in \mathbb{R}_+^l : \Omega - x \in LC(R_2, z_2) \cap UC(R_2', z_2)\} \subset LC^0(\bar{R}_1, z_1')$. By (i) and Lemma 2, $\varphi(\bar{R}_1, R_2) = z$. By the incentive compatibility associated with (R_2, R_2', z), $\varphi_2(\bar{R}_1, R_2') \in LC(R_2, z_2) \cap UC(R_2', z_2)$. Hence by *efficiency,* $\varphi_1(\bar{R}_1, R_2') \in P_1(\bar{R}_1, R_2') \cap \{x \in \mathbb{R}_+^l : \Omega - x \in LC(R_2, z_2) \cap UC(R_2', z_2)\}$. Therefore by (ii), $\varphi_1(\bar{R}_1, R_2') \in LC^0(\bar{R}_1, z_1')$, that is, $\varphi_1(R_1', R_2')\ \bar{P}_1\ \varphi_1(\bar{R}_1, R_2')$, contradicting *strategy-proofness.* *Q.E.D.*

We will show that when a rule gives one agent the whole endowment at a profile, for it to be *efficient* and *strategy-proof,* it should be dictatorial over a certain neighborhood of the initial profile.[11] In this sense, dictatorship at a

[10]Lemma 3 corresponds to Step 4 of Proof of Theorem 1 by Zhou (1991) and Lemmas 2, 3, and Corollary 1 by Schummer (1997). Zhou and Schummer make use of Maskin monotonic transformation in the proofs. Particularly in Schummer (1997), M_i is the line segment between 0 and Ω. He uses a preference which is a Maskin monotonic transformation of both R_i at z_i and R_i at z_i', where $z_i, z_i' \in M_i$. This preference should be kinked as far as it is homothetic and R_i has a different supporting hyperplane at z_i from the supporting hyperplane of R_i' at z_i'. In restricted domains without kinked preferences, the proof in Schummer (1997) does not work. Our proof does not necessarily require such Maskin monotonic transformation. We only use a preference that satisfies (i) and (ii) in the above proof. Our argument is based on strong invariance property established in Lemma 2. Consequently, as we show in Example 2 and in Section 4 later, Lemma 3 applies in a number of domains without kinked preferences.

[11]By using the term "neighborhood" of a profile R, we do not mean an "open" set containing R. It simply means a set containing R.

profile contaminates the choices made for some other profiles. For the formal description, we need the following notation.

Let $R \in \mathcal{D}$ and $i \in N$. Let $\mathcal{S}^i(R) \equiv \{R' \in \mathcal{D} : I'_{-i}(0) \supseteq I_{-i}(0)$ and there exists $R''_i \in \mathcal{D}_i$ such that $R'_{-i} \in \mathcal{D}_{-i}(R''_i)$ and $R''_i \in \mathcal{D}_i(R_{-i})\}$. Note that if $R' \in \mathcal{S}^i(R)$, then R' is reachable from R through the following three unilateral variations: $(R_i, R_{-i}) \to (R''_i, R_{-i}) \to \left(R''_i, R'_{-i}\right) \to \left(R'_i, R'_{-i}\right)$, where $R''_i \in \mathcal{D}_i$ is such that $R'_{-i} \in \mathcal{D}_{-i}(R''_i)$ and $R''_i \in \mathcal{D}_i(R_{-i})$. Since $I'_{-i}(0) \supseteq I_{-i}(0)$, all these unilateral variations are 0-indifference-monotonic for $-i$. Note also that if R' is a unilateral variation of R, which is 0-indifference-monotonic for $-i$, then $R' \in \mathcal{S}^i(R)$.

Let $\bar{\mathcal{S}}^i(R)$ be defined as follows: for all $R' \in \mathcal{D}$, $R' \in \bar{\mathcal{S}}^i(R)$ if and only if there exists a finite sequence $(R^1, \cdots, R^n)$ of profiles in $\mathcal{D}$, $n \geq 2$, such that $R^1 \equiv R$, $R^n \equiv R'$, and $R^2 \in \mathcal{S}^i(R^1), \cdots, R^n \in \mathcal{S}^i(R^{n-1})$. We call $\bar{\mathcal{S}}^i(R)$ the ***contamination set relative to R and i***. Then every $R' \in \bar{\mathcal{S}}^i(R)$ is reachable from R through iterative unilateral variations that are 0-indifference-monotonic for $-i$, and conversely. Note that when $\mathcal{D}$ is *everywhere reachable** and R_{-i} is strictly monotonic, every $R' \in \mathcal{D}$ is reachable from R through iterative unilateral variations that are 0-indifference-monotonic for $-i$; so $\bar{\mathcal{S}}^i(R) = \mathcal{D}$.

Lemma 4: *Let $\varphi \colon \mathcal{D} \to Z$ be efficient and strategy-proof. If there exist $i \in N$ and $R \in \mathcal{D}$ such that $\varphi_i(R) = \Omega$, then φ is dictatorial over $\bar{\mathcal{S}}^i(R)$.*

Proof: Let $R \in \mathcal{D}$, $i \in N$, and $R' \in \mathcal{S}^i(R)$. Suppose $\varphi_i(R) = \Omega$. We only have to show that $\varphi_i(R') = \Omega$. By definition of $\mathcal{S}^i(R)$, $I'_{-i}(0) \supseteq I_{-i}(0)$ and there exists $R''_i \in \mathcal{D}_i$ such that $R'_i \in \mathcal{D}_i(R'_{-i})$, $R'_{-i} \in \mathcal{D}_{-i}(R''_i)$, and $R''_i \in \mathcal{D}_i(R_{-i})$. Since R_i is strictly monotonic over $\mathbb{R}^l_{++}$, then by i's incentive compatibility condition relative to (R_i, R''_i, Ω), $\varphi_i(R''_i, R_{-i}) = \Omega$ and so $\varphi_{-i}(R''_i, R_{-i}) = 0$. By $-i$'s incentive compatibility condition relative to $(R_{-i}, R'_{-i}, 0)$, $\varphi_{-i}\left(R''_i, R'_{-i}\right)\ I_{-i}\ 0$. Since $I'_{-i}(0) \supseteq I_{-i}(0)$, $\varphi_{-i}\left(R''_i, R'_{-i}\right)\ I'_{-i}\ 0$. Therefore, by *efficiency*, $\varphi_{-i}\left(R''_i, R'_{-i}\right) = 0$ and so $\varphi_i\left(R''_i, R'_{-i}\right) = \Omega$. Finally, by i's incentive compatibility condition relative to (R''_i, R'_i, Ω), $\varphi_i\left(R'_i, R'_{-i}\right) = \Omega$. *Q.E.D.*

Let $\mathcal{D}$ be *rich* with respect to $\bar{\mathcal{D}} \subseteq \mathcal{D}$ and $M \subset Z$. Given $(\bar{\mathcal{D}}, M)$, we call $\bigcup_{R \in \{R' \in \bar{\mathcal{D}} \colon P(R') = M\}} \bar{\mathcal{S}}^i(R)$ ***i's minimal contamination set relative to*** $\left(\bar{\mathcal{D}}, M\right)$.

Proposition 1: *Assume that domain $\mathcal{D}$ is rich with respect to $\bar{\mathcal{D}} \subseteq \mathcal{D}$ and $M \subset \mathbb{R}^l_+$. Then if a rule over $\mathcal{D}$ is efficient and strategy-proof, then for some $i \in N$, the rule is dictatorial over i's minimal contamination set relative to $\left(\bar{\mathcal{D}}, M\right)$.*[12]

[12]Proposition 1 corresponds to Steps 2, 5, and 6 in Proof of Theorem 1 by Zhou (1991) and Proof of Theorem 1 by Schummer (1997). Zhou and Schummer construct kinked preferences

Proof: Let $\varphi\colon \mathcal{D} \to Z$ be *efficient* and *strategy-proof.* Let d be an allocation such that for all $R' \in \bar{\mathcal{D}}$, if $P(R') = M$, then $\varphi(R') = d$. By *A1* and Lemma 3, d is well-defined. Let $R \in \bar{\mathcal{D}}$ be such that $P(R) = M$. By Lemmas 3 and 4, we only have to show that $d_1 \in \{0, \Omega\}$. Suppose to the contrary that $d_1 \notin \{0, \Omega\}$. By *double transformability*, there exists $R \in \bar{\mathcal{D}}$ such that (i) $P_1(R) \cap M_1 = \{0, \Omega\}$ and (ii) for all $z \in P(R)$ with $z_1 \neq d_1$, there exists $R'_1 \in \bar{\mathcal{D}}_1(R_2)$ such that if $z'_1 \in P_1(R'_1, R_2) \cap LC(R_1, z_1) \cap UC(R'_1, z_1)$, then either

(ii-1) for some $R''_1 \in \bar{\mathcal{D}}_1(R_2)$,

$$P_1(R''_1, R_2) \cap LC(R'_1, z'_1) \cap UC(R''_1, z'_1) = \{z'_1\} \text{ and } d_1 \; P''_1 \; z'_1$$

or

(ii-2) for some $R'_2 \in \bar{\mathcal{D}}_2(R'_1)$,

$$P_2(R'_1, R'_2) \cap LC(R_2, z'_2) \cap UC(R'_2, z'_2) = \{z'_2\} \text{ and } d_2 \; P'_2 \; z'_2.$$

Let $z \equiv \varphi(R)$. Then clearly $z \in P(R)$ and by (i), $z_1 \neq d_1$. Therefore by *efficiency* and *strategy-proofness* and (ii), there exists $R'_1 \in \bar{\mathcal{D}}_1(R_2)$ such that either (ii-1) or (ii-2) holds at $z'_1 \equiv \varphi_1(R'_1, R_2)$. When (ii-1) holds, there exists $R''_1 \in \mathcal{D}_1(R_2)$ such that $d_1 \; P''_1 \; z'_1$ and $P_1(R''_1, R_2) \cap LC(R'_1, z'_1) \cap UC(R''_1, z'_1) = \{z'_1\}$. By Lemma 2, $\varphi(R''_1, R_2) = z'$. By *A2,* there exists $\bar{R}_1 \in \mathcal{D}_1(R_2)$ such that $P(\bar{R}_1, R_2) = M$. Since $\varphi(\bar{R}_1, R_2) = d$, $\varphi_1(\bar{R}_1, R_2) \; P''_1 \; \varphi_1(R''_1, R_2)$. This contradicts *strategy-proofness.* When (ii-2) holds, we can derive a contradiction, using the same argument as above. *Q.E.D.*

When the domain satisfies *everywhere reachability** in addition, the minimal contamination set in Proposition 1 coincides with the entire domain.

Theorem 1: *Given a rich and everywhere reachable** *domain, a rule is efficient and strategy-proof if and only if it is dictatorial.*

Proof: Let $\mathcal{D}$ be *rich* and *everywhere reachable**. Then there exist $\bar{\mathcal{D}} \subseteq \mathcal{D}$ and $M \subset \mathbb{R}^l_+$ satisfying *A1, A2, A3,* and *double transformability.* Let $\varphi\colon \mathcal{D} \to Z$ be *efficient* and *strategy-proof.* Then by Proposition 1, there exists $i \in N$ such that φ is dictatorial over i's minimal contamination set relative to $(\bar{\mathcal{D}}, M)$. By *A1,* there exists $R \in \bar{\mathcal{D}}$ such that $P(R) = M$ and both R_1 and R_2 are strictly monotonic. Note that by strict monotonicity, $I_{-i}(0) = \{0\}$ and so for all $R' \in \mathcal{D}$, $I_{-i}(0) \subseteq I'_{-i}(0)$. Then by *everywhere reachable**, every other profile is reachable from R through iterative unilateral variations that are 0-indifference monotonic for $-i$. Therefore $\bar{\mathcal{S}}^i(R) = \mathcal{D}$ and φ is dictatorial. *Q.E.D.*

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and make use of the invariance of *strategy-proof* and *efficient* rules with respect to Maskin monotonic transformations. Our proof is simpler and works well without kinked preferences. This is because our proof makes use of the stronger invariance property in Lemma 2.

Since every product domain is *everywhere reachable**, Theorem 1 applies to *rich* product domains.

Remark 1: Both *richness* and *everywhere reachability** are essential in Theorem 1. Example 1 shows that without *richness,* the impossibility does not hold. Without *everywhere reachability**, the impossibility does not hold either. The following example shows this. Let $\mathcal{D}_a$ and $\mathcal{D}_b$ be such that $\mathcal{D} \equiv \mathcal{D}_a \cup \mathcal{D}_b$. Suppose that for all $R \in \mathcal{D}_a$ and all $R' \in \mathcal{D}_b$, R' is not reachable from R through iterative unilateral variations. Now let φ be dictatorial over $\mathcal{D}_a$ and agent 1 be the dictator over $\mathcal{D}_a$. Let φ be dictatorial over $\mathcal{D}_b$ and agent 2 be the dictator over $\mathcal{D}_b$. Then φ is *efficient, strategy-proof,* and *non-dictatorial.*

4 Applications

In this section, we apply our result in Section 3 to "intertemporal exchange problem", "risk sharing problem", and two restricted domains, the "CES domain" and the "quasilinear domain".

4.1 Intertemporal exchange

Let T be the number of periods, $T \geq 2$. For each $t = 1, \cdots, T$, let $\Omega_t > 0$ be the endowment of a single consumption good at period t. Suppose that there exists no saving technology. Then an allocation $(z_i)_N \in \mathbb{R}_+^{T \times N}$ is *feasible* if for all $t = 1, \cdots, T$, $\sum_i z_{it} \leq \Omega_t$.

Each agent $i \in N$ has a preference R_i represented by a *temporal utility function* $u_i \colon \mathbb{R}_+ \to \mathbb{R}$ and a *discount factor* $\delta_i \in (0,1)$ as follows: for all $x, y \in \mathbb{R}_+^T$,

$$x \; R_i \; y \Leftrightarrow \sum_{t=1}^{T} \delta_i^{t-1} u_i(x_t) \geq \sum_{t=1}^{T} \delta_i^{t-1} u_i(y_t) \, .$$

Note that when the temporal utility function u_i is *concave* (respectively, *strictly concave*), R_i is *convex* (respectively, *strictly convex*). Let $\boldsymbol{\mathcal{R}_{\mathrm{IE}}}$ be the class of all such preferences represented by *concave*, *strictly monotonic*, and *continuous* temporal utility functions. We refer to $\boldsymbol{\mathcal{R}_{\mathrm{IE}}^N}$ as the ***intertemporal exchange domain***. Let $\boldsymbol{\mathcal{R}_{\text{IE-s.con}}}$ be the class of all preferences in $\mathcal{R}_{\mathrm{IE}}$ with *strictly concave* temporal utility functions. Let $\boldsymbol{\mathcal{R}_{\text{IE-lin}}}$ be the class of preferences in $\mathcal{R}_{\mathrm{IE}}$ with the *linear* temporal utility function, $u^{\mathrm{lin}}(m) = m$, for all $m \in \mathbb{R}_+$.

In order to show that the intertemporal exchange domain is *rich*, we make use of the following subsets of Z. Let $P^{\lrcorner} \equiv \{z \in Z \colon$ for some $t \in \{1, \cdots, T\}$, $z_1 = (\Omega_1, \cdots, \Omega_{t-1}, z_{1t}, 0, \cdots, 0)$ and $z_2 = (0, \cdots, 0, z_{2t}, \Omega_{t+1}, \cdots, \Omega_T)\}$. Let $P^{\ulcorner} \equiv$

$\{z \in Z\colon$ for some $t \in \{1, \cdots, T\}$, $z_2 = (\Omega_1, \cdots, \Omega_{t-1}, z_{2t}, 0, \cdots, 0)$ and $z_1 = (0, \cdots, 0, z_{1t}, \Omega_{t+1}, \cdots, \Omega_T)\}$. Note that for each $i \in N$, both $P_i^{\lrcorner}$ and $P_i^{\ulcorner}$ are monotonic path from 0 to $(\Omega_1, \cdots, \Omega_T)$.

Proposition 2: *The intertemporal exchange domain $\mathcal{R}^N_{\text{IE}}$ and the two subdomains, $\mathcal{R}^N_{\text{IE-lin}}$ and $\mathcal{R}^N_{\text{IE-s.con}}$, are rich and everywhere reachable**.[13]

Proof: *Everywhere reachability** is obvious. We only have to show that both $\mathcal{R}^N_{\text{IE-lin}}$ and $\mathcal{R}^N_{\text{IE-s.con}}$ are *rich*. In what follows, we show *richness* of $\mathcal{R}^N_{\text{IE-lin}}$. A similar argument applies for $\mathcal{R}^N_{\text{IE-s.con}}$.[14]

Throughout the proof, the linear preference with discount factor $\delta \in (0,1)$ is denoted by R_δ. Then R_δ is represented by the following utility function U_δ: for all $x \in \mathbb{R}^T_+$, $U_\delta(x) \equiv \sum_t \delta^{t-1} x_t$. We make use of the following claim, which states that when both agents have linear preferences, the Pareto set is equal to $P^{\lrcorner}$ (respectively, $P^{\ulcorner}$) if and only if agent 2 is more (respectively, less) patient than agent 1. We omit the proof.

Claim 1: *For all $\delta_1, \delta_2 \in (0,1)$, (i) $P(R_{\delta_1}, R_{\delta_2}) = P^{\lrcorner} \Leftrightarrow \delta_1 < \delta_2$ and (ii) $P(R_{\delta_1}, R_{\delta_2}) = P^{\ulcorner} \Leftrightarrow \delta_1 > \delta_2$.*

Let $M \equiv P^{\lrcorner}$ and $\bar{\mathcal{D}} \equiv \mathcal{R}^N_{\text{lin}}$. Then both *A1* and *A2* follow from Claim 1.

A3: Let $(\delta_1, \delta_2), (\delta_1', \delta_2') \in (0,1)^2$ and $z, z' \in M$ be such that $z \neq z'$ and $P(R_{\delta_1}, R_{\delta_2}) = P(R_{\delta_1'}, R_{\delta_2'}) = M$. Without loss of generality, we assume $z_1 \leq z_1'$. By Claim 1, $\delta_1 < \delta_2$ and $\delta_1' < \delta_2'$. There exists $\delta_1^* \in (0,1)$ such that $\delta_1^* \leq \min\{\delta_1, \delta_1'\}$. Then by Claim 1, $P(R_{\delta_1^*}, R_{\delta_2}) = P^{\lrcorner}$. Since $P^{\lrcorner}$ is a monotonic path, $P_1(R_{\delta_1^*}, R_{\delta_2}) \cap LC(R_{\delta_1}, z_1) \cap UC(R_{\delta_1^*}, z_1) = \{z_1\}$. Also by Claim 1, $P_1(R_{\delta_1^*}, R_{\delta_2'}) =$

[13] In the two period case, $T = 2$, for each agent, there are infinitely many admissible linear preferences in $\mathcal{R}_{\text{IE-lin}}$. Schummer (1997) shows that in the 2-good exchange economy case, given any domain with at least four admissible linear preferences for each agent, dictatorial rules are the only *efficient* and *strategy-proof* rules. Therefore his result applies. Schummer (1997) extends this result for the 2-good case to the l-good case using specific preferences in which commodities are partitioned into two groups with identical marginal utilities. Such preferences are not admissible in $\mathcal{R}^N_{\text{IE-lin}}$, since marginal utility decreases in the rate of discount factor over periods. Therefore, when $T \geq 3$, Schummer's result does not apply.

[14] Fix $\rho \in (0,1)$. Let u such that $u(m) = -e^{-\rho m}$ for all $m \in \mathbb{R}_+$. Let $\bar{\mathcal{D}}$ be the family of profiles of preferences represented by u and a discount factor $\delta \in (0,1)$. Then $\bar{\mathcal{D}} \subseteq \mathcal{R}^N_{\text{IE-s.con}}$. For each discount factor $\delta \in (0,1)$, let $U_\delta\colon \mathbb{R}^2_+ \to \mathbb{R}$ be the utility function associated with u and δ and let R_δ be the corresponding preference. We can establish the following claim, similar to Claim 1 in Proof of Proposition 2.

Claim 1: *For all $\delta_1, \delta_2 \in (0,1)$, (i) $P(R_{\delta_1}, R_{\delta_2}) = P^{\lrcorner} \Leftrightarrow \frac{\delta_2}{\delta_1} \geq e^{\rho(\Omega_1+\Omega_2)}$;*
(ii) $P(R_{\delta_1}, R_{\delta_2}) = P^{\ulcorner} \Leftrightarrow \frac{\delta_1}{\delta_2} \geq e^{\rho(\Omega_1+\Omega_2)}$.

Using this claim and the same argument with a slight modification as in Proof of Proposition 2, we can show that $\bar{\mathcal{D}}$ and $M \equiv P^{\lrcorner}$ satisfy *A1-A3* and *double transformability.*

$P^{\lrcorner}$. Hence $P_1(R_{\delta_1^*}, R_{\delta_2'}) \cap \{x \in Z_0 : \Omega - x \in LC(R_{\delta_2}, z_2) \cap UC(R_{\delta_2'}, z_2)\} = \{z\}$. Since $z_1 \leq z_1'$, $z_1 \in LC^0(R_{\delta_1^*}, z_1')$. $\square$

Double transformability: Let $d \in M$. Let $\delta_1 > \delta_2$. Then $P\left(R_{\delta_1}, R_{\delta_2}\right) = P^{\ulcorner}$ and so $P_i\left(R_{\delta_1}, R_{\delta_2}\right) \cap M_i = \{0, \Omega\}$ for each $i \in N$. Let $z \in P\left(R_{\delta_1}, R_{\delta_2}\right)$ be such that $z_1 \neq d_1$. When $U_{\delta_1}(z_1) < U_{\delta_1}(d_1)$, if we let $R_1' = R_1'' \equiv R_{\delta_1}$, then (ii-1) of *double transformability* holds. When $U_{\delta_2}(z_2) < U_{\delta_2}(d_2)$, if we let $R_1' \equiv R_{\delta_1}$ and $R_2' \equiv R_{\delta_2}$, then (ii-2) of *double transformability* holds.

Now assume $U_{\delta_1}(z_1) \geq U_{\delta_1}(d_1)$ and $U_{\delta_2}(z_2) \geq U_{\delta_2}(d_2)$. Then $d_1 \in P^{\lrcorner}\backslash\{0, \Omega\}$. So d is not *efficient* for $\left(R_{\delta_1}, R_{\delta_2}\right)$. Hence either $U_{\delta_1}(z_1) > U_{\delta_2}(d_1)$ or $U_{\delta_2}(z_2) > U_{\delta_2}(d_2)$. We consider the case $U_{\delta_1}(z_1) > U_{\delta_2}(d_1)$ and $U_{\delta_2}(z_2) \geq U_{\delta_2}(d_2)$ (the same argument applies in the other case). Since $U_{\delta_2}(z_2) = \sum_t \delta_2^{t-1}\left(\Omega_t - z_{1t}\right) \geq \sum_t \delta_2^{t-1}\left(\Omega_t - d_{1t}\right) = U_{\delta_2}(d_2)$, then $U_{\delta_2}(z_1) \leq U_{\delta_2}(d_1)$. Since $U_{\delta_1}(z_1) > U_{\delta_1}(d_1)$, $U_{\delta_2}(z_1) \leq U_{\delta_2}(d_1)$, and $\delta_1 > \delta_2$, then there exists $\delta_2' \in (\delta_2, \delta_1)$ such that $U_{\delta_2'}(z_1) > U_{\delta_2'}(d_1)$. Then, $U_{\delta_2'}(z_2) < U_{\delta_2'}(d_2)$. Now let $R_1' \equiv R_{\delta_1}$ and $R_2' \equiv R_{\delta_2'}$. Then $P\left(R_1', R_2'\right) = P^{\ulcorner}$ and (ii-2) of *double transformability* holds. $\square$ *Q.E.D.*

Remark 2: *Intertemporal exchange domains with bounded difference in agents' discount factors*: It may be the case that both agents share a common cultural background relevant to impatience. Then it is appealing to assume that their impatience levels are not too different; that is, the difference of their discount factors is bounded by a fixed positive number. For each $\mu > 0$, let $\boldsymbol{\mathcal{D}}^{\boldsymbol{\mu}}_{\mathbf{IE}}$ be the family of preference profiles R in $\mathcal{R}^N_{\text{IE}}$ such that the difference between the two discount factors δ_1 and δ_2 for R_1 and R_2 respectively is less than μ, that is, $|\delta_1 - \delta_2| < \mu$. Similarly, we define $\boldsymbol{\mathcal{D}}^{\boldsymbol{\mu}}_{\textbf{IE-lin}} \subseteq \mathcal{R}^N_{\text{IE-lin}}$ and $\boldsymbol{\mathcal{D}}^{\boldsymbol{\mu}}_{\textbf{IE-s.con}} \subseteq \mathcal{R}^N_{\text{IE-s.con}}$. The proof of Proposition 2 can be modified to show *richness* of the three non-product domains for each $\mu > 0$. To show *everywhere reachability**, let $R, R' \in \mathcal{D}^{\mu}_{\text{IE}}$ (or $\mathcal{D}^{\mu}_{\text{IE-lin}}$ or $\mathcal{D}^{\mu}_{\text{IE-s.con}}$) be such that for all $i \in N$, R_i and R_i' are represented by temporal utility functions u_i and u_i', respectively, and discount factors δ_i and δ_i', respectively. Since all preferences in $\mathcal{D}^{\mu}_{\text{IE}}$ are strictly monotonic, we only have to show that R' is reachable from R through iterative unilateral variations, which can be constructed by changing discount factors first as follows $(\delta_1, \delta_2) \to (\delta_1', \delta_2) \to (\delta_1', \delta_2')$ and then changing temporal utility functions as follows $(u_1, u_2) \to (u_1', u_2) \to (u_1', u_2')$.

4.2 Risk sharing

Let S be the number of states, $S \geq 2$. For each $s = 1, \cdots, S$, let $\Omega_s > 0$ be the endowment at state s. We consider the problem of allocating these endowments prior to the realization of state. An allocation is a list of state contingent consumption bundles indexed by agents, $z \equiv (z_i)_{i \in N} \in \mathbb{R}^{S \times N}_+$.

Each agent $i \in N$ has a preference R_i that is represented by a subjective probability distribution, or belief, $\pi_i \equiv (\pi_{is})_s \in \Delta^{S-1}$ and a utility index $u_i \colon \mathbb{R}_+ \to \mathbb{R}$ in the expected utility form as follows: for all $x, y \in \mathbb{R}^S_+$

$$x \; R_i \; y \Leftrightarrow \sum_{s=1}^{S} \pi_{is} u_i (x_s) \geq \sum_{s=1}^{S} \pi_{is} u_i (y_s) \,.$$

We assume that $\pi_i > 0$ and that u_i is *strictly increasing* and *continuous*. We further assume that u_i is *concave*. Let $\boldsymbol{\mathcal{R}}_{\mathbf{RS}}$ be the family of all such expected utility preferences. We refer to $\boldsymbol{\mathcal{R}}^{\boldsymbol{N}}_{\mathbf{RS}}$ as the ***risk sharing domain***. Preference R_i is ***risk averse*** if u_i is *strictly concave*. It is ***risk neutral*** if u_i is the linear function u^{lin}, that is, for all $m \in \mathbb{R}_+$, $u^{\text{lin}} (m) = m$. Let $\boldsymbol{\mathcal{R}}_{\textbf{RS-aver}}$ be the family of all *risk averse* preferences. Let $\boldsymbol{\mathcal{R}}_{\textbf{RS-neut}}$ be the family of all *risk neutral* preferences. Note that in the risk sharing domain both belief and utility index of each agent are private information and so it differs from Example 1.[15]

Let $R_0 \in \mathcal{R}_{\text{IE}}$ be the preference in Section 4.1, which is represented by a temporal convex utility function u_0 and discount factor $\delta \in (0, 1)$. Then R_0 is represented by the following utility function U: for all $x \in \mathbb{R}^T_+$, $U (x) \equiv \sum_t \delta^{t-1} u_0 (x_t)$. Therefore, when $T = S$, R_0 coincides with the preference in $\mathcal{R}_{\text{RS}}$ with utility index u_0 and the following belief,

$$\left(\frac{1}{\sum_t \delta^{t-1}}, \frac{\delta}{\sum_t \delta^{t-1}}, \dots, \frac{\delta^T}{\sum_t \delta^{t-1}} \right).$$

This shows that $\mathcal{R}^N_{\text{IE-lin}} \subseteq \mathcal{R}^N_{\text{RS-neut}} \subseteq \mathcal{R}^N_{\text{RS}}$ and $\mathcal{R}^N_{\text{IE-s.con}} \subseteq \mathcal{R}^N_{\text{RS-aver}}$. Therefore, it follows directly from Propositions 2 that:

Proposition 3: *The risk sharing domain* $\mathcal{R}^N_{\text{RS}}$ *and two subdomains,* $\mathcal{R}^N_{\text{RS-aver}}$ *and* $\mathcal{R}^N_{\text{RS-neut}}$*, are rich and everywhere reachable**.

Remark 3 *Risk sharing domains with bounded difference in agents' beliefs*: When both agents share information on the state space, their beliefs will be affected commonly. Then, agents' beliefs may not be too far from each other. For each $\mu > 0$, let $\boldsymbol{\mathcal{D}}^{\boldsymbol{\mu}}_{\mathbf{RS}}$ be the family of preference profiles $(R_1, R_2) \in \mathcal{R}^N_{\text{RS}}$ such that the difference between the two beliefs π_1 and π_2 for R_1 and R_2 respectively is less than μ, that is, $|\pi_1 - \pi_2| < \mu$. Similarly, we define $\boldsymbol{\mathcal{D}}^{\boldsymbol{\mu}}_{\textbf{RS-neut}} \subseteq \mathcal{R}^N_{\text{RS-neut}}$ and $\boldsymbol{\mathcal{D}}^{\boldsymbol{\mu}}_{\textbf{RS-aver}} \subseteq \mathcal{R}^N_{\text{RS-aver}}$. Then as in Remark 2, these three non-product domains are *rich* and *everywhere reachable**.

[15] Ju (2001) considers the case when all of agents have an identical and revealed belief or an objective distribution as in Example 1.

4.3 Other restricted domains

In this section, we show that the domain of "CES preferences" is *flexible* (so, *rich*) and the domain of "quasilinear", strictly convex, and smooth preferences is *rich.*

A preference R_0 is a ***CES preference*** if there exist $(a_1, \cdots, a_l) \in \mathbb{R}^l_{++}$ and $\rho \in (-\infty, 1)$ such that R_0 is represented by the following utility function u_0: for all $x \in \mathbb{R}^l_{++}$,

$$u_0(x) \equiv \begin{cases} \left(\sum_k a_k x_k^\rho\right)^{1/\rho}, \text{ if } \rho \neq 0; \\ x_1^{a_1} \times \cdots \times x_k^{a_k}, \text{ if } \rho = 0, \end{cases}$$

for all $x \in \mathbb{R}^l_{+} \backslash \mathbb{R}^l_{++}$, if $\rho > 0$, $u_0(x) \equiv \left(\sum_k a_k x_k^\rho\right)^{1/\rho}$; if $\rho \leq 0$, $u_0(x) \equiv 0$. Let $\mathcal{R}_{CES}$ be the class of all *CES* preferences. We refer to $\boldsymbol{\mathcal{R}^N_{CES}}$ as the ***CES-domain***.

Proposition 4: *The CES-domain is flexible and everywhere reachable**.

Proof: *Everywhere reachability** is obvious. Let $M \equiv \{z \in Z : z_1 \in \overline{0, \Omega}\}$. We show that $\mathcal{R}^N_{CES}$ and M satisfy *A1, A2, A3,* F1, and F2. The first four properties can be shown similarly to Example 2. We are left with F2. In what follows, we only consider the 2-good case; our argument can be extended to the l-good case.[16] We use the following property of Pareto set for homothetic preferences.

Fact 1 (Thomson, 1995): *Let $l = 2$. When R_1 and R_2 are homothetic preferences in $\mathcal{R}$, $P(R)$ is "doubly visible", that is, for all $z_1, z'_1 \in \mathbb{R}^l_{++}$, if $z_1, z'_1 \in P_1(R)$ and $z_{11} < z'_{11}$, then either (a) $z_{12}/z_{11} \geq z'_{12}/z'_{11}$ and $z_{22}/z_{21} \geq z'_{22}/z'_{21}$ or (b) $z_{12}/z_{11} \leq z'_{12}/z'_{11}$ and $z_{22}/z_{21} \leq z'_{22}/z'_{21}$.*

Let $i \in N$ and $d_i \in M_i$. Without loss of generality we set $i \equiv 1$. We show that for some $R \in \mathcal{R}^N_{CES}$, (i) $P_1(R) \cap M_1 = \{0, \Omega\}$ and (ii) if $z \in P(R) \backslash \{d\}$ and $p \cdot z_1 = p \cdot d_1$ for all $p \in \nabla R(z)$, then there exists $\bar{R}_1 \in \mathcal{R}_{CES}$ such that for all $\bar{z}_1 \in P_1(\bar{R}_1, R_2) \cap LC(R_1, z_1) \cap UC(\bar{R}_1, z_1)$, $p \cdot \bar{z}_1 \neq p \cdot d_1$ for some $p \in \nabla(\bar{R}_1, R_2)(\bar{z})$.

[16] For the l-good case, we simply use the following relations between some special preference profiles in the l-good case and their counterparts in the 2-good case.

Let $a, b \in \mathbb{R}_{++}$ and $(c_2, \cdots, c_l) \in \mathbb{R}^{l-1}_{++}$. Let $U_1 \colon \mathbb{R}^l_+ \to \mathbb{R}$ and $U_2 \colon \mathbb{R}^l_+ \to \mathbb{R}$ be defined as follows: for all $x \in \mathbb{R}^l_+$, $U_1(x) \equiv (a(x_1/\Omega_1)^{\rho_1} + \sum_{k=2}^l c_k(x_k/\Omega_k)^{\rho_1})^{1/\rho_1}$; $U_2(x) \equiv (b(x_1/\Omega_1)^{\rho_2} + \sum_{k=2}^l c_k(x_k/\Omega_k)^{\rho_2})^{1/\rho_2}$. Let $u_1 \colon \mathbb{R}^2_+ \to \mathbb{R}$, and $u_2 \colon \mathbb{R}^2_+ \to \mathbb{R}$ be defined as follows: for all $(x_1, x_2) \in \mathbb{R}^2_+$, $u_1(x_1, x_2) \equiv (a(x_1/\Omega_1)^{\rho_1} + (\sum_{k=2}^l c_k)(x_2/\Omega_2)^{\rho_1})^{1/\rho_1}$; $u_2(x_1, x_2) \equiv (b(x_1/\Omega_1)^{\rho_2} + (\sum_{k=2}^l c_k)(x_2/\Omega_2)^{\rho_2})^{1/\rho_2}$. Then for all $x \in \mathbb{R}^l_{++}$, (i) if $(x, \Omega - x)$ is efficient in l-good economy (U_1, U_2, Ω), then $\frac{x_2}{\Omega_2} = \frac{x_3}{\Omega_3} = \cdots = \frac{x_l}{\Omega_l}$ and $((x_1, x_2), (\Omega_1 - x_1, \Omega_2 - x_2))$ is efficient in 2-good economy $(u_1, u_2, (\Omega_1, \Omega_2))$, and (ii) if $((x_1, x_2), (\Omega_1 - x_1, \Omega_2 - x_2))$ is efficient in 2-good economy $(u_1, u_2, (\Omega_1, \Omega_2))$, then $((x_1, x_2, \frac{\Omega_3}{\Omega_2} x_2 \cdots, \frac{\Omega_l}{\Omega_2} x_2), (\Omega_1 - x_1, \Omega_2 - x_2, \Omega_3(1 - \frac{x_2}{\Omega_2}), \cdots, \Omega_l(1 - \frac{x_2}{\Omega_2})))$ is efficient in l-good economy (U_1, U_2, Ω).

Clearly, (i) holds. If $d_1 \in \{0, \Omega\}$, (ii) holds vacantly.

Assume that $d_1 \notin \{0, \Omega\}$. Without loss of generality, let $\Omega \equiv (1, \cdots, 1)$. Let $u_1 \colon \mathbb{R}^l_+ \to \mathbb{R}$ and $u_2 \colon \mathbb{R}^l_+ \to \mathbb{R}$ be defined as follows: for all $x \in \mathbb{R}^l_+$, $u_1(x) \equiv (a x_1^{\rho_1} + x_2^{\rho_1})^{1/\rho_1}$ and $u_2(x) \equiv (b x_1^{\rho_2} + x_2^{\rho_2})^{1/\rho_2}$, where $\rho_1, \rho_2 \in (0,1)$ and $a, b \in \mathbb{R}_{++}$. Let R_1 and R_2 be the two preferences represented by u_1 and u_2 respectively. Let $\rho_1, \rho_2 \in (0,1)$ and $a, b \in \mathbb{R}_{++}$ be chosen in such a way that $P_1(R) \cap \overline{0, \Omega} = \{0, \Omega\}$, $P_1(R) \backslash \{0\} \subset \mathbb{R}^l_{++}$, and for all $z'_1 \in P_1(R) \backslash \{0, \Omega\}$, $z'_1 \in \mathbb{R}^l_{++}$ and $z'_{12}/z'_{11} > d_{12}/d_{11}$.

Let $z \in P(R)$ be such that $z_1 \neq d_1$ and for all $p' \in \nabla R(z)$, $p' \cdot z_1 = p' \cdot d_1$. Then clearly, $z_1 \notin \{0, \Omega\}$. Then since $P_1(R) \backslash \{0\} \subset \mathbb{R}^l_{++}$, $z_1 \in \mathbb{R}^l_{++}$ and $z_2 \in \mathbb{R}^l_{++}$. Let $p \equiv \nabla u_1(z_1)$. Since $d_1 \in \overline{0, \Omega}$ and $(\Omega_1, \Omega_2) = (1,1)$, then for all $i \in N$, $d_{i1} = d_{12}$. Since $p \in \mathbb{R}^l_{++}$ and $p \cdot z_1 = p \cdot d_1$, then $z_1 \not\leq d_1$ and $z_1 \not\geq d_1$.

Let $(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2_{++}$ be such that $u_2(1 - \bar{x}_1, 1 - \bar{x}_2) = u_2(z_{21}, z_{22})$ and $\bar{x}_1 < z_{11}$ (since $z_1, z_2 \in \mathbb{R}^l_{++}$, there exists such $(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2_{++}$). Let $(\bar{p}_1, \bar{p}_2) \in \mathbb{R}^2_{++}$ be a vector normal to $\overline{(\bar{x}_1, \bar{x}_2), (d_{11}, d_{12})}$. Then

$$(\star) \quad (\bar{p}_1, \bar{p}_2) \cdot (\bar{x}_1, \bar{x}_2) = (\bar{p}_1, \bar{p}_2) \cdot (d_{11}, d_{12}) < (\bar{p}_1, \bar{p}_2) \cdot (z_{11}, z_{12}).$$

Then there exists a CES function $\bar{u}$ such that $\bar{u}_1(\bar{x}_1, \bar{x}_2) = \bar{u}_1(z_{11}, z_{12})$ and $\nabla \bar{u}_1(\bar{x}_1, \bar{x}_2) = (\bar{p}_1, \bar{p}_2)$. Without loss of generality, we assume that $\bar{u}_1(x_1, x_2) \equiv \left(\bar{a} x_1^{\bar{\rho}_1} + x_2^{\bar{\rho}_1}\right)^{1/\bar{\rho}_1}$, where $\bar{a} \in \mathbb{R}_{++}$ and $\bar{\rho}_1 \in (-\infty, 1)$. Then $\nabla \bar{u}_1(x) \equiv K \cdot (\bar{a} x_1^{\bar{\rho}-1}, x_2^{\bar{\rho}-1})$, where $K \equiv \left(\bar{a} x_1^{\bar{\rho}} + x_2^{\bar{\rho}}\right)^{1/\bar{\rho}-1}$, for all $x \in \mathbb{R}^l_{++}$.

By $(\star)$, $\nabla \bar{u}_1(\bar{x}) \cdot \bar{x} = \nabla \bar{u}_1(\bar{x}) \cdot d_1$. By Fact 1, we can show that for all $\bar{z}_1 \in P_1(\bar{R}_1, R_2) \cap LC(R_1, z_1) \cap UC(\bar{R}_1, z_1)$, $\bar{x}_2/\bar{x}_1 > \bar{z}_{12}/\bar{z}_{11} > d_{12}/d_{11} = 1$. Clearly, $\nabla \bar{u}_1(\bar{x}) \cdot \bar{z}_1 > \nabla \bar{u}_1(\bar{x}) \cdot x = \nabla \bar{u}_1(\bar{x}) \cdot d_1$. Then $\bar{a} \bar{x}_1^{\bar{\rho}-1} \bar{z}_{11} + x_2^{\bar{\rho}-1} \bar{z}_{12} > \bar{a} \bar{x}_1^{\bar{\rho}-1} d_{11} + \bar{x}_2^{\bar{\rho}-1} d_{12}$, that is, $\bar{a} \bar{z}_{11} - \bar{a} d_{11} + (\frac{\bar{x}_1}{\bar{x}_2})^{1-\bar{\rho}} (\bar{z}_{12} - d_{12}) > 0$. Since $\bar{z}_{11} < d_{11}$, $\bar{z}_{12} > d_{12}$. Since $\frac{\bar{x}_1}{\bar{x}_2} < \frac{\bar{z}_{11}}{\bar{z}_{12}}$, $\bar{a} \bar{z}_{11} - \bar{a} d_{11} + (\frac{\bar{z}_{11}}{\bar{z}_{12}})^{1-\bar{\rho}} (\bar{z}_{12} - d_{12}) > 0$. Therefore, $\nabla \bar{u}_1(\bar{z}_1) \cdot \bar{z}_1 > \nabla \bar{u}_1(\bar{z}_1) \cdot d_1$. *Q.E.D.*

A preference $R_0 \in \mathcal{R}$ is ***quasilinear with respect to numeraire good k*** $\in \{1, \cdots, l\}$ if for all $x, y \in \mathbb{R}^l_+$ and all $\alpha \in \mathbb{R}$, whenever $x + \alpha e_k, y + \alpha e_k \in \mathbb{R}^l_+$, where e_k is the unit vector with zero components except at the k^{th} component, $x \; I_0 \; y \Rightarrow (x + \alpha e_k) \; I_0 \; (y + \alpha e_k)$. Let $\mathcal{R}_Q$ be the family of quasilinear, *strictly convex*, and *smooth* preferences with respect to a common numeraire good. We refer to $\boldsymbol{\mathcal{R}_Q^N}$ as the ***quasilinear domain***.

Proposition 5: *The quasilinear domain is rich and everywhere reachable*.*

Proof: *Everywhere reachability** is obvious. Let $\rho \in (0,1)$. For each $a > 0$, let $u^a \colon \mathbb{R}^l_+ \to \mathbb{R}$ be such that: for all $x \in \mathbb{R}^l_+$, $u^a(x) \equiv a \cdot \frac{x_1}{\Omega_1} + \sum_{k=2}^{l} \left(\frac{x_k}{\Omega_k} + 1\right)^{\rho}$.

Let $\mathcal{R}_{Q,\rho}$ be the set of preferences represented by u^a for some $a > 0$. Let $M^1 \equiv \{z \in \bar{Z} : z_{11} = 0 \text{ or } \frac{z_{12}}{\Omega_2} = \frac{z_{13}}{\Omega_3} = \cdots = \frac{z_{1l}}{\Omega_l} = 1\}$, and $M^2 \equiv \{z \in Z : \frac{z_{11}}{\Omega_1} = 1 \text{ or } \frac{z_{12}}{\Omega_2} = \frac{z_{13}}{\Omega_3} = \cdots = \frac{z_{1l}}{\Omega_l} = 0\}$. Let $\bar{\mathcal{D}} \equiv \mathcal{R}^N_{Q,\rho}$ and $M \equiv M^1$. We show that $\bar{\mathcal{D}}$ and M satisfy *A1, A2, A3* and *double transformability.* It is easy to show the following claim.

Claim 1: *Let R_1, R_2 be represented by u^{a_1}, u^{α_2} respectively. Then for all $i \in N$,*

$$P(R) = M^i \Leftrightarrow a_i \le a_{-i} \cdot 2^{1-\rho} .$$

A1 and *A2* are trivial.

A3: Let $R, R' \in \bar{\mathcal{D}}$ and $z_i, z'_i \in M_i$ be such that $P(R) = P(R') = M$ and $z_i \ne z'_i$. Without loss of generality, assume $z_1 \le z'_1$. For all $i \in N$, let R_i be represented by u^{a_i} and R'_i by $u^{a'_i}$, where $a_i, a'_i > 0$. Then by Claim 1, $a_1 \le a_2 \cdot 2^{1-\rho}$ and $a'_1 \le a'_2 \cdot 2^{1-\rho}$. Let $\bar{a}_1 \equiv \min\{a_1, a'_1\}$. Let $\bar{R}_1$ be represented by $u^{\bar{a}_1}$. Then since $\bar{a}_1 \le a_2 \cdot 2^{1-\rho}$ and $\bar{a}_1 \le a'_2 \cdot 2^{1-\rho}$, then by Claim 1, $P(\bar{R}_1, R_2) = P(\bar{R}_1, R'_2) = M$. Since M_1 is a monotone path and all preferences are strictly monotonic, $P_1(\bar{R}_1, R_2) \cap LC(R_1, z_1) \cap UC(\bar{R}_1, z_1) = \{z_1\}$. Hence part (i) of *A3* holds. Similarly, $P_1(\bar{R}_1, R'_2) \cap \{x \in Z_0 : \Omega - x \in LC(R_2, z_2) \cap UC(R'_2, z_2)\} = \{z_1\}$. Since $z_1 \le z'_1$ and $\bar{R}_1$ is strictly monotonic, $z'_1 \; \bar{P}_1 \; z_1$. Therefore part (ii) of *A3* also holds. □

Double transformability: Let $d_1 \in M_1$. Let $R \in \bar{\mathcal{D}}$ be such that $P(R) = M^2$. Let $z \in P(R)$ and $z_1 \ne d_1$. Then clearly, $P_1(R) \cap M_1 = \{0, \Omega\}$. When $d_1 \ge z_1$ or $d_1 \le z_1$, if we let $R'_1 \equiv R_1$, then $P_1(R'_1, R_2) \cap LC(R_1, z_1) \cap UC(R'_1, z_1) = \{z_1\}$. Therefore when $d_1 \ge z_1$, by strict monotonicity of R_1, (ii-1) of *double transformability* is satisfied with respect to $R''_1 \equiv R_1$. When $d_1 \le z_1$, (ii-2) of *double transformability* is satisfied with respect to $R'_2 \equiv R_2$.

Now assume that $d_1 \not\ge z_1$ and $d_1 \not\le z_1$. Then $d_{11} < z_{11}$ and $d_{12} > z_{12}$. Let $a_1, a_2 > 0$ be such that R_1 is represented by u^{a_1} and R_2 is represented by u^{a_2}. Then by Claim 1, $a_2 \le a_1 \cdot 2^{1-\rho}$. Let $a'_2 > (l-1) \cdot \frac{(z_{22}+1)^\rho - (d_{22}+1)^\rho}{d_{21} - z_{21}}$. Let $a'_1 \ge \max\{a_1, \frac{a'_2}{2^{1-\rho}}\}$. Let R'_1 be represented by $u^{a'_1}$ and R'_2 be represented by $u^{a'_2}$. Then since $a'_1 \ge a_1$, $a'_1 \cdot 2^{1-\rho} \ge a_2$. Hence by Claim 1, $P_1(R'_1, R_2) = M^2$. Since M^2_1 is a monotone path through z_1 and all preferences are strictly monotonic, $P_1(R'_1, R_2) \cap LC(R_1, z_1) \cap UC(R'_1, z_1) = \{z_1\}$. Since $a'_1 \ge a'_2/2^{1-\rho}$, by Claim 1, $P(R'_1, R'_2) = M^2$. Since M^2_2 is a monotone path through z_2 and all preferences are strictly monotonic, $P_2(R'_1, R'_2) \cap LC(R_2, z_2) \cap UC(R'_2, z_2) = \{z_2\}$. Since $a'_2 > (l-1) \cdot \frac{(z_{22}+1)^\rho - (d_{22}+1)^\rho}{d_{21} - z_{21}}$, $d_2 \; P'_2 \; z_2$. Therefore, (ii-2) of *double transformability* is satisfied. □ *Q.E.D.*

5 Concluding remarks

1. In several other economic environments, a number of authors have reported the same impossibility results as in the 2-agent exchange economy. Among others are Walker (1980), Zhou (1991b), Schummer (1999), Serizawa (2000), and Le Breton and Weymark (1999). Identification of general domain properties that induce their impossibility results will be an interesting research agenda.

2. Our conclusion makes use of the strong invariance property of *efficient* and *strategy-proof* rules, which is established in Lemma 2. The same invariance property will hold in other economic environments with the addition of "non-bossiness" in more than two agents cases. Examples are exchange economies with more than two agents, classical production economies, public goods economies, etc. Application of the strong invariance property may lead to simpler proofs and extensions of the existing results, for example, by Walker (1980), Satterthwaite and Sonnenschein (1981), Serizawa (2000), and Schummer (1999).

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