# Ranking Profiles of Capability Sets*

Rebeca A. Echávarri[a], Iñaki Permanyer[b] †

[a] *Universidad Pública de Navarra* [b] *Universitat Autònoma de Barcelona*

Abstract

In this paper we present an innovative approach for ranking profiles of capability sets on the basis of equality. We begin by introducing and defining the concept of common capability sets as all those functioning vectors that are shared by certain subgroups of the population under study. This allows us to rank profiles of capability sets without the sometimes stringent requirement of having a complete binary relation that orders the capability sets from the worst-off to the better-off. In order to overcome some of the shortcomings found in similar approaches, we then introduce a capability set ranking that takes into account both the intrinsic and the instrumental value of freedom.

*Keywords:* Capability Set, Capability Profile, Set Ranking, Profile Ranking.

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†Corresponding author. Campus de la Universitat Autònoma de Barcelona. Facultat de Ciències Econòmiques. Edifici B, 08193 Barcelona, Catalunya (Spain). E-mail: *inaki.permanyer@uab.es*; Phone: +34 93 581 17 20; Fax: +34 93 581 20 12.

# 1 Introduction

When it comes to comparing the well-being offered by social states, the influential works of Amartya Sen ([13], [14]) have convincingly argued that conventional evaluative spaces for the analysis of well-being are doomed to failure. This is the case of the space of commodities (opulence), and some of its transformations such as the space of utilities. The focus on commodities, he argues, does not take into account the pervasive human diversity that may prevent two different human beings from enjoying a given bundle of commodities in the same way. A bicycle, for instance, cannot be enjoyed by a disabled person to the same extent as it is by someone who is young, healthy and able-bodied. The problems when using utilities as the evaluative space arise from the fact that utilities are states of mind. With no variation in her living conditions, and without a glimpse of hope of any qualitative improvement in those conditions, a person's utility could increase. The utility could increase as consequence of the person adapting to her living conditions (see Elster [4] for further research on the notion of *adaptive preference formation*).

Well-being would be adequately measured, and therefore evaluated, when dimensions of the space to be used were directly states of being. To provide an appropriate evaluative space for social arrangements, Sen puts forward that of *functionings* (Sen [13],[14]). Each dimension - or functioning - represents a state of being. Typical examples of what a functioning could be are being well-sheltered, well-educated and well-fed. In the current paper, well-being will be evaluated within the space of functionings.

A further issue that arises when evaluating the well-being provided by social states involves the choice between evaluating the achieved well-being and evaluating the freedom to have well-being. In our context, this is the choice between focusing on the person's achieved functionings and on her capabilities (see Sen [13], [14]). Focusing on the achieved functionings by each individual is the same as addressing only the options *actually chosen* by the individual (see Dutta et al. [3], Chakraborty [2]). If, however, we are interested in taking into account the freedom of individuals to achieve well-being, we will need to focus on their respective capability sets, which can be defined as the various combinations of functionings that lie within an individual's reach (see Farina et al.[5], Xu [15]). This paper is concerned with the second approach. Hence, social states will be thought of as *profiles* of capability sets. Each person in the society has her own specific capability set.

In order to rank profiles of capability sets, our main concern will lie with the *equality of capabilities*. Our formal framework for ranking capability profiles has many similarities with the framework for ranking opportunity profiles in terms of *equality of opportunities* (see [7], [8], [9], [10], and Sen [14] for a discussion about the conceptual differences between the notion of equality of capabilities and that of equality of opportunities). Different procedures can be used to capture the notion of equality. For example, in Herrero et al. [7], the focus is placed on the set of opportunities within each individual's reach. This is called the *common opportunity set*. The main difference between our proposal and that of Herrero et al. [7] is that we examine the functionings simultaneously shared by certain population subgroups. Other similarities and differences are discussed throughout the paper.

The ranking of profiles of capability sets is closely related to the general framework of the ranking of sets of objects. As a matter of fact, in order to fully specify how a capability profile ranking works, we will need to specify a way to rank capability sets *vis-à-vis* each other. In this context it is important to recall that such rankings can take two extreme positions: they may focus either on the *instrumental* value of freedom or on its *intrinsic* value. In the former procedure, the set is ranked according to its best element(s), whereas, in the latter, freedom is considered to be valuable in itself and not simply as a means by which to attain a desired state of being (see Pattanaik and Xu [11, 12] as one of the leading pioneers of this literature, or Barberà et al. [1] for a survey on ranking sets of objects). We will consider rankings of capability sets that take into account both the instrumental and the intrinsic value of freedom (see Pattanaik and Xu [12]).

Explicitly dealing with the ranking of capability sets, Xu [15] ranks those that are non-degenerate, compact, comprehensive and convex. In a related paper, Farina et al.[5] also ranks those that are compact, convex, comprehensive but not necessarily non-degenerate. In this paper, we study the ranking of profiles of capability sets in which the respective capability sets are not necessarily convex, but are non-degenerate, compact and comprehensive. This way, we are working in a more general framework than what is done in Xu [15]. Let us finally recall that our ranking of capability sets will not be asked to be necessarily *complete* as is usually done in related papers.

The plan of the paper is as follows. In section 2, we introduce the general notation and definitions. In section 3 we present a binary relation to rank profiles of capability sets, as well as an innovative relation to rank capability sets *vis-à-vis* each other. In section 4 we show an example to better illustrate how the proposed capability profile and capability set rankings work. Section 5 presents the conclusion, while the proofs are included in the appendix.

# 2 General Notation and Definitions.

Let $N = \{1, \ldots, n\}$ be the set of individuals we are going to consider, and let $\{f_1, \ldots, f_m\}$ be a list of functionings representing valued states of being. Let $\mathbb{R}^m_+$ be the non-negative orthant of the Euclidean $m$-dimensional space. Each dimension is used to measure the achieved levels of one of the functionings in the list. Hence, the points in $\mathbb{R}^m_+$ are to be interpreted as *functioning vectors* which will be denoted by $x, y, z, a, b$, and so on. We will write $x = (x_1, \ldots, x_m)$, where each $x_i$ denotes the achieved level of the $i$-th functioning in vector $x$.

For each individual one can consider the set of functioning vectors at his disposal. Such a set is a subset of $\mathbb{R}^m_+$ and is to be interpreted as that individual's *capability set*. We will denote by $\mathcal{C}$ the set of capability sets we will be dealing with. Usually, the elements of $\mathcal{C}$ will be denoted by capital letters: $A, B, C, \ldots$. Every member of $\mathcal{C}$ (that is: every capability set) will be required to satisfy the following properties

**Non-degenerate**: A capability set $C \subset \mathbb{R}^m_+$ is non-degenerate if and only if there exists $x = (x_1, \ldots, x_m) \in C$ such that $x_i > 0$ for all $i = 1, \ldots, m$.

**Compact**: A capability set $C \subset \mathbb{R}^m_+$ is compact if and only if $C$ is closed and bounded.

**Comprehensive**: A capability set $C \subset \mathbb{R}^m_+$ is comprehensive if and only if, for all $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m) \in \mathbb{R}^m_+$, if $x_i \geq y_i$ for all $i = 1, \ldots, m$ and $x \in C$, then $y \in C$.

There is another property that in similar contexts has also been invoked (see Xu [15]): **Convexity** (A capability set $C \subset \mathbb{R}^m_+$ is convex if and only if for all $x, y \in C$ and for all $\lambda \in [0, 1], \lambda x + (1 - \lambda)y \in C$ holds). We contend that in the context of capability sets it does not make much sense to ask for convexity since there are no intuitive or appealing arguments supporting it. On the contrary: if $m = 2$ and an individual is able to achieve $(0, 1)$ and $(1, 0)$, why should she be able to achieve $(\frac{1}{2}, \frac{2}{5})$ or $(\frac{1}{2}, \frac{1}{2})$ but not $(\frac{1}{2}, \frac{3}{5})$? Hence, the framework we will be dealing with is more general and less restrictive, and includes the capability sets introduced by Xu in [15] as a particular case.

Now, if $x = (x_1, \dots, x_m), y = (y_1, \dots, y_m) \in \mathbb{R}^m_+$, we write $x > y$ if $x_i \geq y_i$ for all $i = 1, \dots, m$ and $x_j > y_j$ for some $j \in \{1, \dots, m\}$. In a similar vein, we will write $x \geq y$ if $x_i \geq y_i$ for all $i = 1, \dots, m$. If $C \in \mathcal{C}$ and $\lambda_i > 0$ for all $i = 1, \dots, m$, we define $\lambda_i C := \{x \in \mathbb{R}^m_+ | x_i = \lambda_i y_i$ and $x_j = y_j$ for all $j \neq i$, $\forall y \in C\}$. Thus $\lambda_i C$ is an homothetic expansion or contraction of the capability set $C$ of factor $\lambda_i$ along the $i$th coordinate axis of $\mathbb{R}^m_+$.

In this paper we are mainly concerned with the ranking of social states, where each social state is represented by a *profile of capability sets* or *capability profile*. One can define a capability profile as a vector $\mathbf{A} = (A_1, \dots, A_n)$ where each $A_i$ represents individual's $i$ capability set. We will denote by $\mathcal{C}^n = \prod_{i \in N} \mathcal{C}$ the set of all profiles of capabilities available in a given society. One can define the following subfamilies in $\mathcal{C}^n$. If one has a profile $\mathbf{A} = (A_1, \dots, A_n)$ such that $A_1 \subseteq \dots \subseteq A_n$ we say that $\mathbf{A}$ is a *nested profile*. We will denote by $\mathcal{N}^n$ the family of all nested profiles in $\mathcal{C}^n$. In the special case in which $A_i = A_j \ \forall \ i, j \in N$ we to refer to a *uniform profile* which is denoted by $\mathbf{A} = (A)$. Let $\mathcal{U}^n$ denote the set of all uniform profiles in $\mathcal{C}^n$. Recall that $\mathcal{U}^n \subseteq \mathcal{N}^n \subseteq \mathcal{C}^n$.

For the purpose of ranking profiles of capability sets, let us consider a binary relation $\succeq^n$ defined on $\mathcal{C}^n$ that can satisfy some of the following properties

(2.4) **Reflexivity**: For all $\mathbf{A} \in \mathcal{C}^n$, one has that $\mathbf{A} \succeq^n \mathbf{A}$.

(2.5) **Transitivity**: For all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{C}^n$, if $\mathbf{A} \succeq^n \mathbf{B}$ and $\mathbf{B} \succeq^n \mathbf{C}$, then $\mathbf{A} \succeq^n \mathbf{C}$.

(2.6) **Completeness**: For all $\mathbf{A}$,$\mathbf{B} \in \mathcal{C}^n$ with $\mathbf{A} \neq \mathbf{B}$, then either $\mathbf{A} \succeq^n \mathbf{B}$ or $\mathbf{B} \succeq^n \mathbf{A}$.

When $\succeq^n$ satisfies both Reflexivity and Transitivity we say that it is a *quasi-ordering* or *partial ordering*. If, moreover, $\succeq^n$ also satisfies Completeness we say that it is a *complete ordering*. Given two profiles, $\mathbf{A}, \mathbf{B} \in \mathcal{C}^n$, we will write $\mathbf{A} \succeq^n \mathbf{B}$ when $\mathbf{A}$ is considered to be at least as good as $\mathbf{B}$. $\succ^n$ and $\sim^n$ will denote the usual asymmetric and symmetric parts of $\succeq^n$.

Now, in order to fully specify how the capability profile ranking $\succeq^n$ works we might need to specify how to rank individual capability sets. Then, we will need to talk about a binary relation $\succeq$ defined over $\mathcal{C}$ (the set of all capability sets that are non-degenerate, compact and comprehensive). As before, $\succeq$ could be reflexive, transitive and/or complete. Given any two capability sets $A, B \in \mathcal{C}$, the relation $A \succeq B$ is to be interpreted as 'The well-being offered by capability set $A$ is at least as good as the well-being offered by capability set $B$'. As usual, $\succ$ and $\sim$ will respectively denote the asymmetric and symmetric parts of $\succeq$.

In order to rank profiles of capability sets we will be concerned with certain fairness criteria, such as that of equality of capabilities available to different individuals. An interesting way of capturing the notion of equality consists in giving priority to the common capabilities, *i.e:* those capabilities that are shared by all members in the society (see Herrero et al. [7]). Given a profile of capability sets $\mathbf{A} = (A_1, \ldots, A_n)$ one can define the *common capability set* as the set of functioning vectors that are within each individual's reach, that is

$$A_0 := \bigcap_{i \in N} A_i \tag{1}$$

In the context of non-degenerate, compact and comprehensive capability sets of $\mathbb{R}^m_+$ one can readily verify that $A_0 \in \mathcal{C}$ and that $A_0 \neq \emptyset$.

Whenever $\succeq$ is a complete ordering, for any profile $\mathbf{A} = (A_1, \ldots, A_n) \in \mathcal{C}^n$ one can define a permutation $q_{\mathbf{A}} : N \to N$ such that $A_{q_{\mathbf{A}}(n)} \succeq A_{q_{\mathbf{A}}(n-1)} \succeq \ldots \succeq A_{q_{\mathbf{A}}(1)}$. This way, given a capability profile $\mathbf{A} \in \mathcal{C}^n$, one can define $q(\mathbf{A}) := (A_{q_{\mathbf{A}}(1)}, A_{q_{\mathbf{A}}(2)}, \ldots, A_{q_{\mathbf{A}}(n)})$ an opportunity profile that places the individuals from the worst-off to the better-off in an increasing order. When $\succeq$ is complete, Herrero et al.[7] have defined the *lexmin opportunity relation*, denoted $\succeq^n_{lo}$, as

$$\mathbf{A} \succ^n_{lo} \mathbf{B} \Leftrightarrow (A_0, A_{q_{\mathbf{A}}(1)}, A_{q_{\mathbf{A}}(2)}, \ldots, A_{q_{\mathbf{A}}(n)}) >_L (B_0, B_{q_{\mathbf{B}}(1)}, B_{q_{\mathbf{B}}(2)}, \ldots, B_{q_{\mathbf{B}}(n)}) \tag{2}$$

for any $\mathbf{A}, \mathbf{B} \in \mathcal{C}^n$, where $>_L$ denotes the usual lexicographical relation: $\mathbf{A} >_L \mathbf{B} \Leftrightarrow A_i \sim B_i$ for all $i \in \{1, \ldots, k\}$ and $A_{k+1} \succ B_{k+1}$.

In order to rank two profiles, the rule $\succeq^n_{lo}$ first focuses on the well-being offered by their corresponding common capability set. When both common capability sets offer the same degree of well-being, the rule ranks those profiles by comparing the well-being of the worst-off in each profile. If the well-being of the worse-off is the same in both profiles, the rule then compares the profiles by comparing the well-being of the second worse-off, and so on.

# 3 A ranking of capability profiles and capability sets

In this section we firstly introduce a ranking of profiles in $\mathcal{C}^n$ that incorporates certain egalitarian-distributional criteria. Then, we introduce a capability set ranking that takes into account both the instrumental and the intrinsic value of freedom. Let us start with the following definitions.

**Definition 1.** Given a profile of capability sets $(A_1, \ldots, A_n) \in \mathcal{C}^n$ we denote by $a_i = (a_{i_1}, \ldots, a_{i_m})$ a generic element of $A_i$. Given a functioning vector $p = (p_1, \ldots, p_m) \in \mathbb{R}^m_+$, one defines

$$I(A_1, \ldots, A_n; p) := \frac{\sharp\{i \in N | \exists\ a_i \in A_i \text{ such that } a_i \geq p\}}{n}. \tag{3}$$

Equation (3) represents the number of people in the society that can achieve or improve the functioning vector $p$. This could be intuitively interpreted as a sort of 'probability of achieving or improving the vector $p$ in the society'. Furthermore, it is straightforward to verify that $I(A_1, \ldots, A_n; p) \in [0, 1]$. We can now give the following definition.

**Definition 2.** For a given $\alpha \in [0, 1]$ one defines

$$C(A_1, \ldots, A_n; \alpha) = \{p \in \mathbb{R}^m_+ | I(A_1, \ldots, A_n; p) \geq \alpha\}. \tag{4}$$

Equation (4) represents the set of functioning vectors that can be achieved or improved by at least $\lceil n\alpha \rceil$ individuals. The different sets $C(A_1, \ldots, A_n; \alpha)$ obtained for the different possible values of $\alpha \in [0, 1]$ satisfy some interesting properties.

(3.1.) The sets $C(A_1, \ldots, A_n; \alpha)$ have a clear and intuitive interpretation. Take, for example, the extreme case $C(A_1, \ldots, A_n; 1)$: this is the set of functioning vectors that can be achieved or improved by all the members in the society, *i.e*: $A_0 = \bigcap_{i \in N} A_i$. On the other hand, $C(A_1, \ldots, A_n; 1/n)$ is the set of functioning vectors that can be achieved or improved by at least one individual. This is the set $\bigcup_{i \in N} A_i$. Furthermore, it is straightforward to verify that, for any sequence $\{\alpha_i\}_{i \in \{1, \ldots, k\}}$ such that $0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_k \leq 1$, then one has that

$$C(A_1, \ldots, A_n; \alpha_k) \subseteq C(A_1, \ldots, A_n; \alpha_{k-1}) \subseteq \ldots \subseteq C(A_1, \ldots, A_n; \alpha_1),$$

that is, the $C(A_1, \ldots, A_n; \alpha)$ have a nested structure.

(3.2.) One can verify that, for any $\alpha \in (0,1], C(A_1, \dots, A_n; \alpha) \in \mathcal{C}$, that is, they are capability sets (in the extreme case of $\alpha = 0$, by definition one would have that $C(A_1, \dots, A_n; \alpha_1) = \mathbb{R}^m_+$, which obviously is not a member of $\mathcal{C}$). In order not to burden the text too much we show the proof in the appendix.

(3.3.) By definition, one can easily verify that if $\alpha \in (\frac{i-1}{n}, \frac{i}{n}]$ then $C(A_1, \dots, A_n; \alpha) = C(A_1, \dots, A_n; i/n)$ for all $i = 1, \dots, n$. Thus, instead of considering the $C(A_1, \dots, A_n; \alpha)$ for any $\alpha \in [0,1]$, it will be enough to consider the discrete values $\alpha \in \{\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$. We can now present the following definition.

**Definition 3.** Given any profile of capability sets $\mathbf{A} = (A_1, \dots, A_n) \in \mathcal{C}^n$ and any $k \in \{1, \dots, n\}$ one can define

$$C_k^{\mathbf{A}} := C(A_1, \dots, A_n; \frac{k}{n}) \tag{5}$$

The set $C_k^{\mathbf{A}}$ is called the *Common Capability Set* related to $k$ people out of the total population. Each $C_k^{\mathbf{A}}$ is to be interpreted as the set of functioning vectors that can be achieved or improved by at least $k$ individuals at the same time.

**Definition 4.** Given any profile of capability sets $\mathbf{A} = (A_1, \dots, A_n) \in \mathcal{C}^n$ and its corresponding common capability sets $C_k^{\mathbf{A}}$, we define the vector

$$\mathbf{C}^{\mathbf{A}} := (C_n^{\mathbf{A}}, \dots, C_1^{\mathbf{A}}). \tag{6}$$

The vector in equation (6) is going to be named the *Common Capability Profile* related to the profile $\mathbf{A}$. Given any profile of capability sets $\mathbf{A} \in \mathcal{C}^n$, it is straightforward to verify not only that $\mathbf{C}^{\mathbf{A}} \in \mathcal{C}^n$, but also that, as $C_n^{\mathbf{A}} \subseteq C_{n-1}^{\mathbf{A}} \subseteq \dots \subseteq C_1^{\mathbf{A}}$, then $\mathbf{C}^{\mathbf{A}} \in \mathcal{N}^n$.

One of the main purposes of this paper is to rank profiles in $\mathcal{C}^n$ according to certain egalitarian-distributional criteria. For this purpose, we will pay attention to the well-being offered by the common capability sets. That is, given a pair of capability profiles $(A_1, \dots, A_n), (B_1, \dots, B_n) \in \mathcal{C}^n$, we will consider that

$$(A_1, \dots, A_n) \succeq^n (B_1, \dots, B_n) \Leftrightarrow (C_n^{\mathbf{A}}, \dots, C_1^{\mathbf{A}}) \succeq^n (C_n^{\mathbf{B}}, \dots, C_1^{\mathbf{B}}). \tag{7}$$

This is one of the most important contributions of the current paper: we contend that in order to rank profiles of capability sets the key information to be considered are the functioning vectors that are *shared* by different subgroups of the population (instead of focusing on individual distributions as in similar approaches). It is readily seen that a ranking focused on the common capability sets is more "community-oriented" than "individual-oriented". This may make it specially well-suited for equality measurement purposes.

Another attractive characteristic of this kind of ranking is the following. Having been created with a nested structure, common capability sets are hierarchically ordered from smallest to largest. Thus, it is not necessary to have a complete ordering $\succeq$ over $\mathcal{C}$ *a priori* to rank individuals from worst-off to best-off. This way, one is able, if reasonable or necessary, to introduce a partial ordering $\succeq$ that might not be capable of ordering the individual capability sets in a capability profile but still be conceptually operational. The only methodologically implicit restriction in such a procedure is that the ranking $\succeq$ should be a non necessarily strict extension of the set inclusion relation. That is, if $A, B \in \mathcal{C}$ are such that $A \subseteq B$ then $B \succeq A$.

There are many available options from which the decision-maker may choose the precise $\succeq^n$ and $\succeq$ that will be used. Our proposal pays attention to both the intrinsic and the instrumental value of freedom and is concerned with some egalitarian-distributional criteria. We start by defining $\succeq^n$ and then proceed with $\succeq$.

## 3.1 Ranking profiles with Common Capability Sets

We are interested in ranking profiles in $\mathcal{C}^n$ using information about their corresponding common capability profiles. Hence, we propose a binary relationship of the type shown in (7). For this purpose, we transform the relation $\succeq^n_{lo}$ as follows.

**Definition 5.** Given any pair of capability profiles, $\mathbf{A} = (A_1, \ldots, A_n)$ and $\mathbf{B} = (B_1, \ldots, B_n) \in \mathcal{C}^n$, one defines the *lexmin common capability relation*, denoted $\succeq^n_{lcc}$, as

$$(A_1, \ldots, A_n) \succ^n_{lcc} (B_1, \ldots, B_n) \Leftrightarrow (C^{\mathbf{A}}_n, \ldots, C^{\mathbf{A}}_1) >_L (C^{\mathbf{B}}_n, \ldots, C^{\mathbf{B}}_1), \qquad (8)$$

where $>_L$ denotes the usual lexicographical relation. (Recall that in order to be fully operational, this profile ranking needs to specify how any two common capability sets are to be ranked *vis-à-vis* each other. A specific capability set ranking will be introduced in section 3.2.).

This way, the first piece of information that is taken into account is the set of functioning vectors shared by *all* individuals in the society. If these are ranked as indifferent then one proceeds with the functioning vectors simulataneously shared by at least $n-1$ individuals, and so on. It is straightforward to verify that $\succeq_{lcc}^n$ is a reflexive and transitive binary relation which is not necessarily complete. Recall that, as mentioned before, the fact that are dealing with a non complete capability set ordering $\succeq$ does not prevent us from using $\succeq_{lcc}^n$: Given any pair $(A_1, \ldots, A_n), (B_1, \ldots, B_n) \in \mathcal{C}^n$, if $C_i^{\mathbf{A}} \sim C_i^{\mathbf{B}}$ for $n \geq i > k \geq 1$ and $\succeq$ fails to rank $C_k^{\mathbf{A}}$ *vis-à-vis* $C_k^{\mathbf{B}}$, then we simply say that $(A_1, \ldots, A_n)$ could not be ranked *vis-à-vis* $(B_1, \ldots, B_n)$ by $\succeq_{lcc}^n$ .

Let us now show which are the main properties satisfied by the ranking $\succeq_{lcc}^n$. We will present them one by one.

(3.1.1.) *Anonymity* (**AN**): For any $\mathbf{A} \in \mathcal{C}^n$ and any permutation $p : N \to N, \mathbf{A} \sim^n p(\mathbf{A})$.

Anonymity ensures that no individual in the society is privileged or treated differently from the others.

(3.1.2.) *Strong Pareto Efficiency on Common Capability profiles* (**SPEC**): For all $\mathbf{A} = (A_1, \ldots, A_n), \mathbf{B} = (B_1, \ldots, B_n) \in \mathcal{C}^n$, if $C_i^{\mathbf{A}} \succeq C_i^{\mathbf{B}}$ $\forall i \in \{1, \ldots, n\}$, then $\mathbf{A} \succeq^n \mathbf{B}$. If moreover some preference is strict, then $\mathbf{A} \succ^n \mathbf{B}$.

This is a reasonable property when ranking profiles of capability sets: if every common capability set in the first profile is at least as good as every respective common capability set in the second, then the former is at least as good as the latter. In the present context this is clearly verified because of the lexicographical nature of $\succeq_{lcc}^n$.

(3.1.3.) *Common Improvement* (**CI**): For any $\mathbf{A} \in \mathcal{C}^n$ and any $\mathbf{B} = (B) \in \mathcal{U}^n$, such that $C_n^{\mathbf{A}} \subset B$ and $B \succ C_n^{\mathbf{A}}$, then, $\mathbf{B} \succ^n \mathbf{A}$.

Common improvement prioritizes the set of functioning vectors simultaneously shared by all members of the society. If the set of functioning vectors shared by all members in a society is expanded, then the new situation is socially preferable, no matter how great the well-being offered by the common capability sets related to the various subgroups of the population. The same intuition is captured by *Hammond's equity principle* (see Hammond [6]). The following is a transformation of Hammond's equity principle to the context of common capability profiles.

(3.1.4.) *Strong Hammond Equity on Common Capability Profiles* (**SHEC**): For all $\mathbf{A}, \mathbf{B} \in \mathcal{C}^n$ with $C_j^{\mathbf{A}} = C_{j-1}^{\mathbf{A}} = \ldots = C_{j-s}^{\mathbf{A}}$, $C_j^{\mathbf{B}} = C_{j-1}^{\mathbf{B}} = \ldots = C_{j-s}^{\mathbf{B}}$ for some $j \in \{2, \ldots, n-1\}$ and for some non negative $s$ that $j - s \geq 1$, if $C_j^{\mathbf{B}} \succ C_j^{\mathbf{A}} \succeq C_k^{\mathbf{A}} \succ C_k^{\mathbf{B}}$ for some $n \geq k > j$, and $C_i^{\mathbf{A}} \sim C_i^{\mathbf{B}}$ $\forall i \neq k, j$, then, $\mathbf{A} \succ^n \mathbf{B}$.

This is a property ensuring that if a potentially large number of common capability sets $C_j^{\mathbf{A}}, C_{j-1}^{\mathbf{A}}, \ldots, C_{j-s}^{\mathbf{A}}$ being equal are improved, while a capability set $C_k^{\mathbf{A}}$ with $k > j$ is worsened, then the former social state is socially preferable to the latter. Thus, **SHEC** emphasizes the priority of the "smaller" common capability sets over the "larger".

Up to this point we have seen four important properties that are satisfied by the binary relation $\succeq_{lcc}^n$. Actually, it can be easily proved that, with the addiction of a couple of hypotheses, these four properties do axiomatically characterize the relation $\succeq_{lcc}^n$. The hypotheses we need are labeled as *Weak efficiency on uniform profiles* (**WEUP**) and *Richness* (**RIC**).

(3.1.5.) *Weak Efficiency on Uniform Profiles* **WEUP**: For any $\mathbf{A}, \mathbf{B} \in \mathcal{U}^n, \mathbf{A} = (A), \mathbf{B} = (B)$, if $A \subseteq B$, then $\mathbf{B} \succeq^n \mathbf{A}$.

(3.1.6.) *Richness* **RIC**: For all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{U}^n, \mathbf{A} = (A), \mathbf{B} = (B), \mathbf{C} = (C)$, such that $\mathbf{A} \succ^n \mathbf{C}, \mathbf{B} \succ^n \mathbf{C}$, there exist $\mathbf{D}, \mathbf{E} \in \mathcal{U}^n, \mathbf{D} = (D), \mathbf{E} = (E)$, with $C = D \cap E$ such that $\mathbf{A} \sim^n \mathbf{D}, \mathbf{B} \sim^n \mathbf{E}$.

Now, the characterization result for $\succeq_{lcc}^n$ can be presented:

**Theorem 1.** Let $\succeq^n$ be a binary relation on $\mathcal{C}^n$ fulfilling **WEUP** and **RIC**. Then $\succeq^n$ satisfies **AN**, **SPEC**, **SHEC** and **CI** if and only if $\succeq^n = \succeq_{lcc}^n$.

**Proof:** See the appendix.

## 3.2 The ranking of capability sets.

Our next step is to specify how the different common capability *sets* are to be ranked *vis-à-vis* each other. We present the axioms we would like the binary relation $\succeq$ over $\mathcal{C}$ to satisfy.

(3.2.1) **Monotonicity**: If $A, B \in \mathcal{C}$ and if $A \subseteq B$, then $B \succeq A$. Moreover, if $A \subset B$, then $B \succ A$.

(3.2.2) **Invariance of Scaling effects**: Given any pair $A, B \in \mathcal{C}$ and any $\lambda_i > 0$, then $A \succeq B \Leftrightarrow \lambda_i A \succeq \lambda_i B$.

(3.2.3) **Continuity** : For all $A \in \mathcal{C}$ and a sequence of capability sets $\{B_k\}_{k=1,\ldots,+\infty}$ such that $B_k \in \mathcal{C}$ for all $k$ and $B = \lim_{k\to\infty} B_k \in \mathcal{C}$, if $B_k \succeq A$ for all $k$, then $B \succeq A$.

These axioms try to reflect some of the intuitive ideas that we feel a capability set ranking should satisfy. Monotonicity, for example, states that any expansion of any given capability set is always desirable. It has elsewhere been argued (see Barberà et al. [1]) that the expansion of opportunity sets should not invariably be taken to be in itself desirable, due to the fact that the newly available options may in fact offer the agents no relevant or interesting choices. However, this objection does not apply in our context: by construction, each of the chosen functionings (the $\{f_1, \ldots, f_m\}$) is supposed to measure a certain characteristic considered to be desirable by all $n$ individuals.

The invariance of scaling effects means that the ranking of any two capability sets should not be altered if the units in which one measures a certain functioning are re-scaled while the others are left unchanged. Furthermore, by $m$ successive applications of the same axiom (one for each re-scaled functioning), the ranking remains unaltered even if all the units of measurement of the different functionings are re-scaled. It is worth noting that the invariance of scaling effects does not allow any kind of inter-functioning comparability.

The Continuity axiom states that the ranking $\succeq$ should not be suddenly reversed. If each member of the sequence $\{B_k\}_k$ is considered to be at least as good as $A$, then the limit of the sequence should also be considered to be at least as good as $A$.

Apart from satisfying the aforementioned axioms, we also want the relation $\succeq$ to take into account both the intrinsic and the instrumental value of freedom. On the one hand, in the attempt to rank capability sets according to the intrinsic value of freedom, one might end up with a sort of quantitative approach as in the cardinal relation (see Pattanaik and Xu [11]). In that case,

$$A \succeq B \Leftrightarrow |A| \geq |B|. \tag{9}$$

In this context, a capability set is considered to be good if it offers a great deal of opportunities; the fact of having *many* available options within reach is a valuable asset.

On the other hand, if the ranking of capability sets is done according to the instrumental value of freedom, one ends up by valuing a capability set according to its best element(s). That is, consider a non-negative, increasing in its argument function $V_j : \mathbb{R}_+^m \to \mathbb{R}_+$, where $V_j(x_1, \ldots, x_m)$ represents the value attached to the functioning vector $(x_1, \ldots, x_m)$. Each $V_j$ will be called a *valuation function* and represents a different way of evaluating the functioning vectors $(x_1, \ldots, x_m) \in \mathbb{R}_+^m$. Now, if one has to rank two capability sets $A, B \in \mathcal{C}$ according to $V_j$ then one has

$$A \succeq B \Leftrightarrow \max_{a \in A} V_j(a) \geq \max_{b \in B} V_j(b). \tag{10}$$

Now, we would like to have a ranking that combines the spirit of these two extreme approaches. As in Pattanaik and Xu [12], we propose a procedure whereby the ranking increases with the number of alternatives, and is sensitive to individuals' preferences.

**Definition 6.** Given the capability set $A$, the evaluation of $A$ according to the valuation function $V_j$ is defined as,

$$\int_A V_j(x_1, \ldots, x_m) dx_1 \ldots dx_m. \tag{11}$$

As long as no confusion arises, the last expression can be written as $V_j(A)$. This way of evaluating a capability set has the advantage of taking into account both the intrinsic and the instrumental value of freedom. While the integral is performed over the whole of $A$, taking the amount of available choices into account, at the same time a valuation of the value of each point is also being considered.

With respect to the explicit formulation of $V_j(x_1, \ldots, x_m)$, many possibilities can be given. A simple one would be a Cobb-Douglas type of function. That is, $V_j(x_1, \ldots, x_m) = x_1^{w_1^j} \ldots x_m^{w_m^j}$, where $\sum_{k=1}^{k=m} w_k^j = 1$ and each $w_k^j$ represents the relative weight that the valuation function $V_j$ attaches to the $k$-th functioning. Consider now a set $\mathcal{V}$ of $p$ valuation functions ($p \in \mathbb{N}$), that is: $\mathcal{V} = \{V_1, \ldots, V_p\}$, each of which representing a "reasonable" way of valuating functioning vectors (see Pattanaik and Xu [12] for a discussion about reasonableness). Our proposal for ranking capability sets is defined as follows.

**Definition 7.** Given any two capability sets $A, B \in \mathcal{C}$ and any set $\mathcal{V} = \{V_1, \ldots, V_p\}$ of valuation functions, we define the capability set relation $\succeq_{\mathcal{V}}$ as,

$$A \succeq_{\mathcal{V}} B \Leftrightarrow V_j(A) \geq V_j(B) \text{ for all } j = \{1, \ldots, p\}. \tag{12}$$

Recall that the set $\mathcal{V}$ could be interpreted as the set of preferences that are to be taken into account when ranking the capability sets $A$ and $B$. One possibility is to consider the case in which each individual in the society has a given preference, so $|\mathcal{V}| = n$ and each $V_i$ could be interpreted as individual's $i$ preference. Another is to interpret the set $\mathcal{V}$ as the different preferences that a group of decision-makers (not necessarily the individuals in $N$) could have.

One can readily verify that the ranking $\succeq_{\mathcal{V}}$ is reflexive and transitive, but not necessarily complete (non-completeness appears when the different valuation functions $V_j$ do not all rank $A$ and $B$ equally). This is an example of the *intersection approach*, advocated among others by Sen in [13], [14] as a useful methodology for social evaluative exercises in which different and conflicting opinions are present. Recall that in the specific case in which $|\mathcal{V}| = 1$ (that is, when just one preference is taken into account), the ranking $\succeq_{\mathcal{V}}$ is complete. It is routinely verified that such a ranking (employing the Cobb-Douglas valuation functions) verifies the axioms of Monotonicity, Invariance of Scaling Effects and Continuity.

**Remark.** There is another interesting way of considering how a capability set $A$ can be evaluated through the evaluation functions $V_j$ apart from the one presented above. We could evaluate $A$ with the formula

$$\widetilde{V}_j(A) := \int_{U(A)} V_j(x_1, \ldots, x_m) dx_1 \ldots dx_m \tag{13}$$

where $U(A)$ are the non-dominated elements of $A$, that is, $U(A) := \{x \in A | \nexists\ y \in A \text{ such that } y > x\}$. This way of evaluating a capability set also also has the advantage of considering the quantitative and qualitative approaches at the same time. Moreover, it could be reasonably argued that with $\widetilde{V}_j(A)$, one is focusing specifically on the *relevant* opportunities offered by a given capability set. When the integral is performed all over $A$, most of the opportunities we are taking into account are not relevant (*i.e.*: dominated) and we could well get rid of them.

Alas, although such arguments are strong and convincing, there is a methodological hurdle that deters us from using such formula to evaluate capability sets in the present context. If our capability sets $A$ are non-degenerate, compact and comprehensive subsets of $\mathbb{R}^m_+$, then $U(A)$ can happen to be a set which is the disjoint union of sets that do not necessarily have the same dimensions. For example, one could well have that $U(A) = S_0 \sqcup S_1 \sqcup \ldots \sqcup S_{m-1}$ where $\dim(S_i) = i$ for all $i = 0, \ldots, m-1$. In such cases, it is not clear how the operator $\int_{U(A)}$ should be defined. (In the case in which one considers the additional hypotheses that the capability sets should also be *convex*, this problem would disappear. Indeed, $U(A)$ would be a $m-1$ dimensional subset of $\mathbb{R}^m_+$ and $\int_{U(A)} V_i dx_1 \ldots dx_m$ could be computed without problem. However, as we mentioned before, we do not consider convexity a reasonable property that a capability set should satisfy.) In the face of this and other related problems, this interesting approach has not been adopted. More research would be needed to present another framework in which it might be computationally feasible.

# 4 An example.

In order to illustrate more clearly how our proposed rankings $\succeq_{lcc}^{n}$ and $\succeq_{\mathcal{V}}$ work, we will present a simple example in the context of a three-member society in which only 2 functionings are taken into account. We will present a pair of capability profiles $(A_1, A_2, A_3)$ and $(B_1, B_2, B_3) \in \mathcal{C}^3$ and show how they are ranked.

Before proceeding, let us remark that the different capability sets $C \in \mathcal{C}$ are univocally represented by its non dominated elements $U(C)$. The comprehensiveness of such sets fills the rest of dominated vectors. For example, $C = \{(1,5),(4,2)\}$ represent the members of $\mathbb{R}_+^2$ included in any of the following closed rectangles: $\{(x,y) \in \mathbb{R}^2 |\ x \in [0,1], y \in [0,5]\}$ and $\{(x,y) \in \mathbb{R}^2 |\ x \in [0,4], y \in [0,2]\}$.

Now, let $\mathbf{A} = (A_1, A_2, A_3) = (\{(1,5),(3,1)\},\{(3,4),(4,3)\},\{(2,3),(5,2)\})$ and $\mathbf{B} = (B_1, B_2, B_3) = (\{(1,5),(5,1)\},\{(4,4)\},\{(2,3),(3,2)\})$ be the capability profiles we would like to rank. For this purpose, we need to compute their respective common capability sets. One clearly has that:
$\mathbf{C^A}$=$(C_3^{\mathbf{A}}, C_2^{\mathbf{A}}, C_1^{\mathbf{A}}) = (\{(1,3),(3,1)\},\{(1,4),(2,3),(4,2)\},\{(1,5),(3,4),(4,3),(5,2)\})$
and that
$\mathbf{C^B}$=$(C_3^{\mathbf{B}}, C_2^{\mathbf{B}}, C_1^{\mathbf{B}})$ =$(\{(1,3),(3,1)\},\{(1,4),(2,3),(3,2),(4,1)\},\{(1,5),(4,4),(5,1)\})$.

First, we have to rank $C_3^{\mathbf{A}}$ *vis-à-vis* $C_3^{\mathbf{B}}$ according to $\succeq_{\mathcal{V}}$. These common capability sets are to be evaluated with the valuation functions in $\mathcal{V}$. Let us suppose that one has two decision-makers $d_1, d_2$ with different opinions about the weights to be attached to the functionings $f_1$ and $f_2$. Furthermore, we assume that the respective valuation functions of $d_1$ and $d_2$ are $\mathcal{V} = \{V_1(x,y) = x^{\frac{1}{2}}y^{\frac{1}{2}}, V_2(x,y) = x^{\frac{1}{3}}y^{\frac{2}{3}}\}$.

Then one computes the following,

$\int_{C_3^{\mathbf{A}}} V_1(x,y)dxdy =$

$$\int_{C_3^{\mathbf{A}}} \sqrt{xy}dxdy = \int_0^1 \int_0^3 \sqrt{xy}dydx + \int_1^3 \int_0^1 \sqrt{xy}dydx = \frac{4}{9}(2(3)^{3/2} - 1) \qquad (14)$$

$\int_{C_3^{\mathbf{B}}} V_1(x,y)dxdy =$

$$\int_{C_3^{\mathbf{B}}} \sqrt{xy}dxdy = \int_0^1 \int_0^3 \sqrt{xy}dydx + \int_1^3 \int_0^1 \sqrt{xy}dydx = \frac{4}{9}(2(3)^{3/2} - 1) \tag{15}$$

$\int_{C_3^{\mathbf{A}}} V_2(x,y)dxdy =$

$$\int_{C_3^{\mathbf{A}}} x^{\frac{1}{3}}y^{\frac{2}{3}}dxdy = \int_0^1 \int_0^3 x^{\frac{1}{3}}y^{\frac{2}{3}}dydx + \int_1^3 \int_0^1 x^{\frac{1}{3}}y^{\frac{2}{3}}dydx = \frac{9}{20}(3^{5/3} + 3^{4/3} - 1) \tag{16}$$

$\int_{C_3^{\mathbf{B}}} V_2(x,y)dxdy =$

$$\int_{C_3^{\mathbf{B}}} x^{\frac{1}{3}}y^{\frac{2}{3}}dxdy = \int_0^1 \int_0^3 x^{\frac{1}{3}}y^{\frac{2}{3}}dydx + \int_1^3 \int_0^1 x^{\frac{1}{3}}y^{\frac{2}{3}}dydx = \frac{9}{20}(3^{5/3} + 3^{4/3} - 1) \tag{17}$$

By comparing equations (14) and (15), on one side, and equations (16) and (17) on the other, we obtain that $C_3^{\mathbf{A}}$ and $C_3^{\mathbf{B}}$ are indifferent from the point of view of both $d_1$ and $d_2$. As both decision-makers coincide, $C_3^{\mathbf{A}}$ and $C_3^{\mathbf{B}}$ are declared to be indifferent: $C_3^{\mathbf{A}} \sim_{\mathcal{V}} C_3^{\mathbf{B}}$. According to the $\succeq_{lcc}^3$ ranking, one proceeds to compare and rank $C_2^{\mathbf{A}}$ with $C_2^{\mathbf{B}}$. Then, we need to check whether both decision makers agree with their respective rankings or not. Thus, one should compute the respective integrals and compare the results. One obtains that

$$\int_{C_2^{\mathbf{A}}} \sqrt{xy}dxdy > \int_{C_2^{\mathbf{B}}} \sqrt{xy}dxdy \tag{18}$$

and that

$$\int_{C_2^{\mathbf{A}}} x^{\frac{1}{3}}y^{\frac{2}{3}}dxdy > \int_{C_2^{\mathbf{B}}} x^{\frac{1}{3}}y^{\frac{2}{3}}dxdy. \tag{19}$$

This means that both $d_1$ and $d_2$ consider that $C_2^{\mathbf{A}}$ is strictly preferred to $C_2^{\mathbf{B}}$, so one has that $C_2^{\mathbf{A}} \succ_{\mathcal{V}} C_2^{\mathbf{B}}$. Therefore, one concludes that $(A_1, A_2, A_3) \succ_{lcc}^3 (B_1, B_2, B_3)$.

# 5 Concluding Remarks.

In this paper we have presented an innovative approach for ranking profiles of capability sets. Throughout our description we have presented highly explicit formulations of the capability profile ranking $\succeq^n$ (with the lexicographical relation $>_L$) and of the valuation functions $\{V_j\}_{1 \leq j \leq p}$ (with the Cobb-Douglas type of functions). However, we must point out that these were actually chosen for illustrative purposes, in order to show explicitly how our proposed ranking worked. If deemed necessary, they could be replaced in different context with other more appropriate functions. These would only be minor changes and would not modify the core ideas of our approach; namely, the use of common capability sets to rank profiles and the use of valuation functions to evaluate them.

The main concern of this paper has been to offer an appropriate framework in which to rank profiles of capability sets. We contend that the approach presented here could also be extended to other more general contexts, such as opportunity sets or opportunity profile rankings. In such a general context there is no need for the opportunity sets to be of an explicit topological structure, as in the present case, and different results might emerge. As the present approach could also be insightful and useful in other contexts, we consider that such a generalization merits further research.

We believe that the approach presented here is a potentially useful addition to the social planner's toolkit, since it offers a relatively new and complementary view of how to evaluate and modify a given social state of affairs.

# Appendix.

**Preposition**: For any $\alpha \in (0,1], C(A_1,\ldots,A_n;\alpha) \in \mathcal{C}$.

**Proof**: To prove that, one has to prove that the $C(A_1,\ldots,A_n;\alpha)$ are: a) non-degenerate, b) compact and c) comprehensive. We see them one by one. a) Take $C(A_1,\ldots,A_n;1) = \bigcap_{i\in N} A_i$. As we have already mentioned, in the context of non-degenerate, compact and comprehensive capability sets, $\bigcap_{i\in N} A_i \neq \emptyset$. Moreover, it can be proven that there exists some $x = (x_1,\ldots,x_m) \in \bigcap_{i\in N} A_i$ such that $x_i > 0$ for all $i = 1,\ldots,m$. To see this consider, for each $A_i$, an element $a_i = (a_{i_1},\ldots,a_{i_m})$ such that $a_{i_j} > 0$ for all $j = 1,\ldots,m$. Since the $A_i$ are non-degenerate, such $a_i$ can be found for all $i = 1,\ldots,n$. Now, consider the values $d_i := \min\{a_{i_1},\ldots,a_{i_m}\}$. It is then clear that $[B(0,d_i) \cap \mathbb{R}^m_+] \subseteq A_i$ for all $i = 1,\ldots,n$ (where $B(0,d_i)$

is the set of points in $\mathbb{R}^m_+$ which are at a distance less than $d_i$ from the origin). If we define $\epsilon := \min(d_1, \ldots, d_n)$, then, by comprehensivity of each of the $A_i$, one clearly has that the vector $(\sqrt{\frac{\epsilon}{m}}, \ldots, \sqrt{\frac{\epsilon}{m}}) \in A_i$ for all $i = 1, \ldots, n$. So $(\sqrt{\frac{\epsilon}{m}}, \ldots, \sqrt{\frac{\epsilon}{m}}) \in \bigcap_{i\in N} A_i$ is the non-degenerate element we were looking for. Now, one concludes by observing that each $C(A_1, \ldots, A_n; \alpha) \supseteq C(A_1, \ldots, A_n; 1)$. b) For any $\alpha \in (0, 1]$, one has that $C(A_1, \ldots, A_n; \alpha) \subseteq \bigcup_{i\in N} A_i$ (even for those $\widetilde{\alpha} \in (0, 1/n)$, for which, by definition, $C(A_1, \ldots, A_n; \widetilde{\alpha}) = \bigcup_{i\in N} A_i$). Thus, as each of the $A_i$ is bounded, and the finite union of bounded sets is bounded, the $C(A_1, \ldots, A_n; \alpha)$ are also bounded. Let us now see that they are also closed. Recall that each $C(A_1, \ldots, A_n; \alpha)$ can be interpreted as the set of functioning vectors that can be achieved or improved by at least $\lceil n\alpha \rceil$ individuals at the same time. Thus, one could say that the $C(A_1, \ldots, A_n; \alpha)$ are obtained as the union of all possible intersections of the sets $\{A_i\}_{i\in\{i_1,\ldots,i_{\lceil n\alpha\rceil}\}}$, where $\{i_1, \ldots, i_{\lceil n\alpha\rceil}\} \subseteq \{1, \ldots, n\}$. That is:

$$C(A_1, \ldots, A_n; \alpha) = \bigcup_{\{i_1,\ldots,i_{\lceil n\alpha\rceil}\}\subseteq\{1,\ldots,n\}} \Big[ \bigcap_{i\in\{i_1,\ldots,i_{\lceil n\alpha\rceil}\}} A_i \Big]$$

Since each $A_i$ is closed and the finite unions and intersections of closed sets is closed, one concludes that, as we wanted to see, $C(A_1, \ldots, A_n; \alpha)$ is closed. Thus, the $C(A_1, \ldots, A_n; \alpha)$ are compact. c) Suppose that $x \in C(A_1, \ldots, A_n; \alpha)$ and $x \geq y$ for a certain $y \in \mathbb{R}^m_+$. By definition, it is clear that if $x$ is achieved of improved by $j \leq n$ individuals, then those $j$ individuals will also achieve or improve the vector $y$, so $I(A_1, \ldots, A_n; x) \leq I(A_1, \ldots, A_n; y)$. Thus, one concludes that $y \in C(A_1, \ldots, A_n; \alpha)$. Q.E.D.

**Theorem 1.** Let $\succeq^n$ be a binary relation on $\mathcal{C}^n$ fulfilling **WEUP** and **RIC**. Then $\succeq^n$ satisfies **AN**, **SPEC**, **SHEC** and **CI** if and only if $\succeq^n = \succeq^n_{lcc}$.

**Proof:** Let us start by noting that, for any capability profile $\mathbf{A} = (A_1, \ldots, A_n) \in \mathcal{C}^n$ one has that $\mathbf{C}^\mathbf{A} = (C^\mathbf{A}_n, \ldots, C^\mathbf{A}_1) \in \mathcal{N}^n$, and that every nested profile $\mathbf{B} = (B_1, \ldots, B_n) \in \mathcal{N}^n$ can be thought as a common capability profile (one can readily verify that $\mathbf{C}^\mathbf{B} = \mathbf{B}$). In other words: the space of common capability profiles and the space of nested profiles $\mathcal{N}^n$ is exactly the same.

One can define the following binary relation on $\mathcal{C}^n$: given any $\mathbf{A}, \mathbf{B} \in \mathcal{C}^n$ we say that $\mathbf{A}$ is related to $\mathbf{B}$ (written as $\mathbf{A} \sim \mathbf{B}$) if and only if $\mathbf{C}^\mathbf{A} = \mathbf{C}^\mathbf{B}$. This way one can define a partition of $\mathcal{C}^n$ through the equivalence classes established by $\sim$. Each of the equivalence classes will be denoted by

$[(D_1, \ldots, D_n)]$ for a certain representative $(D_1, \ldots, D_n) \in \mathcal{C}^n$. It is straightforward to verify that there is a one-to-one correspondence $\varphi : (\mathcal{C}^n / \sim) \to \mathcal{N}^n$, where $\varphi([(D_1, \ldots, D_n)])$ is the *only* nested profile included in the equivalence class $[(D_1, \ldots, D_n)]$. We will now show that, given any $\mathbf{A}, \mathbf{B} \in \mathcal{C}^n$, then

$$\mathbf{A} \succ_{lcc}^n \mathbf{B} \Leftrightarrow \varphi([\mathbf{A}]) \succ_{lo}^n \varphi([\mathbf{B}]). \tag{20}$$

From one side, by definition one has that $\mathbf{A} \succ_{lcc}^n \mathbf{B} \Leftrightarrow (C_n^{\mathbf{A}}, \ldots, C_1^{\mathbf{A}}) >_L (C_n^{\mathbf{B}}, \ldots, C_1^{\mathbf{B}})$. From the other side, one has that $\varphi([\mathbf{A}]) = \mathbf{C}^{\mathbf{A}}$ and $\varphi([\mathbf{B}]) = \mathbf{C}^{\mathbf{B}}$. Now, it is easily checked that, by construction, $\mathbf{C}^{\mathbf{A}} \succ_{lo}^n \mathbf{C}^{\mathbf{B}} \Leftrightarrow (C_n^{\mathbf{A}}, \ldots, C_1^{\mathbf{A}}) >_L (C_n^{\mathbf{B}}, \ldots, C_1^{\mathbf{B}})$, so (20) is proven. What this relation is telling us is that ranking capability profiles with $\succeq_{lcc}^n$ is exactly the same as ranking *nested* capability profiles with $\succeq_{lo}^n$. This means that in order to characterize the capability set ranking $\succeq_{lcc}^n$ we will just have to characterize the capability set ranking $\succeq_{lo}^n$ when this one is restricted to $\mathcal{N}^n$. Now, the relation $\succeq_{lo}^n$ has been univocally characterized in [7] with the following axioms:

*Anonymity*: For any $\mathbf{A} \in \mathcal{C}^n$ and any permutation $p : N \to N, \mathbf{A} \sim^n p(\mathbf{A})$.

*Strong Pareto Efficiency*: For all $\mathbf{A} = (A_1, \ldots, A_n), \mathbf{B} = (B_1, \ldots, B_n) \in \mathcal{C}^n$, if $A_i \succeq B_i \; \forall i \in \{0, \ldots, n\}$, then $\mathbf{A} \succeq^n \mathbf{B}$. If moreover some preference is strict, then $\mathbf{A} \succ^n \mathbf{B}$.

*Strong Hammond Equity*: For all $\mathbf{A}, \mathbf{B} \in \mathcal{C}^n, \mathbf{A} = q(\mathbf{A}), \mathbf{B} = q(\mathbf{B})$, such that: $A_j = A_{j+1} = \ldots = A_{j+s}; B_j = B_{j+1} = \ldots = B_{j+s}$ for some $j \in \{2, \ldots, n-1\}$ and $s$ such that $j + s \leq n$, $A_k \succ B_k$ for some $0 < k < j$, $A_i \sim B_i$ for all $i \neq k, j$, and $B_j \succ A_j \succeq A_k \succ B_k$, then $\mathbf{A} \succ^n \mathbf{B}$.

*Common Improvement*: For any $\mathbf{A} \in \mathcal{C}^n, \mathbf{A} = q(\mathbf{A})$, any $B \in \mathcal{C}, \; A_0 \subset B, \; B \succ A_0, \; \mathbf{B} = (B) \in \mathcal{U}^n$, then $\mathbf{B} \succ^n \mathbf{A}$.

Finally, it is routinely verified that when these axioms are restricted to the set of nested profiles they coincide respectively with the properties **AN**, **SPEC**, **SHEC** and **CI** presented in this paper. Q.E.D.

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