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Axiomatizations of the proportional division value

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Abstract

We present axiomatic characterizations of the *proportional division value* for TU-games, which distributes the worth of the grand coalition in proportion to the stand-alone worths of the players. First, a new proportionality principle, called *proportional-balanced treatment*, is introduced by strengthening Shapley's symmetry axiom, which states that if two players make the same contribution to any nonempty coalition, then they receive the amounts in proportion to their stand-alone worths. We characterize the family of values satisfying efficiency, weak linearity, and proportional-balanced treatment. We also show that this family is incompatible with the dummy player property. However, we show that the proportional division value is the unique value in this family that satisfies the *dummifying player property*. Second, we propose appropriate *monotonicity* axioms, and obtain axiomatizations of the proportional division value without both weak linearity and the dummifying player property. Third, from the perspective of a variable player set, we show that the proportional division value is the only one that satisfies *proportional standardness* and *projection consistency*. Finally, we provide a characterization of proportional standardness.

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1 Introduction

A situation in which a finite set of players can generate certain worths by cooperation can be described by a *cooperative game with transferable utility*, or simply a *TU-game*. A TU-game consists of a set of *players* and a *characteristic function* that specifies a worth to each coalition of players. A *single-valued solution* or a *value* on a class of TU-games assigns a unique payoff vector to every game in this class.

Proportionality is an often applied equity principle in allocation problems. The idea of proportionality can be traced at least as far back as Aristotle's celebrated maxim, "*Equals should be treated equally, unequals unequally, in proportion to relevant similarities and differences*" from Nicomachean Ethics. With a natural proportionality consideration, the *proportional rule*[1] (Moriarity 1975; Banker 1981) allocates the worth of the grand coalition in proportion to the stand-alone worths of its members. In this paper, we call this the *proportional division value*, shortly denoted by the PD value, in order to distinguish it from the proportional rule in claims problems, bargaining problems, insurance, law and so on.[2,3]

Moulin (1987) characterizes the PD value for joint venture games, being a class of TU-games where intermediate coalitions are *inessential*, in the sense that the worth of every proper subset of the full player set equals the sum of the worths of its stand-alone coalitions. These are the *quasi-additive games* in Carreras and Owen (2013), where the PD value is discussed by comparing it with the Shapley value (Shapley 1953). Banker (1981) considers the situation that the worth of a coalition is a non-negative strictly increasing function with respect to the sum of the worths of its members. However, for more general TU-games, since the proportionality principle is not obvious, as far as we know, an axiomatic characterization of the PD value is still missing.

In this paper, we axiomatize the PD value on the domain of TU-games in which the worths of all singleton coalitions have the same sign. This restrictive class of TU-games is considered in Béal et al. (2018) who also provide many applications. We focus on some intuitive fairness criteria that are widely used in the theory for TU-games, including *equal treatment of equals* (also known as *symmetry*), *monotonicity*, and *consistency*.

First, we introduce a proportionality principle called *proportional-balanced treatment*, which is a strengthening of Shapley's symmetry axiom. It states that the payoffs to two players whose contribution to every nonempty coalition not containing

[1] The proportional rule is identical to the stand-alone-coalition proportional value in Kamijo and Kongo (2015).

[2] For other proportional solutions, we refer to the proportional value (Ortmann 2000; Khmelnitskaya and Driessen 2003; Kamijo and Kongo 2015), the proper Shapley values (Vorob'ev and Liapunov 1998; van den Brink et al. 2015), the proportional Shapley value (Béal et al. 2018; Besner 2019), and the proportional Harsanyi solution (Besner 2020).

[3] We remark that the proportional division value cannot be considered as a weighted division value (Béal et al. 2016) or a weighted surplus division value (Calleja and Llerena 2017, 2019) since those values are based on exogenous weights, while the weights in the PD value are determined in the game, specifically they are equal to the stand-alone worths.

them is the same (we call this weak symmetric players), are proportional to their stand-alone worths. It well captures the principles of ‘equal treatment of equals’ and ‘unequal treatment of unequals’. Besner (2019) gives a similar axiom for the proportional Shapley value. Interestingly, proportional-balanced treatment together with efficiency and weak linearity as introduced in Béal et al. (2018), give a family of values that have a formula similar as the family of efficient linear and symmetric values (ELS values for short) introduced in Ruiz et al. (1998), but where the role of equal division is replaced by proportional division. While the Shapley value is the only ELS value that satisfies the dummy player property, we reveal that there is *no* value belonging to our family that satisfies the dummy player property. Instead, we adopt the dummifying player property introduced in Casajus and Huettner (2014), and obtain a characterization of the PD value.

Second, we also provide characterizations of the PD value by applying weaker versions of well-known monotonicity axioms. A monotonicity axiom states that the payoff of a player should not decrease if a TU-game changes in certain ways that are ‘advantageous’ for this player. We introduce three such monotonicity axioms that are a relaxation of three existing axioms, by adding restrictions on the stand-alone worths of the players.[4] The three existing axioms are coalitional monotonicity due to van den Brink (2007), and coalitional surplus equivalence and coalitional surplus monotonicity, both axioms due to Casajus and Huettner (2014). Not surprisingly, any of our monotonicity axioms together with efficiency and symmetry cannot characterize a unique value. However, replacing symmetry by proportional-balanced treatment and any of our monotonicity axioms, characterizes the PD value.

For a variable player set, we provide an axiomatization of the PD value using proportional standardness and the well-known projection consistency due to Funaki and Yamato (2001). Proportional standardness requires to apply proportional division for two-player games, and is used in Ortmann (2000); Khmelnitskaya and Driessen (2003); van den Brink and Funaki (2009) and Huettner (2015). Like other standardness axioms, proportional standardness is rather strong since it sets the payoff distribution for two-player games. Therefore, we conclude with characterizing proportional standardness on the class of two-player games.

The paper is organized as follows. Section 2 provides basic definitions and notation. In Sect. 3, we introduce proportional-balanced treatment and provide some results including an axiomatic characterization of the PD value. In Sect. 4, we offer three axiomatic characterizations using some monotonicity axioms. In Sect. 5, we give an axiomatic characterization on variable player sets by employing projection consistency and proportional standardness. In Sect. 6, we characterize proportional standardness for two-player games. Finally, there is an appendix with the proof of Theorem 1 and the independence of the axioms in the characterization results.

[4] This modification is similar in spirit to parameterized monotonicity introduced in Yokote and Funaki (2017).

2 Preliminaries

2.1 Notation and TU-games

We denote by $\mathbb{R}$ and $\mathbb{R}_+$ the sets of all real numbers and positive real numbers, respectively. The cardinality of a set S will be denoted by $|S|$ or, if no ambiguity is possible, appropriate small letter s. The notation $S \subseteq T$ means that S is a subset of T, while the notation $S \subset T$ means that S is a proper subset of T.

Let $\mathcal{N}$ be the universe of potential players, and let $N \in \mathcal{N}$ be a finite set of n players. A cooperative game with transferable utility, or simply a TU-game, is a pair (N, v), where $N \in \mathcal{N}$ is a set of players, and $v : 2^N \to \mathbb{R}$ is a characteristic function assigning a *worth* $v(S)$ to each $S \in 2^N$, with the convention that $v(\emptyset) = 0$. A subset $S \subseteq N$ or $S \in 2^N$ is called a *coalition*, and $v(S)$ is the reward that coalition S can guarantee by itself without the cooperation of the other players.

Denote $\mathcal{G}$ as the class of all TU-games with a finite player set in $\mathcal{N}$, and $\mathcal{G}^N$ the class of TU-games with player set N. Following Béal et al. (2018), a TU-game (N, v) is *individually positive* if $v(\{i\}) > 0$ for all $i \in N$, and *individually negative* if $v(\{i\}) < 0$ for all $i \in N$. Let $\mathcal{G}_{nz}$ denote the class consisting of all individually positive and individually negative TU-games, and let $\mathcal{G}^N_{nz}$ denote the intersection of $\mathcal{G}_{nz}$ and $\mathcal{G}^N$. For brevity, we refer to a TU-game just as a game.

2.2 Values

A value on $\mathcal{G}^N$ (respectively on $\mathcal{G}^N_{nz}$) is a function ψ that assigns a single payoff vector $\psi(N, v) \in \mathbb{R}^N$ to every game $(N, v) \in \mathcal{G}^N$ (respectively $(N, v) \in \mathcal{G}^N_{nz}$).

The **equal division value** is the value ED on $\mathcal{G}^N$ given by

$$ED_i(N, v) = \frac{1}{n} v(N)$$

for all $(N, v) \in \mathcal{G}^N$ and $i \in N$.

The **proportional division value**[5] is the value PD on $\mathcal{G}^N_{nz}$ given by

$$PD_i(N, v) = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N) \tag{1}$$

for all $(N, v) \in \mathcal{G}^N_{nz}$ and $i \in N$.

We employ the following definitions. Player $i \in N$ is a *dummy player* in game (N, v) if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \backslash \{i\}$. Player $i \in N$ is a *dummifying player* in game (N, v) if $v(S) = \sum_{j\in S} v(\{j\})$ for all $S \subseteq N$ with $i \in S$. Notice the difference between a dummy and dummifying player: a dummy player adds its own stand-alone worth when it joins any coalition, while a dummifying player entering a coalition results in the worth of that coalition becoming equal to the

[5] As mentioned in the introduction, this is called the proportional rule in Moriarty (1975) and Banker (1981).

sum of the stand-alone worths of the players in that coalition. Players $i,j \in N$, $i \neq j$, are *symmetric* in (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i,j\}$. For $(N,v),(N,w) \in \mathcal{G}^N_{nz}$ and $a,b \in \mathbb{R}$, the game $(N, av+bw)$ is defined by $(av+bw)(S) = av(S) + bw(S)$ for all $S \subseteq N$.

Next, we state some properties of values for games.

- *Efficiency* $\sum_{i \in N} \psi_i(N,v) = v(N)$ for all $(N,v) \in \mathcal{G}^N_{nz}$.
- *Symmetry* $\psi_i(N,v) = \psi_j(N,v)$ for all $(N,v) \in \mathcal{G}^N_{nz}$ and $i,j \in N$ being symmetric in (N, v).
- *Dummy player property* $\psi_i(N,v) = v(\{i\})$ for all $(N,v) \in \mathcal{G}^N_{nz}$ and $i \in N$ being a dummy player in (N, v).
- *Dummifying player property* $\psi_i(N,v) = v(\{i\})$ for all $(N,v) \in \mathcal{G}^N_{nz}$ and $i \in N$ being a dummifying player in (N, v).
- *Weak linearity* For all $a \in \mathbb{R}$, and all $(N,v),(N,w) \in \mathcal{G}^N_{nz}$ such that there exists $c \in \mathbb{R}_+$ with $w(\{i\}) = cv(\{i\})$ for all $i \in N$, if $(N, av+w) \in \mathcal{G}^N_{nz}$, then $\psi(N, av+w) = a\psi(N,v) + \psi(N,w)$.

The first three axioms are classical, except that they are defined on subclass $\mathcal{G}^N_{nz}$. The dummifying player property, proposed by Casajus and Huettner (2014), states that a dummifying player just earns its own stand-alone worth. Weak linearity, proposed by Béal et al. (2018), states that when taking a linear combination of two games, where the ratio between the stand-alone worths is the same in both games, the payoff allocation equals the corresponding linear combination of the payoff vectors of the two separate games. This axiom is a weak version of the axiom of linearity as proposed by Shapley (1953). If $a = 1$, then weak linearity reduces to **weak additivity**, which is introduced and studied in Besner (2019).

3 Proportionality principle

In this section, we introduce a new axiom, called proportional-balanced treatment, and characterize the proportional division value.

Definition 1 Players $i,j \in N$, $i \neq j$, are *weak symmetric* in (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i,j\}$, $S \neq \emptyset$.

Two players being weak symmetric still allows them to have a different stand-alone worth, but their contribution to any nonempty coalition including neither of them should be equal. Notice that in two-player games, both players are always weak symmetric. We now introduce a proportionality property, comparable to symmetry, which says that the payoffs to two weak symmetric players are in the same proportion as their stand-alone worths. This axiom can be considered as a strengthening of Shapley's symmetry axiom since it implies that any two symmetric players in any game should earn the same payoff.

- *Proportional-balanced treatment* $\frac{\psi_i(N,v)}{v(\{i\})} = \frac{\psi_j(N,v)}{v(\{j\})}$ for all $(N, v) \in \mathcal{G}^N_{nz}$ and $i, j \in N$ being weak symmetric players in (N, v).

Next, we exactly characterize the class of values on $\mathcal{G}^N_{nz}$ that satisfies efficiency, weak linearity, and proportional-balanced treatment.

Theorem 1 *A value ψ on $\mathcal{G}^N_{nz}$ satisfies efficiency, weak linearity, and proportional-balanced treatment if and only if for each $(N, v) \in \mathcal{G}^N_{nz}$ and each $i \in N$,*

$$\psi_i(N,v) = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N) + v(\{i\}) \left[\sum_{\substack{S\,:\, i\in S\neq N \\ |S|\geq 2}} \frac{\lambda_S}{\sum_{j\in S} v(\{j\})} v(S) - \sum_{\substack{S\,:\, i\notin S \\ |S|\geq 2}} \frac{\lambda_S}{\sum_{j\in N\setminus S} v(\{j\})} v(S) \right], \tag{2}$$

where for each $S \subset N$ with $|S| \geq 2$, λ_S is a real number such that

$$\frac{\lambda_S}{\sum_{j\in S} v(\{j\}) \sum_{j\in N\setminus S} v(\{j\})} = \frac{\lambda_T}{\sum_{j\in T} v(\{j\}) \sum_{j\in N\setminus T} v(\{j\})}, \quad \text{if } |S| = |T|. \tag{3}$$

The lengthy proof of Theorem 1 is given in the Appendix. The proof uses the following proposition. Since two players in a two-player game $(N, v) \in \mathcal{G}^N_{nz}$ are always weak symmetric, if value ψ satisfies proportional-balanced treatment, then $\frac{\psi_i(N,v)}{v(\{i\})} = \frac{\psi_j(N,v)}{v(\{j\})}$ for $i, j \in N$. By efficiency, we then obtain that the worth of the grand coalition is allocated proportional to the stand-alone worths.

Proposition 1 *Let $N \in \mathcal{N}$ with $|N| = 2$. The PD value is the unique value on $\mathcal{G}^N_{nz}$ satisfying efficiency and proportional-balanced treatment.*

The values characterized in Theorem 1, can be seen as modifications of the PD value, where to every game they first apply the PD value and then make a 'correction' that is based on the stand-alone worth of a player and the difference between weighted sums of the worths of all other coalitions with and without this player. The weights depend on all stand-alone worths. In this sense, (2) bears some similarity with the family of efficient, linear and symmetric (ELS) values (Lemma 9, Ruiz et al. (1998)) which can be written as:

$$\psi_i(N,v) = \frac{v(N)}{|N|} + \sum_{S: i\in S\neq N} \frac{\rho_s}{|S|} v(S) - \sum_{S: i\notin S} \frac{\rho_s}{|N| - |S|} v(S), \tag{4}$$

where ρ_s, $s \in \{1, 2, \dots, n-1\}$, is a real number. The ELS values can be seen as first applying equal division and then make a correction based on a weighted sum of differences between worths of coalitions with and without a player. Specifically, if $v(\{i\}) = v(\{j\})$ for all $i, j \in N$, (2) coincides with the above equation.

Remark 1 Note that (3) indicates that all coefficients of coalitions of the same size are uniquely determined as soon as any one of them is given. For computational convenience, given $\{\lambda_S \in \mathbb{R} \mid S \subset N, |S| \geq 2\}$, denoting $\lambda_s = \frac{\lambda_S}{\sum_{j \in S} v(\{j\}) \sum_{j \in N \setminus S} v(\{j\})}$, (2) can be rewritten as

$$\psi_i(N, v) = \frac{v(\{i\})v(N)}{\sum_{j \in N} v(\{j\})} + v(\{i\}) \left[\sum_{S: i \in S \neq N} \sum_{j \in N \setminus S} v(\{j\}) \lambda_s v(S) - \sum_{S: i \notin S} \sum_{j \in S} v(\{j\}) \lambda_s v(S) \right], \tag{5}$$

where $\lambda_1 = 0$, and λ_s, $s \in \{2, \dots, n-1\}$, is a function with respect to λ_S and all stand-alone worths. Since λ_s might be different for games in which stand-alone worths are different, (5) cannot be directly used to verify weak linearity.

Remark 2 A family of values derived from the family of ELS values given by (4) with $\rho_1 = 0$, satisfies proportional-balanced treatment as follows. For any ELS value ψ' given by (4) with $\rho_1 = 0$, the value ψ defined by

$$\psi_i(N, v) = \frac{v(\{i\})}{\sum_{k \in N} v(\{k\})} \psi_i'(N, v) + \frac{v(\{i\})}{\sum_{k \in N} v(\{k\})} \left[v(N) - \sum_{h \in N} \frac{v(\{h\}) \psi_h'(N, v)}{\sum_{k \in N} v(\{k\})} \right] \tag{6}$$

satisfies proportional-balanced treatment, and also efficiency and weak linearity. This value can be viewed as a multiplicative normalization of an ELS value.

A next question is whether the class of values characterized in Theorem 1 contains a value that satisfies the dummy player property. It turns out that, for games with at least three players, the dummy player property is incompatible with the three axioms in Theorem 1.

Theorem 2 *Let $N \in \mathcal{N}$ with $|N| \geq 3$. There is no value on $\mathcal{G}^N_{nz}$ satisfying efficiency, weak linearity, proportional-balanced treatment, and the dummy player property.*

Proof Let ψ be a value satisfying the four axioms. First, suppose that $|N| \geq 4$. Consider any game $(N, v) \in \mathcal{G}^N_{nz}$ and $i \in N$ such that i is a dummy player in (N, v). By Theorem 1, we have

$$\begin{aligned}
\psi_i(N,v) &= \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N) + \sum_{\substack{S\,:\,i\in S\neq N\\ |S|\geq 2}} \frac{v(\{i\})\cdot\lambda_S}{\sum_{j\in S} v(\{j\})} v(S) - \sum_{\substack{S\,:\,i\notin S\\ |S|\geq 2}} \frac{v(\{i\})\cdot\lambda_S}{\sum_{j\in N\setminus S} v(\{j\})} v(S)\\
&= \frac{v(\{i\})v(N)}{\sum_{j\in N} v(\{j\})} + \sum_{\substack{S\,:\,i\notin S\\ 1\leq |S|\leq n-2}} \frac{v(\{i\})\cdot\lambda_{S\cup\{i\}}}{\sum_{j\in S\cup\{i\}} v(\{j\})} v(S\cup\{i\}) - \sum_{\substack{S\,:\,i\notin S\\ |S|\geq 2}} \frac{v(\{i\})\cdot\lambda_S}{\sum_{j\in N\setminus S} v(\{j\})} v(S)\\
&= \frac{v(\{i\})[v(N\setminus\{i\})+v(\{i\})]}{\sum_{j\in N} v(\{j\})} + \sum_{j\in N\setminus\{i\}} \frac{v(\{i\})\cdot\lambda_{\{i,j\}}}{v(\{i\})+v(\{j\})} v(\{i,j\}) - \lambda_{N\setminus\{i\}} v(N\setminus\{i\})\\
&\quad + \sum_{\substack{S\,:\,i\notin S\\ 2\leq |S|\leq n-2}} \left[\frac{v(\{i\})\cdot\lambda_{S\cup\{i\}}}{\sum_{j\in S\cup\{i\}} v(\{j\})} v(S\cup\{i\}) - \frac{v(\{i\})\cdot\lambda_S}{\sum_{j\in N\setminus S} v(\{j\})} v(S)\right]\\
&= \frac{v(\{i\})v(\{i\})}{\sum_{j\in N} v(\{j\})} + v(\{i\}) \sum_{j\in N\setminus\{i\}} \lambda_{\{i,j\}} + \left[\frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} - \lambda_{N\setminus\{i\}}\right] v(N\setminus\{i\})\\
&\quad + \sum_{\substack{S\,:\,i\notin S\\ 2\leq |S|\leq n-2}} \frac{v(\{i\})\cdot\lambda_{S\cup\{i\}}}{\sum_{j\in S\cup\{i\}} v(\{j\})} v(\{i\})\\
&\quad + \sum_{\substack{S\,:\,i\notin S\\ 2\leq |S|\leq n-2}} \left[\frac{v(\{i\})\cdot\lambda_{S\cup\{i\}}}{\sum_{j\in S\cup\{i\}} v(\{j\})} - \frac{v(\{i\})\cdot\lambda_S}{\sum_{j\in N\setminus S} v(\{j\})}\right] v(S),
\end{aligned}$$

where the third equality follows from i being a dummy player in (N, v).

Since, by the dummy player property, the payoff of dummy player i should not depend on $v(S)$, $i\notin S$ and $2\leq |S|\leq |N|-1$, the third term and the fifth term of the above equation must be equal to 0, which yields

$$\lambda_{N\setminus\{i\}} = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})}, \tag{7}$$

$$\frac{\lambda_{S\cup\{i\}}}{\lambda_S} = \frac{\sum_{j\in S\cup\{i\}} v(\{j\})}{\sum_{j\in N\setminus S} v(\{j\})} \quad \text{for } S\subset N \text{ with } 2\leq |S|\leq |N|-2. \tag{8}$$

Since λ_S for each $S\subset N$ with $|S|\geq 2$ satisfies (3), then (7) and (3) together imply that

$$\lambda_{N\setminus\{k_1\}} = \frac{v(\{k_1\})\sum_{j\in N\setminus\{k_1\}} v(\{j\})}{\sum_{j\in N\setminus\{i\}} v(\{j\})\sum_{j\in N} v(\{j\})} \quad \text{for any } k_1\in N\setminus\{i\}.$$

By using (8), we have

$$\lambda_{N\setminus\{i,k_1\}} = \frac{\sum_{j\in\{i,k_1\}} v(\{j\})}{\sum_{j\in N\setminus\{k_1\}} v(\{j\})}\lambda_{N\setminus\{k_1\}} = \frac{v(\{k_1\})\sum_{j\in\{i,k_1\}} v(\{j\})}{\sum_{j\in N\setminus\{i\}} v(\{j\})\sum_{j\in N} v(\{j\})}.$$

The above equation together with (3) imply that, for any $k_2 \in N\setminus\{i,k_1\}$,

$$\lambda_{N\setminus\{k_1,k_2\}} = \frac{\sum_{j\in N\setminus\{k_1,k_2\}} v(\{j\})\sum_{j\in\{k_1,k_2\}} v(\{j\})}{\sum_{j\in N\setminus\{i,k_1\}} v(\{j\})\sum_{j\in\{i,k_1\}} v(\{j\})}\lambda_{N\setminus\{i,k_1\}} = \frac{\sum_{j\in N\setminus\{k_1,k_2\}} v(\{j\})\sum_{j\in\{k_1,k_2\}} v(\{j\})v(\{k_1\})}{\sum_{j\in N\setminus\{i,k_1\}} v(\{j\})\sum_{j\in N\setminus\{i\}} v(\{j\})\sum_{j\in N} v(\{j\})}.$$

Now, exchanging the order of k_1 and k_2, we have

$$\lambda_{N\setminus\{k_1,k_2\}} = \frac{\sum_{j\in N\setminus\{k_1,k_2\}} v(\{j\})\sum_{j\in\{k_1,k_2\}} v(\{j\})v(\{k_2\})}{\sum_{j\in N\setminus\{i,k_2\}} v(\{j\})\sum_{j\in N\setminus\{i\}} v(\{j\})\sum_{j\in N} v(\{j\})}.$$

Therefore, it must be that

$$\frac{v(\{k_1\})}{\sum_{j\in N\setminus\{i,k_1\}} v(\{j\})} = \frac{v(\{k_2\})}{\sum_{j\in N\setminus\{i,k_2\}} v(\{j\})},$$

from which it follows that $v(\{k_1\}) = v(\{k_2\})$ for any $k_1, k_2 \in N\setminus\{i\}$. This contradicts the definition of $\mathcal{G}^N_{nz}$.

Next, suppose that $|N| = 3$. Consider $(N, v) \in \mathcal{G}^N_{nz}$ with $N = \{i,j,k\}$ and $i \in N$ such that i is a dummy player in (N, v). By Theorem 1 and i being a dummy player, we have

$$\begin{aligned}
\psi_i(N,v) &= \tfrac{v(\{i\})[v(\{j,k\})+v(\{i\})]}{v(\{i\})+v(\{j\})+v(\{k\})} \\
&\quad + v(\{i\})\Big(\tfrac{\lambda_{\{i,j\}}(v(\{i\})+v(\{j\}))}{v(\{i\})+v(\{j\})} + \tfrac{\lambda_{\{i,k\}}(v(\{i\})+v(\{k\}))}{v(\{i\})+v(\{k\})} - \tfrac{\lambda_{\{j,k\}}v(\{j,k\})}{v(\{i\})}\Big) \\
&= \tfrac{v(\{i\})[v(\{j,k\})+v(\{i\})]}{v(\{i\})+v(\{j\})+v(\{k\})} + v(\{i\})(\lambda_{\{i,j\}} + \lambda_{\{i,k\}}) - \lambda_{\{j,k\}}v(\{j,k\}) \\
&= v(\{j,k\})\Big(\tfrac{v(\{i\})}{v(\{i\})+v(\{j\})+v(\{k\})} - \lambda_{\{j,k\}}\Big) \\
&\quad + v(\{i\})\Big(\tfrac{v(\{i\})}{v(\{i\})+v(\{j\})+v(\{k\})} + \lambda_{\{i,j\}} + \lambda_{\{i,k\}}\Big).
\end{aligned}$$

By the dummy player property, we have $\psi_i(N,v) = v(\{i\})$. Since $\psi_i(N,v)$ should not depend on $v(\{j,k\})$, it must be that $\frac{v(\{i\})}{v(\{i\})+v(\{j\})+v(\{k\})} - \lambda_{\{j,k\}} = 0$, and thus

$$\lambda_{\{j,k\}} = \frac{v(\{i\})}{v(\{i\})+v(\{j\})+v(\{k\})}. \tag{9}$$

But then

$$\frac{v(\{i\})}{v(\{i\})+v(\{j\})+v(\{k\})}+\lambda_{\{i,j\}}+\lambda_{\{i,k\}}=1,$$

implying that

$$\lambda_{\{i,j\}}+\lambda_{\{i,k\}}=\frac{v(\{j\})+v(\{k\})}{v(\{i\})+v(\{j\})+v(\{k\})}. \tag{10}$$

Meanwhile, (3) implies that

$$\frac{\lambda_{\{j,k\}}}{[v(\{j\})+v(\{k\})]v(\{i\})}=\frac{\lambda_{\{i,j\}}}{[v(\{i\})+v(\{j\})]v(\{k\})}=\frac{\lambda_{\{i,k\}}}{[v(\{i\})+v(\{k\})]v(\{j\})}.$$

It follows that

$$\frac{\lambda_{\{j,k\}}}{[v(\{j\})+v(\{k\})]v(\{i\})}=\frac{\lambda_{\{i,j\}}+\lambda_{\{i,k\}}}{[v(\{i\})+v(\{j\})]v(\{k\})+[v(\{i\})+v(\{k\})]v(\{j\})}. \tag{11}$$

Substituting (9) and (10) into (11) yields

$$(v(\{j\}))^2+(v(\{k\}))^2=v(\{i\})v(\{k\})+v(\{i\})v(\{j\}),$$

which does not hold for all games in $\mathcal{G}^N_{nz}$. □

Notice that for $|N|=2$, the PD value satisfies these axioms.

Since the PD value satisfies efficiency, weak linearity, and proportional-balanced treatment on $\mathcal{G}^N_{nz}$, it belongs to the class of values characterized in Theorem 1. In fact, it is the value corresponding to $\lambda_S=0$ for all $S\subseteq N$. Moreover, replacing the dummy player property in Theorem 2 by the dummifying player property, characterizes the PD value (also holds for two-player games).

Theorem 3 *The PD value is the unique value on $\mathcal{G}^N_{nz}$ that satisfies efficiency, weak linearity, proportional-balanced treatment, and the dummifying player property.*

Proof It is obvious that the PD value satisfies efficiency, weak linearity, proportional-balanced treatment, and the dummifying player property. It remains to prove the uniqueness part. Let ψ be a value satisfying these axioms. By Theorem 1, any value satisfying efficiency, weak linearity and proportional-balanced treatment is given by (2) for some λ_S ($S\subseteq N, |S|\geq 2$) satisfying (3). To derive λ_S, we consider a modified game $(N,v_i)\in\mathcal{G}^N_{nz}$ with respect to $(N,v)\in\mathcal{G}^N_{nz}$ and $i\in N$, defined by

$$v_i(S)=\begin{cases} v(\{j\}), & \text{if } S=\{j\} \text{ for all } j\in N,\\ \sum_{j\in S}v(\{j\}), & \text{if } i\in S \text{ and } |S|\geq 2,\\ v(S), & \text{otherwise.}\end{cases}$$

Applying (2) to the game (N,v_i), we have

$$\psi_i(N,v_i)=v(\{i\})+v(\{i\})\Bigg[\sum_{\substack{S\,:\,i\in S\neq N\\|S|\geq 2}}\lambda_S-\sum_{\substack{S\,:\,i\notin S\\|S|\geq 2}}\frac{\lambda_S v(S)}{\sum_{j\in N\setminus S}v(\{j\})}\Bigg].$$

Since i is dummifying in (N,v_i), the dummifying player property requires that $\psi_i(N,v_i)=v(\{i\})$, and thus

$$\sum_{\substack{S\,:\,i\in S\neq N\\|S|\geq 2}}\lambda_S-\sum_{\substack{S\,:\,i\notin S\\|S|\geq 2}}\frac{\lambda_S v(S)}{\sum_{j\in N\setminus S}v(\{j\})}\equiv 0.$$

It follows that

$$\sum_{s=2}^{n-1}\left[\sum_{\substack{S\,:\,i\in S\\|S|=s}}\lambda_S-\sum_{\substack{S\,:\,i\notin S\\|S|=s}}\frac{\lambda_S v(S)}{\sum_{j\in N\setminus S}v(\{j\})}\right]\equiv 0. \tag{12}$$

We will show that $\lambda_S=0$ for all $S\subset N$ in (12). Suppose by contradiction that there exist some $S\subset N$ with $s\in\{2,\dots,n-1\}$ such that $\lambda_S\neq 0$ and $|S|=s$. Let $\mathcal{S}=\{s_1,s_2,\dots,s_m\}$ be the set of such coalitional sizes. Note that (3) implies that if $\lambda_S\neq 0$, then all coefficients of coalitions of the same size s are not equal to zero. We denote by $\mathcal{S}_k=\{S_k^1,S_k^2,\dots,S_k^h\}$, $k=1,\dots,m$, $h=\binom{n}{s_k}$, the set of all coalitions of the same size $s_k\in\mathcal{S}$. Pick any $S_k^r\in\mathcal{S}_k$ with $i\in S_k^r$. By (3), we have $\lambda_{S_k^t}=\frac{\sum_{j\in S_k^t}v(\{j\})\sum_{j\in N\setminus S_k^t}v(\{j\})}{\sum_{j\in S_k^r}v(\{j\})\sum_{j\in N\setminus S_k^r}v(\{j\})}\lambda_{S_k^r}$ for any $S_k^t\in\mathcal{S}_k$ (it obviously holds for the case $S_k^t=S_k^r$). With this equality, (12) can be written as

$$\sum_{s_k\in\mathcal{S}}\left[A(\mathcal{S}_k)-\sum_{S_k^t\in\mathcal{S}_k,i\notin S_k^t}B(S_k^t)v(S_k^t)\right]\lambda_{S_k^r}=0, \tag{13}$$

where $A(\mathcal{S}_k)=\sum_{S_k^t\in\mathcal{S}_k,i\in S_k^t}\frac{\sum_{j\in S_k^t}v(\{j\})\sum_{j\in N\setminus S_k^t}v(\{j\})}{\sum_{j\in S_k^r}v(\{j\})\sum_{j\in N\setminus S_k^r}v(\{j\})}$ and $B(S_k^t)=\frac{\sum_{j\in S_k^t}v(\{j\})}{\sum_{j\in S_k^r}v(\{j\})\sum_{j\in N\setminus S_k^r}v(\{j\})}$.

Now, pick any $s_l\in\{s_1,s_2,\dots,s_m\}$ and any $c\in\mathbb{R}\setminus\{0\}$, and consider the game $(N,v_{i,s_l})\in\mathcal{G}_{nz}^N$ given by

$$v_{i,s_l}(S)=\begin{cases}v_i(S)+c, & \text{if } |S|=s_l \text{ and } i\notin S,\\ v_i(S), & \text{otherwise.}\end{cases}$$

Note that (3) shows that λ_S only depends on the size of S and the worths of all singleton coalitions. Therefore, since i is a dummifying player in (N,v_{i,s_l}), for this game we can obtain an equation similar as (13) but with an additional term that depends on c,

$$\sum_{s_k \in \mathcal{S}} \left[A(\mathcal{S}_k) - \sum_{S_k^t \in \mathcal{S}_k, i \notin S_k^t} B(S_k^t) v(S_k^t) \right] \lambda_{S_k^r} - c\lambda_{S_l^r} \sum_{S_l^t \in \mathcal{S}_l, i \notin S_l^t} B(S_l^t) = 0.$$

Together with this equation and (13), it holds that $-c\lambda_{S_l^r} \sum_{S_l^t \in \mathcal{S}_l, i \notin S_l^t} B(S_l^t) = 0$, yielding $\lambda_{S_l^r} = 0$, which is a contradiction. □

Remark 3 Besner (2019) characterizes the proportional Shapley value by employing a proportionality axiom, which says $\frac{\psi_i(N,v)}{v(\{i\})} = \frac{\psi_j(N,v)}{v(\{j\})}$ for all $(N, v) \in \mathcal{G}_{nz}^N$ and $i, j \in N$ such that $v(S \cup \{k\}) = v(S) + v(\{k\})$, $k \in \{i, j\}$, for all $S \subseteq N \backslash \{i, j\}$. Clearly, this axiom focuses on a pair of weakly dependent players, whereas proportional-balanced treatment considers weak symmetric players.

Remark 4 We conclude this section by comparing our results with the main results in Casajus and Huettner (2014), which show that on the domain of TU-games $\mathcal{G}^N$, the equal surplus division value[6] treats dummifying players in the same way as the Shapley value handles dummy players. Restricting ourselves to the subclass $\mathcal{G}_{nz}^N$, notice that the PD value is a variation of both the equal division value and the equal surplus division value since $PD_i(N, v) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N) = v(\{i\}) + \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})}[v(N) - \sum_{j \in N} v(\{j\})]$. Interestingly, Theorem 3 gives a characterization of the PD value using the dummifying player property, whereas, for $|N| \geq 3$, using the dummy player property instead of the dummifying player property leads to an impossibility, as in Theorem 2.

4 Monotonicity

In this section, we present axiomatic characterizations of the PD value by imposing three appropriate monotonicity axioms being weaker versions of classical monotonicity axioms in the literature.

- *Weak coalitional surplus equivalence*[7]. For all $(N, v), (N, w) \in \mathcal{G}_{nz}^N$ with $v(\{j\}) = w(\{j\})$ for all $j \in N$, and $i \in N$ being a dummifying player in (N, w), we have $\psi_i(N, v + w) = \psi_i(N, v) + w(\{i\})$.
- *Weak coalitional surplus monotonicity* For all $(N, v), (N, w) \in \mathcal{G}_{nz}^N$ with $w(\{j\}) = cv(\{j\})$ for all $j \in N$ and $c \in \mathbb{R}_+$, and $i \in N$ such that

[6] The *equal surplus division value ESD*, also known as the *center-of-gravity of the imputation-set value* in Driessen and Funaki (1991), on $\mathcal{G}_{nz}^N$ is defined for all $(N, v) \in \mathcal{G}_{nz}^N$ and all $i \in N$, by

$$ESD_i(N, v) = v(\{i\}) + \frac{1}{n}[v(N) - \sum_{j \in N} v(\{j\})].$$

[7] Weak coalitional surplus equivalence is a monotonicity principle since it is implied by weak coalitional surplus monotonicity with $c = 2$. See Lemma 1.

$v(S) - \sum_{j \in S} v(\{j\}) \geq w(S) - \sum_{j \in S} w(\{j\})$ for all $S \subseteq N$ with $i \in S$, we have $\psi_i(N, v) - v(\{i\}) \geq \psi_i(N, w) - w(\{i\})$.

- *Weak coalitional monotonicity* For all $(N, v), (N, w) \in \mathcal{G}^N_{nz}$ with $v(\{j\}) = w(\{j\})$ for all $j \in N$, and $i \in N$ such that $v(S) \geq w(S)$ for all $S \subseteq N$ with $i \in S$, we have $\psi_i(N, v) \geq \psi_i(N, w)$.

Weak coalitional surplus equivalence states that the payoff of a player increases by her stand-alone worth if we add a game in which this player is a dummifying player and each stand-alone worth is the same as that of the original game.

Weak coalitional surplus monotonicity states that if two games in which the stand-alone worths of all players are in the same proportion to each other and the surplus of every coalition a player belongs to (measured by the worth of the coalition minus the sum of the stand-alone worths of its players) weakly increases, then the relative payoff of this player (being the difference between the payoff and the stand-alone worth) should not decrease.

Weak coalitional monotonicity states that the payoff of a player should not decrease whenever the worth of every coalition containing this player weakly increases, while the worth of every singleton coalition remains unchanged.

Weak coalitional surplus equivalence (respectively, weak coalitional surplus monotonicity) is a weak version of coalitional surplus equivalence[8] (respectively, coalitional surplus monotonicity[9]) as defined in Casajus and Huettner (2014). Weak coalitional monotonicity is stronger than coalitional monotonicity[10] as defined in Shubik (1962), while it is weaker than coalitional monotonicity[11] as defined in van den Brink (2007).[12] Not surprisingly, any of our monotonicity axioms together with efficiency and symmetry cannot characterize a unique value. Outstandingly, replacing symmetry by proportional-balanced treatment and keeping efficiency, we derive that any of our monotonicity axioms characterizes the PD value.

Notice that weak coalitional monotonicity is a special case of weak coalitional surplus monotonicity taking $c = 1$. In addition, weak coalitional surplus monotonicity implies weak coalitional surplus equivalence.

[8] A value ψ satisfies *coalitional surplus equivalence* if $\psi_i(N, v + w) = \psi_i(N, v) + w(\{i\})$ for all $(N, v), (N, w) \in \mathcal{G}^N$ and $i \in N$ being a dummifying player in (N, w).

[9] A value ψ satisfies *coalitional surplus monotonicity* if $\psi_i(N, v) - v(\{i\}) \geq \psi_i(N, w) - w(\{i\})$ for all $(N, v), (N, w) \in \mathcal{G}^N$, and $i \in N$ such that $v(S) - \sum_{j \in S} v(\{j\}) \geq w(S) - \sum_{j \in S} w(\{j\})$ for all $S \subseteq N$ with $i \in S$.

[10] A value ψ satisfies Shubik's version of *coalitional monotonicity* if $\psi_i(N, v) \geq \psi_i(N, w)$ for all $(N, v), (N, w) \in \mathcal{G}^N$ and $i \in N$ such that $v(S) \geq w(S)$ for all $S \subseteq N$ with $i \in S$, and $v(S) = w(S)$ for all $S \subseteq N \backslash \{i\}$.

[11] A value ψ satisfies van den Brink's version of *coalitional monotonicity* if $\psi_i(N, v) \geq \psi_i(N, w)$ for all $(N, v), (N, w) \in \mathcal{G}^N$ and $i \in N$ such that $v(S) \geq w(S)$ for all $S \subseteq N$ with $i \in S$.

[12] Under efficiency and symmetry, coalitional monotonicity characterizes the equal division value in van den Brink (2007), and either coalitional surplus equivalence or coalitional surplus monotonicity characterizes the equal surplus division value in Casajus and Huettner (2014).

Lemma 1 *On $\mathcal{G}_{nz}^N$, weak coalitional surplus monotonicity implies weak coalitional surplus equivalence.*

Proof Let ψ be a value satisfying weak coalitional surplus monotonicity. Consider a pair of games $(N,v),(N,v+w)\in\mathcal{G}_{nz}^N$, where $v(\{j\})=w(\{j\})$ for all $j\in N$, and $i\in N$ being a dummifying player in (N,w). Since $(v+w)(S)-\sum_{j\in S}(v+w)(\{j\})=v(S)-\sum_{j\in S}v(\{j\})$ for all $S\subseteq N$ with $i\in S$, by weak coalitional surplus monotonicity, we have $\psi_i(N,v+w)-(v+w)(\{i\})=\psi_i(N,v)-v(\{i\})$. It follows that $\psi_i(N,v+w)=\psi_i(N,v)+w(\{i\})$, which shows that ψ satisfies weak coalitional surplus equivalence. □

Considering weak coalitional surplus equivalence and weak coalitional surplus monotonicity, the PD value is characterized by either one of these axioms in addition to efficiency and proportional-balanced treatment.

Theorem 4 *(i) The PD value is the unique value on $\mathcal{G}_{nz}^N$ that satisfies efficiency, proportional-balanced treatment, and weak coalitional surplus equivalence.*

(ii) The PD value is the unique value on $\mathcal{G}_{nz}^N$ that satisfies efficiency, proportional-balanced treatment, and weak coalitional surplus monotonicity.

Proof (i) It is clear that the PD value satisfies efficiency, proportional-balanced treatment, and weak coalitional surplus equivalence. Now let ψ be a value on $\mathcal{G}_{nz}^N$ satisfying the three axioms. For $|N|=1$, (1) is satisfied by efficiency. For $|N|=2$, (1) is obtained from Proposition 1. For $|N|\geq 3$, uniqueness follows by induction on $d(v)=|\{T\subseteq N \mid v(T)-\frac{1}{2}\sum_{j\in T}v(\{j\})\neq 0 \text{ and } |T|\geq 2\}|$. For any $(N,v)\in\mathcal{G}_{nz}^N$, define $(N,v^0)\in\mathcal{G}_{nz}^N$ as follows:

$$v^0(T)=v(T)-\frac{1}{2}\sum_{j\in T}v(\{j\}) \text{ for all } T\subseteq N. \tag{14}$$

Initialization. If $d(v)=0$, then $v(N)=\frac{1}{2}\sum_{j\in N}v(\{j\})$. Notice that, by $d(v)=0$, in this case $v^0(T)=0$ for all $T\subseteq N$ with $|T|\geq 2$. Clearly, all players $i,j\in N$ are weak symmetric in (N,v^0) and $v^0(N)=0$. By efficiency and proportional-balanced treatment, we have $\psi_i(N,v^0)=0$ for all $i\in N$. Notice that $(v-v^0)(\{i\})=v(\{i\})-v(\{i\})+\frac{1}{2}v(\{i\})=\frac{1}{2}v(\{i\})$ for all $i\in N$, and all players are dummifying in $(N,v-v^0)$ since $(v-v^0)(T)=v(T)-v(T)+\frac{1}{2}\sum_{j\in T}v(\{j\})=\frac{1}{2}\sum_{j\in T}v(\{j\})=\sum_{j\in T}(v-v^0)(\{j\})$. It follows from weak coalitional surplus equivalence that $\psi_i(N,v)=\psi_i(N,v^0+(v-v^0))=\psi_i(N,v^0)+\frac{1}{2}v(\{i\})$ for all $i\in N$. Thus, $\psi_i(N,v)=\frac{1}{2}v(\{i\})=PD_i(N,v)$ for all $i\in N$.

Proceeding by induction, assume that $\psi(N,w)=PD(N,w)$ for all $(N,w)\in\mathcal{G}_{nz}^N$ with $d(w)=h$, $0\leq h\leq 2^n-n-2$. Consider $(N,v)\in\mathcal{G}_{nz}^N$ such that $d(v)=h+1$. Let $\mathcal{S}=\{S_1,S_2,\dots,S_{h+1}\}$ be the set of coalitions such that $v(S_k)-\frac{1}{2}\sum_{j\in S_k}v(\{j\})\neq 0$

and $|S_k| \geq 2$. Let S be the intersection of all such coalitions S_k, i.e., $S = \bigcap_{1 \leq k \leq h+1} S_k$. We distinguish between two cases:

- **Case (a)**: $i \in N \backslash S$. Each player $i \in N \backslash S$ is a member of at most h coalitions in $\mathcal{S}$, and at least one $S_k \in \mathcal{S}$ such that $i \notin S_k$ (obviously, $S_k \neq N$). For $(N, v) \in \mathcal{G}_{nz}^N$, define three associated games as follows:

$$v^{S_k,1}(T) = \begin{cases} 0, & \text{if } T = S_k, \\ v(T) - \frac{1}{2}\sum_{j \in T} v(\{j\}), & \text{otherwise.} \end{cases}$$

$$v^{S_k,2}(T) = \begin{cases} v(T), & \text{if } T = S_k, \\ \frac{1}{2}\sum_{j \in T} v(\{j\}), & \text{otherwise.} \end{cases}$$

$$v^3(T) = \frac{1}{2}\sum_{j \in T} v(\{j\}), \quad \text{for all } T \subseteq N.$$

Clearly, $v = v^{S_k,1} + v^{S_k,2}$, $v^{S_k,1}(\{j\}) = v^{S_k,2}(\{j\}) = \frac{1}{2}v(\{j\})$ for all $j \in N$, $v^{S_k,1}(N) = v(N) - \frac{1}{2}\sum_{j \in N} v(\{j\})$ (since $S_k \neq N$), and every player $i \in N \backslash S_k$ is dummifying in $(N, v^{S_k,2})$. By weak coalitional surplus equivalence, $\psi_i(N, v) = \psi_i(N, v^{S_k,1}) + \frac{1}{2}v(\{i\})$ for all $i \in N \backslash S_k$.

Moreover, $d(v^{S_k,1} + v^3) = h$, and every player $i \in N$ is dummifying in (N, v^3). Thus, weak coalitional surplus equivalence and the induction hypothesis imply that $\psi_i(N, v^{S_k,1} + v^3) = \psi_i(N, v^{S_k,1}) + v^3(\{i\}) = \psi_i(N, v^{S_k,1}) + \frac{1}{2}v(\{i\}) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})}[v^{S_k,1}(N) + v^3(N)] = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})}v(N)$ for all $i \in N$. It follows that $\psi_i(N, v^{S_k,1}) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})}v(N) - \frac{1}{2}v(\{i\})$ for all $i \in N$.

Therefore, $\psi_i(N, v) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})}v(N) = PD_i(N, v)$ for all $i \in N \backslash S_k$. Since there exists such a S_k for all $i \in N \backslash S$, we obtain $\psi_i(N, v) = PD_i(N, v)$ for all $i \in N \backslash S$.

- **Case (b)**: $i \in S$. If $S = \{i\}$, we obtain, by efficiency of ψ and PD together with Case (a), $\psi_i(N, v) = PD_i(N, v)$. If $|S| \geq 2$, every player $j \in S$ is a member of all coalitions in $\mathcal{S}$. We consider the game (N, v^0) as defined by (14). Clearly, all players $i, j \in S$ are weak symmetric in (N, v^0), and thus by proportional-balanced treatment, $\frac{\psi_i(N, v^0)}{v(\{i\})} = \frac{\psi_j(N, v^0)}{v(\{j\})}$ for all $i, j \in S$. Since $v = v^0 + (v - v^0)$ and all players are dummifying in $(N, v - v^0)$, by weak coalitional surplus equivalence we have that $\psi_j(N, v) = \psi_j(N, v^0) + \frac{v(\{j\})}{2}$ for all $j \in S$. Hence, $\sum_{j \in S} \psi_j(N, v) = \sum_{j \in S}(\psi_j(N, v^0) + \frac{v(\{j\})}{2}) = \sum_{j \in S} \frac{v(\{j\})}{v(\{i\})}\psi_i(N, v^0) + \frac{\sum_{j \in S} v(\{j\})}{2}$ for any $i \in S$. On the other hand, by efficiency and Case (a), $\sum_{j \in S} \psi_j(N, v) = v(N) - \sum_{j \in N \backslash S} \psi_j(N, v) = v(N) - \sum_{j \in N \backslash S} PD_i(N, v) = \frac{\sum_{j \in S} v(\{j\})}{\sum_{k \in N} v(\{k\})}v(N)$. Therefore, $\sum_{j \in S} \frac{v(\{j\})}{v(\{i\})}\psi_i(N, v^0) + \frac{\sum_{j \in S} v(\{j\})}{2} = \frac{\sum_{j \in S} v(\{j\})}{\sum_{k \in N} v(\{k\})}v(N)$. Since $\sum_{j \in S} v(\{j\}) \neq 0$, then $\psi_i(N, v^0) = \frac{v(\{i\})}{\sum_{k \in N} v(\{k\})}v(N) - \frac{v(\{i\})}{2}$, and thus $\psi_i(N, v) = PD_i(N, v)$ for all $i \in S$.

The proof of (i) is complete.

(ii) Since it is obvious that the PD value satisfies efficiency and proportional-balanced treatment, we only show that the PD value satisfies weak coalitional surplus monotonicity. Clearly, $w(\{j\}) = cv(\{j\})$ for all $j \in N$ and $v(S) - \sum_{j\in S} v(\{j\}) \geq w(S) - \sum_{j\in S} w(\{j\})$ for all $S \subseteq N$ with $i \in S$, imply that $v(N) \geq w(N) - \sum_{j\in N} w(\{j\}) + \sum_{j\in N} v(\{j\}) = w(N) - \sum_{j\in N} w(\{j\}) + \sum_{j\in N} \frac{w(\{j\})}{c} = w(N) - (1 - \frac{1}{c}) \sum_{j\in N} w(\{j\})$. Thus, $PD_i(N, v) = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N) = \frac{w(\{i\})}{\sum_{j\in N} w(\{j\})} v(N) \geq \frac{w(\{i\})}{\sum_{j\in N} w(\{j\})} [w(N) - (1 - \frac{1}{c}) \sum_{j\in N} w(\{j\})]$ $= PD_i(N, w) - w(\{i\}) + \frac{1}{c} w(\{i\}) = PD_i(N, w) - w(\{i\}) + v(\{i\})$.

Uniqueness follows from Theorem 4(i) and Lemma 1. □

The next lemma shows a logical implication between the axioms in Theorems 3 and 4(i), which implies that weak linearity in Theorem 3 can be weakened as weak additivity.

Lemma 2 *Weak additivity and the dummifying player property together imply weak coalitional surplus equivalence.*

Proof Let $(N, v), (N, w) \in \mathcal{G}^N_{nz}$ be two games such that $v(\{j\}) = w(\{j\})$ for all $j \in N$, and $i \in N$ is dummifying in (N, w). The dummifying player property implies that $\psi_i(N, w) = w(\{i\})$. Then weak additivity implies that $\psi_i(N, v + w) = \psi_i(N, v) + \psi_i(N, w) = \psi_i(N, v) + w(\{i\})$, as desired. □

It is easy to verify that the PD value satisfies weak coalitional monotonicity. Interestingly, the PD value is characterized by replacing weak coalitional surplus monotonicity with weak coalitional monotonicity in Theorem 4(ii). In this case, proportional-balanced treatment even can be weakened by requiring the proportionality only for games in which all players are weak symmetric.

- *Weak proportional-balanced treatment* $\frac{\psi_i(N,v)}{v(\{i\})} = \frac{\psi_j(N,v)}{v(\{j\})}$ for all $(N, v) \in \mathcal{G}^N_{nz}$ and $i, j \in N$ if all players are weak symmetric in (N, v).

Theorem 5 *The PD value is the unique value on $\mathcal{G}^N_{nz}$ that satisfies efficiency, weak proportional-balanced treatment, and weak coalitional monotonicity.*

Proof It is clear that the PD value satisfies efficiency, weak proportional-balanced treatment, and weak coalitional monotonicity. To show uniqueness, let ψ be a value satisfying the three axioms. For any game $(N, v) \in \mathcal{G}^N_{nz}$, define the game (N, w) by

$$w(S) = \begin{cases} v(N), & \text{if } S = N, \\ v(\{j\}), & \text{if } S = \{j\} \text{ for all } j \in N, \\ \min\limits_{T \subseteq N, |T| \geq 2} v(T), & \text{otherwise.} \end{cases}$$

Efficiency and weak proportional-balanced treatment imply that $\psi_i(N, w) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N)$ for all $i \in N$. Pick any $i \in N$. Since $v(S) \geq w(S)$ for all $S \subseteq N$ with $i \in S$, then weak coalitional monotonicity implies that $\psi_i(N, v) \geq \psi_i(N, w) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N)$. Efficiency then implies that $\psi_i(N, v) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N)$ for all $i \in N$. □

Notice that by using the monotonicity axioms in Theorems 4 and 5 , we can get rid of weak linearity.

Considering the relationship between our monotonicity axioms and the stronger versions introduced in Casajus and Huettner (2014) and van den Brink (2007) (to characterize the ESD value or the ED value), from Theorems 4 and 5 , we obtain the following corollary.

Corollary 1 *Let* $|N| \geq 2$. *There is no value on* $\mathcal{G}^N_{nz}$ *satisfying*

(i) *efficiency, proportional-balanced treatment, and coalitional surplus equivalence*.
(ii) *efficiency, proportional-balanced treatment, and coalitional surplus monotonicity*.
(iii) *efficiency, weak proportional-balanced treatment, and coalitional monotonicity*.

As shown before, weak coalitional surplus monotonicity is stronger than both weak coalitional surplus equivalence and weak coalitional monotonicity. We conclude this section by mentioning two values to show logical independence of weak coalitional surplus equivalence and weak coalitional monotonicity. The value $\psi_i(N, v) = v(\{i\}) - \frac{1}{n}[v(N) - \sum_{j \in N} v(\{j\})]$, $i \in N$, satisfies weak coalitional surplus equivalence, but not weak coalitional monotonicity; the ED value $\psi_i(N, v) = \frac{v(N)}{n}$, $i \in N$, satisfies weak coalitional monotonicity, but not weak coalitional surplus equivalence.

5 Consistency

In this section, we consider a variable player set, and characterize the PD value by *proportional standardness* used in Ortmann (2000), Khmelnitskaya and Driessen (2003) and Huettner (2015), and *projection consistency* used in Funaki and Yamato (2001), van den Brink and Funaki (2009), and Calleja and Llerena (2017, 2019).

If a player $j \in N$ leaves game (N, v) with a certain payoff, then the projection reduced game is a game on the remaining player set that assigns to every proper subset of $N \setminus \{j\}$ its worth in the original game, and to coalition $N \setminus \{j\}$ assigns its worth in (N, v) minus the payoff assigned to player j.

Definition 2 Given a game $(N, v) \in \mathcal{G}_{nz}$ with $|N| \geq 2$, a player $j \in N$ and a payoff vector $x \in \mathbb{R}^N$, the *projection reduced game with respect to j and x* is the game $(N\backslash\{j\}, v^x)$ given by

$$v^x(S) = \begin{cases} v(N) - x_j & \text{if } S = N\backslash\{j\}, \\ v(S) & \text{if } S \subset N\backslash\{j\}. \end{cases}$$

Projection consistency requires that the payoffs assigned to the remaining players in $N \setminus \{j\}$, after player j leaving the game with its payoff according to a value, is the same in the reduced game as in the original game.

Definition 3 A value ψ satisfies **projection consistency** if for every game $(N, v) \in \mathcal{G}_{nz}$ with $|N| \geq 3$, $j \in N$, and $x = \psi(N, v)$, it holds that $(N\backslash\{j\}, v^x) \in \mathcal{G}_{nz}$, and $\psi_i(N\backslash\{j\}, v^x) = \psi_i(N, v)$ for all $i \in N\backslash\{j\}$.

Proportional standardness requires that in two-player games we allocate the worth of the grand coalition over the two players proportional to their stand-alone worths. This is equivalent to saying that every player in a two-player game earns its own stand-alone worth, and the remainder of the worth is shared proportionally based on their stand-alone worths.

Definition 4 A value ψ satisfies **proportional standardness** if for every game $(N, v) \in \mathcal{G}_{nz}$ with $|N| = 2$, it holds that

$$\psi_i(N, v) = v(\{i\}) + \frac{v(\{i\})}{v(\{i\}) + v(\{j\})}[v(N) - v(\{i\}) - v(\{j\})] \quad \text{for all } i, j \in N.$$

Proportional standardness is called "proportional for two person games" in Ortmann (2000).

Proposition 2 *The PD value satisfies projection consistency on the class of all games* $\mathcal{G}_{nz}$.

Proof For every $(N,v) \in \mathcal{G}_{nz}$ with $|N| \geq 3$ and any $j \in N$, $(N\backslash\{j\}, v^x) \in \mathcal{G}_{nz}$.[13] For $x = PD(N,v)$ and $i \in N\backslash\{j\}$, we have

$$\begin{aligned}
PD_i(N\backslash\{j\}, v^x) &= \frac{v^x(\{i\})}{\sum_{k\in N\backslash\{j\}} v^x(\{k\})} v^x(N\backslash\{j\}) \\
&= \frac{v(\{i\})}{\sum_{k\in N\backslash\{j\}} v(\{k\})}[v(N) - PD_j(N,v)] \\
&= \frac{v(\{i\})}{\sum_{k\in N\backslash\{j\}} v(\{k\})}[v(N) - \frac{v(\{j\})}{\sum_{k\in N} v(\{k\})} v(N)] \\
&= \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N) \\
&= PD_i(N,v).
\end{aligned}$$

□

Projection consistency together with proportional standardness for two-player games characterizes the PD value on the class of games with at least two players. We denote the class of games in $\mathcal{G}_{nz}$ with at least two players by $\hat{\mathcal{G}}_{nz}$.

Theorem 6 *The PD value is the unique value on $\hat{\mathcal{G}}_{nz}$ that satisfies proportional standardness and projection consistency.*

Proof It is straightforward to show that the PD value satisfies proportional standardness. Projection consistency follows from Proposition 2. To show uniqueness, let ψ be a value on $\hat{\mathcal{G}}_{nz}$ satisfying proportional standardness and projection consistency.

If $|N| = 2$, then $\psi(N,v) = PD(N,v)$ follows from proportional standardness.

Proceeding by induction, for $|N| \geq 3$, suppose that $\psi(N',w) = PD(N',w)$ whenever $|N'| = |N| - 1$. Take any $i,j \in N$ such that $i \neq j$. Let $x = \psi(N,v)$ and $y = PD(N,v)$. For the two reduced games $(N\backslash\{j\}, v^x)$ and $(N\backslash\{j\}, v^y)$, by the induction hypothesis, we have

$$\begin{aligned}
x_i - y_i &= \psi_i(N\backslash\{j\}, v^x) - PD_i(N\backslash\{j\}, v^y) \\
&= PD_i(N\backslash\{j\}, v^x) - PD_i(N\backslash\{j\}, v^y).
\end{aligned} \tag{15}$$

By definition of the PD value and the projection reduced game, we have

[13] Notice that, if $(N,v) \in \mathcal{G}_{nz}$ with $|N| = 2$ and $v(N) = 0$, then for $x = PD(N,v)$, we have that $x_i = x_j = 0$, and thus $(N\backslash\{j\}, v^x) \notin \mathcal{G}_{nz}$ for any $j \in N$. In case $v(N) \neq 0$, for $x = PD(N,v)$, we have $(N\backslash\{j\}, v^x) \in \mathcal{G}_{nz}$, since $[v(\{i\}) > 0 \Rightarrow PD_j(N,v) < v(N) \Rightarrow v^x(\{i\}) > 0]$ (similar if $v(\{i\}) < 0$).

$$
\begin{aligned}
&PD_i(N\backslash\{j\}, v^x) - PD_i(N\backslash\{j\}, v^y)\\
&= \frac{v^x(\{i\})}{\sum_{k\in N\backslash\{j\}} v^x(\{k\})}\left(v(N) - x_j) - \frac{v^y(\{i\})}{\sum_{k\in N\backslash\{j\}} v^y(\{k\})}(v(N) - y_j\right)\\
&= \frac{v(\{i\})}{\sum_{k\in N\backslash\{j\}} v(\{k\})}\big(y_j - x_j\big).
\end{aligned}
$$

Together with (15), this implies that, for all $i, j \in N$ with $i \neq j$,

$$
x_i - y_i = \frac{v(\{i\})}{\sum_{k\in N\backslash\{j\}} v(\{k\})}(y_j - x_j). \tag{16}
$$

Summing (16) over all $i \in N\backslash\{j\}$ yields

$$
\sum_{i\in N\backslash\{j\}} (x_i - y_i) = \frac{\sum_{i\in N\backslash\{j\}} v(\{i\})}{\sum_{k\in N\backslash\{j\}} v(\{k\})}(y_j - x_j) = y_j - x_j. \tag{17}
$$

On the other hand, (16) can be written as $v(\{i\})(y_j - x_j) = \sum_{k\in N\backslash\{j\}} v(\{k\})(x_i - y_i)$. Summing this equality over all $j \in N\backslash\{i\}$, we have

$$
\begin{aligned}
&\sum_{j\in N\backslash\{i\}} \sum_{k\in N\backslash\{j\}} v(\{k\})(x_i - y_i) = \sum_{j\in N\backslash\{i\}} v(\{i\})(y_j - x_j)\\
\Leftrightarrow &\sum_{j\in N\backslash\{i\}} \Big(v(\{i\})(x_i - y_i) + \sum_{k\in N\backslash\{i,j\}} v(\{k\})(x_i - y_i)\Big) = v(\{i\}) \sum_{j\in N\backslash\{i\}} (y_j - x_j)\\
\Leftrightarrow &[(n-1)v(\{i\}) + (n-2)\sum_{j\in N\backslash\{i\}} v(\{j\})](x_i - y_i) = v(\{i\}) \sum_{j\in N\backslash\{i\}} (y_j - x_j).
\end{aligned} \tag{18}
$$

Together with (17) and (18), it holds that $(n-2)(x_i - y_i)\sum_{j\in N} v(\{j\}) = 0$. Thus, $x_i - y_i = 0$ for all $i \in N$. This shows that $\psi(N, v) = PD(N, v)$. □

Replacing proportional standardness by standardness in Theorem 6 yields a characterization of the equal surplus division value, as a special case of Theorem 4.4 in van den Brink et al. (2016).

Proposition 1 and Theorem 6 together imply the following corollary.

Corollary 2 *The PD value is the unique value on $\hat{\mathcal{G}}_{nz}$ that satisfies efficiency, proportional-balanced treatment, and projection consistency.*

Due to efficiency, this corollary also holds on $\mathcal{G}_{nz}$.

6 Proportional standardness: characterization for two-player games

In the previous section we imposed proportional standardness to characterize the PD value for any player set. Note that proportional standardness, as other two-player standardness axioms, is a quite strong axiom since it coincides with not only the definition for the PD value, but also those of all proportional values of TU-games mentioned in Footnote 2. In this section, we support proportional standardness by showing how the PD value can be characterized on the class of two-player games. We first characterize the PD value for rational numbers, and then apply continuity to obtain a characterization for real worths. Denote $\mathcal{G}^2_{nz} = \{(N,v) \in \mathcal{G}_{nz} \mid |N| = 2\}$ and $\mathcal{G}^2_{nz\mathbb{Q}} = \{(N,v) \in \mathcal{G}^2_{nz} \mid v(S) \in \mathbb{Q} \text{ for all } S \subseteq N\}$, so the worths of coalitions for games in $\mathcal{G}^2_{nz\mathbb{Q}}$ are rational numbers.

First, we introduce two additional axioms, the first on $\mathcal{G}^2_{nz\mathbb{Q}}$ and the second on $\mathcal{G}^2_{nz}$.

- *Grand worth additivity* For games $(N,v),(N,w) \in \mathcal{G}^2_{nz\mathbb{Q}}$ with $N = \{i,j\}$ such that $v(\{i\}) = w(\{i\})$ and $v(\{j\}) = w(\{j\})$, it holds that $\psi(N,v) + \psi(N,w) = \psi(N, v \oplus w)$, where $(N, v \oplus w)$ is defined as: $(v \oplus w)(\{i\}) = v(\{i\})$, $(v \oplus w)(\{j\}) = v(\{j\})$ and $(v \oplus w)(N) = v(N) + w(N)$.
- *Inessential game property* For every game $(N,v) \in \mathcal{G}^2_{nz}$ with $N = \{i,j\}$ such that $v(\{i\}) + v(\{j\}) = v(\{i,j\})$, it holds that $\psi_i(N,v) = v(\{i\})$ and $\psi_j(N,v) = v(\{j\})$.

Grand worth additivity[14] states that for two games in which all worths are rational numbers and the stand-alone worths are the same, we consider the game where the stand-alone worths are the same as in the original game, and the worth of the grand coalition equals the sum of the worth of the grand coalition in the two games, then the payoff to each player equals the sum of the payoffs in the two separate games. The inessential game property is a well-known axiom requiring that players earn their stand-alone payoff in an inessential game. First, we show that these two axioms characterize the PD value on the class of two-player games with rational worths.

Proposition 3 *The PD value is the unique value on $\mathcal{G}^2_{nz\mathbb{Q}}$ that satisfies grand worth additivity and the inessential game property.*

Proof It is obvious that the PD value satisfies grand worth additivity and the inessential game property. To show uniqueness, let ψ be a value on $\mathcal{G}^2_{nz\mathbb{Q}}$ satisfying the two axioms. Let $(N,v) \in \mathcal{G}^2_{nz\mathbb{Q}}$ be an arbitrary game with $N = \{i,j\}$. For any $\alpha \in \mathbb{Q}$, let (N, v^α) be the game defined by $v^\alpha(\{i\}) = v(\{i\})$, $v^\alpha(\{j\}) = v(\{j\})$ and $v^\alpha(N) = \alpha v(N)$. Clearly, $(N, v^\alpha) \in \mathcal{G}^2_{nz\mathbb{Q}}$.

If $\alpha = 0$, then grand worth additivity implies that $\psi(N, v^\alpha) = 0$. For any $\alpha \in \mathbb{Z}_+$[15], since $(N, v^\alpha) = (N, v^{\alpha-1} \oplus v) = \cdots = (N, \underbrace{v \oplus \cdots \oplus v}_{\alpha})$, grand worth additivity implies $\psi(N, v^\alpha) = \alpha\psi(N,v)$. For any $\alpha \in \mathbb{Z}_-$, since $(N, v^\alpha \oplus \underbrace{v \oplus \cdots \oplus v}_{|\alpha|}) = (N, v^0)$, grand

[14] This axiom is similar to additivity in Moulin (1987) and Chun (1988) for bankruptcy problems.

[15] $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{Z}_-$ denote the sets of integers, positive integers and negative integers, respectively.

worth additivity and $\psi(N, v^0) = 0$ imply $\psi(N, v^\alpha) = -|\alpha|\psi(N, v) + \psi(N, v^0) = \alpha\psi(N, v)$. Similarly, considering (N, v), for any $\alpha \in \mathbb{Z}_+$, $(N, v) = (N, \underbrace{v^{\frac{1}{\alpha}} \oplus \cdots \oplus v^{\frac{1}{\alpha}}}_{\alpha})$ implies that $\psi(N, v) = \alpha\psi(N, v^{\frac{1}{\alpha}})$; for any $\alpha \in \mathbb{Z}_-$, $(N, v \oplus \underbrace{v^{\frac{1}{\alpha}} \oplus \cdots \oplus v^{\frac{1}{\alpha}}}_{|\alpha|}) = (N, v^0)$ implies that $\psi(N, v) = \alpha\psi(N, v^{\frac{1}{\alpha}})$.

Next, take any $\alpha \in \mathbb{Q}$ and consider the game (N, v^α). Since any rational number can be written as a fraction, we suppose that $\alpha = \frac{k}{m}$ with $k \in \mathbb{Z}$ and $m \in \mathbb{Z}\backslash\{0\}$. Therefore, $\psi(N, v^\alpha) = \psi(N, v^{\frac{k}{m}}) = k\psi(N, v^{\frac{1}{m}}) = \frac{k}{m}\psi(N, v) = \alpha\psi(N, v)$.

Take any game $(N, v) \in \mathcal{G}^2_{nz\mathbb{Q}}$. Taking $\alpha = \frac{v(\{i\})+v(\{j\})}{v(N)}$, v^α is an inessential game, and thus by the inessential game property, $\psi_i(N, v^\alpha) = v^\alpha(\{i\}) = v(\{i\})$. Since $\psi(N, v^\alpha) = \alpha\psi(N, v)$, we have $\psi_i(N, v) = \frac{1}{\alpha}\psi_i(N, v^\alpha) = \frac{v(N)v(\{i\})}{v(\{i\})+v(\{j\})} = PD_i(N, v)$. □

Next, adding continuity, which states that if two games are almost the same then their payoffs are almost the same, we can extend this result from rational numbers to real numbers.

- *Continuity* For all sequences of games $\{(N, w_k)\}$ and game (N, v) in $\mathcal{G}^2_{nz}$ such that $(N, w_k) \to (N, v)$, $\lim\limits_{(N,w_k)\to(N,v)} \psi(N, w_k) = \psi(N, v)$.

Theorem 7 *The PD value is the unique value on $\mathcal{G}^2_{nz}$ that satisfies grand worth additivity, the inessential game property, and continuity.*

Proof It is clear that the PD value satisfies the three axioms. To show uniqueness, let ψ be a value on $\mathcal{G}^2_{nz}$ satisfying the three axioms. From Proposition 3, we already know $\psi(N, v) = PD(N, v)$ for all $(N, v) \in \mathcal{G}^2_{nz\mathbb{Q}}$. Now, take any game $(N, v) \in \mathcal{G}^2_{nz}$, and let $\{(N, v_m)\}$ be a sequence of games in the class $\mathcal{G}^2_{nz\mathbb{Q}}$ such that $\lim\limits_{m\to\infty}(N, v_m) = (N, v)$. Using continuity we have $\psi(N, v) = \lim\limits_{(N,v_m)\to(N,v)} \psi(N, v_m) = \lim\limits_{(N,v_m)\to(N,v)} PD(N, v_m) = PD(N, v)$, where the last equality holds since $PD(N, v)$ is a continuous function with respect to $(N, v) \in \mathcal{G}^2_{nz}$. □

By Theorems 6 and 7 , we immediately obtain the following corollary.

Corollary 3 *The PD value is the unique value on $\hat{\mathcal{G}}_{nz}$ that satisfies grand worth additivity, the inessential game property, continuity, and projection consistency.*

This corollary is valid on $\mathcal{G}_{nz}$ if we require the inessential game property for all games in $\mathcal{G}_{nz}$, i.e., for every game $(N, v) \in \mathcal{G}_{nz}$ such that $v(S) = \sum_{i\in S} v(\{i\})$, it holds that $\psi_i(N, v) = v(\{i\})$ for all $i \in N$.

Remark 5 Ortmann (2000) introduced his proportional value that can be characterized by proportional standardness and consistency due to Hart and Mas-Colell (1989). As a consequence, a characterization of this proportional value can be obtained by replacing proportional standardness by the axioms in Theorem 7. Notice that we cannot use the axiomatization as given by Proposition 1, since proportional-balanced treatment is not satisfied by this proportional value for games with more than two players.

7 Conclusion

In this paper, we have provided characterizations of the PD value for TU-games using axioms, such as *proportional-balanced treatment*, *monotonicity*, and *consistency*. It is worth noticing that proportional-balanced treatment, one of our main axioms, in some sense reflects not only *equal treatment of equals* but also *unequal treatment of unequals*. This axiom captures this feature of the PD value. For games with at least three players, our axiomatic characterizations are similar to the characterizations of the equal division value due to van den Brink (2007) and van den Brink and Funaki (2009), and the characterizations of the equal surplus division value due to Casajus and Huettner (2014). That is, most of them are obtained by weakening one axiom while strengthening the other axiom. This shows that the proportional division value is axiomatically closely related to these two equal surplus sharing values.

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Appendix

This 'appendix' contains the lengthy proof of Theorem 1 (Appendix 1) and the logical independence of the axioms used in characterization results (Appendix 2).

Appendix 1: Proof of Theorem 1

Existence It is straightforward to show that any value defined by (2) satisfies efficiency and weak linearity. Next, we show that it also satisfies proportional-balanced treatment. Let $i, k \in N$ be two players such that $v(S \cup \{i\}) = v(S \cup \{k\})$ for all $S \subseteq N \backslash \{i, k\}$, $S \neq \emptyset$. We have

$$
\begin{aligned}
&\frac{\psi_i(N,v)}{v(\{i\})}-\frac{\psi_k(N,v)}{v(\{k\})}\\
&=\sum_{\substack{S\,:\,i\in S,k\notin S\\|S|\geq 2}}\frac{\lambda_S}{\sum_{j\in S}v(\{j\})}v(S)-\sum_{\substack{S\,:\,i\notin S,k\in S\\|S|\geq 2}}\frac{\lambda_S}{\sum_{j\in N\setminus S}v(\{j\})}v(S)\\
&\quad-\left[\sum_{\substack{S\,:\,k\in S,i\notin S\\|S|\geq 2}}\frac{\lambda_S}{\sum_{j\in S}v(\{j\})}v(S)-\sum_{\substack{S\,:\,k\notin S,i\in S\\|S|\geq 2}}\frac{\lambda_S}{\sum_{j\in N\setminus S}v(\{j\})}v(S)\right]\\
&=\sum_{\substack{S\,:\,i,k\notin S\\|S|\geq 1}}\frac{\lambda_{S\cup\{i\}}}{\sum_{j\in S}v(\{j\})+v(\{i\})}v(S\cup\{i\})-\sum_{\substack{S\,:\,i,k\notin S\\|S|\geq 1}}\frac{\lambda_{S\cup\{k\}}}{\sum_{j\in N\setminus S}v(\{j\})-v(\{k\})}v(S\cup\{k\})\\
&\quad-\left[\sum_{\substack{S\,:\,i,k\notin S\\|S|\geq 1}}\frac{\lambda_{S\cup\{k\}}}{\sum_{j\in S}v(\{j\})+v(\{k\})}v(S\cup\{k\})-\sum_{\substack{S\,:\,i,k\notin S\\|S|\geq 1}}\frac{\lambda_{S\cup\{i\})}}{\sum_{j\in N\setminus S}v(\{j\})-v(\{i\})}v(S\cup\{i\}\right]\\
&=\sum_{\substack{S\,:\,i,k\notin S\\|S|\geq 1}}\frac{\lambda_{S\cup\{i\}}}{\sum_{j\in S}v(\{j\})+v(\{i\})}v(S\cup\{i\})+\sum_{\substack{S\,:\,i,k\notin S\\|S|\geq 1}}\frac{\lambda_{S\cup\{i\}}}{\sum_{j\in N\setminus S}v(\{j\})-v(\{i\})}v(S\cup\{i\})\\
&\quad-\left[\sum_{\substack{S\,:\,i,k\notin S\\|S|\geq 1}}\frac{\lambda_{S\cup\{k\}}}{\sum_{j\in N\setminus S}v(\{j\})-v(\{k\})}v(S\cup\{k\})+\sum_{\substack{S\,:\,i,k\notin S\\|S|\geq 1}}\frac{\lambda_{S\cup\{k\}}}{\sum_{j\in S}v(\{j\})+v(\{k\})}v(S\cup\{k\})\right]\\
&=\sum_{\substack{S\,:\,i,k\notin S\\|S|\geq 1}}\frac{\sum_{j\in N}v(\{j\})\lambda_{S\cup\{i\}}v(S\cup\{i\})}{\left[\sum_{j\in S}v(\{j\})+v(\{i\})\right]\left[\sum_{j\in N\setminus S}v(\{j\})-v(\{i\})\right]}\\
&\quad-\sum_{\substack{S\,:\,i,k\notin S\\|S|\geq 1}}\frac{\sum_{j\in N}v(\{j\})\lambda_{S\cup\{k\}}v(S\cup\{k\})}{\left[\sum_{j\in S}v(\{j\})+v(\{k\})\right]\left[\sum_{j\in N\setminus S}v(\{j\})-v(\{k\})\right]}\\
&=\sum_{\substack{S\,:\,i,k\notin S\\|S|\geq 1}}\frac{\sum_{j\in N}v(\{j\})\lambda_{S\cup\{k\}}v(S\cup\{i\})}{\left[\sum_{j\in S}v(\{j\})+v(\{k\})\right]\left[\sum_{j\in N\setminus S}v(\{j\})-v(\{k\})\right]}\\
&-\sum_{\substack{S\,:\,i,k\notin S\\|S|\geq 1}}\frac{\sum_{j\in N}v(\{j\})\lambda_{S\cup\{k\}}v(S\cup\{k\})}{\left[\sum_{j\in S}v(\{j\})+v(\{k\})\right]\left[\sum_{j\in N\setminus S}v(\{j\})-v(\{k\})\right]}\\
&=0,
\end{aligned}
$$

where the fifth equality follows by (3). Thus, (2) satisfies proportional-balanced treatment.

Uniqueness Let ψ be a value satisfying efficiency, weak linearity, and proportional-balanced treatment. For $|N| = 1$ and $|N| = 2$, uniqueness follows from efficiency and Proposition 1 respectively. Now let $(N, v) \in \mathcal{G}^N_{nz}$ be an arbitrary game with $|N| \geq 3$. In order to use the property of weak linearity, we decompose (N, v) into the unique combination of the following two kinds of games (N, w^S) and (N, w)[16]. For any coalition $S \subseteq N$ with $|S| \geq 2$, the game (N, w^S) is defined as follows:

$$w^S(T) = \begin{cases} v(\{i\}), & \text{if } T = \{i\} \text{ for all } i \in N, \\ 1, & \text{if } T = S, \\ 0, & \text{otherwise.} \end{cases}$$

The game (N, w) is defined as follows:

$$w(T) = \begin{cases} v(\{i\}), & \text{if } T = \{i\} \text{ for all } i \in N, \\ 0, & \text{otherwise.} \end{cases}$$

One can easily check that (N, v) can be written as $v = I(v)w + \sum\limits_{S \subseteq N, |S| \geq 2} v(S)w^S$, where $I(v) = 1 - \sum_{S \subseteq N, |S| \geq 2} v(S)$. By using weak linearity[17] of ψ, we have

$$\psi_i(N, v) = I(v)\psi_i(N, w) + \sum_{S \subseteq N, |S| \geq 2} v(S)\psi_i(N, w^S), \quad \text{for all } i \in N.$$

Now, by proportional-balanced treatment, for each $S \subset N$ with $|S| \geq 2$, since all players in S are weak symmetric in (N, w^S), and the same for all players in $N \setminus S$, there must exist some λ_S and μ_S such that

$$\psi_i(N, w^S) = \begin{cases} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \lambda_S, & \text{if } i \in S, \\ \frac{v(\{i\})}{\sum_{j \in N \setminus S} v(\{j\})} \mu_S, & \text{if } i \notin S. \end{cases} \tag{19}$$

By efficiency, it must be $\sum_{i \in S} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \lambda_S + \sum_{i \in N \setminus S} \frac{v(\{i\})}{\sum_{j \in N \setminus S} v(\{j\})} \mu_S = 0$, which shows $\lambda_S = -\mu_S$. Similarly, $\psi_i(N, w^N) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})}$ for all $i \in N$. Meanwhile, we have $\psi(N, w) = 0$. Putting all together we have the expression of ψ as given by (2).

We prove that λ_S only depends on the size of S ($S \neq N$) and the worths of all singleton coalitions $\{i\}$, $i \in N$. Let $S \subset N$, $S \neq \emptyset$ with $i, j \notin S$, and consider the game $(N, w^{S \cup \{i\}} + w^{S \cup \{j\}})$. In this game, since i and j are weak symmetric, by

[16] For any $(N, v) \in \mathcal{G}^N_{nz}$, the collection of games $\{(N, w), (N, w^S)_{S \subseteq N, |S| \geq 2}\}$ is a basis of the class of games $\mathcal{G}^N_v = \{(N, v') \in \mathcal{G}^N_{nz} \mid \exists c \in \mathbb{R} \text{ such that } v'(\{i\}) = cv(\{i\}) \text{ for all } i \in N\} \cup \{(N, v) \in \mathcal{G}^N \mid v(\{i\}) = 0 \text{ for all } i \in N\}$. The dimension of $\mathcal{G}^N_v$ is $2^n - n$. Another interesting basis can be found in the proof of Proposition 5 in Béal et al. (2018) or in van den Brink et al. (2020).

[17] To ensure that we stay in the class $\mathcal{G}^N_{nz}$, we should consider the games in which their coefficients are nonzero in a suitable ordering, just like the technical approach as given by Lemma 5 in Béal et al. (2018).

proportional-balanced treatment, it must be that $\frac{1}{v(\{i\})}\psi_i(N, w^{S\cup\{i\}} + w^{S\cup\{j\}}) = \frac{1}{v(\{j\})}\psi_j(N, w^{S\cup\{i\}} + w^{S\cup\{j\}})$. With (19) and weak linearity, we have that

$$\begin{aligned}&\frac{1}{\sum_{k\in S\cup\{i\}} v(\{k\})}\lambda_{S\cup\{i\}} - \frac{1}{\sum_{k\in N\setminus(S\cup\{j\})} v(\{k\})}\lambda_{S\cup\{j\}}\\ &= \frac{1}{\sum_{k\in S\cup\{j\}} v(\{k\})}\lambda_{S\cup\{j\}} - \frac{1}{\sum_{k\in N\setminus(S\cup\{i\})} v(\{k\})}\lambda_{S\cup\{i\}},\end{aligned}$$

from which it immediately follows that

$$\frac{\lambda_{S\cup\{i\}}}{\sum_{k\in S\cup\{i\}} v(\{k\})\sum_{k\in N\setminus(S\cup\{i\})} v(\{k\})} = \frac{\lambda_{S\cup\{j\}}}{\sum_{k\in S\cup\{j\}} v(\{k\})\sum_{k\in N\setminus(S\cup\{j\})} v(\{k\})}.$$

Therefore, whenever S and T are of the same size, replacing player by player, we can form a sequence with at most $s+1$ coalitions, such that the first one is S, and any of them is the result of replacing a player of S by a player of $N\setminus S$. In this way, we conclude the relationship between λ_S and λ_T given by (3). □

Appendix 2: Logical independence of the axioms

Logical independence of the axioms in Theorem 3 can be shown by the following alternative values:

(i) The value ψ on $\mathcal{G}^N_{nz}$ defined for all $(N, v) \in \mathcal{G}^N_{nz}$ and all $i \in N$, by

$$\psi_i(N, v) = v(\{i\}) \tag{20}$$

satisfies all axioms, but not efficiency.

(ii) Let D_v be the set of all dummy players and all dummifying players in (N, v). The value ψ on $\mathcal{G}^N_{nz}$ defined for all $(N, v) \in \mathcal{G}^N_{nz}$ and all $i \in N$, by

$$\psi_i(N, v) = \begin{cases} v(\{i\}), & \text{if } i \in D_v,\\ \frac{v(\{i\})}{\sum_{j\in N\setminus D_v} v(\{j\})}\left[v(N) - \sum_{j\in D_v} v(\{j\})\right], & \text{otherwise.}\end{cases}$$

satisfies all axioms, but not weak linearity.

(iii) The ESD value (see Footnote 4) satisfies all axioms, but not proportional-balanced treatment.

(iv) The value defined for all $(N, v) \in \mathcal{G}^N_{nz}$ and all $i \in N$, by

$$\psi_i(N, v) = \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}\left[v(N) - v(N\setminus\{i\}) + \sum_{h\in N}\frac{v(\{h\})v(N\setminus\{h\})}{\sum_{k\in N} v(\{k\})}\right] \tag{21}$$

satisfies all axioms, but not the dummifying player property. Clearly, (21) coincides with (2) by taking $\lambda_S = \frac{\sum_{j\in S} v(\{j\})\sum_{j\in N\setminus S} v(\{j\})}{(\sum_{k\in N} v(\{k\}))^2}$ for all $S \subset N$ with $|S| = n-1$, and $\lambda_S = 0$ otherwise. Notice that (21) also coincides with (6) by taking the EANC value (Moulin 1985) given by $\psi'_i(N,v) = EANC_i(N,v) = SC_i + \frac{1}{n}[v(N) - \sum_{j\in N} SC_j]$, where $SC_i = v(N) - v(N\backslash\{i\})$.

Logical independence of the axioms in Theorem 4 can be shown by the following alternative values:

(i) The value defined by (20) satisfies all axioms, but not efficiency.
(ii) The ESD value on $\mathcal{G}^N_{nz}$ satisfies all axioms, but not proportional-balanced treatment.
(iii) The value defined by (21) satisfies all axioms, but neither weak coalitional surplus equivalence nor weak coalitional surplus monotonicity.

Logical independence of the axioms in Theorem 5 can be shown by the following alternative values:

(i) The value defined by (20) satisfies all axioms, but not efficiency.
(ii) The ED value on $\mathcal{G}^N_{nz}$ satisfies all axioms, but not weak proportional-balanced treatment.
(iii) The value defined by (21) satisfies all axioms, but not weak coalitional monotonicity.

Logical independence of the axioms in Theorem 7 can be shown by the following alternative values:

(i) The value defined by (20) satisfies all axioms, but not grand worth additivity.
(ii) The ED value on $\mathcal{G}^2_{nz}$ satisfies all axioms, but not the inessential game property.
(iii) The value ψ defined for all $(N,v) \in \mathcal{G}^N_{nz}$ and all $i \in N$, by

$$\psi_i(N,v) = \begin{cases} PD_i(N,v), & \text{if } v(N) \in \mathbb{Q}, \\ ESD_i(N,v), & \text{if } v(N) \notin \mathbb{Q}. \end{cases}$$

satisfies all axioms, but not continuity.

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