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ORIGINAL PAPER

Collective decision under ignorance

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Abstract
This paper studies aggregation of preferences under ignorance, in which everybody knows that the true probability distribution over outcomes lies in some objective set but knows nothing about which one in it is true or which one in it is more likely to be true. We consider two decision models which express the precautionary principle under ignorance, the maximin criterion and the α-maximin criterion. We show that the Pareto axiom implies dictatorship, in each decision model. The impossibility results force us to choose between two options, one is to give up the precautionary principle as modelled at the social level, the other is to weaken the Pareto axiom. We provide possibility results for each of the options.

1 Introduction

This paper studies aggregation of preferences under ignorance, in the sense that everybody knows that the true probability distribution lies in some objective set but knows nothing about which one in it is true or which one in it is more likely to be true.

When a policy or collective choice is being considered, the society and its members try to calculate the resulting probability distribution over outcomes. Although certain scientific knowledge and institutional restriction (such as law and contract) allow us to narrow down the range of possible probability distributions, due to lack of complete knowledge, typically we can conclude only that the true probability distribution belongs to some set. Throughout the paper, we call such set a probability possibility set.[1] The so-called precautionary principle states that in such situation the society should focus on the worst-case scenario and should try to respond to it.

[1] An even more radical view about ignorance would say that people are unsure even about such set. This paper restricts attention to the case in which there are objective bounds that determine sets of probability distributions, due to physical natures or due to institutional natures such as law and contracts, as assumed in the objective ambiguity approach (Olszewski 2007; Ahn 2008).

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It is not obvious, however, what the "worst-case scenario" is at a social level, even when the individuals agree on what the probability possibility set is. Indeed, the individuals may disagree on what the worst-case scenario is. For example, under the current situation of COVID-19, although it is natural that the worst-case scenario for many people is further spread of the virus, it is also natural that the worst-case for other many people is damage to their economic activity and usual life and loss of jobs due to lockdown.

We consider the problem of aggregating preferences over probability possibility sets. We adopt two decision models as expression of the precautionary principle, one is the maximin criterion and the other is the α-maximin criterion, where the latter is a generalization of the former as it allows intermediate degrees of precaution. Thus we consider that individual preferences following the (α-)maximin criterion are aggregated into a social ranking which satisfies the (α-)maximin criterion.

We point out that requiring the social ranking to satisfy the (α-)maximin criterion has a conflict with the Pareto axiom, an equally appealing requirement, which states that if everybody ranks one probability possibility set over another so should the society. We show that Pareto implies dictatorship in each case of maximin and α-maximin.

The impossibility result forces us to choose between two options. One is to give up the (α-)maximin criterion at the social level, the other is to weaken the Pareto axiom. We provide possibility characterization results for each of the options.

The Pareto axiom has been criticized when subjective uncertainty is relevant, since double disagreements in beliefs and tastes may lead to spurious unanimity (Mongin 2016; Gilboa et al. 2004).

There is no such disagreements in beliefs in our setting, as everybody faces the same probability-possibility set. However, there may be disagreements in *supporting beliefs* and that makes the Pareto axiom questionable, or at least awkward. Note that this type of disagreement happens also when individuals have a common multiple-prior belief over a state space (Gajdos et al. 2008).

To illustrate, consider that there are two individuals, A and B. They face a choice between two policies, one yields a possible set of outcomes $\{x, y\}$, where we do not know anything about which is one is true or which one is more likely to be true, the other yields a singleton set $\{z\}$, where we are sure that the outcome is z. Suppose A prefers x over z and z over y, B prefers y over z and z over x. When we apply the precautionary principle to each individual's preference, both A and B would prefer $\{z\}$ over $\{x, y\}$, because for A the worst-case scenario from $\{x, y\}$ is y and for B the worst-case scenario from $\{x, y\}$ is x. But then, the society is endorsing two *mutually exclusive* scenarios when it ranks $\{z\}$ over $\{x, y\}$.

The paper proceeds as follows. Section 2 describes the setting and present the main axiom, Pareto. Section 3 presents results in the problem of aggregating maximin preferences. Section 4 presents results in the problem of aggregating α -maximin preferences. Section 5 concludes by explaining the connection to the existing literature.

2 Setting and the main axiom

Let X be the set of pure outcomes, which is assumed to be finite for simplicity. Let $\Delta(X)$ be the set of lotteries (probability distributions) over X, which is a compact and convex subset of a finite-dimensional Euclidian space. Let $\mathcal{K}$ be the set of compact subsets of $\Delta(X)$, which is a compact set with regard to the Hausdorff metric and a mixture set according to the operation

$$rA + (1-r)B = \left\{ rl + (1-r)l' \,:\, l \in A,\ l' \in B \right\}$$

for $A, B \in \mathcal{K}$ and $r \in [0,1]$.

When $A \in \mathcal{K}$ is given the society and its members know that the true probability distribution lies in A but know nothing about which one in it is true or which one in it is more likely to be true. In this sense we call it a *probability possibility set*.

Let I be a finite set of individuals. Each individual $i \in I$ has preference $\succsim_i$ over $\mathcal{K}$. The social ranking over $\mathcal{K}$ is denoted by $\succsim_0$. Thus the profile of social ranking and individual preferences is denoted by $(\succsim_i)_{\{0\}\cup I}$.

Our primary axiom is Pareto, which says that if everybody ranks one probability possibility set over another so should the society.

Pareto: For all $A, B \in \mathcal{K}$, if $A \succ_i B$ for all $i \in I$ then $A \succ_0 B$.

3 Aggregation of maximin preferences

First we consider that both individual preferences and the social ranking follow the maximin criterion, which is the simplest form of the precautionary principle.

Definition 1 A profile of social ranking and individual preferences $(\succsim_i)_{\{0\}\cup I}$ is said to follow the maximin criterion if there exists a list of von-Neumann and Morgenstern utility functions $(u_i)_{\{0\}\cup I}$ such that for every $i \in \{0\} \cup I$ the function $u_i : \Delta(X) \to \mathbb{R}$ is mixture-linear and $\succsim_i$ is represented in the form

$$U_i(A) = \min_{l \in A} u_i(l)$$

where $A \in \mathcal{K}$.

In contrary to the setting of subjective uncertainty (Milnor 1951; Gilboa and Dravid 1989), an axiomatization of the maximin criterion on the domain of probability possibility sets has not been published to our knowledge, perhaps because it is too straightforward. It is easy to verify that the maximin criterion is characterized by the following axioms, in any case.

Weak Order: $\succsim$ is complete and transitive.
Continuity: $\succsim$ is continuous with respect to the Hausdorff metric.

Independence: For all $A, B, C \in \mathcal{K}$ and $r \in (0, 1)$ it holds

$$A \gtrsim B \iff (1-r)A + rC \gtrsim (1-r)B + rC.$$

Minimum: For all $A, B \in \mathcal{K}$, it holds

$$A \gtrsim B \implies A \cup B \sim B.$$

We consider that the profile of individual preferences is rich, which is generically true in the space of expected utility preferences over lotteries.

Richness: (i) There exist $l, l' \in \Delta(X)$ such that $\{l\} \succ_i \{l'\}$ for all $i \in I$. (ii) For all $i \in I$, there exist $l, l' \in \Delta(X)$ such that $\{l\} \prec_i \{l'\}$ and $\{l\} \succ_j \{l'\}$ for all $j \in I \setminus \{i\}$.

Here is our first result.

Theorem 1 *Let $(\gtrsim_i)_{\{0\}\cup I}$ be a profile of social ranking and individual preferences which follow the maximin criterion and satisfy Richness.*

Then, it satisfies Pareto if and only if there is $i \in I$ such that

$$A \gtrsim_0 B \iff A \gtrsim_i B$$

for all $A, B \in \mathcal{K}$.

The result could follow from a result by Gajdos et al. (2008), which is stated in the setting of subjective ambiguity over a state space, by applying it to the case of common multiple priors and translating to the current domain by looking at sets of probability distributions over outcomes induced by the common prior set and Savage/Anscombe-Aumann acts. We provide a direct and specific proof instead, to make the argument more direct and transparent along with the current domain.

Proof Fix a profile of representations $(U_i, u_i)_{i\in\{0\}\cup I}$.

Note that $\mathcal{K}$ is a mixture set. From Richness-(i) there exist $l, l' \in \Delta(X)$ such that $\{l'\} \succ_i \{l\}$ for all $i \in I$. Hence we can take an analogue of the proof of Harsanyi's theorem (Harsanyi 1955; De Meyer and Philippe 1995), so that there exist $\lambda \in \mathbb{R}^I_+ \setminus \{\mathbf{0}\}$ and $\gamma \in \mathbb{R}$ such that

$$U_0(A) = \sum_{i\in I} \lambda_i U_i(A) + \gamma$$

for all $A \in \mathcal{K}$.

By restricting attention to singleton sets, we obtain

$$u_0(l) = \sum_{i\in I} \lambda_i u_i(l) + \gamma$$

for all $l \in \Delta(X)$.

Thus it holds

$$\sum_{i\in I}\lambda_i \min_{l\in A} u_i(l) = \min_{l\in A}\sum_{i\in I}\lambda_i u_i(l)$$

for all $l \in \Delta(X)$.

Pick any $i \in I$. From Richness (ii) there exist $l, l' \in \Delta(X)$ such that $u_i(l) < u_i(l')$ and $u_j(l) > u_j(l')$ for all $j \in I \setminus \{i\}$.

By applying $A = \{l, l'\}$ we obtain

$$\lambda_i u_i(l) + \sum_{j\in I\setminus\{i\}} \lambda_j u_j(l') = \min\left\{ \lambda_i u_i(l) + \sum_{j\in I\setminus\{i\}} \lambda_j u_j(l), \lambda_i u_i(l') + \sum_{j\in I\setminus\{i\}} \lambda_j u_j(l') \right\}.$$

There are two possible cases now.

Case 1: If $\lambda_i u_i(l) + \sum_{j\in I\setminus\{i\}} \lambda_j u_j(l) \geqq \lambda_i u_i(l') + \sum_{j\in I\setminus\{i\}} \lambda_j u_j(l')$, we obtain

$$\lambda_i u_i(l) + \sum_{j\in I\setminus\{i\}} \lambda_j u_j(l') = \lambda_i u_i(l') + \sum_{j\in I\setminus\{i\}} \lambda_j u_j(l'),$$

which leads to

$$\lambda_i(u_i(l) - u_i(l')) = 0,$$

which implies $\lambda_i = 0$.

Case 2: If $\lambda_i u_i(l) + \sum_{j\in I\setminus\{i\}} \lambda_j u_j(l) \leqq \lambda_i u_i(l') + \sum_{j\in I\setminus\{i\}} \lambda_j u_j(l')$, we obtain

$$\lambda_i u_i(l) + \sum_{j\in I\setminus\{i\}} \lambda_j u_j(l') = \lambda_i u_i(l) + \sum_{j\in I\setminus\{i\}} \lambda_j u_j(l),$$

which leads to

$$\sum_{j\in I\setminus\{i\}} \lambda_j(u_j(l') - u_j(l)) = 0,$$

which implies $\lambda_j = 0$ for all $j \in I \setminus \{i\}$.

Thus we deduce that λ_i can be positive for only one $i \in I$, which concludes the proof of the theorem. □

The above impossibility result forces us to choose between two options. One is to give up imposing the maximin criterion at the social level, the other is to weaken the Pareto axiom.

3.1 Giving up the maximin criterion at the social level

In the first option, for the social ranking we drop one of the four axioms for the maximin criterion, Minimum. It still allows that the social ranking $\succsim_0$ has a mixture-linear representation $U_0 : \mathcal{K} \to \mathbb{R}$, and that the same argument for the proof of Harsanyi-type aggregation goes through.

Thus we have the following proposition.

Proposition 1 *Let $(\succsim_i)_{\{0\}\cup I}$ be a profile of social ranking and individual preferences, where the social ranking satisfies Order, Continuity and Independence, and the individual preferences follow the maximin criterion and Richness. Fix a representation of $\succsim_0$ denoted by $U_0 : \mathcal{K} \to \mathbb{R}$, and a profile of representations of individual preferences $(U_i, u_i)_{i\in I}$.*

Then $(\succsim_i)_{\{0\}\cup I}$ satisfies Pareto if and only if there exist $\lambda \in \mathbb{R}^I_+ \setminus \{\mathbf{0}\}$ and $\gamma \in \mathbb{R}$ such that

$$U_0(A) = \sum_{i\in I} \lambda_i U_i(A) + \gamma = \sum_{i\in I} \lambda_i \min_{l\in A} u_i(l) + \gamma$$

for all $A \in \mathcal{K}$.

3.2 Weakening Pareto

Next we consider weakening Pareto to the following.

Complete Knowledge Pareto: For all $l, l' \in \Delta(X)$, if $\{l\} \succ_i \{l'\}$ for all $i \in I$ then $\{l\} \succ_0 \{l'\}$

Because Complete Knowledge Pareto still allows the Harsanyi-type aggregation over the domain $\Delta(X)$, we obtain the following proposition.

Proposition 2 *Let $(\succsim_i)_{\{0\}\cup I}$ be a profile of social ranking and individual preferences which follow the maximin criterion and satisfy Richness. Fix a profile of representations of social ranking and individual preferences denoted by $(U_i, u_i)_{i\in\{0\}\cup I}$.*

Then $(\succsim_i)_{\{0\}\cup I}$ satisfies Complete Knowledge Pareto if and only if there exist $\lambda \in \mathbb{R}^I_+ \setminus \{\mathbf{0}\}$ and $\gamma \in \mathbb{R}$ such that

$$u_0(l) = \sum_{i\in I} \lambda_i u_i(l) + \gamma$$

for all $l \in \Delta(X)$.

Proof The result follows from applying the Harsanyi aggregation argument to the subdomain $\Delta(X)$. □

4 Aggregation of α-maximin preferences

The preceding argument assumed that both the society and its members have an extreme attitude in precaution. Here we extend the argument to the setting which allows intermediate and diverse degrees of precaution.

Definition 2 A profile of social ranking and individual preferences $(\succsim_i)_{\{0\}\cup I}$ is said to follow the α-maximin criterion if there exists a list $(u_i, \alpha_i)_{\{0\}\cup I}$ such that for every $i \in \{0\} \cup I$ the function $u_i : \Delta(X) \to \mathbb{R}$ is mixture-linear, $\alpha_i \in [0, 1]$, and $\succsim_i$ is represented in the form

$$U_i(A) = \alpha_i \min_{l\in A} u_i(l) + (1 - \alpha_i) \max_{l\in A} u_i(l)$$

where $A \in \mathcal{K}$.

The α-maximin was proposed by Hurwicz (1951). Its axiomatic characterization in the setting of probability possibility sets is provided by Olszewski (2007), and by Ghirardato et al. (2004) in the Savage/Anscombe-Aumann setting. The criterion is characterized by the following four axioms, while the last one is presented in a simpler version here.

Weak Order: $\succsim$ is complete and transitive.
Continuity: $\succsim$ is continuous with respect to the Hausdorff metric.
Independence: For all $A, B, C \in \mathcal{K}$ and $r \in (0, 1)$ it holds

$$A \succsim B \iff (1 - r)A + rC \succsim (1 - r)B + rC.$$

Betweeness: For all $A, B \in \mathcal{K}$, it holds

$$A \succsim B \implies A \succsim A \cup B \succsim B.$$

When all the individuals and the society have the most extreme degree of precaution we go back to the previous impossibility result. Hence we consider an alternative richness condition below, which is again generically true.

Richness*: (i) There exist $l_1, l_1', l_2, l_2', \ldots, l_m, l_m' \in \Delta(X)$ such that $\{l_k'\} \succ_i \{l_k\}$ for all $i \in I$ and $k = 1, \ldots, m$, and vectors $(u_i(l_1') - u_i(l_1), \ldots, u_i(l_m') - u_i(l_m))$, $i \in I$ are linearly independent. (ii) One of the followings is true. (a) $(\alpha_i)_{i\in I}$ are distinct. (b) For all $i \in I$ there exist $l, l', l'' \in \Delta(X)$ such that $\{l''\} \succ_i \{l'\} \succ_i \{l\}$ and $\{l'\} \succ_j \{l''\} \succ_j \{l\}$ for all $j \in I \setminus \{i\}$. (c) For all $i \in I$ there exist $l, l', l'' \in \Delta(X)$ such that $\{l''\} \prec_i \{l'\} \prec_i \{l\}$ and $\{l'\} \prec_j \{l''\} \prec_j \{l\}$ for all $j \in I \setminus \{i\}$.

Theorem 2 *Let $(\succsim_i)_{\{0\}\cup I}$ be a profile of social ranking and individual preferences which follow the α-maximin criterion and satisfy Richness*.*

Then, it satisfies Pareto if and only if there is $i \in I$ such that

$$A \succsim_0 B \iff A \succsim_i B$$

for all $A, B \in \mathcal{K}$.

The result could follow from a result by Gajdos et al. (2008), which is stated in the setting of subjective ambiguity over a state space, by applying it to the case of α

-maximin model with common multiple priors and translating to the current domain by looking at sets of probability distributions over outcomes induced by the common prior set and Savage/Anscombe-Aumann acts. We provide a direct and specific proof instead, to make the argument more direct and transparent along with the current domain.

Proof Fix a profile of representations $(U_i, u_i, \alpha_i)_{i\in\{0\}\cup I}$.

Note that $\mathcal{K}$ is a mixture set. From Richness*-(i) there exist $l, l' \in \Delta(X)$ such that $\{l'\} \succ_i \{l\}$ for all $i \in I$. Hence we can again take an analogue of the proof of Harsanyi's theorem (Harsanyi 1955; De Meyer and Philippe 1995), so that exist $\lambda \in \mathbb{R}^I_+ \setminus \{\mathbf{0}\}$ and $\gamma \in \mathbb{R}$ such that there

$$U_0(A) = \sum_{i\in I} \lambda_i U_i(A) + \gamma$$

for all $A \in \mathcal{K}$.

By restricting attention to singleton sets, we obtain

$$u_0(l) = \sum_{i\in I} \lambda_i u_i(l) + \gamma$$

for all $l \in \Delta(X)$.

From Richness* (i), there exist $l_1, l'_1, l_2, l'_2, \ldots, l_m, l'_m \in \Delta(X)$ such that $\{l'_k\} \succ_i \{l_k\}$ for all $i \in I$ and $k = 1, \ldots, m$, and vectors $(u_i(l'_1) - u_i(l_1), \ldots, u_i(l'_m) - u_i(l_m))$, $i \in I$ are linearly independent.

Thus, for each $k = 1, \ldots, m$, by applying $A = \{l_k, l'_k\}$ we obtain

$$\sum_{i\in I} \lambda_i \left[\alpha_i u_i(l_k) + (1-\alpha_i) u_i(l'_k)\right] = \alpha_0 \sum_{i\in I} \lambda_i u_i(l_k) + (1-\alpha_0) \sum_{i\in I} \lambda_i u_i(l'_k)$$

which reduces to

$$\sum_{i\in I} \lambda_i (\alpha_i - \alpha_0)\left(u_i(l'_k) - u_i(l_k)\right) = 0.$$

Since $(u_i(l'_1) - u_i(l_1), \cdots, u_i(l'_m) - u_i(l_m))$, $i \in I$ are linearly independent, we obtain $\lambda_i(\alpha_i - \alpha_0) = 0$ for all $i \in I$. Hence it holds $\lambda_i = 0$ or $\alpha_i = \alpha_0$ for all $i \in I$.

Suppose the set $I_+ = \{i \in I : \lambda_i > 0\}$ has two or more elements. Then, for any $i, j \in I_+$ it holds $\alpha_i = \alpha_0 = \alpha_j$. We obtain a contradiction at this point if Richness*-(ii)-(a) is met.

Suppose Richness*-(ii)-(b) is met. Then there exist $l, l', l'' \in \Delta(X)$ such that $\{l''\} \succ_i \{l'\} \succ_i \{l\}$ and $\{l'\} \succ_j \{l''\} \succ_j \{l\}$ for all $j \in I \setminus \{i\}$.

By restricting attention to set $\{l', l''\}$ we obtain

$$\lambda_i\left\{\alpha_0 u_i(l') + (1-\alpha_0) u_i(l'')\right\} + \sum_{j\in I_+\setminus\{i\}} \lambda_j\left\{\alpha_0 u_j(l'') + (1-\alpha_0) u_j(l')\right\}$$

$$= \alpha_0 \min\left\{\sum_{j\in I_+} \lambda_j u_j(l'), \sum_{j\in I_+} \lambda_j u_j(l'')\right\} + (1-\alpha_0) \max\left\{\sum_{j\in I_+} \lambda_j u_j(l'), \sum_{j\in I_+} \lambda_j u_j(l'')\right\}$$

Suppose $\sum_{j\in I_+}\lambda_j u_j(l') \leq \sum_{j\in I_+}\lambda_j u_j(l'')$. Then we obtain

$$(1-2\alpha_0)\left\{\sum_{j\in I_+\setminus\{i\}}\lambda_j u_j(l') - \sum_{j\in I_+\setminus\{i\}}\lambda_j u_j(l'')\right\} = 0,$$

which implies $\alpha_0 = 1/2$.

Suppose $\sum_{j\in I_+}\lambda_j u_j(l') \geq \sum_{j\in I_+}\lambda_j u_j(l'')$. Then we obtain

$$(1-2\alpha_0)\lambda_i\left\{u_i(l'') - u_i(l')\right\} = 0,$$

which implies $\alpha_0 = 1/2$.

Now by restricting attention to set $\{l, l', l''\}$ we obtain

$$\lambda_i\left\{\frac{1}{2}u_i(l) + \frac{1}{2}u_i(l'')\right\} + \sum_{j\in I_+\setminus\{i\}}\lambda_j\left\{\frac{1}{2}u_j(l) + \frac{1}{2}u_j(l')\right\}$$

$$= \frac{1}{2}\sum_{j\in I_+}\lambda_j u_j(l) + \frac{1}{2}\max\left\{\sum_{j\in I_+}\lambda_j u_j(l'), \sum_{j\in I_+}\lambda_j u_j(l'')\right\}$$

Suppose $\sum_{j\in I_+}\lambda_j u_j(l') \geq \sum_{j\in I_+}\lambda_j u_j(l'')$. Then we obtain

$$\lambda_i(u_i(l'') - u_i(l')) = 0,$$

which implies $\lambda_i = 0$, a contradiction.

Suppose $\sum_{j\in I_+}\lambda_j u_j(l') \leq \sum_{j\in I_+}\lambda_j u_j(l'')$. Then we obtain

$$\sum_{j\in I_+\setminus\{i\}}\lambda_j\left(u_j\left(l'\right) - u_j\left(l''\right)\right) = 0,$$

which implies $\lambda_j = 0$ for all $j \in I_+ \setminus \{i\}$, a contradiction.

We reach a similar contradiction when Richness*-(ii)-(c) is met. □

Again the impossibility result forces us to choose between two options. One is to give up imposing the α-maximin criterion at the social level, the other is to weaken the Pareto axiom.

In the first option, for the social ranking we drop one of the four axioms for the α-maximin criterion, Betweeness. It still allows that social ranking $\succsim_0$ has a mixture-linear representation $U_0 : \mathcal{K} \to \mathbb{R}$, and that the same argument for the proof of Harsanyi-type aggregation goes through.

Thus we have the following proposition.

Proposition 3 *Let $(\succsim_i)_{\{0\}\cup I}$ be a profile of social ranking and individual preferences, where the social ranking satisfies Order, Continuity and Independence, and the individual preferences follow the α-maximin criterion and Richness. Fix a representation of $\succsim_0$ denoted by $U_0 : \mathcal{K} \to \mathbb{R}$, and a profile of representations of individual preferences $\left(U_i, u_i, \alpha_i\right)_{i\in I}$.*

Then $(\gtrsim_i)_{\{0\}\cup I}$ *satisfies Pareto if and only if there exist* $\lambda \in \mathbb{R}^I_+ \setminus \{\mathbf{0}\}$ *and* $\gamma \in \mathbb{R}$ *such that*

$$U_0(A) = \sum_{i\in I} \lambda_i U_i(A) + \gamma = \sum_{i\in I} \lambda_i \left[\alpha_i \min_{l\in A} u_i(l) + (1-\alpha_i) \max_{l\in A} u_i(l)\right] + \gamma$$

for all $A \in \mathcal{K}$.

The second option is to weaken Pareto. Because Complete Knowledge Pareto still allows the Harsanyi-type aggregation over the domain $\Delta(X)$, we obtain the following result.

Proposition 4 *Let* $(\gtrsim_i)_{\{0\}\cup I}$ *be a profile of social ranking and individual preferences which follow the* α*-maximin criterion and satisfy Richness***. Fix a profile of representations of social ranking and individual preferences denoted by* $\left(U_i, u_i, \alpha_i\right)_{i\in\{0\}\cup I}$.

Then $(\gtrsim_i)_{\{0\}\cup I}$ *satisfies Complete Knowledge Pareto if and only if there exist* $\lambda \in \mathbb{R}^I_+ \setminus \{\mathbf{0}\}$ *and* $\gamma \in \mathbb{R}$ *such that*

$$u_0(l) = \sum_{i\in I} \lambda_i u_i(l) + \gamma$$

for all $l \in \Delta(X)$.

Proof The result follows from applying the Harsanyi aggregation argument to the subdomain $\Delta(X)$. □

One might think that Complete Knowledge Pareto is now too weak, as it leaves us silent about what the social degree of precaution should be. Thus we add the following Pareto condition which is disjoint to Complete Knowledge Pareto.

Intermediate Pareto: For all $\bar{l}, \underline{l} \in \Delta(X)$ such that $\bar{l} \succ_i \underline{l}$ for all $i \in I$ and for all $l \in \Delta(X)$, if $\{\bar{l}, \underline{l}\} \succ_i \{l\}$ for all $i \in I$ then $\{\bar{l}, \underline{l}\} \succ_0 \{l\}$, and if $\{\bar{l}, \underline{l}\} \prec_i \{l\}$ for all $i \in I$ then $\{\bar{l}, \underline{l}\} \prec_0 \{l\}$.

Proposition 5 *Let* $(\gtrsim_i)_{\{0\}\cup I}$ *be a profile of social ranking and individual preferences which follow the* α*-maximin criterion and satisfy Richness***. Fix a profile of representations of social ranking and individual preferences denoted by* $(U_i, u_i, \alpha_i)_{i\in\{0\}\cup I}$.

Then $(\gtrsim_i)_{\{0\}\cup I}$ *satisfies Complete Knowledge Pareto and Intermediate Pareto if and only if there exist* $\lambda \in \mathbb{R}^I_+ \setminus \{\mathbf{0}\}$ *and* $\gamma \in \mathbb{R}$ *such that*

$$u_0(l) = \sum_{i\in I} \lambda_i u_i(l) + \gamma$$

for all $l \in \Delta(X)$ *and*

$$\alpha_0 \in \left[\min_{i\in I} \alpha_i, \max_{i\in I} \alpha_i\right].$$

Proof It suffices to prove the necessity of $\alpha_0 \in \left[\min_{i\in I} \alpha_i, \max_{i\in I} \alpha_i\right]$. Suppose $\alpha_0 > \max_{i\in I} \alpha_i$. From Richness*-(i) there exist $\bar{l}, \underline{l} \in \Delta(X)$ such that $\{\bar{l}\} \succ_i \{\underline{l}\}$ for all $i \in I$.

Then it holds

$$\alpha_0 u_i(\underline{l}) + (1-\alpha_0) u_i\left(\bar{l}\right) < \alpha_i u_i(\underline{l}) + (1-\alpha_i) u_i\left(\bar{l}\right)$$

for all $i \in I$. By taking $r \in \left(\max_{i\in I} \alpha_i, \alpha_0\right)$ we obtain

$$\alpha_0 u_i(\underline{l}) + (1-\alpha_0) u_i\left(\bar{l}\right) < u_i\left(r\underline{l} + (1-r)\bar{l}\right) < \alpha_i u_i(\underline{l}) + (1-\alpha_i) u_i\left(\bar{l}\right)$$

for all $i \in I$.

Hence we obtain

$$\alpha_0 \sum_{i\in I} \lambda_i u_i(\underline{l}) + (1-\alpha_0) \sum_{i\in I} \lambda_i u_i\left(\bar{l}\right) < \sum_{i\in I} \lambda_i u_i\left(r\underline{l} + (1-r)\bar{l}\right),$$

but this contradicts to Intermediate Pareto. Thus $\alpha_0 \leqq \max_{i\in I} \alpha_i$.

Likewise, we can prove $\alpha_0 \geqq \min_{i\in I} \alpha_i$. □

5 Conclusion

We have considered the problem of aggregating preferences over probability possibility sets, in which the society and its member know only that the true probability distribution over outcomes resulting from each policy choice lies in some set and know nothing about which one in it is true or which one in it is more likely to be true. The primary axiom is Pareto, which states that if everybody ranks one probability possibility set over another so should the society. We have considered the maximin criterion and the α-maximin criterion as expressions of the precautionary principle.

We have shown that the Pareto axiom implies dictatorship, both in aggregating maximin preferences and in aggregating α-maximin preferences. This contrasts to the possibility results in aggregating preferences over precisely known lotteries (Harsanyi 1955). The negative results force us to choose between two options, one is to give up the precautionary principle as modelled, the other is to weaken the Pareto axiom. We have provided possibility characterizations for each of the options.

We conclude by connecting our arguments to the existing studies of aggregating preferences in the setting with state space (Savage 1972; Anscombe and Aumann 1963). Existing researches show that the social ranking cannot satisfy both the subjective expected utility theory and the Pareto axiom, which is due to double disagreements in beliefs and tastes (Hylland and Zeckhauser 1979;

Mongin 1995, 2016; Mongin and Pivato 2015; Zuber 2016). The literature following these classic impossibility results pursue various ways to handle this double disagreements (see Gilboa et al. (2004), Alon and Gayer (2016), Gayer et al. (2014), among many).

The most related paper is Gajdos et al. (2008) (GTV hereafter, see their Theorem 1, Corollary 1 and discussion after those). In the Savage/Anscombe-Aumann setting with a state space, they show an impossibility result even under absence of the double disagreements. They show that when the individuals are ambiguity-sensitive the social objective satisfying (ex-ante) Pareto must be ambiguity-neutral. In particular, this implies that impossibility is obtained even under *common* multiple-priors.

When translated to the state-space setting our result tells such impossibility of Paretian aggregation under common multiple-priors, and hence it is seen as a special case of the GTV result. Also, when we start from the problem of aggregation under common multiple-priors and define the corresponding rankings over probability possibility sets induced by the common prior set and Savage/Anscombe-Aumann acts, our impossibility results could follow from their result.

We chose to provide direct and specific proofs, although, to make the argument more direct and transparent along with the current domain. Also, in the problem of aggregating α-maximin preferences in particular, we have provided a weakened Pareto condition which is yet stronger than imposing Pareto only under complete probabilistic knowledge, and provided a specific possibility result on aggregating vNM indices and degrees of precaution.

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