



The average-of-awards rule for claims problems

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Abstract

Given a claims problem, the average-of-awards rule (AA) selects the expected value of the uniform distribution over the set of awards vectors. The AA rule is the center of gravity of the core of the coalitional game associated with a claims problem, so it corresponds to the core-center. We show that this rule satisfies a good number of properties so as to be included in the inventory of division rules. We also provide several representations of the AA rule and a procedure to compute it in terms of the parameters that define the problem.

1 Introduction

A claims problem arises when a scarce resource has to be shared among a set of claimants and the endowment falls short of the sum of individual claims. The question is, how to select a division among the claimants of the amount available. The definition of division rules and the study of different approaches to evaluate and compare them started with O’Neill (1982) and has generated a vast literature. The model has many applications that include bankruptcy problems, taxation systems, rationing problems, or the distribution of the carbon budget.

A division rule must satisfy three natural requirements: no claimant is asked to pay; no claimant receives more than her claim; and the entire endowment is allocated. The set of all the allocations that meet these basic properties is the set of awards vectors for the problem. Therefore, formally, a rule is a function that associates with each problem an awards vector. For an inventory of the principal rules, their properties, and a comprehensive survey on claims problems refer to Thomson (2019).

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An intuitive and simple way of selecting an allocation from the set of awards vectors for a problem is to assume that all the awards vectors are equally likely and therefore choosing their “average”. For each problem, the division recommended by the average-of-awards rule (AA) is the expected value of the uniform distribution over the set of awards vectors.

Following O’Neill (1982), to each problem one can associate a coalitional game. Curiel et al. (1987) note that the coalitional game associated with a problem is a convex game so its core is nonempty. In fact, the set of awards vectors for a problem coincides with the core of its associated coalitional game. A division rule corresponds to a solution to coalitional games if the choice made by the rule for each problem coincides with the choice made by the solution to coalitional games when applied to the associated coalitional game.

The core-center was defined and characterized for the general class of games with nonempty core by González-Díaz and Sánchez-Rodríguez (2007, 2009) and was studied on the domain of airport games by González-Díaz et al. (2015, 2016) and Mirás Calvo et al. (2016). Since, geometrically, the AA rule is the center of gravity (centroid) of the set of awards vectors, it corresponds to the core-center. There are other rules that select central points of the set of awards vectors. The random arrival rule (RA), which corresponds to the Shapley value, is the Steiner point of the set of awards vectors (Pechersky 2015). The adjusted proportional rule (APRO), which corresponds to the τ -value, is a weighted average of the extreme points of the set of awards vectors (González-Díaz et al. 2005). The Tal-mud rule (T), which corresponds to the nucleolus, is the lexicographic center of the set of awards vectors (Schmeidler 1969).

Analyzing the AA rule demands a detailed examination of the structure of the set of awards vectors. This structure is particularly simple: it is a nonempty compact and convex polytope, the intersection of a rectangle with a hyperplane. Many of the geometric features of the set of awards vectors are established in the companion paper Mirás Calvo et al. (2022b). Making use of these characteristics, and providing some decompositions of the set of awards vectors, we prove that the AA rule satisfies a good number of the standard properties for division rules. In this sense, we can say that the rule is well-behaved. Mirás Calvo et al. (2022a) incorporate the average of awards rule to the ranking of the standard rules with the Lorenz order.

There are two particular qualities of this rule that are worth noting. Certainly, the AA rule is a continuous rule: small changes in the parameters of the problem do not lead to large changes in the recommendation made by the rule. In particular, a variation in the endowment, no matter how small, produces a change in the corresponding set of awards vectors. The AA rule is highly sensitive to such changes. In fact, we prove that the recommendation made by the AA rule, for problems with at least three claimants, not only varies continuously with respect to the endowment but also that the rate at which the recommendation changes varies also with continuity, that is, the AA rule is endowment differentiable. Secondly, the AA rule coincides with the concede-and-divide rule (CD) for two claimants but, for larger populations, it differs from the standard rules. Therefore, the AA rule is an endowment differentiable extension of the CD rule.

There are simple mechanisms to compute some of the basic rules. But, in general, computing the allocation selected by a rule for a given problem can be computationally hard when the population is large. For example, Aziz (2013) shows that the allocation returned by the RA rule is #P-complete to compute. There are general algorithms to obtain the centroid of higher dimensional polyhedra that can be applied to calculate the AA rule. Here we present a procedure to exactly compute this rule in terms of the parameters of the problem, and provide an analytic expression for three-claimant problems.

The paper is organized as follows. Section 2 introduces the AA rule. In Sect. 3 we define some properties of division rules and we summarize which of them are satisfied or violated by the AA rule. The specific results are presented in Sect. 4. The computation of the AA rule is addressed in Sect. 5. The purely technical results are left to the Appendix.

2 The average-of-awards rule

Let \mathcal{N} be the set of all finite nonempty subsets of the natural numbers \mathbb{N} . Given $N \in \mathcal{N}$, $z \in \mathbb{R}^N$, and $S \in 2^N$ let $|M|$ be the number of elements of N and $z(S) = \sum_{i \in S} z_i$. Given $N' \subset N \in \mathcal{N}$, let $z_{N'} = (z_i)_{i \in N'} \in \mathbb{R}^{N'}$ be the projection of z onto $\mathbb{R}^{N'}$. In particular denote $z_{-i} = z_{N \setminus \{i\}} \in \mathbb{R}^{N \setminus \{i\}}$ the vector obtained by neglecting the i th-coordinate of z . For simplicity, we will write $z = (z_{-i}, z_i)$. Given $z, w \in \mathbb{R}^N$, the notation $z \leq w$ means that $z_i \leq w_i$ for all $i \in N$.

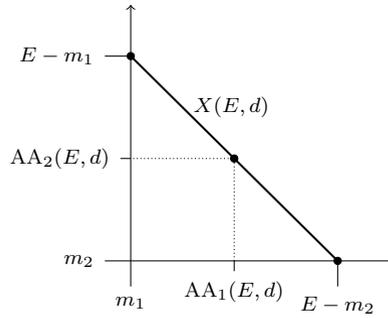
A **claims problem** (O’Neill 1982) with set of claimants $N \in \mathcal{N}$ is a pair (E, d) where $E \geq 0$ is the **endowment** and $d \in \mathbb{R}^N$ is the **vector of claims** satisfying $d \geq 0$ and $E \leq d(N)$. We denote the class of claims problems with set of claimants N by C^N . Let us assume throughout the paper that $N = \{1, \dots, n\}$.

The **minimal right** of claimant $i \in N$ in $(E, d) \in C^N$ is the quantity $m_i(E, d) = \max\{0, E - d(N \setminus \{i\})\}$. It is what is left after every one else has been fully compensated, or 0 if that is not possible. The **truncated claim** of claimant $i \in N$ in $(E, d) \in C^N$ is $t_i(E, d) = \min\{E, d_i\}$. It is the claim truncated by the amount to divide. To simplify, sometimes we write $m_i = m_i(E, d)$ and $t_i = t_i(E, d)$. Let $m(E, d) = (m_i(E, d))_{i \in N}$ and $t(E, d) = (t_i(E, d))_{i \in N}$.

A vector $x \in \mathbb{R}^N$ is an **awards vector** for $(E, d) \in C^N$ if $0 \leq x \leq d$ and $x(N) = E$. Let $X(E, d)$ be the **set of awards vectors** for $(E, d) \in C^N$. Clearly, $X(E, d)$ is the intersection of the rectangle $\prod_{i \in N} [0, d_i]$ with the hyperplane $H(E, d) = \{x \in \mathbb{R}^N : x(N) = E\}$. Then, the set of awards vectors for (E, d) is a non-empty compact convex polytope that has, at most, dimension $n - 1$.

A **division rule** is a function $\mathcal{R} : C^N \rightarrow \mathbb{R}^N$ assigning to each problem $(E, d) \in C^N$ an awards vector $\mathcal{R}(E, d) \in X(E, d)$, that is, a way of associating with each problem a division of the amount available among the claimants. In our definition of a division rule we include three requirements: **nonnegativity** (no claimant is asked to pay); **claims boundedness** (no claimant receives more than her claim); and **balance** (the entire endowment is allocated). Certainly, there are applications in which one (or more) of these requirements could be dropped but they are very natural in most situations.

Fig. 1 The set of awards vectors and the AA rule for a two-claimant problem



An intuitive and simple way of selecting an allocation from the set of awards vectors for a problem is to assume that all the awards vectors are equally likely and therefore choosing their “average”.

Definition 2.1 The average-of-awards rule is the rule $AA : C^N \rightarrow \mathbb{R}^N$ that assigns to each $(E, d) \in C^N$ the value $AA(E, d)$ given by the expected value of the (continuous) uniform distribution over the set of awards vectors $X(E, d)$.

In geometrical terms, $AA(E, d)$ is the centroid of the set of awards vectors for $(E, d) \in C^N$. Since $X(E, d)$ is convex, $AA(E, d) \in X(E, d)$ so, in fact, AA is a division rule.

Suppose that $N = \{1, 2\}$. Let $(E, d) \in C^N$ with $d = (d_1, d_2) \in \mathbb{R}^N$ such that $0 \leq d_1 \leq d_2$. Then, $X(E, d)$ is the line segment with endpoints $(m_1, E - m_1)$ and $(E - m_2, m_2)$, where $m_1 = \max\{0, E - d_2\}$ and $m_2 = \max\{0, E - d_1\}$ (see Fig. 1). The AA rule selects the middle point of the segment:

$$AA(E, d) = \begin{cases} \left(\frac{E}{2}, \frac{E}{2}\right) & \text{if } 0 \leq E \leq d_1 \\ \left(\frac{d_1}{2}, E - \frac{d_1}{2}\right) & \text{if } d_1 \leq E \leq d_2 \\ \left(\frac{E+d_1-d_2}{2}, \frac{E-d_1+d_2}{2}\right) & \text{if } d_2 \leq E \leq d_1 + d_2 \end{cases} \quad (1)$$

Then, the AA rule satisfies the concede-and-divide principle, or contested garment principle (Aumann and Maschler 1985).¹

Example 2.2 Let $N = \{1, 2, 3\}$ and consider the problem $(E, d) \in C^N$ with $E = 3$ and $d = (1, 2, 2)$. Then the set of awards vectors $X(E, d) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 2, 0 \leq x_3 \leq 2, x_1 + x_2 + x_3 = 3\}$ is the quadrilateral, depicted in Fig. 2, with vertices $(1, 0, 2)$, $(0, 1, 2)$, $(0, 2, 1)$, and

¹ Therefore, the AA, T, APRO, RA, and minimal overlap (MO) rules coincide for problems with two claimants.

Fig. 2 The set of awards vectors $X(3, (1, 2, 2))$ and its centroid

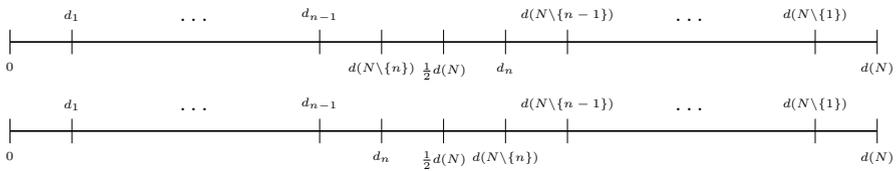
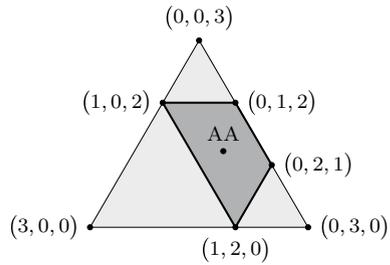


Fig. 3 Claims arranged in ascending order on the interval $[0, d(N)]$

$(1, 2, 0)$. The centroid of $X(E, d)$ can be easily computed using elementary geometry,² in fact, $AA(E, d) = (\frac{5}{9}, \frac{11}{9}, \frac{11}{9})$.

Let \mathbb{R}_{\leq}^n be the set of nonnegative n -dimensional vectors $x = (x_1, \dots, x_n)$ such that $0 \leq x_1 \leq \dots \leq x_n$. In what follows we assume that given a problem $(E, d) \in C^N$, the claims $d = (d_1, \dots, d_n) \in \mathbb{R}^n$ are arranged from small to large, i.e., $d \in \mathbb{R}_{\leq}^n$. As a consequence we have that $d_i \leq d(N \setminus \{i\})$, $d(N \setminus \{i\}) \geq d(N \setminus \{i+1\})$, and $m_i(E, d) \leq m_{i+1}(E, d)$ for all $i \in N \setminus \{n\}$. Then, either $d_n \leq d(N \setminus \{n\})$ or $d(N \setminus \{n\}) \leq d_n$, but in both situations $\frac{1}{2}d(N)$ is the middle point of the line segment with endpoints d_n and $d(N \setminus \{n\})$. The two cases are illustrated in Fig. 3. In fact, $\frac{1}{2}d(N)$ is also the middle point of the intervals $[d_i, d(N \setminus \{i\})]$ for all $i \in N \setminus \{n\}$.

Observe that if $E = 0$ or $d = (0, \dots, 0)$ then $X(E, d) = \{(0, \dots, 0)\}$ so $AA(E, d) = (0, \dots, 0)$. If $E = d(N)$ then $X(E, d) = \{d\}$ and $AA(E, d) = d$. Now, if some claims (but not all) are null and $N' = \{j \in N : d_j = 0\}$ then $X(E, d) = 0_{N'} \times X(E, d_{N \setminus N'})$ so $AA_{N'}(E, d) = 0_{N'}$. Therefore, $X(E, d)$ is not full dimensional if and only if either $E = 0$, or $E = d(N)$, or there is $j \in N$ with $d_j = 0$. So, in order to compute the AA rule we can remove the claimants in N' and apply the rule to the problem $(E, d_{N \setminus N'}) \in C^{N \setminus N'}$ whose set of awards vectors is full dimensional.

Let λ be the $(n - 1)$ -dimensional Lebesgue measure. Fix a vector of claims $d \in \mathbb{R}_{\leq}^n$. Then $(E, d) \in C^N$ for each $E \in [0, d(N)]$. We define the **volume function** $\tilde{V}(\cdot, d) : [0, d(N)] \rightarrow \mathbb{R}$ as $V(E, d) = \lambda(X(E, d))$, the $(n - 1)$ -measure of the set of awards vectors $X(E, d)$. Naturally, $X(E, d)$ is full dimensional if and only if

² Alternatively, compute $AA(E, d)$ using the explicit formulae given in Lemma 5.2.

$V(E, d) > 0$. Then, according to Definition 2.1, for each $(E, d) \in C^N$ such that $V(E, d) > 0$ and each $i \in N$, we have that:

$$AA_i(E, d) = \frac{1}{V(E, d)} \int_{X(E, d)} x_i d\lambda.$$

There are situations where the AA rule is easy to compute. For instance, if all claims are bigger than the endowment, that is $E \leq d_1$, then $X(E, d)$ is the regular simplex spanned by the points $a^i = (a_1^i, \dots, a_n^i), i \in N$, where $a_j^i = E$ if $i = j$ and 0 otherwise. Therefore $AA(E, d)$, the arithmetic mean of these points, coincides with the egalitarian division, that is, $AA_i(E, d) = \frac{E}{n}$ for all $i \in N$. Another interesting case occurs when the largest claim exceeds the aggregate claim of the other claimants and the endowment lies between these two quantities, that is, if $E \in [d(N \setminus \{n\}), d_n]$. Then the set of awards vectors has a very simple structure:

$$X(E, d) = \left\{ x \in \mathbb{R}^n : x_{-n} \in \prod_{i=1}^{n-1} [0, d_i], x_n = E - x(N \setminus \{n\}) \right\}.$$

Consequently, the AA rule gives to all the claimants, except for the one with the largest claim, the geometric center of the $(n - 1)$ -rectangle $\prod_{i=1}^{n-1} [0, d_i]$, and to the other claimant what is left:³

$$AA(E, d) = \left(\frac{d_1}{2}, \dots, \frac{d_{n-1}}{2}, E - \frac{1}{2}d(N \setminus \{n\}) \right). \tag{2}$$

Observe that if $N = \{1, 2\}$ and $d = (d_1, d_2) \in \mathbb{R}_{\leq}^2$ then always $d(N \setminus \{2\}) = d_1 \leq d_2$, so $AA(E, d) = \left(\frac{d_1}{2}, E - \frac{d_1}{2} \right)$ if $d_1 \leq E \leq d_2$.

A **coalitional game** is an ordered pair (N, v) where $N \in \mathcal{N}$ is a finite set of players and $v : 2^N \rightarrow \mathbb{R}$, the **characteristic function**, satisfies $v(\emptyset) = 0$. Let G^N be the set of all coalitional games with player set N . Given a game $v \in G^N$, a vector $x \in \mathbb{R}^N$ is said to be an **efficient allocation** if $x(N) = v(N)$. A **(single-valued) solution** is a mapping that associates with each game v in some admissible class of games an efficient allocation. The **core** of a game $v \in G^N$ is the set

$$C(v) = \{x \in \mathbb{R}^N : x(N) = v(N), x(S) \geq v(S) \text{ for all } S \subset N\}.$$

Following O’Neill (1982), the **coalitional game associated with** the claims problem $(E, d) \in C^N$ is the game $v \in G^N$ defined by $v(S) = \max\{0, E - d(N \setminus S)\}$ for $S \in 2^N$. If for each problem, the recommendation made by a given division rule coincides with the recommendation made by a given solution to coalitional games when applied to the associated coalitional game, we say that the **rule corresponds to the solution**. The set of awards vectors for a problem coincides with the core of the associated coalitional game [see, for instance, Thomson (2019)], that is, $X(E, d)$

³ If $(E, d) \in C^N$ with $E \in [d(N \setminus \{n\}), d_n]$, then $AA(E, d) = T(E, d) = RA(E, d) = APRO(E, d)$.

consists of all allocations satisfying the balance condition that are bounded from below by the minimal rights and bounded from above by the truncated claims:

$$X(E, d) = \{x \in \mathbb{R}^N : m(E, d) \leq x \leq t(E, d), x(N) = E\}.$$

Therefore, the AA rule corresponds to the core-center introduced by González-Díaz and Sánchez-Rodríguez (2007).

3 Summary of properties

The axiomatic approach has dominated the study of rules. Rules are examined, classified, and characterized according to the properties that they satisfy (or violate). We present some standard properties. We say that a rule \mathcal{R} satisfies:

- **minimal rights first**, if for each $(E, d) \in C^N$ we have $\mathcal{R}(E, d) = m(E, d) + \mathcal{R}(E - \sum_{i \in N} m_i(E, d), d - m(E, d))$.
- **claims truncation invariance**, if for each $(E, d) \in C^N$ we have that $\mathcal{R}(E, d) = \mathcal{R}(E, t(E, d))$.
- $\frac{1}{|N|}$ -**truncated-claims lower bounds on awards**, if for each $(E, d) \in C^N$ we have $\mathcal{R}(E, d) \geq \frac{1}{|N|}t(E, d)$.
- $\frac{1}{|N|}$ -**min-of-claim-and-deficit lower bounds on losses**, if for each $(E, d) \in C^N$ we have $d - \mathcal{R}(E, d) \geq \frac{1}{|N|}t(d(N) - E, d)$.
- **min-of-claim-and-equal-division lower bounds on awards**, if for each $(E, d) \in C^N$ we have $\mathcal{R}(E, d) \geq t(\frac{E}{|N|}, d)$.
- **equal treatment of equals**, if for each $(E, d) \in C^N$ and each $\{i, j\} \subset N$, if $d_i = d_j$ we have $\mathcal{R}_i(E, d) = \mathcal{R}_j(E, d)$.
- **anonymity**, if for each $(E, d) \in C^N$, each bijection f from N into itself, and each $i \in N$, we have that $\mathcal{R}_i(E, d) = \mathcal{R}_{f(i)}(E, (d_{f(i)})_{i \in N})$.
- **order preservation**, if for each $(E, d) \in C^N$ and each $\{i, j\} \subset N$, if $d_i \leq d_j$ we have $\mathcal{R}_i(E, d) \leq \mathcal{R}_j(E, d)$ and $d_i - \mathcal{R}_i(E, d) \leq d_j - \mathcal{R}_j(E, d)$.
- **claim monotonicity**, if for each $(E, d) \in C^N$, each $i \in N$, and each $d'_i \geq d_i$, we have $\mathcal{R}_i(E, (d_{-i}, d'_i)) \geq \mathcal{R}_i(E, d)$.
- **linked claim-endowment monotonicity**, if for each $(E, d) \in C^N$, each $i \in N$, and each $\delta > 0$, we have $\mathcal{R}_i(E + \delta, (d_{-i}, d_i + \delta)) - \mathcal{R}_i(E, d) \leq \delta$.
- **other-regarding claim monotonicity**, if for each $(E, d) \in C^N$, each $i \in N$, and each $d'_i \geq d_i$, we have $\mathcal{R}_j(E, (d_{-i}, d'_i)) \leq \mathcal{R}_j(E, d)$ for all $j \in N, j \neq i$.
- **endowment monotonicity**, if for each $(E, d) \in C^N$ and each $E' \geq 0$, if $d(N) \geq E' \geq E$ we have $\mathcal{R}(E', d) \geq \mathcal{R}(E, d)$.
- **homogeneity**, if for each $(E, d) \in C^N$ and each $\rho > 0$, we have $\mathcal{R}(\rho E, \rho d) = \rho \mathcal{R}(E, d)$.
- **composition down**, if for each $(E, d) \in C^N$ and each $E' < E$, we have $\mathcal{R}(E', d) = \mathcal{R}(E', \mathcal{R}(E, d))$.

- **composition up**, if for each $(E, d) \in C^N$ and each $E' \geq 0$, if $d(N) \geq E' > E$ then $\mathcal{R}(E', d) = \mathcal{R}(E, d) + \mathcal{R}(E' - E, d - \mathcal{R}(E, d))$.
- **continuity**, if for each sequence $\{(E^v, d^v)\}$ of elements of C^N and each $(E, d) \in C^N$, if $\{(E^v, d^v)\}$ converges to (E, d) we have $\{\mathcal{R}(E^v, d^v)\}$ converges to $\mathcal{R}(E, d)$.
- **self-duality**, if for each $(E, d) \in C^N$ we have $\mathcal{R}(E, d) = d - \mathcal{R}(d(N) - E, d)$.

Some of these properties are stronger requirements than others. For instance, *min-of-claim-and-equal-division lower bounds on awards* implies $\frac{1}{|N|}$ -truncated-claims lower bounds on awards, and *other-regarding claim monotonicity* implies *claim monotonicity*. *Anonymity* implies *equal treatment of equals*. *Order preservation* also implies *equal treatment of equals*. Moreover, *composition down* (*composition up*) implies *endowment monotonicity*.

With each rule \mathcal{R} we can associate a unique **dual rule** \mathcal{R}^* , the one defined by the right-hand side of the expression in the statement of the *self-duality* property: $\mathcal{R}^*(E, d) = d - \mathcal{R}(d(N) - E, d)$. A rule \mathcal{R} is **self-dual** if $\mathcal{R} = \mathcal{R}^*$. Two properties are **dual** if whenever a rule satisfies one of them then its dual satisfies the other. The following are pairs of dual properties: *claims truncation invariance* and *minimal rights first*; $\frac{1}{|N|}$ -truncated-claims lower bounds on awards and $\frac{1}{|N|}$ -min-of-claim-and-deficit lower bounds on losses; *composition down* and *composition up*; and *claim monotonicity* and *linked claim-endowment monotonicity*. Note that *order preservation* is, in fact, the conjunction of two dual properties: **order preservation in awards** and **order preservation in losses**.

We also consider situations in which the population of claimants involved may vary. In this case, a problem is defined by first specifying $N \in \mathcal{N}$, then a pair $(E, d) \in C^N$. We still denote the class of all problems with claimant set N by C^N . So, a rule is a function defined on $\bigcup_{N \in \mathcal{N}} C^N$ that associates with each $N \in \mathcal{N}$ and each $(E, d) \in C^N$ an awards vector for (E, d) . We say that a rule \mathcal{R} satisfies:

- **population monotonicity**, if for each pair $\{N, N'\} \subset \mathcal{N}$ such that $N' \subset N$, and each $(E, d) \in C^N$ we have $\mathcal{R}_{N'}(E, d) \leq \mathcal{R}(E, d_{N'})$.
- **null claims consistency**, if for each $N \subset \mathcal{N}$, each $(E, d) \in C^N$, and each $N' \subset N$, if $d(N \setminus N') = 0$ we have $\mathcal{R}_{N'}(E, d) = \mathcal{R}(E, d_{N'})$.
- **consistency**, if for each pair $\{N, N'\} \subset \mathcal{N}$ such that $N' \subset N$, and each $(E, d) \in C^N$ if $x = \mathcal{R}(E, d)$ we have $x_{N'} = \mathcal{R}(x(N'), d_{N'})$. **Bilateral consistency** is the weaker property obtained by considering only subgroups of two remaining agents, that is, when $|N'| = 2$.
- **converse consistency**, if for each $N \in \mathcal{N}$, each $(E, d) \in C^N$, and each $x \in X(E, d)$, if for each $N' \subset N$ with $|N'| = 2$ we have $x_{N'} = \mathcal{R}(x(N'), d_{N'})$ then $x = \mathcal{R}(E, d)$.
- **replication invariance**,⁴ if for each $N \in \mathcal{N}$, each $(E, d) \in C^N$, each $N' \supset N$, and each $(E', d') \in C^{N'}$, if $(E', d') \in C^{N'}$ is a k -replica of (E, d) with asso-

⁴ Given $(E, d) \in C^N$ and $k \in \mathbb{N}$, we say that $(E', d') \in C^{N'}$ is a k -replica of (E, d) if $E' = kE$, $N' \supset N$, $|N'| = k|N|$, and there is a partition $(N^i)_{i \in N}$ of N' such that for each $i \in N$ and each $j \in N^i$, $|N^i| = k$ and $d'_j = d_i$.

Table 1 Properties satisfied or violated by the AA rule

Property	AA	Propositions
Minimal rights first	✓	4.1
Claims truncation invariance	✓	4.1
$\frac{1}{ N }$ -truncated-claims lower bounds on awards	✓	4.9
$\frac{1}{ N }$ -min-of-claim-and-deficit lower bounds on losses	✓	4.9
Min-of-claim-and-equal-division lower bounds on awards	–	4.3
Equal treatment of equals	✓	4.2
Anonymity	✓	4.1
Order preservation	✓	4.4
Claim monotonicity	✓	4.8
Linked claim-endowment monotonicity	✓	4.8
Other-regarding claim monotonicity	✓	4.8
Endowment monotonicity	✓	4.6
Continuity	✓	4.2
Homogeneity	✓	4.2
Composition down	–	4.3
Composition up	–	4.3
Self-duality	✓	4.1
Population monotonicity	✓	4.7
Null claims consistency	✓	4.1
Consistency	–	4.3
Converse consistency	–	4.3
Replication invariance	–	4.3

ciated partition $(N^i)_{i \in N}$, then for each $i \in N$ and each $j \in N^i$ we have $\mathcal{R}_j(E', d') = \mathcal{R}_i(E, d)$.

Table 1 summarizes which of the properties listed above are satisfied or violated by the AA rule. A check mark, *mark*, in a cell means that the property is satisfied by the rule and a minus sign, *–*, means the opposite. The last column indicates the specific result in the next section where the corresponding mark is established. As Table 1 shows, the AA rule satisfies a good number of properties so as to be part of the inventory of rules.

In the next section, we use the following general property of the centroid. Let $(E, d) \in C^N$ and K be a convex polytope contained in the hyperplane $H(E, d)$. Denote its centroid by $\bar{c}(K)$.⁵ Then $\bar{c}(a + \alpha K) = a + \alpha \bar{c}(K)$ for all $a \in \mathbb{R}^N$ and $\alpha \in \mathbb{R}$. One of the methods used in obtaining the centroid of a compound shape consists in dividing

⁵ Of course, $AA(E, d) = \bar{c}(X(E, d))$.

the shape into a number of parts, that share no common volume, and then finding the overall centroid as the average of the centroid of each part weighted by its relative measure. Formally, if $\lambda(K) > 0$, $K = K_1 \cup K_2$, $\lambda(K_1 \cap K_2) = 0$, and $\rho = \frac{\lambda(K_1)}{\lambda(K)}$, then

$$\bar{c}(K) = \rho\bar{c}(K_1) + (1 - \rho)\bar{c}(K_2). \quad (3)$$

4 Results

We begin by establishing some properties that are a direct consequence of simple geometrical features of the set of awards vectors.

Proposition 4.1 *The AA rule satisfies anonymity, null claims consistency, self-duality, claims truncation invariance, and minimal rights first.*

Proof Let $N = \{1, \dots, n\}$ and $(E, d) \in C^N$. It is obvious that the AA rule, being the centroid of the set of awards vectors $X(E, d)$, satisfies *anonymity*. We have seen that if $N' = \{j \in N : d_j = 0\}$ then $AA_{N'}(E, d) = 0_{N'}$ and $AA_{N \setminus N'}(E, d) = AA(E, d_{N \setminus N'})$, so the AA rule satisfies *null claims consistency*. *Self-duality* follows from the equality $X(E, d) = d - X(d(N) - E, d)$. Since $X(E, d) = X(E, t(E, d))$ we conclude that AA satisfies *claims truncation invariance*. The AA rule also satisfies *minimal rights first* because it is *self-dual* and *claims truncation invariance* and *minimal rights first* are dual properties. \square

Self-duality implies that the AA rule satisfies the **midpoint property**, that is, $AA(\frac{1}{2}d(N), d) = \frac{1}{2}d$.

Now, some general properties of the core-center for coalitional games are easy to translate to properties of the AA rule.

Proposition 4.2 *The AA rule satisfies equal treatment of equals, homogeneity, and continuity.*

Proof The AA rule corresponds to the core-center for coalitional games with non-empty core. González-Díaz and Sánchez-Rodríguez (2007) show that the core-center treats symmetric players symmetrically and that it is a homogeneous and continuous function of the values of the characteristic function. The AA rule satisfies *equal treatment of equals* because claimants with the same claims are symmetric players in the associated coalitional game. Since, given $\alpha > 0$ and $S \in 2^N$ we have that $v_\alpha(S) = \alpha v(S)$, where v_α is the coalitional game associated with the problem $(\alpha E, \alpha d) \in C^N$, the AA rule satisfies *homogeneity*. The values, $v(S)$, $S \in 2^N$, of the characteristic function are continuous with respect to the endowment E and the

claims d . Then, the AA rule satisfies *continuity* because it is a composition of continuous functions. □

Next, we consider the properties listed in Table 1 that the AA rule fails.

Proposition 4.3 *The AA rule violates composition down, composition up, consistency, min-of-claim-and-equal-division lower bounds on awards, replication invariance, and converse consistency.*

Proof Dagan (1996) shows that the constrained equal awards rule (CEA) is the only rule satisfying *equal treatment of equals*, *claims truncation invariance* and *composition up*. Then, the AA rule does not satisfy *composition up*. Since *composition down* and *composition up* are dual properties and the AA rule is *self-dual*, it violates *composition down*. Aumann and Maschler (1985) prove that the T rule is the only rule to agree with concede-and-divide for two claimants and to be *bilateral consistent*. Then the AA rule fails *bilateral consistency* and therefore *consistency*.⁶

Recall from Example 2.2, that if $N = \{1, 2, 3\}$ and $(E, d) \in C^N$, with $E = 3$ and $d = (1, 2, 2)$, then $AA(E, d) = (\frac{5}{9}, \frac{11}{9}, \frac{11}{9})$. Observe that the truncated claims vector is $t(\frac{E}{3}, d) = (1, 1, 1)$ and that $AA_1(E, d) < 1$, so the AA rule violates *min-of-claim-and-equal-division lower bounds on awards*.

Let $N = \{1, 2\}$ and $(2, (2, 4)) \in C^N$. Then $AA(2, (2, 4)) = (1, 1)$. Now let $N' = \{1, 2, 3, 4\}$ and $(4, (2, 4, 2, 4)) \in C^{N'}$. Then $AA(4, (2, 4, 2, 4)) = (\frac{5}{6}, \frac{7}{6}, \frac{5}{6}, \frac{7}{6})$.⁷ Claimants 1 and 2 are not getting the same amounts in $AA(2, (2, 4))$ and in its 2-replica $(4, (2, 4, 2, 4))$. We conclude that the AA rule violates *replication invariance*. In fact, we have proved that it is not *invariant under replication of two claimant problems*. *Anonymity* and *converse consistency* imply *invariance under replication of two claimant problems*. Therefore, the AA rule violates *converse consistency*.⁸ □

Let us turn our attention to *order preservation*. We show that the awards, and losses, recommended by the AA rule are ordered as claims are.

Proposition 4.4 *The AA rule satisfies order preservation.*

Proof We start by proving that the AA rule satisfies *order preservation in awards*. Let $(E, d) \in C^N$, $d \in \mathbb{R}_{\leq}^n$, and $i \in N \setminus \{n\}$. We have to prove that $AA_i(E, d) \leq AA_{i+1}(E, d)$. Since the AA rule satisfies *minimal rights first* and $m_i(E, d) \leq m_{i+1}(E, d)$ for all $i \in N \setminus \{n\}$, it suffices to prove the result when $m(E, d) = 0$. Assume that $V(E, d) > 0$. If $d_i = d_{i+1}$ then, by *equal treatment of equals*, $AA_i(E, d) = AA_{i+1}(E, d)$. Suppose that $d_i < d_{i+1}$. If $E \leq d_i$ then, by *claims truncation invariance*, $AA_i(E, d) = AA_{i+1}(E, d)$. So,

⁶ A direct proof that the AA rule violates *composition down*, *composition up*, and *consistency* can easily be obtained.

⁷ To compute $AA(4, (2, 4, 2, 4))$ one can use the algorithm of Sect. 5.

⁸ The AA rule is not a Young's parametric rule because all of them are *converse consistent*.

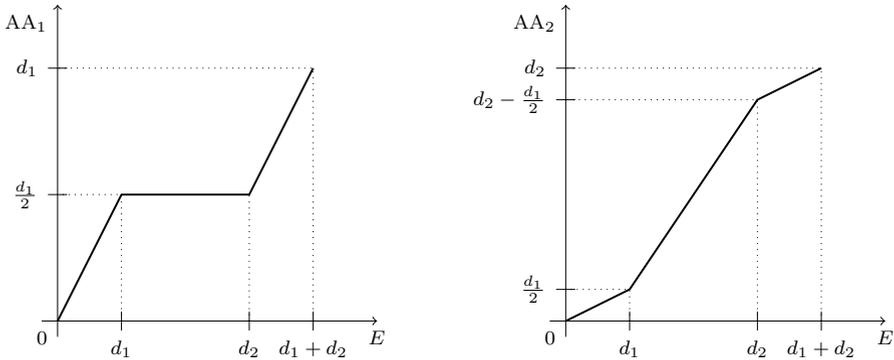


Fig. 4 The schedules of awards of the AA rule for two-claimant problems

assume that $E > d_i$. Take $a = (d_{-(i+1)}, d_i)$, $b = (0_{-(i+1)}, d_i)$, and $h = d - b$. Then $(E, a) \in C^N$, $(E - d_i, h) \in C^N$, and $X(E, d) = X(E, a) \cup (b + X(E - d_i, h))$. Moreover, $\lambda(X(E, a) \cap (b + X(E - d_i, h))) = 0$, because $X(E, a)$ and $b + X(E - d_i, h)$ are separated by the hyperplane $x_{i+1} = d_i$. Take $\rho = \frac{V(E, a)}{V(E, d)}$. By *equal treatment of equals*, $AA_{i+1}(E, a) = AA_i(E, a)$. Moreover, $\bar{c}_{i+1}(b + X(E - d_i, h)) = d_i + AA_{i+1}(E - d_i, h) \geq d_i \geq AA_i(E - d_i, h) = \bar{c}_i(b + X(E - d_i, h))$ where $\bar{c}_i(b + X(E - d_i, h))$ is the centroid of the set $b + X(E - d_i, h)$. Then, applying (3),

$$AA_{i+1}(E, d) = \rho AA_{i+1}(E, a) + (1 - \rho)\bar{c}_{i+1}(b + X(E - d_i, h)) \geq \rho AA_i(E, a) + (1 - \rho)AA_i(E - d_i, h) = AA_i(E, d).$$

The result also holds if $V(E, d) = 0$. Therefore, the AA rule satisfies *order preservation in awards*. Since *order preservation in awards* and *order preservation in losses* are dual properties and the AA rule is *self-dual*, we conclude that the AA rule satisfies *order preservation*. □

So far, we have relied on simple features or decompositions of the set of awards vectors to establish some properties of the AA rule. Our approach is quite different to deal with *endowment monotonicity*. We want to prove that if the amount to divide increases, each claimant should receive at least as much as she did initially. Then, we have to understand how the set of awards changes when the endowment increases. Given a rule \mathcal{R} and a vector of claims $d \in \mathbb{R}^N$, the path followed by the awards vector chosen by \mathcal{R} as the endowment increases from 0 to $d(N)$, that is, the function $\mathcal{R}(\cdot, d) : [0, d(N)] \rightarrow \mathbb{R}^N$, is called the **path of awards** of the rule for the claims vector. The plots of the functions $\mathcal{R}_i(\cdot, d)$, $i \in N$, are called the **schedules of awards** of the rule for the claims vector. A rule \mathcal{R} satisfies **endowment continuity** if the path of awards of the rule is continuous for all claims vector. Naturally, *continuity* implies *endowment continuity*. **Endowment differentiability** requires the path of awards $\mathcal{R}(\cdot, d)$ of the rule to be differentiable on $[0, d(N)]$ for all claims vector $d \in \mathbb{R}^N$. Some of Young’s parametric rules (Young 1987), for instance the PRO rule and Cassel’s rule, are *endowment differentiable*.

As Fig. 4 illustrates, for problems with two claimants, the CD rule (and therefore the AA rule) violates *endowment differentiability*. Nevertheless, as we argue next, the AA rule is an endowment differentiable extension of the CD rule to larger populations.

Let $|N| \geq 3$ and fix $d \in \mathbb{R}_{\leq}^n$ with $0 < d_1$. Now, assume that claimant $i \in N$ receives an award $s \in [m_i(E, d), t_i(E, d)]$ and leaves. The remaining claimants, the members of $N \setminus \{i\}$, face the reduced problem $(E - s, d_{-i}) \in C^{N \setminus \{i\}}$. If claimant $i \in N$ gets her minimal right $m_i(E, d)$ then the remainder $R_i(E, d) = E - m_i(E, d) = \min\{E, d(N \setminus \{i\})\}$ is the maximal aggregate award to the other agents. Analogously, if claimant $i \in N$ gets her truncated claim $t_i(E, d)$ then the remainder $r_i(E, d) = E - t_i(E, d) = \max\{0, E - d_i\}$ is the minimal award for the other claimants. Obviously, if $s \in [m_i(E, d), t_i(E, d)]$ then $E - s \in [r_i(E, d), R_i(E, d)]$. For each $i \in N$ consider the **weight function** $g_i : (0, d(N)) \times [0, d(N \setminus \{i\})] \rightarrow \mathbb{R}$ defined as:

$$g_i(E, u) = \frac{\sqrt{n}}{\sqrt{n} - 1} \frac{V(u, d_{-i})}{V(E, d)}, \text{ for all } (E, u) \in (0, d(N)) \times [0, d(N \setminus \{i\})].$$

Clearly, $g_i(E, u) \geq 0$ for all $(E, u) \in (0, d(N)) \times [0, d(N \setminus \{i\})]$ so indeed $g(E, \cdot)$ is a weight function for all $E \in (0, d(N))$. Furthermore, in the Appendix (see Theorem A.1), we show that $g_i(E, \cdot)$ is a probability density function on the interval $[r_i(E, d), R_i(E, d)]$ and that, for all $j \in N \setminus \{i\}$, the following integral representations hold:

$$\begin{aligned} \text{AA}_j(E, d) &= \int_{m_i(E, d)}^{t_i(E, d)} \text{AA}_j(E - s, d_{-i}) g_i(E, E - s) ds \\ &= \int_{r_i(E, d)}^{R_i(E, d)} \text{AA}_j(u, d_{-i}) g_i(E, u) du. \end{aligned} \tag{4}$$

In words, for problems with at least three claimants, the awards chosen by the AA rule for a problem are a weighted average of the choices made by the rule in each of the reduced problems that result when a claimant receives an award, between her minimal right and her truncated claim, and leaves. We prove in the Appendix (see Theorem A.2) the following result:

Proposition 4.5 *If $|N| \geq 3$ then the AA rule is endowment differentiable.*

The integral representation of the AA rule given by equality (4) allows us to “lift” some properties satisfied by the CD rule to larger populations. For instance, the CD rule is *endowment monotonic* (see Fig. 4). We illustrate how the integral representation given by (4) allows us to prove that the AA rule is *endowment monotonic*. An *endowment continuous* rule \mathcal{R} satisfies *endowment monotonicity* if and only if for each $i \in N$ the function $\mathcal{R}_i(\cdot, d)$ is monotonically increasing

on $[0, d(N)]$ for all claims vector $d \in \mathbb{R}^N$. If a rule is *endowment differentiable* then *endowment monotonicity* can be expressed in terms of the derivatives of the rule with respect to the endowment. Indeed, if $\mathcal{R}(\cdot, d)$ is a differentiable function then \mathcal{R} satisfies *endowment monotonicity* if, for each $d \in \mathbb{R}^N$, the derivatives of its coordinate functions with respect to E are positive, that is, $\frac{\partial R_i}{\partial E}(E, d) \geq 0$ for all $E \in [0, d(N)]$ and for all $i \in N$.

Proposition 4.6 *The AA rule satisfies endowment monotonicity. Moreover, if $|N| \geq 3$, for each $(E, d) \in C^N$ with $d \in \mathbb{R}_{\leq}^n$, and each $i \in N$:*

$$AA(r_i(E, d), d_{-i}) \leq AA_{N \setminus \{i\}}(E, d) \leq AA(R_i(E, d), d_{-i}).$$

Proof The proof is divided into three cases.

(i) $|N| = 2$

The AA rule, whose awards for each claimant are the continuous piecewise linear functions depicted in Fig. 4, satisfies *endowment monotonicity*.

(ii) $|N| = 3$

Let $d \in \mathbb{R}_{\leq}^3$ with $0 < d_1$ and $i \in N$. Then $d_{-i} \in \mathbb{R}^2$ and the function $AA_j(\cdot, \vec{d}_{-i})$ is monotonically increasing for all $j \in N \setminus \{i\}$. So, $AA_j(r_i(E, d), d_{-i}) \leq AA_j(u, d_{-i}) \leq AA_j(R_i(E, d), d_{-i})$ for all $u \in [r_i(E, d), R_i(E, d)]$. Integrating with respect to u on the interval $[r_i(E, d), R_i(E, d)]$, using equality (4) and taking into account the fact that $g_i(E, \cdot)$ is a density function, $AA(r_i(E, d), d_{-i}) \leq AA_{N \setminus \{i\}}(E, d) \leq AA(R_i(E, d), d_{-i})$. Now, according to Theorem A.2, in order to see that $AA_j(\cdot, d)$ is monotonically increasing on $[0, d(N)]$ it suffices to prove that $AA_j(\cdot, d)$ is monotonically increasing on the interval $J = [0, \min\{\frac{1}{2}d(N), d(N \setminus \{n\})\}]$ or, equivalently, that $\frac{\partial AA_j}{\partial E}(E, d) \geq 0$ for all $E \in J$. If $E < d_1$ the result is trivial. But, using the expression for the derivatives of the AA rule given in Theorem A.2 it follows that, for all $E \in [d_1, \min\{\frac{1}{2}d(N), d(N \setminus \{n\})\}]$, and all $j \in N \setminus \{n\}$,

$$\begin{aligned} \frac{\partial AA_j}{\partial E}(E, d) &= g_n(E, E)(AA_j(E, d_{-n}) - AA_j(E, d)) \\ &\quad + \chi_n(E, d)g_n(E, E - d_n)(AA_j(E, d) - AA_j(E - d_n, d_{-n})) \geq 0 \\ \frac{\partial AA_n}{\partial E}(E, d) &= g_1(E, E)(AA_n(E, d_{-1}) - AA_n(E, d)) \\ &\quad + g_1(E, E - d_1)(AA_n(E, d) - AA_n(E - d_1, d_{-1})) \geq 0. \end{aligned}$$

Therefore, $AA_j(\cdot, d)$ is monotonically increasing for all $j \in N$.

(iii) $|N| > 3$

Let $d \in \mathbb{R}_{\leq}^n$ with $0 < d_1$, and $i \in N$. We proceed by induction on the number of claimants, so assume that $AA_j(r_i(E, d), d_{-i}) \leq AA_j(u, d_{-i}) \leq AA_j(R_i(E, d), d_{-i})$ for all $u \in [r_i(E, d), R_i(E, d)]$ and all $j \in N \setminus \{i\}$. Therefore,

$$\int_{r_i(E,d)}^{R_i(E,d)} AA_j(r_i(E,d), d_{-i})g_i(E, u)du \leq \int_{r_i(E,d)}^{R_i(E,d)} AA_j(u, d_{-i})g_i(E, u)du$$

$$\leq \int_{r_i(E,d)}^{R_i(E,d)} AA_j(R_i(E,d), d_{-i})g_i(E, u)du,$$

or, equivalently, by (4), $AA_j(r_i(E,d), d_{-i}) \leq AA_j(E,d) \leq AA_j(R_i(E,d), d_{-i})$. From these inequalities and the derivatives obtained in Theorem A.2, we conclude that $AA_j(\cdot, d)$ is monotonically increasing. □

Population monotonicity states that if the population of claimants enlarges but the amount to divide stays the same, each of the claimants initially present should receive at most as much as she did initially.

Proposition 4.7 *The AA rule satisfies population monotonicity.*

Proof Let $\{N, N'\} \subset \mathcal{N}$ such that $N' \subset N$. We have to prove that $AA_{N'}(E, d) \leq AA(E, d_{N'})$. First, let $N' = N \setminus \{i\}$ for some $i \in N$. Then, by the inequalities in Proposition 4.6 and *endowment monotonicity*, $AA_{N'}(E, d) \leq AA(R_i(E, d), d_{-i}) \leq AA(E, d_{-i})$. The general case follows applying repeatedly this result. □

Recall that, by the balance condition, if a rule satisfies *other-regarding claim monotonicity* then it also satisfies *claim monotonicity*.

Proposition 4.8 *The AA rule satisfies other-regarding claim monotonicity, claim monotonicity, and linked claim-endowment monotonicity.*

Proof We prove *other-regarding claim monotonicity*. Let $(E, d) \in C^N$ with $d \in \mathbb{R}_{\leq}^n$, $i \in N$, and $d_{i+1} \geq d'_i \geq d_i$. Denote $d' = (d_{-i}, d'_i) \in \mathbb{R}^N$. If $E \leq d_i$ then $X(E, d) = X(E, d')$ and $AA(E, d) = AA(E, d')$. Assume that $E > d_i$. Let $b = (0_{-i}, d_i)$, $h = d' - b$, $\rho = \frac{V(E,d)}{V(E,d')}$, and $j \in N \setminus \{i\}$. Then $(E, d') \in C^N$, $(E - d_i, h) \in C^N$, and $X(E, d') = X(E, d) \cup (b + X(E - d_i, h))$. Moreover, $\lambda(X(E, d) \cap (b + X(E - d_i, h))) = 0$, because $X(E, d)$ and $b + X(E - d_i, h)$ are separated by the hyperplane $x_i = d_i$. Then, applying (3), $AA_j(E, d') = \rho AA_j(E, d) + (1 - \rho)AA_j(E - d_i, h)$. But the AA rule satisfies *population monotonicity*, then applying Proposition 4.6, we have $AA_j(E - d_i, h) \leq AA_j(E - d_i, d_{-i}) \leq AA_j(E, d)$. Therefore, $AA_j(E, d') = \rho AA_j(E, d) + (1 - \rho)AA_j(E - d_i, h) \leq \rho AA_j(E, d) + (1 - \rho)AA_j(E, d) = AA_j(E, d)$.

Other-regarding claim monotonicity is a stronger requirement than *claim monotonicity*. So the AA rule satisfies *claim monotonicity*. Since *claim monotonicity* and *linked claim-endowment monotonicity* are dual properties and the AA rule is *self-dual*, the AA rule satisfies *linked claim-endowment monotonicity*. \square

Finally, let us show that the AA rule guarantees a minimal share to every agent equal to one n th her claim truncated at the amount to be divided.

Proposition 4.9 *The AA rule satisfies $\frac{1}{|N|}$ -truncated-claims lower bounds on awards and $\frac{1}{|N|}$ -min-of-claim-and-deficit lower bounds on losses.*

Proof Let $(E, d) \in C^N$ be a problem with $d \in \mathbb{R}_{\leq}^n$. If $E \in [0, d_1]$ then we know that $AA_j(E, d) = \frac{E}{n} = \frac{1}{n} \min\{E, d_j\}$ for all $j \in N$. If $E \in [d_1, d_2]$ then by Lemma A.3, $AA_j(E, d) \geq \frac{E}{n} \min\{E, d_j\}$ for all $j \in N$. Now, by repeatedly applying Lemma A.4 it is easy to see that $AA_j(E, d) \geq \frac{1}{n} \min\{E, d_j\}$ for all $j \in N$ whenever $E \in [d_2, d_n]$. But if $E \geq d_n$ and $j \in N \setminus \{n\}$ then, by *other-regarding claim monotonicity*, $AA_j(E, d) \geq AA_j(E, (d_1, \dots, d_{n-1}, E))$, and we have already shown that $AA_j(E, (d_1, \dots, d_{n-1}, E)) \geq \frac{1}{n} \min\{E, d_j\}$. Lastly, it is clear that $AA_n(E, d) \geq \frac{E}{n} \geq \frac{d_n}{n} = \frac{1}{n} \min\{E, d_n\}$. Therefore, the AA rule satisfies $\frac{1}{|N|}$ -truncated-claims lower bounds on awards. Naturally, the AA rule, being *self-dual*, also satisfies the corresponding dual property, namely, $\frac{1}{|N|}$ -min-of-claim-and-deficit lower bounds on losses. \square

5 Computation of the AA rule

The definition of the AA rule, though natural, does not provide a mechanism to compute it in terms of the $n + 1$ parameters that define a problem: the endowment E and the vector of claims $d \in \mathbb{R}^N$. Here we describe a particular procedure to compute the centroid of the set of awards vectors for a problem.

We can assume, by *anonymity*, that given $(E, d) \in C^N$, the claims are sorted in ascending order, that is, $d \in \mathbb{R}_{\leq}^n$. By *null claims consistency*, the agents whose claims are 0 can be removed so we just have to consider problems whose set of awards vectors is full dimensional, that is, $0 < d_1$ and $0 < E < d(N)$. Moreover, the AA rule is *self-dual* so we can restrict the algorithm to problems for which the endowment does not exceed the half-sum of claims, $E \leq \frac{1}{2}d(N)$. In addition, we can also assume that $E \leq d(N \setminus \{n\})$ because if $d(N \setminus \{n\}) \leq E \leq d_n$ then $AA(E, d)$ is given by expression (2). Taking all these considerations into account, and exploiting the special structure of the set of awards vectors, Mirás Calvo et al. (2022b) obtain a formula for the AA rule. The following procedure is a direct application of their results.

Procedure to compute the AA rule: Let $(E, d) \in C^N$ such that $d \in \mathbb{R}_{\leq}^n$, $0 < d_1$, and $0 < E \leq \min\{\frac{1}{2}d(N), d(N \setminus \{n\})\}$. In order to compute $AA(E, d)$ follow these steps:

1. Determine the set $\mathcal{I} = \{i \in N : d_i < E\}$.
2. Compute $p = V(E, d)$.
3. For each $i \in \mathcal{I}$, compute $p_i = V(E - d_i, (d_{-i}, E - d_i))$.⁹
4. Compute $E^* = \frac{1}{n} \left(E + \sum_{i \in \mathcal{I}} \frac{p_i}{p} d_i \right)$.
5. Finally, $AA_i(E, d) = \begin{cases} E^* - \frac{p_i}{p} d_i & \text{if } d_i < E \\ E^* & \text{otherwise} \end{cases}$.

The key step in the above algorithm is the determination of the ratios $\frac{p_i}{p}$ for each $i \in \mathcal{I}$. Therefore, we need a method to compute the volume of the set of awards vectors for a problem. Of course, one can rely on any general algorithm for the volume of a convex polyhedron [such as Lasserre (1983)]. Nevertheless, we use the simple expression for the volume of the set of awards vectors in terms of the endowment and the claims provided by Mirás Calvo et al. (2022b):

$$V(E, d) = \frac{\sqrt{n}}{(n-1)!} \left(E^{n-1} + \sum_{S \in \mathcal{F}} (-1)^{|S|} (E - d(S))^{n-1} \right) \tag{5}$$

where $\mathcal{F} = \{S \in 2^N : d(S) < E\}$. Let us illustrate how to combine the algorithm and the volume formula to compute the AA rule.¹⁰

Example 5.1 Let $N = \{1, 2, 3, 4\}$ and consider the problem $(E, d) \in C^N$ with $E = 7$ and $d = (2, 4, 6, 8) \in \mathbb{R}_{\leq}^4$. Now, $\mathcal{I} = \{i \in N : d_i < 7\} = \{1, 2, 3\}$. Therefore, we consider the problems:

$(E - d_1, (d_{-1}, E - d_1))$	$(E - d_2, (d_{-2}, E - d_2))$	$(E - d_3, (d_{-3}, E - d_3))$
$(5, (5, 4, 6, 8))$	$(3, (2, 3, 6, 8))$	$(1, (2, 4, 1, 8))$

We compute the volumes $p = V(E, d)$ and $p_i = V(E - d_i, (d_{-i}, E - d_i))$ for $i \in \{1, 2, 3\}$ using formula (5). Then,¹¹

- For the problem $(7, (2, 4, 6, 8))$ we have that $\mathcal{F} = \{S \in 2^N : d(S) < 7\} = \{\{1\}, \{2\}, \{3\}, \{1, 2\}\}$. Then:

⁹ If $i \in N$ is a claimant such that $d_i < E$ then $(E - d_i, (d_{-i}, E - d_i)) \in C^N$ represents the problem where claimant i gets her claim d_i and remains in the problem claiming all that is left, $E - d_i$.

¹⁰ This is the procedure implemented in the *ClaimsProblems* R package (Núñez Lugilde et al. 2021) to compute the AA rule.

¹¹ Since we need to compute the ratios $\frac{p_i}{p}$, the factor $\frac{\sqrt{n}}{(n-1)!}$ can be ignored. Nevertheless, for clarity, we carry this factor in the example.

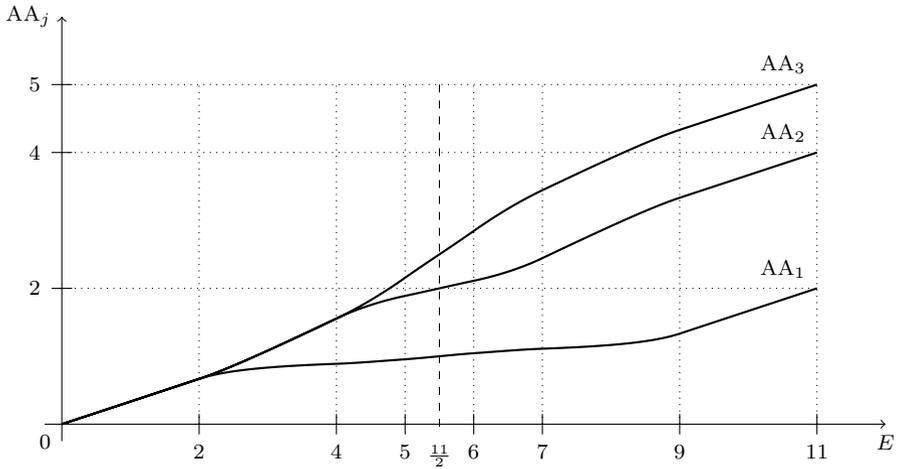


Fig. 5 The schedules of awards of the AA rule for $d = (2, 4, 5)$

$$p = V(E, d) = \frac{1}{3} (7^3 + (7 - 6)^3 - (7 - 6)^3 - (7 - 4)^3 - (7 - 2)^3) = \frac{191}{3}.$$

- $p_1 = \frac{1}{3} (5^3 - (5 - 4)^3) = \frac{124}{3}$, because for this problem $\mathcal{F} = \{\{2\}\}$.
- $p_2 = \frac{1}{3} (3^3 - (3 - 2)^3) = \frac{26}{3}$, because for this problem $\mathcal{F} = \{\{1\}\}$.
- $p_3 = \frac{1}{3} (1^3) = \frac{1}{3}$, because for this problem $\mathcal{F} = \{\emptyset\}$.

Now, we compute E^* :

$$E^* = \frac{1}{n} \left(E + \sum_{i \in \mathcal{I}} \frac{p_i}{p} d_k \right) = \frac{1}{4} \left(7 + \frac{124}{191} 2 + \frac{26}{191} 4 + \frac{1}{191} 6 \right) = \frac{1695}{764}.$$

Finally,

$$\begin{aligned} AA_1(E, d) &= \frac{1695}{764} - \frac{124}{191} 2 = \frac{703}{764}, & AA_2(E, d) &= \frac{1695}{764} - \frac{26}{191} 4 = \frac{1279}{764}, \\ AA_3(E, d) &= \frac{1695}{764} - \frac{1}{191} 6 = \frac{1671}{764}, & AA_4(E, d) &= \frac{1695}{764}. \end{aligned}$$

In summary, $AA(E, d) = (0.9202, 1.6741, 2.1872, 2.2186)$.

Naturally, when $|N| = 2$, the algorithm produces formula (1), the expression for the CD rule. For three-claimant problems, the procedure leads to an analytic expression for the AA rule.¹²

¹² Lemma 5.2 is a particular case of the general formula given in Mirás Calvo et al. (2021) to compute the core-center for three-player convex games.

Lemma 5.2 Let $N = \{1, 2, 3\}$ and $(E, d) \in C^N$ such that $d \in \mathbb{R}_{\leq}^3$ and $E \in [0, \frac{1}{2}d(N)]$. Then,

$$\begin{aligned}
 AA_1(E, d) &= \begin{cases} \frac{1}{3}E & \text{if } E \in [0, d_1] \\ \frac{E^3 - (E-d_1)^3 - 3d_1(E-d_1)^2}{3(E^2 - (E-d_1)^2)} & \text{if } E \in [d_1, d_2] \\ \frac{E^3 - (E-d_1)^3 - (E-d_2)^3 - 3d_1(E-d_1)^2}{3(E^2 - (E-d_1)^2 - (E-d_2)^2)} & \text{if } E \in [d_2, \min\{d_1 + d_2, d_3\}] \\ \frac{1}{2}d_1 & \text{if } E \in [d_1 + d_2, \frac{1}{2}(d(1) + d(2) + d(3))] \\ \frac{E^3 - (E-d_1)^3 - (E-d_2)^3 - (E-d_3)^3 - 3d_1(E-d_1)^2}{3(E^2 - (E-d_1)^2 - (E-d_2)^2 - (E-d_3)^2)} & \text{if } E \in [d_3, \frac{1}{2}(d(1) + d(2) + d(3))] \end{cases} \\
 AA_2(E, d) &= \begin{cases} \frac{1}{3}E & \text{if } E \in [0, d_1] \\ \frac{E^3 - (E-d_1)^3}{3(E^2 - (E-d_1)^2)} & \text{if } E \in [d_1, d_2] \\ \frac{E^3 - (E-d_1)^3 - (E-d_2)^3 - 3d_2(E-d_2)^2}{3(E^2 - (E-d_1)^2 - (E-d_2)^2)} & \text{if } E \in [d_2, \min\{d_1 + d_2, d_3\}] \\ \frac{1}{2}d_2 & \text{if } E \in [d_1 + d_2, \frac{1}{2}(d(1) + d(2) + d(3))] \\ \frac{E^3 - (E-d_1)^3 - (E-d_2)^3 - (E-d_3)^3 - 3d_2(E-d_2)^2}{3(E^2 - (E-d_1)^2 - (E-d_2)^2 - (E-d_3)^2)} & \text{if } E \in [d_3, \frac{1}{2}(d(1) + d(2) + d(3))] \end{cases} \\
 AA_3(E, d) &= \begin{cases} \frac{1}{3}E & \text{if } E \in [0, d_1] \\ \frac{E^3 - (E-d_1)^3}{3(E^2 - (E-d_1)^2)} & \text{if } E \in [d_1, d_2] \\ \frac{E^3 - (E-d_1)^3 - (E-d_2)^3}{3(E^2 - (E-d_1)^2 - (E-d_2)^2)} & \text{if } E \in [d_2, \min\{d_1 + d_2, d_3\}] \\ E - \frac{1}{2}(d_1 + d_2) & \text{if } E \in [d_1 + d_2, \frac{1}{2}(d(1) + d(2) + d(3))] \\ \frac{E^3 - (E-d_1)^3 - (E-d_2)^3 - (E-d_3)^3 - 3d_3(E-d_3)^2}{3(E^2 - (E-d_1)^2 - (E-d_2)^2 - (E-d_3)^2)} & \text{if } E \in [d_3, \frac{1}{2}(d(1) + d(2) + d(3))] \end{cases}
 \end{aligned}$$

Though the scope of Lemma 5.2 is restricted to three-claimant problems, it helps to illustrate the schedules of awards of the AA rule as the next example shows.

Example 5.3 Let $N = \{1, 2, 3\}$. Consider the claims vector $d = (2, 4, 5) \in \mathbb{R}^N$ so that $d(N) = 11$ and $d_3 = 5 \leq \frac{1}{2}d(N) = 5.5 \leq d(N \setminus \{3\}) = 6$. The schedules of awards $AA_j(\cdot, d)$, $j \in N$, are depicted in Fig. 5. We can see that they are monotonically increasing and that $AA_1(E, d) \leq AA_2(E, d) \leq AA_3(E, d)$ for all $E \in [0, 11]$. The self-duality property corresponds with the special symmetry of the graphs with respect to $E = 5.5$. Also, by claims truncation invariance, the three curves coincide on the interval $[0, 2]$ and $AA_2(\cdot, d) = AA_3(\cdot, d)$ on $[2, 4]$. By endowment differentiability, the three curves are smooth.

Appendix

Analyzing the centroid of the set of awards vectors for a problem requires a thorough study of the volume function. That examination was carry out in the companion paper Mirás Calvo et al. (2022b). Let us state the results that are needed in our analysis.

Let $d \in \mathbb{R}_{\leq}^n$ such that $0 < d_1$. The volume function $V(\cdot, d) : [0, d(N)] \rightarrow \mathbb{R}$ is a continuous function that is symmetric with respect to $E = \frac{1}{2}d(N)$. In fact, if $d(N \setminus \{n\}) > d_n$ then $V(\cdot, d)$ is strictly increasing on $[0, \frac{1}{2}d(N)]$ and $V(\cdot, d)$ is strictly

decreasing on $[\frac{1}{2}d(N), d(N)]$, so $V(\cdot, d)$ attains its maximum at $E = \frac{1}{2}d(N)$. On the other hand, if $d(N \setminus \{n\}) \leq d_n$ then $V(\cdot, d)$ is strictly increasing on $[0, d(N \setminus \{n\})]$, it is strictly decreasing on $[d_n, d(N)]$ and it is constant on $[d(N \setminus \{n\}), d_n]$. Moreover, if $|N| \geq 3$ then $V(\cdot, d)$ is a continuously differentiable function. For each $i \in N$ let $\chi_i(E, d) = 0$ if $E \leq d_i$ and $\chi_i(E, d) = 1$ otherwise. Then, if $E \in [0, \frac{1}{2}d(N)]$ and $i \in N$,

$$\frac{\partial V}{\partial E}(E, d) = \frac{\sqrt{n}}{\sqrt{n-1}} \left(V(E, d_{-i}) - \chi_i(E, d)V(E - d_i, d_{-i}) \right). \tag{6}$$

For each claimant $i \in N$ denote $I_i = [m_i(E, d), t_i(E, d)]$. Recall that $R_i(E, d) = E - m_i(E, d) = \min\{E, d(N \setminus \{i\})\}$ and $r_i(E, d) = E - t_i(E, d) = \max\{0, E - d_i\}$. It is easy to see that

$$X(E, d) = \bigcup_{x_i \in I_i} \{x_i\} \times X(E - x_i, d_{-i}). \tag{7}$$

For each $i \in N$ consider the weight function $g_i : (0, d(N)) \times [0, d(N \setminus \{i\})] \rightarrow \mathbb{R}$ defined, for all $(E, u) \in (0, d(N)) \times [0, d(N \setminus \{i\})]$ as:

$$g_i(E, u) = \frac{\sqrt{n}}{\sqrt{n-1}} \frac{V(u, d_{-i})}{V(E, d)}.$$

Theorem A.1 *Let $|N| \geq 3$. If $(E, d) \in \mathbb{R}^N$ with $d \in \mathbb{R}_{\leq}^n$ and $i \in N$, then*

$$\int_{r_i(E, d)}^{R_i(E, d)} g_i(E, u) du = 1 \text{ and, for all } j \in N \setminus \{i\},$$

$$AA_j(E, d) = \int_{r_i(E, d)}^{R_i(E, d)} AA_j(u, d_{-i}) g_i(E, u) du.$$

Proof Let us simplify the notation by writing $I_i = [m_i, t_i]$, $X = X(E, d)$, and $X_{x_i} = X(E - x_i, d_{-i})$. The transformation

$\pi_n(x_1, \dots, x_{n-1}) = (x_1, \dots, x_{n-1}, E - x_1 - \dots - x_{n-1})$ defines a parametrization of the hyperplane $x_1 + \dots + x_n = E$. The vector $(1, 1, \dots, 1) \in \mathbb{R}^n$ is normal to the hyperplane and it has length \sqrt{n} . For each $x_i \in I_i$ the transformation

$h_{x_i}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n-1}, (E - x_i) - x_1 - \dots - x_{i-1} - x_{i+1} - \dots - x_{n-1})$ defines a parametrization¹³ of the hyperplane $x_1 + \dots + x_{i-1} + x_{i+1} + \dots + x_{n-1} = E - x_i$. The vector $(1, 1, \dots, 1) \in \mathbb{R}^{n-1}$ is normal to the hyperplane and it has length $\sqrt{n-1}$.

From (7), we have that $\pi_n^{-1}(X) = \bigcup_{x_i \in I_i} \{x_i\} \times h_{x_i}^{-1}(X_{x_i})$. If λ_{n-1} and λ_{n-2} denote the $(n-1)$ -dimensional and $(n-2)$ -dimensional Lebesgue measures respectively, then:

¹³ If $i = n$ take $h_{x_n}(x_2, \dots, x_n) = (E - x_2 - \dots - x_n, x_2, \dots, x_n)$.

$$\lambda_{n-1}(X) = \int_X d\lambda_{n-1} = \sqrt{n} \int_{\pi_n^{-1}(X)} d\lambda_{n-1} = \sqrt{n} \int_{m_i}^{t_i} \left(\int_{h_{x_i}^{-1}(X_{x_i})} d\lambda_{n-2} \right) dx_i$$

$$\lambda_{n-2}(X_{x_i}) = \int_{X_{x_i}} d\lambda_{n-2} = \sqrt{n-1} \int_{h_{x_i}^{-1}(X_{x_i})} d\lambda_{n-2}.$$

Combining these two expressions we obtain that $\lambda_{n-1}(X) = \frac{\sqrt{n}}{\sqrt{n-1}} \int_{m_i}^{t_i} \lambda_{n-2}(X_{x_i}) dx_i$,

that is $\int_{m_i}^{t_i} g_i(E, E - x_i) dx_i = 1$. Now, if $j \neq i$, then

$$\int_X x_j d\lambda_{n-1} = \frac{\sqrt{n}}{\sqrt{n-1}} \int_{m_i}^{t_i} \left(\int_{X_{x_i}} x_j d\lambda_{n-2} \right) dx_i.$$

Therefore,

$$\begin{aligned} AA_j(E, d) = \bar{c}_j(X) &= \frac{1}{\lambda_{n-1}(X)} \int_X x_j d\lambda_{n-1} = \frac{1}{\lambda_{n-1}(X)} \frac{\sqrt{n}}{\sqrt{n-1}} \int_{m_i}^{t_i} \left(\int_{X_{x_i}} x_j d\lambda_{n-2} \right) dx_i \\ &= \frac{1}{\lambda_{n-1}(X)} \frac{\sqrt{n}}{\sqrt{n-1}} \int_{m_i}^{t_i} \bar{c}_j(X_{x_i}) \lambda_{n-2}(X_{x_i}) dx_i \\ &= \int_{m_i}^{t_i} AA_j(E - x_i, d_{-i}) g_i(E, E - x_i) dx_i, \end{aligned}$$

because, by definition, $AA_j(E - x_i, d_{-i}) = \bar{c}_j(X_{x_i}) = \frac{1}{\lambda_{n-2}(X_{x_i})} \int_{X_{x_i}} x_j d\lambda_{n-2}$. Finally, applying the change of variable $u = E - x_i$, the result follows immediately. \square

A first consequence of Theorem A.1 is that, for problems with at least three claimants, the AA rule is *endowment differentiable*.

Theorem A.2 *Let $d \in \mathbb{R}_{\leq}^n$ with $0 < d_1$. If $|N| \geq 3$ then $AA(\cdot, d)$ is a continuously differentiable function on $[0, d(N)]$. Moreover:*

1. *If $E \in [0, d_1]$ then $\frac{\partial AA_j}{\partial E}(E, d) = \frac{1}{n}$ for all $j \in N$.*
2. *If $E \in [d(N \setminus \{n\}), d_n]$ then $\frac{\partial AA_j}{\partial E}(E, d) = 0$ for $j \in N \setminus \{n\}$ and $\frac{\partial AA_n}{\partial E}(E, d) = 1$.*
3. *If $E \in [d_1, \min\{\frac{1}{2}d(N), d(N \setminus \{n\})\}]$ then, for all $j \in N \setminus \{n\}$,*

$$\begin{aligned} \frac{\partial AA_j}{\partial E}(E, d) &= g_n(E, E)(AA_j(E, d_{-n}) - AA_j(E, d)) \\ &\quad + \chi_n(E, d)g_n(E, E - d_n)(AA_j(E, d) - AA_j(E - d_n, d_{-n})) \\ \frac{\partial AA_n}{\partial E}(E, d) &= g_1(E, E)(AA_n(E, d_{-1}) - AA_n(E, d)) \\ &\quad + g_1(E, E - d_1)(AA_n(E, d) - AA_n(E - d_1, d_{-1})). \end{aligned}$$

4. If $E \in [\frac{1}{2}d(N), d(N)]$ then $\frac{\partial AA_j}{\partial E}(E, d) = \frac{\partial AA_j}{\partial E}(d(N) - E, d)$ for all $j \in N$.

Proof If $E \in [0, d_1]$ then $AA(E, d) = \frac{E}{2}$ so $\frac{\partial AA_j}{\partial E}(E, d) = \frac{1}{2}$ for all $j \in N$. If $E \in [d(N \setminus \{n\}), d_n]$ then $AA(E, d) = (\frac{d_1}{2}, \dots, \frac{d_{n-1}}{2}, E - \frac{1}{2}d(N \setminus \{n\}))$ so $\frac{\partial AA_j}{\partial E}(E, d) = 0$ for all $j \in N \setminus \{n\}$ and $\frac{\partial AA_n}{\partial E}(E, d) = 1$. Next, let us prove that $AA(\cdot, d)$ is differentiable on the interval $[d_1, \min\{\frac{\partial \bar{H}}{2}d(N), d(N \setminus \{n\})\}]$. We distinguish two cases.

Case 1: $d(N \setminus \{n\}) < \frac{\partial \bar{H}}{2}d(N)$.

Take $E \in [d_1, d(N \setminus \{n\})]$. Then $r_n(E, d) = \max\{0, E - d_n\} = 0$ and $R_n(E, d) = \min\{E, d(N \setminus \{n\})\} = E$. By Theorem A.1, we have, for all $j \in N \setminus \{n\}$, $AA_j(E, d) = \int_0^E AA_j(u, d_{-n})g_n(E, u)du$. Now, applying Leibniz’s rule for differentiation under the integral sign and using expression (6) we obtain that $AA_j(\cdot, d)$ is differentiable at E for all $j \in N \setminus \{n\}$ and

$$\begin{aligned} \frac{\partial AA_j}{\partial E}(E, d) &= \int_0^E AA_j(u, d_{-n}) \frac{\partial g_n}{\partial E}(E, u)du + AA_j(E, d_{-n})g_n(E, E) \\ &= -\frac{n}{n-1} \int_0^E AA_j(u, d_{-n}) \frac{V(u, d_{-n})V(E, d_{-n})}{(V(E, d))^2} du + AA_j(E, d_{-n}) \frac{V(E, d_{-n})}{V(E, d)} \frac{\sqrt{n}}{\sqrt{n-1}} \\ &= \frac{\sqrt{n}}{\sqrt{n-1}} \frac{V(E, d_{-n})}{V(E, d)} \left(- \int_0^E AA_j(u, d_{-n}) \frac{V(u, d_{-n})}{V(E, d)} \frac{\sqrt{n}}{\sqrt{n-1}} du + AA_j(E, d_{-n}) \right) \\ &= g_n(E, E)(AA_j(E, d_{-n}) - AA_j(E, d)). \end{aligned}$$

Since $r_1(E, d) = E - d_1$ and $R_1(E, d) = E$, from Theorem A.1,

$$AA_n(E, d) = \int_{E-d_1}^E AA_n(u, d_{-1})g_1(E, u)du.$$

Applying Leibniz’s rule and the chain rule we have that $AA_n(\cdot, d)$ is differentiable at E and

$$\begin{aligned} \frac{\partial AA_n}{\partial E}(E, d) &= \int_{E-d_1}^E AA_n(u, d_{-1}) \frac{\partial g_1}{\partial E}(E, u) du + AA_n(E, d_{-1}) g_1(E, E) \\ &\quad - AA_n(E - d_1, d_{-1}) g_1(E, E - d_1). \end{aligned}$$

From expression (6) we have

$$\begin{aligned} \frac{\partial AA_n}{\partial E}(E, d) &= -AA_n(E, d)(g_1(E, E) - g_1(E, E - d_1)) + AA_n(E, d_{-1}) g_1(E, E) \\ &\quad - AA_n(E - d_1, d_{-1}) g_1(E, E - d_1) \\ &= g_1(E, E)(AA_n(E, d_{-1}) - AA_n(E, d)) \\ &\quad + g_1(E, E - d_1)(AA_n(E, d) - AA_n(E - d_1, d_{-1})). \end{aligned}$$

Case 2: $d(N \setminus \{n\}) > \frac{1}{2}d(N)$.

If $E \in [d_1, d_n]$ then $r_n(E, d) = 0$, $R_n(E, d) = E$, $r_1(E, d) = E - d_1$ and $R_1(E, d) = E$. Applying Theorem A.1, $AA_j(E, d) = \int_0^E AA_j(u, d_{-n}) g_n(E, u) du$ for all $j \in N \setminus \{n\}$ and $AA_n(E, d) = \int_{E-d_1}^E AA_n(u, d_{-1}) g_1(E, u) du$. By Leibniz’s rule, as in the previous case, we conclude that $AA(\cdot, d)$ is differentiable at E and we obtain the same expressions for the derivatives $\frac{\partial AA_j}{\partial E}(E, d)$ for all $j \in N$. Finally, if $E \in [d_n, \frac{1}{2}d(N)]$ then $r_n(E, d) = E - d_n$, $R_n(E, d) = E$, $r_1(E, d) = E - d_1$ and $R_1(E, d) = E$. Thus, $AA_j(E, d) = \int_{E-d_n}^E AA_j(u, d_{-n}) g_n(E, u) du$ for all $j \in N \setminus \{n\}$ and $AA_n(E, d) = \int_{E-d_1}^E AA_n(u, d_{-1}) g_1(E, u) du$.

By continuity, we know that $AA(\cdot, d)$ is a continuous function on $[0, d(N)]$. We have just seen that $AA(\cdot, d)$ is also a differentiable function on $[0, \frac{1}{2}d(N)]$ except perhaps at the points d_1, d_n and $d(N \setminus \{n\})$. It is easy to check that, in fact, $AA(\cdot, d)$ is also differentiable at those points. Therefore $AA(\cdot, d)$ is differentiable on $[0, \frac{1}{2}d(N)]$. But, since the AA rule satisfies *self-duality*, if $E \in [\frac{1}{2}d(N), d(N)]$ then $d(\bar{N}) - E \in [0, \frac{1}{2}d(N)]$ and $AA(E, d) = d - AA(d(N) - E, d)$, so $AA(\cdot, d)$ is differentiable at E and $\frac{\partial AA_j}{\partial E}(E, d) = \frac{\partial AA_j}{\partial E}(d(N) - E, d)$ for all $j \in N$. \square

Finally we establish two lemmas that are needed in the proof of Proposition 4.9.

Lemma A.3 *Let $(E, d) \in C^N$ with $d \in \mathbb{R}_{\leq}^n$. If $d_1 < E \leq d_2$ then $AA_1(E, d) \geq \frac{d_1}{n}$ and $AA_j(E, d) \geq \frac{E}{n}$ for all $j \in N \setminus \{1\}$.*

Proof Clearly, the result holds when $|N| = 2$. By Theorem A.1, since $0 \leq d_1 < E$,

$$AA_1(E, d) = \int_0^{d_1} AA_1(u, d_{-n})g_n(E, u)du + \int_{d_1}^E AA_1(u, d_{-n})g_n(E, u)du. \tag{8}$$

But, by the definition of the weight function g_n ,

$$\begin{aligned} \int_0^{d_1} AA_1(u, d_{-n})g_n(E, u)du &= \frac{V(d_1, d)}{V(E, d)} \int_0^{d_1} AA_1(u, d_{-n})g_n(d_1, u)du \\ &= \frac{V(d_1, d)}{V(E, d)} AA_1(d_1, d) = \frac{V(d_1, d)}{V(E, d)} \frac{d_1}{n}. \end{aligned} \tag{9}$$

Let $u \in (d_1, E]$. By *endowment monotonicity*, $AA_1(u, d_{-n}) \geq AA_1(d_1, d_{-n}) = \frac{d_1}{n-1}$ and, from Eq. (6), $\frac{\partial V}{\partial u}(u, d) = \frac{\sqrt{n}}{\sqrt{n-1}}V(u, d_{-n})$. Therefore,

$$\int_{d_1}^E AA_1(u, d_{-n})g_n(E, u)du \geq \frac{d_1}{n-1} \int_{d_1}^E g_n(E, u)du = \frac{d_1}{n-1} \left(1 - \frac{V(d_1, d)}{V(E, d)}\right). \tag{10}$$

Combining (8), (9), and (10), and since $V(d_1, d) \leq V(E, d)$, we have that

$$AA_1(E, d) \geq \frac{d_1}{n} \frac{V(d_1, d)}{V(E, d)} + \frac{d_1}{n-1} \left(1 - \frac{V(d_1, d)}{V(E, d)}\right) \geq \frac{d_1}{n}.$$

Finally, $AA_n(E, d) \geq \frac{E}{n}$ by *order preservation of awards*. But then, by *claims truncation invariance*, $AA_j(\frac{E}{n}, d) = AA_n(E, d) \geq \frac{E}{n}$ for all $j \in N \setminus \{1\}$. □

Lemma A.4 *Let $i \in N \setminus \{1\}$ and let $d, c \in \mathbb{R}^n$. If $AA_j(E', c) \geq \frac{1}{n} \min\{E', c_j\}$ for all $j \in N$ and $E' \in [c_{i-1}, c_i]$ then $AA_j(E, d) \geq \frac{1}{n} \min\{E, d_j\}$ for all $j \in N$ and $E \in [d_i, d_{i+1}]$.*

Proof Let $j \in N \setminus \{i\}$ and $E \in [d_i, d_{i+1}]$. Then, because the AA rule satisfies *other-regarding claim monotonicity* and the assumption,

$$\begin{aligned} AA_j(E, d) &= AA_j(E, (d_1, \dots, d_i, E, \dots, E)) \geq AA_j(E, (d_1, \dots, d_{i-1}, E, \dots, E)) \\ &\geq \frac{1}{n} \min\{E, d_j\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} AA_i(E, d) &= AA_i(E, (d_1, \dots, d_i, E, \dots, E)) \geq AA_i(E, (d_1, \dots, d_{i-2}, E, d_i, E, \dots, E)) \\ &= AA_{i-1}(E, (d_1, \dots, d_{i-2}, d_i, E, \dots, E)) \geq \frac{1}{n} \min\{E, d_i\}, \end{aligned}$$

where we have applied *other-regarding claim monotonicity*, *anonymity*, and the hypothesis. □

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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