

Figure 1: (a) A phylogenetic tree $T = (V, E)$ on $X = \{x_1, x_2, \dots, x_6\}$. (b) A non-negative edge-weighting ω for the phylogenetic tree T in (a). The edges in the smallest subtree of T containing the set $Y := \{x_1, x_4, x_5\}$ are drawn bold and their total weight is $D_{(T,\omega)}^3(Y) = 11$. (c) A weighted, rooted phylogenetic tree (T, ρ, ω) on $X = \{x_1, x_2, \dots, x_5\}$ with an equidistant edge-weighting ω . The edges whose weight contribute to $D_{(T,\omega)}^3(\{x_1, x_2, x_3\}) = 3$ are drawn bold.

dissimilarities are commonly called “tree metrics” and there is an extensive literature concerning their properties (see e.g. Semple and Steel (2003) and Gordon (1987) for overviews).

Various methods have been proposed for constructing trees that exploit this observation. These essentially work by projecting an arbitrary pairwise dissimilarity onto some “nearby” tree metric (see e.g. Felsenstein (2003); de Soete (1983)). Even so, it is well-known that such methods can suffer from the fact that pairwise distance estimates involve some loss of information (see e.g. page 176 in Felsenstein (2003)). As a potential solution to this problem, Pachter and Speyer (2004) proposed using k -wise distance estimates, $k \geq 3$, to reconstruct trees, an approach which they subsequently implemented in Levy, Yoshida, and Pachter (2006) (see also Grishin (1999) where a related idea was investigated). Their rationale was that k -wise estimates are potentially more accurate since they can capture more information than pairwise distances, a point that was also made in Chapter 12 of Felsenstein (2003).

To describe Pachter and Speyer’s approach, recall that a *phylogenetic tree* (on X) is a graph-theoretical tree $T = (V, E)$ in which every non-leaf vertex has degree at least three and whose leaf-set is X (cf. Figure 1(a)). In case a real-valued weight $\omega(e)$ is associated to every edge e of T , we call T a *weighted phylogenetic tree*, and we usually denote such a tree by (T, ω) . Now, for any k -element subset $Y \subseteq X$, $k \geq 2$, let $D_{(T,\omega)}^k(Y)$ denote the total edge-weight of the smallest subtree of T with leaf-set Y (cf. Figure 1(b)). Note that this quantity is sometimes called the “phylogenetic diversity” of Y (see e.g. Faith (1992) and Steel (2005)) and that, for $k = 2$, $D_{(T,\omega)}^2(\{x, y\}) = d_{(T,\omega)}(x, y)$ for all $x, y \in X$.

In Pachter and Speyer (2004), the following result is proven:

Theorem 1. *Let (T, ω) be a weighted phylogenetic tree on X with ω non-negative and $\omega(e) > 0$ for every edge e of T that is not incident to a leaf, $|X| = n$, and $k \geq 2$ be some integer. If $n \geq 2k - 1$, then (T, ω) is determined by the map $D_{(T,\omega)}^k$ (and it is not if $2k - 2 = n > 2$).*

In other words, just as in the case $k = 2$, for $k \geq 3$ it is possible to recover (T, ω)

from the function $D_{(T,\omega)}^k$ that maps the set of subsets of X of size k (denoted $\binom{X}{k}$) to \mathbb{R} . Here we call any map $D : \binom{X}{k} \rightarrow \mathbb{R}$ a k -dissimilarity. Note that 3-dissimilarities have been investigated, for example, in Hayashi (1972), Joly and Le Calvé (1995) and Heiser and Bennani (1997), and arbitrary k -dissimilarities in Deza and Rosenberg (2000) and Warrens (2010), under names such as k -way dissimilarities, k -way distances and k -semimetrics (see also Bandelt and Dress (1994) for related work).

In this paper we shall provide a solution to the following problem raised in Pachter and Speyer (2004):

“However, if we are simply given a k -dissimilarity map $D : \binom{X}{k} \rightarrow \mathbb{R}$, we do not know how to test whether this map comes from a phylogenetic tree.”

Note that Dress and Steel (2007) study the related problem of characterizing when a map D from the set of subsets of X of size *at most* k into some Abelian group G can be represented by a phylogenetic tree on X whose edges are assigned elements from G . However, we consider subsets of X of size *precisely* k , leading to a quite different characterization.

In order to state the main result of this paper, we first recall some more definitions concerning phylogenetic trees. A *rooted* phylogenetic tree (on X), is a tree $T = (V, E)$ with (i) a distinguished vertex ρ , called the *root* of T , that has degree at least 2, (ii) leaf-set X and (iii) no vertex in $V \setminus (X \cup \{\rho\})$ with degree less than 3. In case a real-valued weight $\omega(e)$ is associated to every edge $e \in E$, we call T a *weighted, rooted* phylogenetic tree, and denote it by (T, ρ, ω) . Note that, for such a tree, we define the maps $d_{(T,\omega)}$ and $D_{(T,\omega)}^k$ in the same way as for (unrooted) phylogenetic trees (cf. Figure 1(c)). In addition, we call an edge-weighting ω of T *equidistant* if (i) $d_{(T,\omega)}(x, \rho) = d_{(T,\omega)}(x', \rho)$ for all $x, x' \in X$, and (ii) $d_{(T,\omega)}(x, u) \leq d_{(T,\omega)}(x, v)$ for all $x \in X$ and any $u, v \in V$ that lie on the path from x to ρ in T which first meets u and then v (cf. Figure 1(c)). Such weightings commonly arise when modeling sequence evolution assuming a molecular clock (see e.g. Felsenstein (2003)).

Now, we call a k -dissimilarity D *treelike* if there exists a weighted phylogenetic tree (T, ω) with ω non-negative such that $D = D_{(T,\omega)}^k$ holds, and we call D *equidistant* if there exists a weighted, rooted phylogenetic tree (T, ρ, ω) on X with ω equidistant such that $D = D_{(T,\omega)}^k$ holds. In this paper, we shall prove the following:

Theorem 2. *Let $k \geq 2$ and D be a k -dissimilarity map on a set X with $|X| \geq 2k$. Then D is treelike/equidistant if and only if the restriction of D to every $2k$ -element subset of X is treelike/equidistant. Moreover, for all $k \geq 3$ there exist k -dissimilarity maps whose restrictions to every $(2k - 1)$ -element subset of X are treelike/equidistant but that are not treelike/equidistant.*

Note that for the case $k = 2$ this result is well-known (see e.g. Semple and Steel (2003, Theorem 7.2.5 and Corollary 7.2.7)).

After presenting some preliminaries in the next section, we prove Theorem 2 in Sections 3 and 4. As a corollary of Theorem 2, we also show that, for fixed $k \geq 2$, there

is an algorithm with run-time that is bounded by a polynomial in $|X|$ to decide if an arbitrary k -dissimilarity D is treelike (Corollary 1). It would be interesting to know if such algorithms can be found that have good run-time bounds for $k \geq 3$, such as those that have been devised for $k = 2$ (see e.g. Culberson and Rudnicki (1989), Bandelt (1990)). More generally, it might also be of interest to use Theorem 2 to help devise new methods to construct trees from k -dissimilarities such as the one described in Levy et al. (2006).

Note that for $k = 2$ the bound $2k = 4$ given in the second sentence of Theorem 2 is sharp for treelike dissimilarities, but that it can be improved to $2k - 1 = 3$ for equidistant dissimilarities (see e.g. Semple and Steel (2003, Theorem 7.2.5)). Although this is not the case for $k \geq 3$, in Section 5 we shall prove that under certain circumstances it may still be possible to recover a tree from a k -dissimilarity D on X in case it is equidistant on every $(2k - 1)$ -element subset of X (see Theorem 5).

We conclude the paper by considering 3-dissimilarities in more detail. It is well-known (see e.g. Gordon (1987) and Semple and Steel (2003)) that treelike and equidistant 2-dissimilarities can be characterized in terms the *4-point* and *ultrametric* condition, respectively (for more details see Section 6). Thus, for $k \geq 3$, we can ask for similar “ m -point” conditions that characterize treelike/equidistant k -dissimilarities. This question has been studied in Rubei (2011) for the case $k = 3$, where a recursive characterization is provided, and related problems are considered in Bocci and Cools (2009) in the context of tropical geometry. In addition, a necessary (but not sufficient) $(k + 2)$ -point condition is given for the general case in (Pachter and Speyer, 2004, p. 618).

In the last section, we provide explicit 6-point characterizations for 3-dissimilarities that are treelike/equidistant, which can be regarded as generalizations of the 4-point/ultrametric conditions (Theorem 7). We conclude with a short discussion as to why finding similar conditions for $k \geq 4$ appears to be somewhat more challenging.

2 Preliminaries on phylogenetic trees

For the remainder of this paper, X will always denote a non-empty, finite set. Also, for a k -dissimilarity $D : \binom{X}{k} \rightarrow \mathbb{R}$ and $\{x_1, x_2, \dots, x_k\} \in \binom{X}{k}$, we will write $D(x_1, x_2, \dots, x_k)$ instead of $D(\{x_1, x_2, \dots, x_k\})$.

We now recall some further definitions concerning phylogenetic trees (for more details see Semple and Steel (2003)). Let $T = (V, E)$ be a phylogenetic tree on X . A vertex $v \in V \setminus X$ is called an *interior* vertex of T . The set of leaves of T , that is, the set X , is also denoted by $L(T)$. Recall that it is assumed that all interior vertices have degree at least three. An edge $e \in E$ is called *pendant* if it is incident to a leaf of T . All other edges are called *interior* edges.

Now, two phylogenetic trees $T_1 = (V_1, E_1)$ and $T_2 = (V_2, E_2)$ on the same set X are *isomorphic* if there exists a bijective map $\iota : V_1 \rightarrow V_2$ such that $\iota(x) = x$ holds for all $x \in X$ and $\{u, v\} \in E_1$ if and only if $\{\iota(u), \iota(v)\} \in E_2$ for any two distinct $u, v \in V_1$. In case we also have edge-weightings $\omega_i : E_i \rightarrow \mathbb{R}$, $i \in \{1, 2\}$, the weighted phylogenetic trees

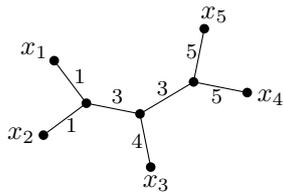


Figure 2: The weighted phylogenetic tree $(T(M), \omega(M))$ arising from the weighted, rooted phylogenetic tree in Figure 1(c) for $M = 2$.

(T_1, ω_1) and (T_2, ω_2) are isomorphic if, in addition, $\omega_1(\{u, v\}) = \omega_2(\{\iota(u), \iota(v)\})$ holds for every edge $\{u, v\} \in E_1$. Note that interior edges with weight 0 can give rise to non-isomorphic weighted phylogenetic trees that induce the same k -dissimilarity. Therefore, in the following we will always implicitly assume that in any weighted phylogenetic tree interior edges are assigned positive weights. We call such edge-weightings *interior-positive*, for short.

We also apply the above terminology to (weighted) rooted phylogenetic trees with the following minor adaptations. For two rooted phylogenetic trees T_1 and T_2 with roots ρ_1 and ρ_2 to be *isomorphic* we require, in addition, that $\iota(\rho_1) = \rho_2$ holds. Note that in a weighted, rooted phylogenetic tree (T, ρ, ω) with ω equidistant, every interior edge has a non-negative weight while pendant edges might have negative weights (cf. Figure 1(c)). Again, to avoid non-isomorphic weighted, rooted phylogenetic trees giving rise to the same k -dissimilarity, we always assume that the edge-weightings are interior-positive. A rooted phylogenetic tree $T = (V, E)$ on X with root ρ is *binary* if every vertex in $V \setminus (X \cup \{\rho\})$ has degree precisely three and ρ has degree two.

Next note that for every weighted, rooted phylogenetic tree (T, ρ, ω) with, not necessarily non-negative, equidistant edge-weighting ω there exists a constant $M \geq 0$ such that the edge-weighting ω_M , that assigns weight $\omega(e)$ to every interior edge e and weight $\omega(e) + M$ to every pendant edge e , is also equidistant and non-negative. Thus, given a weighted, rooted phylogenetic tree (T, ρ, ω) on X with ω equidistant, we can construct, for any sufficiently large constant $M \geq 0$, a weighted phylogenetic tree $(T(M), \omega(M))$ on X with $\omega(M)$ non-negative and interior-positive as follows: If ρ has degree at least three, then put $T(M) = T$ and $\omega(M) = \omega_M$. Otherwise, delete ρ and connect the two vertices u and v adjacent to ρ by a new edge with weight $\omega_M(\{\rho, u\}) + \omega_M(\{\rho, v\})$ (cf. Figure 2). Note that if M is known, we can completely recover (T, ρ, ω) from $(T(M), \omega(M))$.

For every rooted phylogenetic tree $T = (V, E)$ on X with root ρ there is a natural partial ordering \leq_T on V with unique minimal element ρ defined by $v \leq_T w$ if and only if v is a vertex of the unique path from w to ρ in T . The rooted subtree T_v of T induced by $v \in V$ has vertex set $\{u \in V : v \leq_T u\}$ and root v . In addition, for any equidistant edge-weighting ω of T , we define the *height* $h_{(T, \omega)}(v)$ of v , also referred to as the height of T_v , as the value $d_{(T, \omega)}(v, x)$ for any leaf x of T_v . Note that this height is well-defined in view of the fact that ω is equidistant.

Finally, for rooted, as well as unrooted, phylogenetic trees T on X we denote, for any subset $Y \subseteq X$, the smallest subtree of T containing the vertices in Y by $T|_Y$ and

refer to it as the *restriction* of T to Y . (To formally view $T|_Y$ as a phylogenetic tree on Y , we suppress any vertices of degree 2.) In case T is rooted, we also consider $T|_Y$ as a rooted phylogenetic tree where we distinguish the minimal element in the vertex set of $T|_Y$ with respect to the partial order \leq_T as the root of the restriction. And, in case T has an edge-weighting ω , we consider $T|_Y$ as a weighted tree with edge-weighting $\omega|_Y$ obtained by restricting ω to the edge set of $T|_Y$.

3 Determining trees

We begin this section by stating a uniqueness theorem that will be useful later:

Theorem 3. *For every integer $k \geq 2$ and every set X with at least $2k - 1$ elements we have:*

- (i) *Two weighted phylogenetic trees (T_1, ω_1) and (T_2, ω_2) on X with ω_i non-negative and interior-positive, $i \in \{1, 2\}$, are isomorphic if and only if $D_{(T_1, \omega_1)}^k = D_{(T_2, \omega_2)}^k$ holds.*
- (ii) *Two weighted, rooted phylogenetic trees (T_1, ρ_1, ω_1) and (T_2, ρ_2, ω_2) on X with ω_i equidistant and interior-positive, $i \in \{1, 2\}$, are isomorphic if and only if $D_{(T_1, \omega_1)}^k = D_{(T_2, \omega_2)}^k$ holds.*

Note that for $k = 2$ parts (i) and (ii) of this theorem are well-known (see e.g. Semple and Steel (2003, Theorem 7.1.8)). Moreover, part (i) is just a restatement of Theorem 1 above due to Pachter and Speyer, and part (ii) immediately follows from part (i) by considering the weighted phylogenetic trees $(T_1(M), \omega_1(M))$ and $(T_2(M), \omega_2(M))$ for some sufficiently large constant $M \geq 0$.

We now prove the first part of Theorem 2:

Theorem 4. *Let $k \geq 2$ and D be a k -dissimilarity map on a set X , $|X| \geq 2k$.*

- (i) *D is treelike if and only if the restriction of D to every $2k$ -element subset of X is treelike.*
- (ii) *D is equidistant if and only if the restriction of D to every $2k$ -element subset of X is equidistant.*

Proof. (i) For $k = 2$ this well-known (see e.g. Semple and Steel (2003)). So we shall assume in the following that $k \geq 3$ holds. Clearly, if D is treelike, then also the restriction to every $2k$ -element subset of X is treelike.

Conversely, assume that the restriction of D to every $2k$ -element subset of X is treelike. Note that this implies that the restriction of D to every i -element subset Y of X , $k \leq i \leq 2k$, is treelike, that is, there exists a weighted phylogenetic tree (T_Y, ω_Y) on Y with ω_Y non-negative and interior-positive such that $D|_Y = D_{(T_Y, \omega_Y)}^k$ holds.

Now consider an arbitrary pair of elements $\{a, b\} \in \binom{X}{2}$. We claim that in any weighted phylogenetic tree (T_Z, ω_Z) , $Z \in \binom{X}{2k-1}$, $\{a, b\} \subseteq Z$, the induced distance

$d_{(T_Z, \omega_Z)}(a, b)$ is the same. To show this, it suffices to consider such sets Z, Z' with $Z' = (Z \setminus \{x\}) \cup \{y\}$ for two distinct elements $x, y \in X \setminus \{a, b\}$. We now consider the weighted phylogenetic tree (T_Y, ω_Y) for the $2k$ -element set $Y := Z \cup \{y\}$. Since $|Z| = |Z'| = 2k - 1$, it follows by Theorem 3(i) that (T_Z, ω_Z) is isomorphic to $(T_Y|_Z, \omega_Y|_Z)$ and $(T_{Z'}, \omega_{Z'})$ is isomorphic to $(T_Y|_{Z'}, \omega_Y|_{Z'})$. This implies that the induced distance between a and b is the same for (T_Z, ω_Z) and $(T_{Z'}, \omega_{Z'})$, as claimed.

As a consequence, for every pair $\{a, b\} \in \binom{X}{2}$, the restriction of D to any $(2k - 1)$ -element subset of X containing a and b yields the same distance between a and b , which we denote by $\delta(a, b)$. Note that the restriction of the so-defined 2-dissimilarity δ on X to every 4-element subset of X is treelike: For any four distinct elements $a, b, c, d \in X$ we can select an arbitrary $Z \in \binom{X}{2k-1}$ with $\{a, b, c, d\} \subseteq Z$ and in the weighted phylogenetic tree (T_Z, ω_Z) the induced distances between a, b, c, d will equal the corresponding values of δ . Hence, (since the theorem holds for $k = 2$) there exists a unique weighted phylogenetic tree (T, ω) on X with ω non-negative and interior-positive such that $D_{(T, \omega)}^2 = \delta$ holds. Moreover, the restriction of (T, ω) to any $2k$ -element subset $Z \subseteq X$ is isomorphic to (T_Z, ω_Z) . Hence, the k -dissimilarity $D_{(T, \omega)}^k$ must be D .

(ii) Again, if a k -dissimilarity on X is equidistant, so is its restriction to each $2k$ -subset of X . So, let D be a k -dissimilarity on X such that its restriction to each $Y \in \binom{X}{2k}$ is represented by a weighted, rooted phylogenetic tree (T_Y, ρ_Y, ω_Y) on Y with ω_Y equidistant and interior-positive. Then, for some sufficiently large $M \geq 0$, all the weighted phylogenetic trees $(T_Y(M), \omega_Y(M))$ are such that $\omega_Y(M)$ is non-negative and interior-positive. Therefore, by the first part of the theorem, there exists a unique weighted phylogenetic tree (T, ω) on X with ω non-negative and interior-positive such that $D_{(T, \omega)}^k(A) = D(A) + kM$ holds for all $A \in \binom{X}{k}$. Moreover, (T, ω) must be isomorphic to $(T'(M), \omega'(M))$ for some weighted, rooted phylogenetic tree (T', ρ', ω') on X with ω' equidistant and interior-positive, since otherwise there would exist some $Y \in \binom{X}{2k}$ such that ω_Y is not equidistant in view of the fact that, for all $Y \in \binom{X}{2k}$, $(T|_Y, \omega|_Y)$ is isomorphic to $(T_Y(M), \omega_Y(M))$. Hence D must equal $D_{(T', \omega')}^k$, as required. \square

Using this theorem we now show that, for fixed $k \geq 3$, it is possible to efficiently check when a k -dissimilarity is treelike/equidistant. Note that any algorithm to check whether a given k -dissimilarity D is treelike/equidistant needs to read D first. Assuming that D is given as the list of values it takes on for each k -element subset of X , this yields a lower bound of $|X|^k$ on the run-time of any such algorithm.

Corollary 1. *For any fixed $k \geq 3$ and any k -dissimilarity D on X , there is an algorithm with run-time in $O(f(k) \cdot |X|^{2k})$ to decide whether D is treelike/equidistant or not, where f is a function that does not depend on $|X|$.*

Proof. Given a k -dissimilarity D on X , it suffices to check for every $Z \in \binom{X}{2k}$ whether $D|_Z$ is treelike/equidistant. To do this, one can enumerate all (isomorphism classes of) unweighted phylogenetic trees (rooted or unrooted) with $2k$ leaves labeled by the elements in Z . Note that the number of these trees depends on k but not on $|X|$. For each of those trees T , it remains to check if there exists an edge-weighting ω with

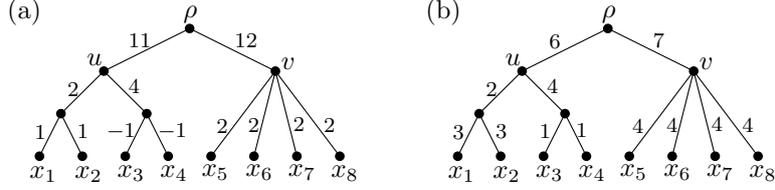


Figure 3: (a) A weighted, rooted phylogenetic tree (T, ρ, ω) with an equidistant and interior-positive edge-weighting ω . (b) The same rooted phylogenetic tree but with the equidistant edge-weighting $\omega_{(k, \alpha)}$ for $k = 5$ and $\alpha = 10$. Note that $D_{(T, \omega)}^5 = D_{(T, \omega_{(5, 10)})}^5$.

certain properties so that $D_{(T, \omega)}^k = D|_Z$ holds. The latter can be phrased as a test whether a system of linear equations and inequalities has a solution, a problem for which a polynomial time algorithm is known (see e.g. Schrijver (1986)). Therefore, one can check in $O(f(k))$ time whether $D|_Z$ is treelike/equidistant where f is a function that does not depend on $|X|$. Since the number of $2k$ -element subsets of X is in $\mathcal{O}(|X|^{2k})$, this establishes the required run-time bound. \square

4 Sharpness of the bounds

In this section, we shall prove the second part of Theorem 2, that is, we shall prove that the bounds presented in the theorem are indeed sharp. More specifically, for each $k \geq 3$, we will present an example of a k -dissimilarity D whose restriction to every $(2k - 1)$ -element subset is treelike/equidistant while D is not treelike/equidistant. These examples will be presented in Examples 1 and 2 below.

We begin by presenting a useful lemma. Assume we have $k \geq 3$, $|X| \geq k$ and that $(T = (V, E), \rho, \omega)$ is a weighted, rooted phylogenetic tree on X with ω equidistant and interior-positive. In addition, assume that ρ is adjacent to precisely two vertices u and v (cf. Figure 3(a)). Put $a = \omega(\{\rho, u\})$ and $b = \omega(\{\rho, v\})$. Now define, for each $\alpha \in \mathbb{R}$ with $\alpha < 2 \min\{a, b\}$, a new equidistant edge-weighting $\omega_{(k, \alpha)}$ for T (cf. Figure 3(b)) by putting, for all $e \in E$,

$$\omega_{(k, \alpha)}(e) = \begin{cases} \omega(e) + \alpha/k, & \text{if } e \text{ is incident to a leaf,} \\ \omega(e) - \alpha/2, & \text{if } e \text{ is incident to } \rho, \\ \omega(e), & \text{else.} \end{cases}$$

Lemma 1. *Suppose $k \geq 3$, $|X| \geq k$ and that $(T = (V, E), \rho, \omega)$ is a weighted, rooted phylogenetic tree on X with ω equidistant and interior-positive such that ρ is adjacent with precisely two vertices u and v and put $a = \omega(\{\rho, u\})$ and $b = \omega(\{\rho, v\})$. Then for each $\alpha \in \mathbb{R}$ with $\alpha < 2 \min\{a, b\}$ and $A \in \binom{X}{k}$ we have*

$$D_{(T, \omega_{(k, \alpha)})}^k(A) = \begin{cases} D_{(T, \omega)}^k(A) + \alpha, & \text{if } A \subseteq L(T_u) \text{ or } A \subseteq L(T_v), \\ D_{(T, \omega)}^k(A), & \text{else.} \end{cases}$$

In particular, (T, ρ, ω) and $(T, \rho, \omega_{(k,\alpha)})$ induce the same k -dissimilarity if and only if we have $\alpha = 0$ or $|L(T_u)| \leq k - 1$ and $|L(T_v)| \leq k - 1$ hold.

Proof. Let $A \subseteq \binom{X}{k}$. The restriction $T|_A$ contains k pendant edges. Moreover, $T|_A$ contains the two edges incident in T with ρ if and only if we have $A \cap L(T_u) \neq \emptyset$ and $A \cap L(T_v) \neq \emptyset$. Hence we have $D_{(T, \omega_{(k,\alpha)})}^k(A) = D_{(T, \omega)}^k(A) + k \frac{\alpha}{k}$ in case $A \subseteq L(T_u)$ or $A \subseteq L(T_v)$ holds and we have $D_{(T, \omega_{(k,\alpha)})}^k(A) = D_{(T, \omega)}^k(A) + k \frac{\alpha}{k} - 2 \frac{\alpha}{2} = D_{(T, \omega)}^k(A)$ otherwise, as claimed. The second assertion trivially holds if $\alpha = 0$. If $\alpha \neq 0$, then it holds since all $A \in \binom{X}{k}$ contain leaves from both $L(T_u)$ and $L(T_v)$ if and only if $|L(T_u)| \leq k - 1$ and $|L(T_v)| \leq k - 1$ hold. \square

Example 1 (Equidistant). Let $k \geq 3$ and (T, ρ, ω) be a weighted, rooted phylogenetic tree on X with ω equidistant and interior-positive such that the root ρ of T is adjacent with precisely two vertices u and v . Put $a = \omega(\{\rho, u\})$ and $b = \omega(\{\rho, v\})$. Assume that $|L(T_u)|, |L(T_v)| \geq k$. Now, choose some non-zero $\alpha < 2 \min\{a, b\}$ and define a k -dissimilarity D on X via

$$D(A) := \begin{cases} D_{(T, \omega_{(k,\alpha)})}^k(A), & \text{if } A \subseteq L(T_u), \\ D_{(T, \omega)}^k(A), & \text{else,} \end{cases}$$

for all $A \in \binom{X}{k}$. We first show that the restriction of D to any $(2k - 1)$ -element subset of X is equidistant: For any $Y \in \binom{X}{2k-1}$ we define a weighted, rooted phylogenetic tree (T_Y, ρ_Y, ω_Y) on Y with ω_Y equidistant and interior-positive by setting

$$(T_Y, \rho_Y, \omega_Y) := \begin{cases} (T, \rho, \omega_{(k,\alpha)})|_Y, & \text{if } |Y \cap L(T_u)| \geq k, \\ (T, \rho, \omega)|_Y, & \text{else.} \end{cases}$$

By the definition of D it follows that $D|_Y = D_{(T_Y, \omega_Y)}^k$ holds.

We now show that D is not equidistant. It suffices to show that there exists some subset Z of X such that $D|_Z$ is not equidistant. So, let $Z \subseteq X$ be such that $|Z \cap L(T_u)| = |Z \cap L(T_v)| = k$ (such a subset exists since $|L(T_u)|, |L(T_v)| \geq k$), and suppose there exists some weighted, rooted phylogenetic tree (T_Z, ρ_Z, ω_Z) on Z with ω_Z equidistant and interior-positive such that $D|_Z = D_{(T_Z, \omega_Z)}^k$ holds. For every $a \in Z \cap L(T_u)$ we define the weighted, rooted phylogenetic tree $(T_a, \rho_a, \omega_a) := (T_Z, \rho_Z, \omega_Z)|_{Z \setminus \{a\}}$ on $Z \setminus \{a\}$. Choose distinct $x, y \in Z \cap L(T_u)$ which are not adjacent to a common vertex of degree 3 (this is possible since $|Z \cap L(T_u)| \geq k > 2$). By Theorem 3(ii), (T_x, ρ_x, ω_x) is isomorphic to $(T_{Z \setminus \{x\}}, \rho_{Z \setminus \{x\}}, \omega_{Z \setminus \{x\}})$ and $(T, \rho, \omega)|_{Z \setminus \{x\}}$, and (T_y, ρ_y, ω_y) is isomorphic to $(T_{Z \setminus \{y\}}, \rho_{Z \setminus \{y\}}, \omega_{Z \setminus \{y\}})$ and $(T, \rho, \omega)|_{Z \setminus \{y\}}$.

By our choice of x and y , up to isomorphism, there exists only one possible weighted, rooted phylogenetic tree on Z whose restriction to $Z \setminus \{x\}$ and $Z \setminus \{y\}$ is T_x and T_y , respectively. Hence (T_Z, ρ_Z, ω_Z) is isomorphic to $(T, \rho, \omega)|_Z$. However, this contradicts the fact that (T_Z, ρ_Z, ω_Z) induces $D|_Z$ since $D(Z \cap L(T_u)) = D_{(T, \omega_{(k,\alpha)})}^k(Z \cap L(T_u)) \neq D_{(T, \omega)}^k(Z \cap L(T_u)) = D_{(T_Z, \omega_Z)}^k(Z \cap L(T_u))$, where the first equality is by definition and the second follows from the fact that (T_Z, ρ_Z, ω_Z) and $(T, \rho, \omega)|_Z$ are isomorphic.

Example 2 (Treelike). An example for the case $k = 3$ was given by Chepoi and Fichet (2007). Here we give an example for general k . Based on the weighted, rooted phylogenetic tree (T, ρ, ω) in Example 1, consider $(T(M), \omega(M))$ for some sufficiently large constant $M \geq 0$ and choose $\alpha > 0$. Then, using the same arguments as in Example 1, it is straight-forward to show that the k -dissimilarity D constructed in the same way as in Example 1 is not treelike while its restriction to every $(2k - 1)$ -element subset of X is treelike.

5 The case $2k - 1$ for equidistant k -dissimilarities

It is well-known that if the restriction of a 2-dissimilarity D on X to every subset of X of size 3 is equidistant, then D is equidistant (Semple and Steel, 2003, Theorem 7.2.5). In contrast, in Example 1, we have seen that for $k \geq 3$ a k -dissimilarity D is not necessarily equidistant if its restriction to every $(2k - 1)$ -element subset is equidistant. However, we shall now prove that we can still recover a tree if the restriction of such a D to all $(2k - 1)$ -element subsets $Y \subseteq X$ is induced by a weighted, rooted phylogenetic tree (T, ρ, ω) on Y with ω equidistant and interior-positive such that (T, ρ, ω) is *generic*, that is, T is binary and no two distinct interior vertices have the same height.

Theorem 5. *Let $k \geq 3$ and D be a k -dissimilarity map on X such that, for all $Y \in \binom{X}{2k-1}$, there exists a generic weighted, rooted phylogenetic tree (T_Y, ρ_Y, ω_Y) on Y with ω_Y equidistant and interior-positive such that $D|_Y = D_{(T_Y, \omega_Y)}$ holds. Then there exists a binary rooted phylogenetic tree T on X such that, for all $Y \in \binom{X}{2k-1}$, the unweighted, rooted phylogenetic trees T_Y and $T|_Y$ are isomorphic.*

To prove this theorem, we shall use a well-known result about collections of rooted phylogenetic trees each having three leaves, that will allow us to “merge” trees, which we now recall. A *triplet* on X is a pair $(\{a, b\}, c)$ with $a, b, c \in X$ distinct, which we denote also by $ab|c$. The set of all triplets on X is denoted by $\mathcal{R}(X)$, and a subset \mathcal{R} of $\mathcal{R}(X)$ is called a *triplet system* on X . Given a rooted phylogenetic tree T on X , the triplet system \mathcal{R}_T of T is the set of all triplets $ab|c$ on X such that the path from a to b in T is vertex-disjoint from the path from c to the root ρ in T . It is easily seen that for a rooted phylogenetic tree T on X and a rooted phylogenetic tree T' on $Y \subseteq X$, we have $\mathcal{R}_{T'} \subseteq \mathcal{R}_T$ if T' is isomorphic to $T|_Y$. We now state the aforementioned result:

Theorem 6 (Theorem 9.2 (ii) in Dress, Huber, Koolen, Moulton, and Spillner (2011)). *A rooted phylogenetic tree T on X is, up to isomorphism, uniquely determined by the triplet system \mathcal{R}_T . Moreover, given a triplet system $\mathcal{R} \subseteq \mathcal{R}(X)$ there exists a rooted phylogenetic tree on X with $\mathcal{R}_T = \mathcal{R}$ if and only if \mathcal{R} satisfies the following two conditions:*

(R1) *For any three elements $a, b, c \in X$ at most one of the triplets $ab|c$, $bc|a$ and $ca|b$ is contained in \mathcal{R} .*

(R2) *For any four elements $a, b, c, d \in X$, $ab|c \in \mathcal{R}$ implies $ad|c \in \mathcal{R}$ or $ab|d \in \mathcal{R}$.*

To prove Theorem 5 we will also use the following rather technical result:

Lemma 2. *Let $k \geq 3$ be an integer, X a set with $|X| = 2k - 2$ and (T_1, ρ_1, ω_1) and (T_2, ρ_2, ω_2) be two generic weighted, rooted phylogenetic trees on X with ω_i equidistant and interior-positive, $i \in \{1, 2\}$. If $D_{(T_1, \omega_1)}^k = D_{(T_2, \omega_2)}^k$ then T_1 and T_2 are isomorphic as unweighted, rooted phylogenetic trees.*

Proof. We distinguish two cases. First consider the case that at least one of T_1 and T_2 , say T_1 , contains a vertex v_1 such that the set B of leaves of the rooted subtree $(T_1)_{v_1}$ has cardinality $k - 1$. Define $A := X \setminus B$. We claim that T_2 must contain a vertex v_2 such that the set of leaves of $(T_2)_{v_2}$ is B . To establish this, first note that $D_{(T_1, \omega_1)}^k(A \cup \{b\}) = D_{(T_1, \omega_1)}^k(A \cup \{b'\})$ and, therefore, in view of $D_{(T_1, \omega_1)}^k = D_{(T_2, \omega_2)}^k$, also $D_{(T_2, \omega_2)}^k(A \cup \{b\}) = D_{(T_2, \omega_2)}^k(A \cup \{b'\})$ must hold for all $b, b' \in B$. This implies that, for every $b \in B$, the height of the vertex w_b where the path from b to the root of T_2 first meets the subtree $T_2|_A$ must be the same. Hence, since T_2 is generic, $w_b = w_{b'}$ holds for all $b, b' \in B$. But then the tree $T_2|_B$ must equal the tree $(T_2)_{v_2}$ for some vertex v_2 of T_2 , as claimed.

Next, we claim that $T_1|_A$ and $T_2|_A$ as well as $T_1|_B$ and $T_2|_B$ are isomorphic as unweighted, rooted phylogenetic trees. By Theorem 6 it suffices to show that $\mathcal{R}(T_1|_A) = \mathcal{R}(T_2|_A)$ and $\mathcal{R}(T_1|_B) = \mathcal{R}(T_2|_B)$ holds. In the following we will focus on the set A . A completely analogous argument yields $\mathcal{R}(T_1|_B) = \mathcal{R}(T_2|_B)$. So, consider three arbitrary distinct elements a, b and c in A and an arbitrary $(k - 2)$ -element subset C of B . Up to relabeling, Figure 4 depicts the possible cases for the structure of the tree $T_1|_{C \cup \{a, b, c\}}$. Note that, by the assumption that T_1 is generic, cases (b), (d), (g), (i), (j) and (k) are ruled out. In the remaining cases we have:

- (a) $D_{(T_1, \omega_1)}^k(C \cup \{a, b\}) = D_{(T_1, \omega_1)}^k(C \cup \{a, c\}) < D_{(T_1, \omega_1)}^k(C \cup \{b, c\})$
- (c) (and, similarly, (e) and (f))
 $D_{(T_1, \omega_1)}^k(C \cup \{b, c\}) < D_{(T_1, \omega_1)}^k(C \cup \{a, b\}) = D_{(T_1, \omega_1)}^k(C \cup \{a, c\})$
- (h) $D_{(T_1, \omega_1)}^k(C \cup \{b, c\}) < D_{(T_1, \omega_1)}^k(C \cup \{a, c\}) < D_{(T_1, \omega_1)}^k(C \cup \{a, b\})$

So, we have $ab|c \in \mathcal{R}(T_1|_A)$ if and only if either

- (1) $D_{(T_1, \omega_1)}^k(C \cup \{a, c\}) = D_{(T_1, \omega_1)}^k(C \cup \{b, c\}) \neq D_{(T_1, \omega_1)}^k(C \cup \{a, b\})$ holds, or
- (2) $D_{(T_1, \omega_1)}^k(C \cup \{a, b\})$, $D_{(T_1, \omega_1)}^k(C \cup \{a, c\})$ and $D_{(T_1, \omega_1)}^k(C \cup \{b, c\})$ are pairwise distinct and $D_{(T_1, \omega_1)}^k(C \cup \{a, b\})$ is the smallest value among them.

But this implies, in view of the fact that $D_{(T_1, \omega_1)}^k = D_{(T_2, \omega_2)}^k$ holds, that we have $\mathcal{R}(T_1|_A) = \mathcal{R}(T_2|_A)$, as required.

Now, to show that the trees T_1 and T_2 are isomorphic as unweighted, rooted phylogenetic trees, it remains to show that $T_1|_B$ and $T_2|_B$ are attached to $T_1|_A$ and $T_2|_A$, respectively, at the same position. To establish this, consider the set A^* containing all $a \in A$ such that $D_{(T_1, \omega_1)}^k(B \cup \{a\})$ is minimal. Note that there must exist vertices w_1 in

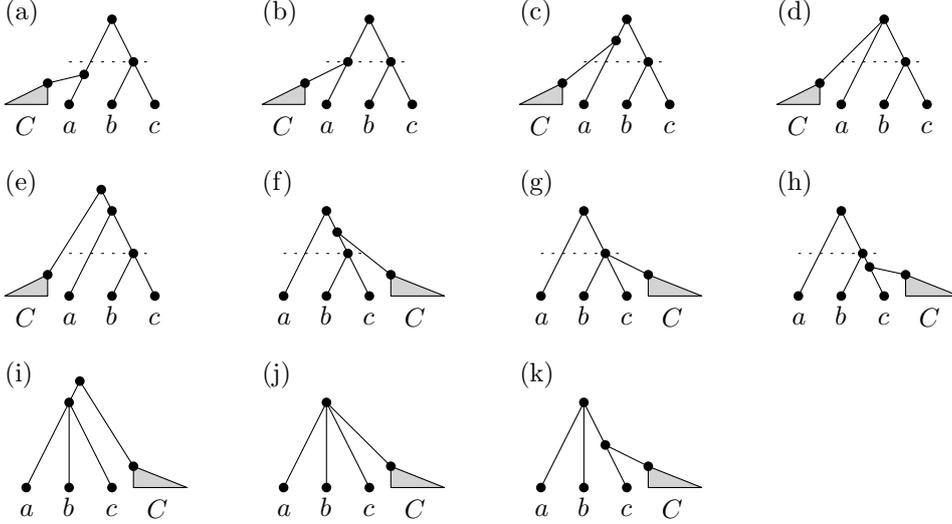


Figure 4: Schematic representations of the cases considered in the proof of Lemma 2 in the context of establishing that $\mathcal{R}(T_1|_A) = \mathcal{R}(T_2|_A)$ holds.

T_1 and w_2 in T_2 such that $T_1|_{A^*} = (T_1)_{w_1}$ and $T_2|_{A^*} = (T_2)_{w_2}$ hold. Moreover, note that v_1 and w_1 as well as v_2 and w_2 must be adjacent to a common vertex, namely the vertex where $T_1|_B$ and $T_2|_B$ are attached to $T_1|_A$ and $T_2|_A$, respectively. But this implies that T_1 and T_2 are isomorphic as unweighted, rooted phylogenetic trees.

Next consider the case that neither T_1 nor T_2 contains a vertex such that the subtree induced by that vertex has precisely $k - 1$ leaves. Choose some $M \in \mathbb{R}$ large enough and consider the weighted phylogenetic trees $(T_1(M), \omega_1(M))$ and $(T_2(M), \omega_2(M))$. Note that, for every edge e of $T_j(M)$, $j \in \{1, 2\}$, removing e from $T_j(M)$ yields two subtrees, one of which has at least k leaves. But this is the crucial property used in the proof of Theorem 1 presented in Pachter and Speyer (2004), and the lower bound $|X| \geq 2k - 1$ stated in this theorem is only needed to ensure that this property holds. Hence, even in case $|X| = 2k - 2$ the proof can be applied as long as we have this property. This implies that $T_1(M)$ and $T_2(M)$ are isomorphic (even as weighted phylogenetic trees!) and, hence, also T_1 and T_2 , as required. \square

Proof of Theorem 5. Define $\mathcal{R} := \bigcup_{Y \in \binom{X}{2k-1}} \mathcal{R}_{T_Y}$. We have to show that Conditions (R1) and (R2) hold for \mathcal{R} .

To show that (R1) holds, it suffices to show that for all $Z \in \binom{X}{2k-2}$, and distinct $x, y \in X \setminus Z$, $a, b, c \in Z$ at most one of the triplets $ab|c$, $bc|a$ and $ca|b$ is contained in $\mathcal{R}_{T_{Z \cup \{x\}}} \cup \mathcal{R}_{T_{Z \cup \{y\}}}$. Lemma 2 implies that the phylogenetic trees $T^* := T_{\{Z \cup \{x\}}|_Z}$ and $T_{Z \cup \{y\}}|_Z$ are isomorphic. By Theorem 6, Condition (R1) holds for \mathcal{R}_{T^*} , as required.

We now show that (R2) holds. Let a, b, c, d be distinct elements of X and suppose $ab|c \in \mathcal{R}$. Then there exists some $Y \in \binom{X}{2k-1}$ such that $ab|c \in \mathcal{R}_{T_Y}$. If $d \in Y$, then we have $ad|c \in \mathcal{R}$ or $ab|d \in \mathcal{R}$ since (R2) holds for \mathcal{R}_{T_Y} . If $d \notin Y$, take some $x \in Y \setminus \{a, b, c\}$ and define $Y' := (Y \setminus \{x\}) \cup \{d\}$. Again, by Lemma 2, the phylogenetic trees $T_Y|_{Y \setminus \{x\}}$ and $T_{Y'}|_{Y \setminus \{x\}}$ are isomorphic, hence $ab|c$ is also an element of $\mathcal{R}_{T_{Y'}}$ and hence $ad|c \in \mathcal{R}$.

or $ab|d \in \mathcal{R}$ since $(\mathcal{R}2)$ holds for \mathcal{R}_{T_Y} . \square

Remark 1. We suspect, but have not been able to prove, that, for $k \geq 3$, if a k -dissimilarity on X is induced by an *arbitrary* weighted, rooted phylogenetic tree (T_Y, ρ_Y, ω_Y) with ω_Y equidistant and interior-positive for all $Y \in \binom{X}{2k-1}$, then it determines a rooted phylogenetic tree T on X such that, for all $Y \in \binom{X}{2k-1}$, the tree T_Y is isomorphic to $T|_Y$. Furthermore, depending on the topology of the unweighted phylogenetic tree T arising in this way, it might even still be possible to assign weights to the edges of T so that the edge-weighting is equidistant and the induced k -dissimilarity is D . For example, if the number of leaves in one of the subtrees induced by the vertices adjacent to the root of T is smaller than k , then one can extend the arguments in the proof of Theorem 4 to show that one can indeed construct a suitable edge-weighting for T , and hence D is equidistant in this case.

6 3-dissimilarities

In this section, we prove that treelike and equidistant 3-dissimilarities can be characterized by certain 6-point conditions. We begin by recalling some conditions for characterizing treelike and equidistant 2-dissimilarities (see e.g. Smolenskii (1962); Zaretsky (1965); Buneman (1971); Gordon (1987); and Semple and Steel (2003)).

It is well-known that a 2-dissimilarity D on X is treelike if and only if D is non-negative, it satisfies the *triangle inequality* (i.e., $D(x_1, x_3) \leq D(x_1, x_2) + D(x_2, x_3)$ holds for any three distinct elements $x_1, x_2, x_3 \in X$) and

$$D(x, x') + D(y, y') \leq \max\{D(x, y) + D(x', y'), D(x, y') + D(x', y)\} \quad (1)$$

holds for any four distinct $x, x', y, y' \in X$. Similarly, it is known that D is equidistant if and only if

$$D(x, y) \leq \max\{D(x, z), D(z, y)\} \quad (2)$$

holds for any three distinct $x, y, z \in X$.

Inequalities (1) and (2) are commonly called the *4-point* and *ultrametric* conditions, respectively. Note that non-negativity of D and the triangle inequality follow from the 4-point condition if one defines $D(a, a) = 0$ for all $a \in X$ and then drops the requirement that the elements are pairwise distinct. However, as we view a k -dissimilarity as being a map from $\binom{X}{k}$ into \mathbb{R} , we need to explicitly require these additional properties.

We now present similar conditions that characterize treelike/equidistant 3-dissimilarities that are obtained by associating with every 3-dissimilarity a suitable 2-dissimilarity. The construction of this 2-dissimilarity is similar to the approach followed in the context of so-called *perimeter models* considered, for example, in Heiser and Bennani (1997) and Chepoi and Fichet (2007).

Theorem 7. *Let D be a 3-dissimilarity on a set X with $|X| \geq 5$.*

(i) D is treelike if and only if for all $\{a, b, c, d, e\} \in \binom{X}{5}$

$$\begin{aligned} D(a, c, d) + D(a, c, e) + D(a, d, e) + D(b, c, d) + D(b, c, e) + D(b, d, e) \\ \leq 2(D(a, b, c) + D(a, b, d) + D(a, b, e) + D(c, d, e)), \end{aligned} \quad (3)$$

$$\begin{aligned} 2(D(a, c, d) + D(a, c, e) + D(b, d, e)) \leq D(a, b, c) + D(a, b, d) + D(a, b, e) \\ + D(a, d, e) + D(b, c, d) + D(b, c, e) + D(c, d, e), \end{aligned} \quad (4)$$

$$D(a, c, d) + D(a, c, e) + D(b, d, e) \leq \max \left\{ \begin{array}{l} D(a, b, d) + D(a, b, e) + D(c, d, e) \\ D(a, d, e) + D(b, c, d) + D(b, c, e) \end{array} \right\}, \quad (5)$$

and for all $\{a, b, c, d, e, e'\} \in \binom{X}{6}$

$$\begin{aligned} 2D(a, b, e) - D(a, c, e) - D(a, d, e) - D(b, c, e) - D(b, d, e) + 2D(c, d, e) = \\ 2D(a, b, e') - D(a, c, e') - D(a, d, e') - D(b, c, e') - D(b, d, e') + 2D(c, d, e'). \end{aligned} \quad (6)$$

(ii) D is equidistant if and only if for all $\{a, b, c, d, e\} \in \binom{X}{5}$

$$D(a, b, e) + D(c, d, e) \leq \max \left\{ \begin{array}{l} D(a, c, e) + D(b, d, e) \\ D(a, d, e) + D(b, c, e) \end{array} \right\} \quad (7)$$

and for all $\{a, b, c, d, e, e'\} \in \binom{X}{6}$ Equation (6) holds.

Proof. For any $Y = \{a, b, c, d, e\} \in \binom{X}{5}$ we define a map $\delta_Y : \binom{Y}{2} \rightarrow \mathbb{R}$ as follows. First define the vector

$$v_Y := (D(a, b, c), D(a, b, d), D(a, b, e), D(a, c, d), \dots, D(c, d, e))^T$$

as well as the following matrix and its inverse (note that A has full rank):

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \quad A^{-1} = \frac{1}{6} \cdot \begin{pmatrix} 2 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & 2 \\ 2 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & 2 & -1 \\ -1 & 2 & -1 & 2 & -1 & 2 & 2 & -1 & 2 & -1 \\ -1 & -1 & 2 & -1 & 2 & 2 & 2 & -1 & -1 & -1 \\ 2 & -1 & -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 \\ -1 & 2 & -1 & -1 & 2 & -1 & 2 & -1 & 2 & -1 \\ -1 & -1 & 2 & 2 & -1 & -1 & -1 & 2 & 2 & -1 \\ -1 & -1 & 2 & 2 & -1 & -1 & 2 & -1 & -1 & 2 \\ -1 & 2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 & 2 \\ 2 & -1 & -1 & -1 & -1 & 2 & -1 & -1 & 2 & 2 \end{pmatrix}$$

Then, using the notation

$$u_Y = (\delta_Y(a, b), \delta_Y(a, c), \delta_Y(a, d), \delta_Y(a, e), \delta_Y(b, c), \dots, \delta_Y(d, e))^T,$$

the map δ_Y is defined by the unique solution of the system of linear equations

$$2v_Y = A \cdot u_Y. \quad (8)$$

In particular we have:

$$\begin{aligned}
3\delta_Y(a, b) &= 2D(a, b, c) + 2D(a, b, d) + 2D(a, b, e) - D(a, c, d) - D(a, c, e) - D(a, d, e) \\
&\quad - D(b, c, d) - D(b, c, e) - D(b, d, e) + 2D(c, d, e), \\
3\delta_Y(a, c) &= 2D(a, b, c) - D(a, b, d) - D(a, b, e) + 2D(a, c, d) + 2D(a, c, e) - D(a, d, e) \\
&\quad - D(b, c, d) - D(b, c, e) + 2D(b, d, e) - D(c, d, e), \\
3\delta_Y(a, d) &= -D(a, b, c) + 2D(a, b, d) - D(a, b, e) + 2D(a, c, d) - D(a, c, e) + 2D(a, d, e) \\
&\quad - D(b, c, d) + 2D(b, c, e) - D(b, d, e) - D(c, d, e), \tag{9} \\
3\delta_Y(b, c) &= 2D(a, b, c) - D(a, b, d) - D(a, b, e) - D(a, c, d) - D(a, c, e) + 2D(a, d, e) \\
&\quad + 2D(b, c, d) + 2D(b, c, e) - D(b, d, e) - D(c, d, e), \\
3\delta_Y(b, d) &= -D(a, b, c) + 2D(a, b, d) - D(a, b, e) - D(a, c, d) + 2D(a, c, e) - D(a, d, e) \\
&\quad + 2D(b, c, d) - D(b, c, e) + 2D(b, d, e) - D(c, d, e), \\
3\delta_Y(c, d) &= -D(a, b, c) - D(a, b, d) + 2D(a, b, e) + 2D(a, c, d) - D(a, c, e) - D(a, d, e) \\
&\quad + 2D(b, c, d) - D(b, c, e) - D(b, d, e) + 2D(c, d, e).
\end{aligned}$$

It is not hard to see that $D|_Y$ is treelike/equidistant if and only if δ_Y is treelike/equidistant. This is the key observation that will allow us to translate the 4-point/ultrametric condition characterizing when a 2-dissimilarity is treelike/equidistant into conditions for when a 3-dissimilarity is treelike/equidistant.

Before we do this, we need some condition that ensures that, for any two distinct $a, b \in X$ and any two distinct $Z, Z' \in \binom{X}{5}$ with $\{a, b\} \subseteq Z \cap Z'$, we have $\delta_Z(a, b) = \delta_{Z'}(a, b)$. Clearly, it suffices to consider such sets $Z, Z' \in \binom{X}{5}$ with $|Z \cap Z'| = 4$. In particular, for $Z = \{a, b, c, d, e\}$ and $Z' = (Z \setminus \{e\}) \cup \{e'\}$ we obtain Equation (6) which ensures that the map $\delta : \binom{X}{2} \rightarrow \mathbb{R}$ defined by putting $\delta(a, b) := \delta_Z(a, b)$ for an arbitrary $Z \in \binom{X}{5}$ with $\{a, b\} \subseteq Z$ is well-defined. And in this case δ_Z is treelike/equidistant for all $Z \in \binom{X}{5}$ if and only if δ is treelike/equidistant if and only if D is treelike/equidistant.

We now prove the two assertions of the theorem: (i) Using the equations in (9), it is not hard to check that the conditions for δ (to be non-negative, to satisfy the triangle inequality and the 4-point condition) translate into Inequalities (3), (4) and (5), respectively. (ii) Again it is not hard to check that, using the equations in (9), the ultrametric condition on δ translates into Inequality (7). \square

Note that Bocci and Cools (2009, Theorem 3.2) showed that there exists a family of maps ϕ_k from the set of all treelike 2-dissimilarities to the set of all treelike k -dissimilarities that maps a 2-dissimilarity $D_{(T, \omega)}^2$ induced by a weighted phylogenetic tree (T, ω) on X with ω non-negative to the treelike k -dissimilarity $\phi_k(D_{(T, \omega)}^2) = D_{(T, \omega)}^k$. For $k = 3$ this map can be thought of as multiplication with the matrix A as considered in the proof of Theorem 7, but for $k \geq 4$ it appears that no such simple representation is possible.

Indeed, the key observation used in proving the above result is that the restriction of any treelike/equidistant 3-dissimilarity D to every 5-element subset $Y \subseteq X$ can be related to a unique treelike/equidistant 2-dissimilarity on Y by a system of linear

equations (as this allowed the straight-forward translation of the 4-point/ultrametric condition into a 5-point condition). Unfortunately, it seems that there are problems when we try to apply this idea in case $k \geq 4$, even for $k = 4$.

More specifically, first note that, although the restriction of any treelike 4-dissimilarity D to every 6-element subset $Y \subseteq X$ can be related to a treelike 2-dissimilarity on Y by a system of linear equations, to do this one has to select a suitable ordering of the elements in Y (see also Bocci and Cools (2009, Theorem 2.2)). In contrast, in the case $k = 3$ any ordering works. Moreover, the system of linear equations does not need to have a unique solution, that is, even after fixing a suitable ordering, there can be more than one 2-dissimilarity on Y associated to the 4-dissimilarity D . This further complicates the translation of the 4-point condition into some form of 8-point condition for when D is treelike.

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