On b-perfect chordal graphs*

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Abstract

The b-chromatic number of a graph G is the largest integer k such that G has a coloring of the vertices in k color classes such that every color class contains a vertex that has a neighbour in all other color classes. We characterize the class of chordal graphs for which the b-chromatic number is equal to the chromatic number for every induced subgraph.

1 Introduction

We deal here with finite undirected graphs. Given a graph G and an integer $k \geq 1$, a coloring of G with k colors is a mapping $c: V(G) \rightarrow \{1, \ldots, k\}$ such that any two adjacent vertices u, v in G satisfy $c(u) \neq c(v)$. For every vertex v, the integer c(v) is called the color of v. The sets $c^{-1}(1), \ldots, c^{-1}(k)$ that are not empty are called the color classes of c. A b-coloring is a coloring such that every color class contains a vertex that has a neighbour in each color class other than its own, and we call any such vertex a b-vertex. The b-chromatic number b(G) of a graph G is the largest integer k such that G admits a b-coloring with exactly k colors. The concept of b-coloring was introduced in [6] and has been studied among others in [2, 4, 7, 8, 9]. Let $\omega(G)$ be the maximum size of a clique in a graph G, and let $\chi(G)$ be the chromatic number of G. It is easy to see that every coloring of G with $\chi(G)$ colors is a b-coloring, and so every graph satisfies $\chi(G) \leq b(G)$. Hoàng and Kouider [4] call a graph G b-perfect if every induced subgraph H of G satisfies $b(H) = \chi(H)$. Also a graph G is b-imperfect if it is not b-perfect, and minimally b-imperfect if it is b-imperfect and every proper subgraph

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of G is b-perfect. Hoàng, Linhares Sales and Maffray [5] found a list \mathcal{F} of twenty-two minimally b-imperfect graphs shown in Figure 1, and posed the following conjecture.

Conjecture 1 ([5]). A graph is b-perfect if and only if it does not contain any member of \mathcal{F} as an induced subgraph.

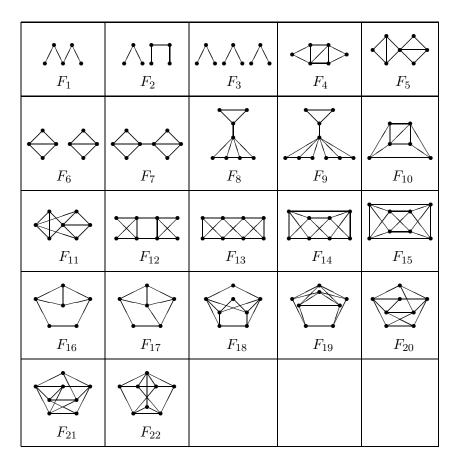


Figure 1: Class $\mathcal{F} = \{F_1, ..., F_{22}\}$

Given a collection \mathcal{H} of graphs, a graph G is usually called \mathcal{H} -free if no induced subgraph of G is a member of \mathcal{H} . When \mathcal{H} consists of only one graph H, we may write H-free instead of $\{H\}$ -free. We let P_k and C_k respectively denote the graph that consists of a path (resp. cycle) on k vertices. We use + to denote the disjoint union of graphs, and nF is the graph which has n components all isomorphic to F. For example, $2K_2$ is the graph with two

components of size 2, and the first three graphs in \mathcal{F} are P_5 , $P_4 + P_3$ and $3P_3$. We say that two vertices x, y in a graph G are *twins* if every vertex of $G \setminus \{x, y\}$ that is adjacent to any of x, y is adjacent to both. Note that two twins may be adjacent or not.

It is a routine matter to check that the graphs in class \mathcal{F} are b-imperfect and minimally so. More precisely, for i=1,2,3, we have $\chi(F_i)=2$ and $b(F_i)=3$, and F_i admits a b-coloring with 3 colors in which its three vertices of degree 3 have color 1,2,3 respectively; and for $i=4,\ldots,22$, we have $\chi(F_i)=3$ and $b(F_i)=4$.

We will prove the conjecture in the case of chordal graphs. Recall that a graph G is chordal [3, 10] if every cycle of length at least four in G has a chord (an edge between non-consecutive vertices of the cycle). We call hole any chordless cycle of length at least four. In these terms, a graph is chordal if and only if it is hole-free.

Theorem 1. Every \mathcal{F} -free chordal graph is b-perfect.

Proof of Theorem 1. Suppose that the theorem is false, and let G be a counterexample to the theorem for which |V(G)| + |E(G)| is minimal. Recall that, since G is chordal, it satisfies $\chi(G) = \omega(G)$ (see [1, 3]). Since G is a counterexample to the theorem, it admits a b-coloring c with $k \geq \chi(G) + 1 = \omega(G) + 1$ colors. For $i = 1, \ldots, k$, let u_i be any b-vertex of color i, that is, a vertex that has a neighbour of each color other than i. Let $U = \{u_1, \ldots, u_k\}$. Note that, since $k > \omega(G)$, the set U does not induce a clique. As usual, we say that a vertex is simplicial if its neighbourhood induces a clique.

1.1. For i = 1, ..., k, vertex u_i is not simplicial.

Proof. Suppose on the contrary and up to symmetry that u_1 is simplicial. Since u_1 is a b-vertex, it has a neighbour v_i of each color $i=2,\ldots,k$. Then the set $\{u_1,v_2,\ldots,v_k\}$ induces a clique of size $k>\omega(G)$, a contradiction. So Claim 1.1 holds.

1.2. G contains a $2K_2$.

Proof. Suppose that G contains no $2K_2$. Since U is not a clique, we may assume up to symmetry that u_1, u_2 are not adjacent. By Claim 1.1, vertex u_1 has two neighbours v, v' that are not adjacent, and vertex u_2 has two neighbours w, w' that are not adjacent. Suppose that u_1 is adjacent to w. Then u_1 is not adjacent to w', for otherwise u_1, w, u_2, w' induce a hole. One of v, v' is not equal to w, say $v \neq w$. Also $v \neq w'$ since u_1 is adjacent to v and not to v'. If v is not adjacent to v, then v is adjacent to v', for

otherwise $\{u_1, v, u_2, w'\}$ induces a $2K_2$; but then either $\{u_1, v, w', u_2, w\}$ or $\{u_1, v, u_2, w\}$ induce a hole. So v is adjacent to u_2 . Then u_2 is not adjacent to v', for otherwise $\{u_1, v, v', u_2\}$ induces a hole. Then v' is adjacent to w', for otherwise $\{u_1, v', u_2, w'\}$ induces a $2K_2$. But then either $\{u_1, v', u_2, w, w'\}$ (if v', w are not adjacent) or $\{v', u_2, w, w'\}$ (if v', w are adjacent) induces a hole. Therefore u_1 is not adjacent to w. Similarly, u_1 is not adjacent to w', and u_2 is not adjacent to any of v, v'. Now v must be adjacent to w, for otherwise $\{u_1, v, u_2, w\}$ induces a $2K_2$, and by symmetry, to w' as well. But then $\{v, u_2, w, w'\}$ induces a hole, a contradiction. So Claim 1.2 holds.

We say that a subgraph of G is big if it contains at least two vertices. Since G contains a $2K_2$, it contains a set S that induces a subgraph with at least two big components and is maximal with this property. Let $R = V(G) \setminus S$.

1.3. Every vertex of R has a neighbour in every big component of S.

Proof. Suppose on the contrary that some vertex x of R has no neighbour in some big component C of S. Then $S \cup \{x\}$ induces a subgraph with at least two big components (of which C is one), which contradicts the maximality of S. So Claim 1.3 holds.

1.4. R is a clique.

Proof. Suppose on the contrary that there are two non-adjacent vertices u, v in R. Consider two big components Z_1, Z_2 of S. By Claim 1.3, for each i = 1, 2, u has a neighbour u_i in Z_i and v has a neighbour v_i in Z_i . Since Z_i is connected, we may choose u_i, v_i and a path $u_i cdots cdots v_i$ in Z_i such that this path is as short as possible (possibly $u_i = v_i$). So no interior vertex of this path is adjacent to u or v. But then the union of the two paths $u_1 cdots c$

1.5. There is a big component Z of S such that every vertex of R is adjacent to every vertex of every big component of $S \setminus Z$.

Proof. Suppose the contrary, that is, there are two big components Z_1, Z_2 of S and vertices x_1, x_2 of R such that x_1 has a non-neighbour in Z_1 and x_2 has a no-neighbour in Z_2 . For each i = 1, 2, since Z_i is connected and by Claim 1.3, there are adjacent vertices y_i, z_i in Z_i such that x_i is adjacent to y_i and not to z_i . If $x_1 = x_2$, then $z_1-y_1-x_1-y_2-z_2$ is a P_5 in G, which contradicts that G is \mathcal{F} -free. So $x_1 \neq x_2$, and by the same argument we

may assume that x_1 is adjacent to all of Z_2 and that x_2 is adjacent to all of Z_1 . By Claim 1.4, vertices x_1, x_2 are adjacent. Then $\{x_1, x_2, y_1, y_2, z_1, z_2\}$ induces an F_4 , which contradicts that G is \mathcal{F} -free. So Claim 1.5 holds.

Let Z be a big component of S as described in Claim 1.5. Let $T = S \setminus Z$. So T contains a big component of S. Put $U_Z = U \cap Z$ and $U_T = U \cap T$.

1.6. For every vertex $a \in R$ and every set $Y \subset Z$ that induces a connected subgraph and contains no neighbour of a, there exists a vertex of Z that is adjacent to all of $Y \cup \{a\}$.

Proof. Pick any vertex y in Y. Since Z is connected, and a has a neighbour in Z by Claim 1.3, there is a shortest path z_0 - z_1 -···- z_p in Z such that z_0 is adjacent to a and $z_p = y$. Let t be any vertex in a big component of T. By Claim 1.5, vertices a, t are adjacent. Then p = 1, for otherwise z_2 - z_1 - z_0 -a-t is a P_5 . Thus z_0 is adjacent to both a, y. We show that z_0 is adjacent to all of Y. In the opposite case, since Y is connected there are adjacent vertices y', y'' such that z_0 is adjacent to y' and not to y''; but then y''-y'- z_0 -a-t is a P_5 , a contradiction. So Claim 1.6 holds.

1.7.
$$|R| \leq \omega(G) - 2$$
.

Proof. By the definition of S, the set T contains two adjacent vertices a, b. By Claim 1.4, $R \cup \{a, b\}$ is a clique. So Claim 1.7 holds.

1.8.
$$U_Z \neq \emptyset$$
.

Proof. Suppose on the contrary that Z contains no vertex of U. Consider the graph $G' = G \setminus Z$. Clearly, G' is a chordal and \mathcal{F} -free graph, and |V(G')| + |E(G')| < |V(G)| + |E(G)|. We show that c is a b-coloring of G'. To establish this, consider vertex u_i for each $i = 1, \ldots, k$ and consider any color $j \neq i$. If u_i is not in R, then u_i has the same neighbours in G and in G', so u_i is a b-vertex in G'. Now suppose that u_i is in R. If u_j is in a component of G of cardinality 1, then $G(u_j) \subseteq G$, so $G(u_i)$ is a simplicial vertex by Claim 1.4, which contradicts Claim 1.1. Thus $G(u_i)$ is in a big component of $G(u_i)$. Then $G(u_i)$ is a neighbour of $G(u_i)$ by Claim 1.5 and the definition of $G(u_i)$. Thus every $G(u_i)$ is a b-vertex for $G(u_i)$ by Claim 1.5 and the definition of $G(u_i)$. Thus every $G(u_i)$ is a b-vertex for $G(u_i)$ by Claim 1.5 and the definition of $G(u_i)$.

1.9. T contains no P_4 and no $2P_3$.

Proof. Suppose on the contrary that T contains a set Q of vertices that induces a P_4 or a $2P_3$. Therefore Z contains no P_3 , for otherwise taking

a P_3 in Z plus Q would give an induced F_2 or F_3 . Since Z is connected and contains no P_3 , it is a clique. By Claim 1.8, we may assume that u_1 is in Z. For $j=2,\ldots,k$, let v_j be a neighbour of u_1 of color j. Since $\{u_1,v_2,\ldots,v_k\}$ is not a clique, we may assume that v_2,v_3 are not adjacent. Since $N(u_1) \subset R \cup Z$ and both R,Z are cliques, we may assume that $v_2 \in R$ and $v_3 \in Z$. By Claim 1.7, R contains at most k-3 of the v_j 's; so we may assume that $v_4 \in Z$. Now, if v_2 is not adjacent to v_4 , then $W \cup \{v_1,v_2,v_3,v_4\}$ induces an F_8 or F_9 ; while if v_2 is adjacent to v_4 then the same set contains an induced F_5 . So Claim 1.9 holds.

1.10. $U_T \neq \emptyset$.

Proof. Suppose on the contrary that T contains no vertex of U. Let G' be the graph obtained from G by removing all edges whose two endvertices are in T. Graph G' satisfies |V(G')| + |E(G')| < |V(G)| + |E(G)| since we have removed at least one edge because T contains a big component of S. We will show that (a) c is a b-coloring of G', (b) G' is a chordal graph, and (c) G' is \mathcal{F} -free. These facts will imply that G' is a counterexample to the theorem, which will contradict the minimality of G and complete the proof of the claim.

To prove (a), it suffices to observe that every vertex of U is a b-vertex for c in G', because the edges we have removed from G to obtain G' are not incident with any vertex of U.

To prove (b), observe that in G' all vertices of T are simplicial (because their neighbourhood is in R) and thus cannot lie in a hole of G'. Moreover, $G' \setminus T = G \setminus T$. So G' contains no hole and is chordal.

Now we prove (c). Suppose on the contrary that G' contains a member F of \mathcal{F} . Note that G' does not contain F_i for $i=10,\ldots,22$, because every such F_i contains a hole of length 4 or 5, while G' is chordal. Thus F must be one of F_1,\ldots,F_9 . Graph F must contain two vertices of T that are adjacent in G, for otherwise F would be an induced subgraph of G. Let x,y be two vertices of T in F that are adjacent in G. So x,y lie in the same big component of T, and it follows from Claim 1.5 that the neighbourhood of each of them in G' is R. In particular, in F they are non-adjacent twins. This immediately implies that F cannot be F_1 , F_4 or F_8 since such graphs do not have twins. Thus F must be one of F_2 , F_3 , F_5 , F_6 , F_7 , F_9 . Note that, in each of these six cases, there is up to symmetry only one pair of non-adjacent twins.

Suppose that F is either F_2 or F_3 . So F has vertices $x, y, a, z_1, \ldots, z_p$, edges xa, ya, and either (if F is F_2) p = 4 and $\{z_1, \ldots, z_4\}$ induces a P_4 , or (if F is F_3) p = 6 and $\{z_1, \ldots, z_6\}$ induces a $2P_3$ with edges $z_1z_2, z_2z_3, z_4z_5, z_5z_6$.

As observed above, we may assume that $x, y \in T$ and consequently $a \in R$; then vertices z_1, \ldots, z_p are in a big component of S, and, by Claim 1.5, they cannot be in T, so they are in Z. Let p = 4. By Claim 1.6, Z contains a vertex z that is adjacent in G to a, z_1, \ldots, z_4 . Then $\{z, z_1, \ldots, z_4, a, x, y\}$ induces an F_8 in G, a contradiction. Now let p = 6. By Claim 1.6, Z contains a vertex z that is adjacent in G to a, z_1, z_2, z_3 and a vertex z' that is adjacent in G to a, z_4, z_5, z_6 . If $z \neq z'$, then $\{z, z', z_1, \ldots, z_6\}$ induces an F_6 or F_7 in G, a contradiction. So z = z'. But then $\{z, z_1, \ldots, z_6, a, x, y\}$ induces an F_9 in G, a contradiction.

Suppose that F is either F_5 or F_9 . So F has vertices $x, y, a, b, z_1, \ldots, z_p$, edges $xa, xb, ya, yb, ab, az_1, z_1z_2, z_1z_3, z_2z_3$ and either (if F is F_5) p=3 and az_2 is an edge, or (if F is F_9) p=6 and vertices z_4, z_5, z_6 induce a P_3 and are adjacent to a. As observed above, we may assume that $x, y \in T$ and consequently $a, b \in R$, and so $z_1, \ldots, z_p \in Z$. By Claim 1.6, Z contains a vertex z that is adjacent in G to b, z_1, z_2, z_3 . Then z is adjacent to a, for otherwise $\{z, a, b, z_1\}$ induces a hole in G. But then $\{z, a, b, z_1, z_3, x\}$ induces an F_4 in G, a contradiction.

Finally suppose that F is either F_6 or F_7 . So F has vertices x,y,a,b,z_1,\ldots,z_4 and edges $xa,xb,ya,yb,ab,z_1z_2,z_1z_3,z_1z_4,z_2z_3,z_2z_4$ and possibly (if F is F_7) the edge az_1 . As observed above, we may assume that $x,y\in T$ and consequently $a,b\in R$ and $z_1,\ldots,z_4\in Z$. By Claim 1.6, Z contains a vertex z that is adjacent in G to a,z_2,z_3,z_4 . Vertex z is also adjacent to z_1 , for otherwise $\{z,z_1,z_3,z_4\}$ induces a hole. By Claim 1.6, Z contains a vertex z' that is adjacent in G to b,z_1,\ldots,z_4 . If none of z,z' is adjacent to both a,b, then either $\{z,z',a,b\}$ or $\{z,z',a,b,z_2\}$ induces a hole. So we may assume, up to symmetry, that z is adjacent to both a,b. But then $\{z,a,b,x,z_2,z_3,z_4\}$ induces an F_5 in G, a contradiction. Thus Claim 1.10 holds.

1.11. U_T is a clique.

Proof. Suppose on the contrary that u_1, u_2 are non adjacent vertices of U_T . By Claim 1.1, vertex u_1 has two neighbours v, v' that are not adjacent, and vertex u_2 has two neighbours w, w' that are not adjacent. By Claims 1.4 and 1.5 we have $v, v', w, w' \in T$. If u_1 is adjacent to w, then $\{u_1, w, u_2, w'\}$ induces a P_4 or a hole, which contradicts Claim 1.9 or the chordality of G. So u_1 is not adjacent to w, and by symmetry it is not adjacent to w', and u_2 is not adjacent to any of v, v'. If v is adjacent to w, then $\{v, u_1, v', w\}$ induce a P_4 or a hole, a contradiction. So v is not adjacent to w, and by symmetry it is not adjacent to w', and v' is not adjacent to any of w, w'.

But now $\{u_1, v, v', u_2, w, w'\}$ induces a $2P_3$, which contradicts Claim 1.9. So Claim 1.11 holds.

By Claim 1.10, there is a vertex u of U in T. By Claim 1.1, vertex u has two neighbours t, t' that are not adjacent. By Claims 1.4 and 1.5, we have $t, t' \in T$. In other words, there is a P_3 t-u-t' in T.

1.12. Z contains no P_4 and no $2P_3$.

Proof. In the opposite case, a P_4 or $2P_3$ from Z plus the P_3 t-u-t' from T form an induced F_2 or F_3 in G, a contradiction. So Claim 1.12 holds.

1.13. U_Z is a clique.

Proof. Suppose on the contrary that u_1, u_2 are non adjacent vertices of U_Z . Since Z is connected, it contains a path from u_1 to u_2 , and since, by Claim 1.12, Z contains no P_4 , such a path has length 2, that is, Z contains a vertex x adjacent to both u_1, u_2 . Suppose that some neighbour $y \neq x$ of u_1 is not adjacent to x. Then y is also not adjacent to u_2 , for otherwise $\{y, u_1, x, u_2\}$ would induce a hole; and so $y-u_1-x-u_2$ is a P_4 . If $y \in Z$ this contradicts Claim 1.12, and if $y \in R$ then u_2 -x- u_1 -y-t is a P_5 , another contradiction. Therefore, x is adjacent to every neighbour of u_1 different from x, and similarly it is adjacent to every neighbour of u_2 different from x. By Claim 1.1, u_1 has neighbours v, v' that are not adjacent. Suppose that one of v, v', say v, is in R. Then, since R is a clique, v' is in Z, and, by the preceding argument, we have $x \neq v'$ and x is adjacent to v, v'. But then $\{v, u_1, v', x, t, u, t'\}$ induces an F_5 , a contradiction. Thus v, v' are both in Z. Likewise, u_2 has neighbours w, w' that are not adjacent, and they are both in Z. If u_2 is adjacent to v, then u_2, v, u_1, v' induce either a P_4 or a hole, a contradiction. Thus u_2 is not adjacent to v, and similarly not to v', and u_1 is not adjacent to any of w, w'. Then v is not adjacent to w, for otherwise u_1 -v-w- u_2 is a P_4 . Similarly, v is not adjacent to w', and v' is not adjacent to any of w, w'. But now $\{u, t, t', u_1, v, v', u_2, w, w'\}$ induces a $3P_3$ in G, a contradiction. So Claim 1.13 holds.

Let C_T be the set of colors that appear in U_T . By Claim 1.10, we have $|C_T| = |U_T| \ge 1$. Let C_Z be the set of colors that do not appear in $R \cup U_T$. By Claim 1.1, a member of U must be in a big component of T, and so, by Claims 1.4, 1.5 and 1.11, $R \cup U_T$ is a clique; thus $|C_Z| \ge 1$. Consider any color $j \in C_Z$. By the definition of U, every member of U_T must have a neighbour of color j, and by the definition of C_Z , any such neighbour

must be in T. Let w_j be one vertex of color j that is adjacent to the most members of U_T . So $w_j \in T$. Suppose that w_j has a non-neighbour u' in U_T . Let w'_j be a neighbour of u' of color j. So $w'_j \in T$. Since u' is adjacent to w'_j and not to w_j , the choice of w_j implies the existence of a vertex u'' of U_T that is adjacent to w_j and not to w'_j . But then w_j -u''-u'- w'_j is a P_4 , which contradicts Claim 1.9. Thus w_j is adjacent to all of U_T . Now $R \cup U_T \cup \{w_j\}$ is a clique, which implies $|C_Z| \geq 2$. Let $W = \{w_j \mid j \in C_Z\}$. Note that W is not a clique, for otherwise $R \cup U_T \cup W$ would be a clique of size k (because it contains a vertex of each color).

For each color $j \in C_Z$, the definition of C_Z implies that u_j is in Z. So

$$|U_Z| \ge |C_Z| \ge 2$$
.

Consider any color $h \in C_T$. By the definition of U, every member of U_Z must have a neighbour of color h, and by the definition of C_T and by Claim 1.5, any such neighbour must be in Z. Let y_h be one vertex of color h that is adjacent to the most members of U_Z . So $y_h \in Z$. Suppose that y_h has a non-neighbour u' in U_Z . Let y'_h be a neighbour of u' of color h. So $y'_h \in Z$. Since u' is adjacent to y'_h and not to y_h , the choice of y_h implies the existence of a vertex u'' of U_Z that is adjacent to y_h and not to y'_h . But then y_h -u''-u'- y'_h is a P_4 , which contradicts Claim 1.12. Thus y_h is adjacent to all of U_Z . Let $Y = \{y_h \mid h \in C_T\}$. So $|Y| = |C_T|$. Suppose that Y is not a clique. So there are non-adjacent vertices y_g, y_h in Y. Thus $|C_T| \ge 2$, and we have $u_g, u_h \in U_T$. Recall that W is not a clique, so it contains two non-adjacent vertices w_i, w_j , and by the definition of W we have $u_i, u_j \in U_T$. But then $\{y_g, y_h, u_i, u_j, w_i, w_j, u_g, u_h\}$ induces an F_6 , a contradiction. Thus Y is a clique, and so

$$Y \cup U_Z$$
 is a clique of size at least $|C_T| + |C_Z| \ge 3$.

Let R_1 be the set of vertices of R that have at most one neighbour in $Y \cup U_Z$, and let $R_2 = R \setminus R_1$. If some vertex $a \in R_2$ has a non-neighbour v in $Y \cup U_Z$, then, since a has two neighbours z, z' in $Y \cup U_Z$, we see that $\{a, z, z', v, t, u, t'\}$ induces an F_5 , a contradiction (recall that t-u-t' is a P_3 in T). Thus every vertex of R_2 is adjacent to every vertex of $Y \cup U_Z$. This implies $R_1 \neq \emptyset$, for otherwise $R \cup Y \cup U_Z$ would be a clique of size k (because it contains a vertex of each color).

Consider any color ℓ that appears in R_1 , and let a_{ℓ} be the vertex of R_1 of color ℓ . By the definition of U and R_1 , every vertex of U_Z , except

possibly one, must have a neighbour of color ℓ in Z. Let x_{ℓ} be one vertex of Z of color ℓ that is adjacent to the most members of U_Z . By the same argument as above concerning y_h , using the fact that Z contains no P_4 , we obtain that x_{ℓ} is adjacent to every vertex of U_Z that has a neighbour of color ℓ in Z. Now we show that x_{ℓ} is adjacent to all of $Y \cup U_Z$. Suppose on the contrary that x_{ℓ} has a non-neighbour v in $Y \cup U_Z$. If x_{ℓ} has two neighbours z, z' in $Y \cup U_Z$, then either t- a_{ℓ} -v-z- x_{ℓ} is a P_5 (if a_{ℓ} is adjacent to v), or $\{v, z, z', x_{\ell}, a_{\ell}, t, u, t'\}$ induces an F_6 or F_7 , a contradiction. So x_{ℓ} has only one neighbour z in $Y \cup U_Z$. By the definition of x_ℓ , this implies that $U_Z = \{z, z'\}$ where z' has no neighbour of color ℓ in T. Since z' is in U, it must have a neighbour of color ℓ , and this can only be a_{ℓ} . But then x_{ℓ} -z-z'- a_{ℓ} -t is a P_5 , a contradiction. Thus x_{ℓ} is adjacent to all of $Y \cup U_Z$. Now we show that x_{ℓ} is adjacent to all of R_2 . For suppose that x_{ℓ} is not adjacent to some vertex a of R_2 . Let z, z' be any two vertices in $Y \cup U_Z$. Then $\{x_{\ell}, z, z', a, t, u, t'\}$ induces an F_5 , a contradiction. In summary, x_{ℓ} is adjacent to all of $Y \cup U_Z \cup R_2$.

Let $X = \{x_{\ell} \mid \text{color } \ell \text{ appear in } R_1\}$. So $X \neq \emptyset$. Suppose that there are two non-adjacent vertices x_{ℓ}, x_m in X. Let a_{ℓ} be a vertex of color ℓ in R_1 . Let z, z' be any two vertices in $Y \cup U_Z$. Then a_{ℓ} is adjacent to x_m , for otherwise $\{x_{\ell}, x_m, z, z', a_{\ell}, t, u, t'\}$ induces an F_6 or F_7 . Then a_{ℓ} is adjacent to z', for otherwise x_{ℓ} -z'- x_m - a_{ℓ} -t is a P_5 . But then $\{x_m, z, z', a_{\ell}, t, u, t'\}$ induces an F_5 , a contradiction. Therefore X is a clique. But now, $X \cup Y \cup U_Z \cup R_2$ is a clique of size k (because it contains a vertex of each color), a contradiction. This completes the proof of the theorem. \square

Theorem 1 can be generalized slightly as follows.

Theorem 2. Every \mathcal{F} -free C_4 -free graph is b-perfect.

Proof. Let G be an \mathcal{F} -free C_4 -free graph. Since G contains no P_5 , it contains no hole C_k with $k \geq 6$. We prove that $b(G) = \chi(G)$ by induction on the number of C_5 's contained in G. If G contains no G_5 , then it is chordal and the result follows from Theorem 1. So we may now assume that G contains a G_5 . Let G_5 be five vertices such that, for G_5 is adjacent to G_6 be five vertices such that, for G_6 be a vertex of G_6 be a vertex of G_6 be a non-neighbour in G_6 . Let G_6 be a vertex of G_6 be that G_6 contains a set that induces either a G_6 or a G_6 or an G_6 a contradiction. Thus G_6 is adjacent to all of G_6 . Let G_6 be the set of vertices that are adjacent to G_6 . Note that G_6 is a clique, for if it contained two non-adjacent vertices G_6 , then G_6 is a clique, for if it contained two non-adjacent vertices G_6 , then it is a clique, for if

a C_4 . Suppose that G admits a b-coloring c with $k > \chi(G)$ colors. We may assume that the colors of c that appear in Z are $1, \ldots, \ell$, with $3 \leq \ell \leq 5$. So only the colors $\ell + 1, \ldots, k$ may appear in X.

If $\ell = 3$, let G' be the graph obtained from $G \setminus Z$ by adding three new vertices a_1, a_2, a_3 that are pairwise adjacent and all adjacent to all of X. If $\ell = 4$ or 5, let G' be the graph obtained from $G \setminus Z$ by adding ℓ new vertices a_1, \ldots, a_ℓ that are pairwise not adjacent and all adjacent to all of X. In either case, since X is a clique the new vertices a_1, \ldots, a_ℓ are simplicial, so they cannot belong to any hole, and so G' has strictly fewer C_5 's than G.

2.1.
$$b(G') \geq b(G)$$
.

Proof. Let c' be the coloring of the vertices of G' defined by c'(x) = c(x) if x is a vertex of $G \setminus Z$ and $c'(a_i) = i$ for $i = 1, \ldots, \ell$. Clearly, c' is a coloring with k colors. For each $i = 1, \ldots, k$, let u_i be a b-vertex of color i for c in G. Suppose that u_i is in $G \setminus Z$. Consider a neighbour v_j of u_i of color j in G for any $j \neq i$. Then either v_j is in $G \setminus Z = G' \setminus Z$, and in this case v_j is a neighbour of u_i of color j in G'; or v_j is in Z, and in this case $j \in \{1, \ldots, \ell\}$ and a_j is a neighbour of u_i of color j in G'. So u_i is a b-vertex for G'. Now suppose that u_i is in Z. Then u_i must have a neighbour of every color $1, \ldots, \ell$ different from i, and since such colors do not appear in X, they must appear in Z, and so $\ell = 3$ and all colors $4, \ldots, k$ appear in X. Then a_i is a b-vertex of color i in G'. Thus c' has a b-vertex of every color $i = 1, \ldots, k$. So Claim 2.1 holds.

2.2.
$$\chi(G') \leq \chi(G)$$
.

Proof. Consider any coloring γ of G with $\chi(G)$. We may assume that the colors of γ that appear in Z are $1, \ldots, h$, with $3 \leq h \leq 5$. Let γ' be defined as follows. For $x \in G \setminus Z$, set $\gamma'(x) = \gamma(x)$. If $\ell = 3$, set $\gamma'(a_i) = i$ for i = 1, 2, 3. If $\ell = 4$ or 5, set $\gamma'(a_i) = 1$ for $i = 1, \ldots, \ell$. In either case, γ' is a coloring of G' with at most $\chi(G)$ colors. So Claim 2.2 holds.

2.3. G' is \mathcal{F} -free and C_4 -free.

Proof. Suppose on the contrary that G' contains a subgraph F which is either a member of \mathcal{F} or a C_4 . Let $A = \{a_1, \ldots, a_\ell\}$. If F contains at most two vertices of A, then, since Z has two adjacent vertices and also two non-adjacent vertices, we can replace the vertices of $F \cap A$ by an appropriate choice of vertices of Z and we find a subgraph of G that is isomorphic to F, a contradiction. So F must contain at least three vertices of A. Note that in F, the neighbourhood of any of these vertices is equal to $F \cap X$, i.e., they

are pairwise twins. But this is impossible, because no member of $\mathcal{F} \cup \{C_4\}$ has three vertices that are pairwise twins. Thus Claim 2.3 holds.

By Claims 2.1–2.3, G' is an \mathcal{F} -free, C_4 -free graph with $b(G') \geq b(G) > \chi(G) \geq \chi(G')$ and G' has strictly fewer C_5 's than G, a contradiction. This completes the proof of Theorem 2. \square

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