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SYMMETRIC IDENTITIES FOR EULER POLYNOMIALS

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ABSTRACT. In this paper we establish two symmetric identities on sums of products of Euler polynomials.

1. Introduction

The Bernoulli numbers B_0, B_1, B_2, \ldots are rational numbers given by

$$B_0 = 1$$
, and $\sum_{k=0}^{n} {n+1 \choose k} B_k = 0$ for $n = 1, 2, 3, \dots$

The Euler numbers E_0, E_1, E_2, \ldots are integers determined by

$$E_0 = 1$$
, and $\sum_{\substack{k=0\\2|n-k}}^{n} \binom{n}{k} E_k = 0$ for $n = 1, 2, 3, \dots$

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. The Bernoulli polynomials $B_n(x)$ $(n \in \mathbb{N})$ and the Euler polynomials $E_n(x)$ $(n \in \mathbb{N})$ are defined by

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k} \text{ and } E_n(x) = \sum_{k=0}^n \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k}.$$

It is well known that

$$\Delta(B_n(x)) = nx^{n-1}$$
 and $\Delta^*(E_n(x)) = 2x^n$

for all $n \in \mathbb{N}$, where we set

$$\Delta(P(x)) = P(x+1) - P(x)$$
 and $\Delta^*(P(x)) = P(x+1) + P(x)$

for any polynomial P(x). Bernoulli and Euler numbers and polynomials play important roles in many fields including number theory and combinatorics.

In 2006 Z. W. Sun and H. Pan [6] established the following theorem which unifies many curious identities concerning Bernoulli and Euler numbers and polynomials.

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Theorem 1.1 (Sun and Pan, 2006). Let n be a positive integer and let x+y+z=1. (i) If r, s, t are complex numbers with r+s+t=n, then we have the symmetric relation

$$r\begin{bmatrix} s & t \\ x & y \end{bmatrix}_n + s\begin{bmatrix} t & r \\ y & z \end{bmatrix}_n + t\begin{bmatrix} r & s \\ z & x \end{bmatrix}_n = 0$$

where

$$\begin{bmatrix} s & t \\ x & y \end{bmatrix}_n := \sum_{k=0}^n (-1)^k \binom{s}{k} \binom{t}{n-k} B_{n-k}(x) B_k(y).$$

(ii) If r + s + t = n - 1, then

$$\frac{r}{2} \sum_{l=0}^{n-1} (-1)^l \binom{s}{l} \binom{t}{n-1-l} E_l(y) E_{n-1-l}(x)
= \sum_{k=0}^n (-1)^k \binom{r}{k} \binom{s}{n-k} B_k(x) E_{n-k}(z)
-(-1)^n \sum_{k=0}^n (-1)^k \binom{r}{k} \binom{t}{n-k} B_k(y) E_{n-k}(z).$$

Recently, by a sophisticated application of the generating function method, A. M. Fu, H. Pan and F. Zhang [2] extended Theorem 1.1(i) of Sun and Pan to an identity on sums of products of $m \geq 2$ Bernoulli polynomials.

In this paper we obtain a general identity only involving Euler polynomials and also give an extension of Theorem 1.1(ii) which involves both Bernoulli and Euler polynomials.

Theorem 1.2. Let m and n be positive integers, and let r_0, r_1, \ldots, r_m be complex numbers with $r_0 + r_1 + \cdots + r_m = n - 1$.

(i) If m is odd, then we have the symmetric relation

$$\sum_{\substack{k_1, \dots, k_m \ge 0 \\ k_1 + \dots + k_m = n}} \prod_{j=1}^m \binom{r_j}{k_j} E_{k_j}(x_j)$$

$$= -\sum_{i=1}^m (-1)^i \sum_{\substack{k_1, \dots, k_m \ge 0 \\ k_1 + \dots + k_m = n}} \binom{r_0}{k_i} E_{k_i}(1 - x_i) \prod_{\substack{1 \le j \le m \\ j \ne i}} \binom{r_j}{k_j} E_{k_j}(x_j - x_i + \mathbf{1}_{j>i}), \quad (1.1)$$

where $\mathbf{1}_{j>i}$ takes 1 or 0 according as j>i or not.

(ii) If m is even, then

$$\frac{r_0}{2} \sum_{\substack{k_1, \dots, k_m \ge 0 \\ k_1 + \dots + k_m = n-1}} \prod_{j=1}^m {r_j \choose k_j} E_{k_j}(x_j)$$

$$= \sum_{i=1}^m (-1)^i \sum_{\substack{k_1, \dots, k_m \ge 0 \\ k_1 + \dots + k_m = n}} {r_0 \choose k_i} B_{k_i}(1-x_i) \prod_{\substack{1 \le j \le m \\ j \ne i}} {r_j \choose k_j} E_{k_j}(x_j - x_i + \mathbf{1}_{j>i}). \tag{1.2}$$

Remark 1.1. If r + s + t = n - 1, then (1.2) in the case m = 2 gives

$$\frac{r}{2} \sum_{k=0}^{n-1} {s \choose k} E_k (1-y) {t \choose n-1-k} E_{n-1-k}(x)$$

$$= -\sum_{k=0}^{n} {r \choose k} B_k (1-(1-y)) {t \choose n-k} E_{n-k}(x-(1-y)+1)$$

$$+\sum_{k=0}^{n} {r \choose k} B_k (1-x) {s \choose n-k} E_{n-k}((1-y)-x)$$

$$= -(-1)^n \sum_{k=0}^{n} (-1)^k {t \choose n-k} E_{n-k}(1-x-y) {r \choose k} B_k(y)$$

$$+\sum_{k=0}^{n} (-1)^k {r \choose k} B_k(x) {s \choose n-k} E_{n-k}(1-x-y),$$

which is equivalent to the identity of Sun and Pan in Theorem 1.1(ii) since $E_k(1-x) = (-1)^k E_k(x)$.

Our proof of Theorem 1.2 given in the next section involves the difference operator Δ and its companion operator Δ^* . We can also show Theorem 1.2 via the generating function approach.

Let k be any nonnegative integer. It is well known that $B_k = 0$ if k is odd and greater than one. By [1, pp. 804-808],

$$B_k\left(\frac{1}{2}\right) = (2^{1-k} - 1)B_k \text{ and } E_k(x) = \frac{2}{k+1}\left(B_{k+1}(x) - 2^{k+1}B_{k+1}\left(\frac{x}{2}\right)\right).$$

Thus

$$(-1)^k E_k(1) = E_k(0) = 2(1 - 2^{k+1}) \frac{B_{k+1}}{k+1}.$$

In view of these, Theorem 1.2 in the case $x_1 = \cdots = x_m = 1/2$ yields the following consequence involving Euler numbers and Bernoulli numbers.

Corollary 1.1. Let m and n be positive integers, and let r_0, r_1, \ldots, r_m be complex numbers with $r_0 + r_1 + \cdots + r_m = n - 1$.

(i) If m is odd, then

$$(-1)^{n} \sum_{\substack{k_{1}, \dots, k_{m} \geq 0 \\ k_{1} + \dots + k_{m} = n}} \prod_{j=1}^{m} {r_{j} \choose k_{j}} E_{k_{j}}$$

$$= \sum_{i=1}^{m} (-1)^{i} \sum_{\substack{k_{1}, \dots, k_{m} \geq 0 \\ k_{1} + \dots + k_{m} = n}} (-1)^{|\{i < j \leq m: \ k_{j} > 0\}|} {r_{0} \choose k_{i}} E_{k_{i}} \prod_{\substack{1 \leq j \leq m \\ j \neq i}} {r_{j} \choose k_{j}} \tilde{B}_{k_{j}+1}, \qquad (1.3)$$

where $\tilde{B}_k = 2^k(2^k - 1)B_k/k$ for k = 1, 2, 3, ...

(ii) If m is even, then

$$(-1)^{n} r_{0} \sum_{\substack{k_{1}, \dots, k_{m} \geq 0 \\ k_{1} + \dots + k_{m} = n - 1}} \prod_{j=1}^{m} {r_{j} \choose k_{j}} E_{k_{j}}$$

$$= \sum_{i=1}^{m} (-1)^{i} \sum_{\substack{k_{1}, \dots, k_{m} \geq 0 \\ k_{1} + \dots + k_{m} = n}} (-1)^{|\{i < j \leq m: \ k_{j} > 0\}|} {r_{0} \choose k_{i}} (2^{k_{i}} - 2) B_{k_{i}} \prod_{\substack{1 \leq j \leq m \\ j \neq i}} {r_{j} \choose k_{j}} \tilde{B}_{k_{j}+1}. \quad (1.4)$$

2. Proof of Theorem 1.2

As usual we let \mathbb{C} denote the field of complex numbers. By [4, Lemma 3.1], for $P(x), Q(x) \in \mathbb{C}[x]$, we have P(x) = Q(x) if $\Delta^*(P(x)) = \Delta^*(Q(x))$. This property will play a central role in our proof of Theorem 1.2.

Lemma 2.1. Let $P_1(x), \dots, P_m(x) \in \mathbb{C}[x]$. Then

$$P_{1}(x) \sum_{1 < i \le m} (-1)^{i} \Delta^{*}(P_{i}(x)) \prod_{\substack{1 < j \le m \\ j \ne i}} P_{j}(x+1_{j < i})$$

$$= \begin{cases} \Delta^{*}(P_{1}(x) \cdots P_{m}(x)) - \Delta^{*}(P_{1}(x)) P_{2}(x+1) \cdots P_{m}(x+1) & \text{if } 2 \nmid m, \\ \Delta^{*}(P_{1}(x) \cdots P_{m}(x)) - \Delta(P_{1}(x)) P_{2}(x+1) \cdots P_{m}(x+1) & \text{if } 2 \mid m. \end{cases}$$

Proof. Observe that

$$\sum_{1 < i \le m} (-1)^i \Delta^*(P_i(x)) \prod_{\substack{1 < j \le m \\ j \ne i}} P_j(x + \mathbf{1}_{j < i})$$

$$= \sum_{1 < i \le m} \left((-1)^i \prod_{\substack{1 < j \le m \\ 1 < j \le m}} P_j(x + \mathbf{1}_{j < i}) - (-1)^{i+1} \prod_{\substack{1 < j \le m \\ 1 < j \le m}} P_j(x + \mathbf{1}_{j < i+1}) \right)$$

$$= (-1)^2 \prod_{\substack{1 < j \le m \\ 1 < j \le m}} P_j(x) - (-1)^{m+1} \prod_{\substack{1 < j \le m \\ 1 < j \le m}} P_j(x + 1).$$

Therefore

$$P_{1}(x) \sum_{1 < i \le m} (-1)^{i} \Delta^{*}(P_{i}(x)) \prod_{\substack{1 < j \le m \\ j \ne i}} P_{j}(x + \mathbf{1}_{j < i})$$

$$= P_{1}(x) \cdots P_{m}(x) + (-1)^{m} P_{1}(x) \prod_{1 < j \le m} P_{j}(x)$$

$$= \Delta^{*}(P_{1}(x) \cdots P_{m}(x)) - (P_{1}(x + 1) + (-1)^{m-1} P_{1}(x)) \prod_{1 < j \le m} P_{j}(x).$$

This proves the desired identity.

Lemma 2.2. Let $a_0, \bar{a}_0, a_1, \bar{a}_1, \dots, a_n, \bar{a}_n$ be complex numbers, and set

$$A_k(t) = \sum_{l=0}^k \binom{k}{l} (-1)^l a_l t^{k-l}$$
 and $\bar{A}_k(t) = \sum_{l=0}^k \binom{k}{l} (-1)^l \bar{a}_l t^{k-l}$

for k = 0, ..., n. Let $r_0 + r_1 + \cdots + r_m = n - 1$. Then

$$\sum_{\substack{k_1, \dots, k_m \ge 0 \\ k_1 + \dots + k_m = n}} {r_0 \choose k_1} (-x_1)^{k_1} \prod_{j=2}^m {r_j \choose k_j} A_{k_j} (x_j - x_1)$$

$$= \sum_{\substack{k_1, \dots, k_m \ge 0 \\ k_1 + \dots + k_m = n}} {r_1 \choose k_1} x_1^{k_1} \prod_{j=2}^m {r_j \choose k_j} A_{k_j} (x_j). \tag{2.1}$$

Also, for any i = 2, ..., m we have

$$\sum_{\substack{k_1,\dots,k_m\geq 0\\k_1+\dots+k_m=n}} {r_0 \choose k_1} A_{k_1} (-x_1) {r_i \choose k_i} (x_i - x_1)^{k_i} \prod_{\substack{2\leq j\leq m\\j\neq i}} {r_j \choose k_j} \bar{A}_{k_j} (x_j - x_1)$$

$$= \sum_{\substack{k_1,\dots,k_m\geq 0\\k_1+\dots+k_m=n}} {r_1 \choose k_1} (x_1 - x_i)^{k_1} {r_0 \choose k_i} A_{k_i} (-x_i) \prod_{\substack{2\leq j\leq m\\j\neq i}} {r_j \choose k_j} \bar{A}_{k_j} (x_j - x_i). \tag{2.2}$$

Proof. By Remark 1.1 of Sun [5],

$$A_k(x+y) = \sum_{l=0}^k {k \choose l} x^{k-l} A_l(y)$$
 and $\bar{A}_k(x+y) = \sum_{l=0}^k {k \choose l} x^{k-l} \bar{A}_l(y)$

for every $k = 0, \ldots, n$. Observe that

$$\begin{split} & \sum_{\substack{k_1,\dots,k_m\geq 0\\k_1+\dots+k_m=n}} \binom{r_0}{k_1} (-x_1)^{k_1} \prod_{j=2}^m \binom{r_j}{k_j} A_{k_j} (x_j-x_1) \\ &= \sum_{\substack{k_1,\dots,k_m\geq 0\\k_1+\dots+k_m=n}} \binom{r_0}{k_1} (-x_1)^{k_1} \prod_{j=2}^m \binom{r_j}{k_j} \sum_{l_j=0}^{k_j} \binom{k_j}{l_j} (-x_1)^{k_j-l_j} A_{l_j} (x_j) \\ &= \sum_{\substack{l_1,\dots,l_m\geq 0\\l_1+\dots+l_m=n}} (-x_1)^{l_1} \prod_{j=2}^m \binom{r_j}{l_j} A_{l_j} (x_j) \sum_{\substack{k_1\geq 0,\ k_j\geq l_j\ (1< j\leq m)\\k_1+\dots+k_m=n}} \binom{r_0}{k_1} \prod_{j=2}^m \binom{r_j-l_j}{k_j-l_j}. \end{split}$$

Given $l_1, \ldots, l_m \in \mathbb{N}$ with $l_1 + \cdots + l_m = n$, by the Chu-Vandermonde convolution identity (cf. [3, (5.22)]), we have

$$\sum_{\substack{k_1 \ge 0, k_j \ge l_j \ (1 < j \le m) \\ k_1 + \dots + k_m = n}} {r_0 \choose k_1} \prod_{j=2}^m {r_j - l_j \choose k_j - l_j}$$

$$= {r_0 + (r_2 - l_2) + \dots + (r_m - l_m) \choose n - l_2 - \dots - l_m} = {l_1 - 1 - r_1 \choose l_1} = (-1)^{l_1} {r_1 \choose l_1}.$$

So (2.1) follows.

(2.2) can be proved similarly. Let Σ denote the left-hand side of (2.2). Then

$$\begin{split} \Sigma &= \sum_{\substack{k_1, \dots, k_m \geq 0 \\ k_1 + \dots + k_m = n}} \binom{r_0}{k_1} \sum_{l_i = 0}^{k_1} \binom{k_1}{l_i} (x_i - x_1)^{k_1 - l_i} A_{l_i} (-x_i) \binom{r_i}{k_i} (x_i - x_1)^{k_i} \\ &\times \prod_{\substack{1 < j \leq m \\ j \neq i}} \binom{r_j}{k_j} \sum_{l_j = 0}^{k_j} \binom{k_j}{l_j} (x_i - x_1)^{k_j - l_j} \bar{A}_{l_j} (x_j - x_i) \\ &= \sum_{\substack{l_1, \dots, l_m \geq 0 \\ l_1 + \dots + l_m = n}} (x_i - x_1)^{l_1} \binom{r_0}{l_i} A_{l_i} (-x_i) \prod_{\substack{1 < j \leq m \\ j \neq i}} \binom{r_j}{l_j} \bar{A}_{l_j} (x_j - x_i) \\ &\times \sum_{\substack{k_j \geq l_j \ (1 \leq j \leq m \ \& \ j \neq i) \\ k_i \geq 0, \ k_1 + \dots + k_m = n}} \binom{r_0 - l_i}{k_1 - l_i} \binom{r_i}{k_i} \prod_{\substack{1 < j \leq m \\ j \neq i}} \binom{r_j - l_j}{k_j - l_j} \\ &= \sum_{\substack{l_1, \dots, l_m \geq 0 \\ l_1 + \dots + l_m = n}} (x_i - x_1)^{l_1} \binom{r_0}{l_i} A_{l_i} (-x_i) \prod_{\substack{1 < j \leq m \\ j \neq i}} \binom{r_j}{l_j} \bar{A}_{l_j} (x_j - x_i) \times (-1)^{l_1} \binom{r_1}{l_1}. \end{split}$$

This concludes the proof.

Remark 2.1. If we set $a_l = (-1)^l B_l$ and $\bar{a}_l = (-1)^l E_l(0)$ for $l = 0, \ldots, n$ in Lemma 2.2, then $A_k(t) = B_k(t)$ and $\bar{A}_k(t) = E_k(t)$ for any $k = 0, \ldots, n$.

Proof of Theorem 1.2. We fix x_2, \ldots, x_m .

(i) Suppose that m is odd. Set

$$P(x_1) = \sum_{\substack{k_1, \dots, k_m \ge 0 \\ k_1 + \dots + k_n = n}} \binom{r_0}{k_1} E_{k_1} (1 - x_1) \prod_{j=2}^m \binom{r_j}{k_j} E_{k_j} (x_j - x_1 + 1).$$

Applying Lemma 2.1, we get

$$\Delta^{*}(P(x_{1})) = \sum_{i=2}^{m} (-1)^{i} \sum_{\substack{k_{1}, \dots, k_{m} \geq 0 \\ k_{1} + \dots + k_{m} = n}} {r_{0} \choose k_{1}} E_{k_{1}} (1 - x_{1}) {r_{i} \choose k_{i}} 2(x_{i} - x_{1})^{k_{i}} \prod_{\substack{2 \leq j \leq m \\ j \neq i}} {r_{j} \choose k_{j}} E_{k_{j}} (x_{j} - x_{1} + \mathbf{1}_{j>i})$$

$$+ \sum_{\substack{k_{1}, \dots, k_{m} \geq 0 \\ k_{1} + \dots + k_{m} = n}} {r_{0} \choose k_{1}} 2(-x_{1})^{k_{1}} \prod_{j=2}^{m} {r_{j} \choose k_{j}} E_{k_{j}} (x_{j} - x_{1}).$$

With the help of Lemma 2.2, we have

$$\Delta^{*}(P(x_{1}))$$

$$=2\sum_{i=2}^{m}(-1)^{i}\sum_{\substack{k_{1},\ldots,k_{m}\geq 0\\k_{1}+\cdots+k_{m}=n}}\binom{r_{1}}{k_{1}}(x_{1}-x_{i})^{k_{1}}\binom{r_{0}}{k_{i}}E_{k_{i}}(1-x_{i})\prod_{\substack{2\leq j\leq m\\j\neq i}}\binom{r_{j}}{k_{j}}E_{k_{j}}(x_{j}-x_{i}+\mathbf{1}_{j>i})$$

$$+2\sum_{\substack{k_{1},\ldots,k_{m}\geq 0\\k_{1}+\cdots+k_{m}=n}}\binom{r_{1}}{k_{1}}x_{1}^{k_{1}}\prod_{j=2}^{m}\binom{r_{j}}{k_{j}}E_{k_{j}}(x_{j}).$$

It follows that $\Delta^*(P(x_1)) = \Delta^*(Q(x_1))$, where

$$Q(x_1) = \sum_{1 < i \le m} (-1)^i \sum_{\substack{k_1, \dots, k_m \ge 0 \\ k_1 + \dots + k_m = n}} \binom{r_0}{k_i} E_{k_i} (1 - x_i) \prod_{\substack{1 \le j \le m \\ j \ne i}} \binom{r_j}{k_j} E_{k_j} (x_j - x_i + \mathbf{1}_{j>i})$$

$$+ \sum_{\substack{k_1, \dots, k_m \ge 0 \\ k_1 + \dots + k_m = n}} \prod_{j=1}^m \binom{r_j}{k_j} E_{k_j} (x_j).$$

Therefore $P(x_1) = Q(x_1)$ by [4, Lemma 3.1]. This proves (1.1).

(ii) Now assume that m is even. Define

$$P(x_1) = \sum_{\substack{k_1, \dots, k_m \ge 0 \\ k_1 + \dots + k_n = n}} {r_0 \choose k_1} B_{k_1} (1 - x_1) \prod_{j=2}^m {r_j \choose k_j} E_{k_j} (x_j - x_1 + 1).$$

For $k_1 = 0, 1, 2, ...$, clearly

$$\binom{r_0}{k_1}(B_{k_1}(1-(x_1+1))-B_{k_1}(1-x_1))=-\binom{r_0}{k_1}k_1(-x_1)^{k_1-1}=-r_0\binom{r_0-1}{k_1-1}(-x_1)^{k_1-1}.$$

(As usual $\binom{x}{-1}$ is regarded as 0.) Thus, by Lemma 2.1 we have

$$\Delta^*(P(x_1))$$

$$=2\sum_{i=2}^{m}(-1)^{i}\sum_{\substack{k_{1},\dots,k_{m}\geq 0\\k_{1}+\dots+k_{m}=n}} \binom{r_{0}}{k_{1}}B_{k_{1}}(1-x_{1})\binom{r_{i}}{k_{i}}(x_{i}-x_{1})^{k_{i}}\prod_{\substack{1< j\leq m\\j\neq i}} \binom{r_{j}}{k_{j}}E_{k_{j}}(x_{j}-x_{1}+\mathbf{1}_{j>i})$$

$$-r_{0}\sum_{\substack{k_{1},\dots,k_{m}\geq 0\\k_{1}+\dots+k_{m}=n-1}} \binom{r_{0}-1}{k_{1}}(-x_{1})^{k_{1}}\prod_{1< j\leq m} \binom{r_{j}}{k_{j}}E_{k_{j}}(x_{j}-x_{1}).$$

With the help of Lemma 2.2,

$$\Delta^*(P(x_1))$$

$$=2\sum_{i=2}^{m}(-1)^{i}\sum_{\substack{k_{1},\dots,k_{m}\geq 0\\k_{1}+\dots+k_{m}=n}}\binom{r_{1}}{k_{1}}(x_{1}-x_{i})^{k_{1}}\binom{r_{0}}{k_{i}}B_{k_{i}}(1-x_{i})\prod_{\substack{1< j\leq m\\j\neq i}}\binom{r_{j}}{k_{j}}E_{k_{j}}(x_{j}-x_{i}+\mathbf{1}_{j>i})$$

$$-r_{0}\sum_{\substack{k_{1},\dots,k_{m}\geq 0\\k_{j},\dots,k_{m}\geq n}}\binom{r_{1}}{k_{1}}x_{1}^{k_{1}}\prod_{\substack{1< j\leq m}}\binom{r_{j}}{k_{j}}E_{k_{j}}(x_{j}).$$

So we have $\Delta^*(P(x_1)) = \Delta^*(Q(x_1))$, where

$$Q(x_1) = \sum_{i=2}^{m} (-1)^i \sum_{\substack{k_1, \dots, k_m \ge 0 \\ k_1 + \dots + k_m = n}} {r_0 \choose k_i} B_{k_i} (1 - x_i) \prod_{\substack{1 \le j \le m \\ j \ne i}} {r_j \choose k_j} E_{k_j} (x_j - x_i + \mathbf{1}_{j>i})$$
$$- \frac{r_0}{2} \sum_{\substack{k_1, \dots, k_m \ge 0 \\ k_1 + \dots + k_m = m-1}} \prod_{j=1}^{m} {r_j \choose k_j} E_{k_j} (x_j).$$

Therefore, $P(x_1)$ coincides with $Q(x_1)$ by [4, Lemma 3.1]. So (1.2) holds. This concludes the proof.

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